VIP Cheatsheet: Second-order ODE

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General case

❒ **General form** – The general form of a second-order ODE can be written as a function F of x, y, y' and y'' as follows:

$$\boxed{F(x,y,y',y'')=0}$$

❒ **Methods of resolution** – The table below summarizes the general tricks to apply when the ODE has the following classic forms:

Old form	Trick	New form
$F\left(x,y',y''\right)=0$	$y'\triangleq u,\quad y''=\frac{du}{dx}$	$G\left(x,u,\frac{du}{dx}\right)=0$
$F\left(y,y',y''\right)=0$	$y'\triangleq u,\quad y''=u\frac{du}{dy}$	$G\left(y,u,\frac{du}{dy}\right)=0$
$F\left(y',y''\right)=0$	$y'\triangleq u,\quad y''=\frac{du}{dx}$ $y'\triangleq u,\quad y''=u\frac{du}{dy}$	Missing-y approach $G\left(u,\frac{du}{dx}\right)=0$ Missing-x approach $G\left(u,\frac{du}{dy}\right)=0$

❒ **Standard form of a linear ODE** – The standard form of a second-order linear ODE is expressed with p, q and r known functions of x such that:

$$\boxed{y''+p(x)y'+q(x)y=r(x)}$$

for which the total solution y is the sum of a homogeneous solution y_h and a particular solution y_p:

$$\boxed{y=y_h+y_p}$$

Remark: if $r=0$, then the ODE is homogeneous (and we have $y_p=0$). If $r\neq 0$, then the ODE is said to be inhomogeneous.

❒ **Linear dependency** – Two functions y_1, y_2 are said to be *linearly dependent* if $\frac{y_2}{y_1}=C$ constant. Conversely, they are *linearly independent* if $\frac{y_2}{y_1}\neq C$.

Linear homogeneous – Variable coefficients

❒ **Method of reduction of order** – Let y_1 be a solution to the equation $y''+p(x)y'+q(x)y=0$. By noting C_1, C_2 constants, the global solution y_h is written as:

$$\boxed{y_h=C_1y_1+C_2y_1\int\frac{e^{-\int pdx}}{y_1^2}dx}$$

Remark: Here, for any function p, the notation $\int pdx$ denotes the primitive of p without additive constant.

Linear homogeneous – Constant coefficients

❒ **General form** – The general form of a linear homogeneous second-order ODE with a,b,c constant coefficients is:

$$\boxed{ay''+by'+cy=0}$$

❒ **Resolution** – Based on the types of solution of the characteristic equation $\boxed{a\lambda^2+b\lambda+c=0}$, and by noting $\boxed{\Delta=b^2-4ac}$ its discriminant, we distinguish the following cases:

Name	Case	Roots	Solution
Two distinct real roots	$\Delta>0$	$\lambda_1=\frac{-b+\sqrt{\Delta}}{2a}$ $\lambda_2=\frac{-b-\sqrt{\Delta}}{2a}$	$y_h=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$
Double real root	$\Delta=0$	$\lambda=-\frac{b}{2a}$	$y_h=[C_1+C_2x]e^{\lambda x}$
Complex conjugate roots	$\Delta<0$	$\lambda_1=\alpha+i\beta$ $\lambda_2=\alpha-i\beta$ where $\alpha=-\frac{b}{2a}$ and $\beta=\frac{\sqrt{\lvert\Delta\rvert}}{2a}$	$y_h=\left[C_1\cos(\beta x)+C_2\sin(\beta x)\right]e^{\alpha x}$

A special case: the Euler-Cauchy equation

❒ **General form** – The Euler-Cauchy equation is a special case of linear homogeneous ODEs and has the following general form, where each $a_i\in\mathbb{R}$ is a constant coefficient:

$$\boxed{a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+...+a_1xy'+a_0y=0}$$

❒ **Second-order case** – For $n=2$, by noting $y=x^m$, the ODE provides the indicial equation:

$$\boxed{am^2+(b-a)m+c=0}$$

with discriminant $\boxed{\Delta=(b-a)^2-4ac}$ and where the resolution of the ODE depends on the cases summarized in the table below.

Name	Case	Roots	Solution
Two distinct real roots	$\Delta > 0$	$m_1 = \dfrac{-b+a+\sqrt{\Delta}}{2a}$ $m_2 = \dfrac{-b+a-\sqrt{\Delta}}{2a}$	$y_h = C_1 x^{m_1} + C_2 x^{m_2}$
Double real root	$\Delta = 0$	$m = -\dfrac{b-a}{2a}$	$y_h = [C_1 + C_2 \ln\|x\|]x^m$
Complex conjugate roots	$\Delta < 0$	$m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$ where $\alpha = -\frac{b-a}{2a}$ and $\beta = \frac{\sqrt{\|\Delta\|}}{2a}$	$y_h = \Big[C_1 \cos(\beta \ln\|x\|)$ $+C_2 \sin(\beta \ln\|x\|)\Big] x^\alpha$

Linear inhomogeneous – Variable coefficients

❒ **Wronskian** – Given y_1 and y_2 the two solutions of the homogeneous equation, we define the Wronskian W as follows:

$$\boxed{W = y_1 y_2' - y_2 y_1'}$$

❒ **Method of Variation of Parameters** – The particular solution y_p of the inhomogeneous ODE is given by:

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

Linear inhomogeneous – Constant coefficients

❒ **Undetermined coefficients method** – The particular solution y_p of the inhomogeneous ODE $ay'' + by' + cy = r(x)$ is determined from the correspondance table below:

Form of r	**Form of y_p**
C	A
x^n, $n \in \mathbb{N}^*$	$A_0 + A_1 x + ... + A_n x^n$
$e^{\gamma(x)}$	$Ae^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$x^n e^{\gamma x}\cos(\omega x)$ or $x^n e^{\gamma x}\sin(\omega x)$	$(A_0 + A_1 x + ... + A_n x^n)\cos(\omega x)e^{\gamma x} +$ $(B_0 + B_1 x + ... + B_n x^n)\sin(\omega x)e^{\gamma x}$

Remark: all new constants are determined after plugging back y_p into the ODE.

❒ **Modification rule** – If the particular solution y_p picked from the above table matches either y_1 or y_2, then has to be multiplied by the lowest power of x such that it is no more the case.

❒ **Sum rule** – If $r(x)$ is a sum of functions of the first column of the above table, then y_p is the sum of its associated particular solutions.