# VIP Cheatsheet: Applications

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September 8, 2020

## Physics Laws

❒ **Gravitational force** – A mass $m$ is subject to the gravitational force $\vec{F_g}$, which is expressed with respect to $\vec{g}$ of magnitude 9.81 m $\cdot$ s$^{-2}$ and directed towards the center of the Earth, as follows:

$$\boxed{\vec{F_g} = m\vec{g}}$$

❒ **Spring force** – A spring of constant $k$ and of relaxed position $\vec{x}_0$ attached a mass $m$ of position $\vec{x}$ has a force $\vec{F_s}$ expressed as follows:

$$\boxed{\vec{F_s} = -k(\vec{x} - \vec{x}_0)}$$

❒ **Friction force** – The friction force $F_f$ of constant coefficient $\beta$ applied on a mass of velocity $\vec{v}$ is written as:

$$\boxed{\vec{F_f} = -\beta\vec{v}}$$

❒ **Mass moment of inertia** – The mass moment of inertia of a system of mass $m_i$ located at distance $r_i$ from point $O$, expressed in point $O$ is written as:

$$\boxed{J_0 = \sum_i m_i r_i^2}$$

❒ **Torque** – The torque $\vec{T}$ of a force $\vec{F}$ located at $\vec{r}$ from the reference point $O$ is written as:

$$\boxed{\vec{T} = \vec{r} \times \vec{F}}$$

❒ **Newton's second law** – A mass $m$ of acceleration $\vec{a}$ to which forces $\vec{F_i}$ are applied verifies the following equation:

$$\boxed{m\vec{a} = \sum_i \vec{F_i}}$$

– In the 1-D case along the $x$ axis, we can write it as $mx'' = \sum_i F_i$.

– In the rotationary case, around point $O$, we can write it as $J_0\theta'' = \sum_i T_i$.

## Spring-mass system

❒ **Free undamped motion** – A free undamped spring-mass system of mass $m$ and spring coefficient $k$ follows the ODE $x'' + \frac{k}{m}x = 0$, which can be written as a function of the natural frequency $\omega$ as:

$$\boxed{x'' + \omega^2 x = 0} \quad \text{with} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

❒ **Free damped motion** – A free damped spring-mass system of mass $m$, of spring coefficient $k$ and subject to a friction force of coefficient $\beta$ follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$, which can be written as a function of the damping parameter $\lambda$ and the natural frequency $\omega$ as:

$$\boxed{x'' + 2\lambda x' + \omega^2 x = 0} \quad \text{with} \quad \boxed{\lambda = \frac{\beta}{2m}} \quad \text{and} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

which has the following cases summed up in the table below:

| Condition | Type of motion |
|---|---|
| $\lambda > \omega$ | Over damped |
| $\lambda = \omega$ | Critically damped |
| $\lambda < \omega$ | Under damped |

❒ **Forcing frequency** – A forcing function $F(t)$ is often modeled with a periodic function of the form $F(t) = F_0 \sin(\gamma t)$, where $\gamma$ is called the forcing frequency.

❒ **Forced undamped motion** – A forced undamped spring-mass system of mass $m$ and spring coefficient $k$ follows the ODE $x'' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the natural frequency $\omega$ as:

$$\boxed{x'' + \omega^2 x = F_0 \sin(\gamma t)} \quad \text{with} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

which has the following cases summed up in the table below:

| Condition | Type of motion |
|---|---|
| $\gamma \neq \omega$ | General response |
| $\gamma \approx \omega$ | Beats |
| $\gamma = \omega$ | Resonance |

❒ **Forced damped motion** – A forced damped spring-mass system of mass $m$, of spring coefficient $k$ and subject to a friction force of coefficient $\beta$ follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the damping parameter $\lambda$ and the natural frequency $\omega$ as:

$$\boxed{x'' + 2\lambda x' + \omega^2 x = F_0 \sin(\gamma t)} \quad \text{with} \quad \boxed{\lambda = \frac{\beta}{2m}} \quad \text{and} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

## Boundary Value Problems

❒ **Types of boundary conditions** – Given a numerical problem between 0 and $L$, we distinguish the following types of boundary conditions:

| Name | Boundary values |
|---|---|
| Dirichlet | $y(0)$ and $y(L)$ |
| Neumann | $y(0)$ and $y'(L)$ |
| Robin | $y(0)$ and $\alpha y(L)+\beta y'(L)$ |

❒ **Numerical differentiation** – The table below sums up the approximation of the derivatives of $y$ at point $x_j$, knowing the values of $y$ at each point of a uniformly spaced set of grid points.

| Order of derivative | Name | Formula | Order of error |
|---|---|---|---|
| First derivative | Forward difference | $y'_j=\dfrac{y_{j+1}-y_j}{h}$ | $O(h)$ |
| | Backward difference | $y'_j=\dfrac{y_j-y_{j-1}}{h}$ | $O(h)$ |
| | Central difference | $y'_j=\dfrac{y_{j+1}-y_{j-1}}{2h}$ | $O(h^2)$ |
| Second derivative | Central difference | $y''_j=\dfrac{y_{j+1}-2y_j+y_{j-1}}{h^2}$ | $O(h^2)$ |

❒ **Direct method** – The direct method can solve linear ODEs by reducing the problem to the resolution of a linear system $Ay=f$, where $A$ is a tridiagonal matrix.

❒ **Shooting method** – The shooting method is an algorithm that can solve ODEs through an iterative process. It uses a numerical scheme, such as Runge-Kutta, and converges to the right solution by iteratively searching for the missing initial condition $y'(0)$.

*Remark: in the linear case, the shooting method converges after the first two initial guesses.*