VIP Cheatsheet: First-order ODE

Afshine Amidi and Shervine Amidi

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Introduction

❒ **Differential Equations** – A differential equation is an equation containing derivatives of a dependent variable y with respect to independent variables x. In particular,

- Ordinary Differential Equations (ODE) are differential equations having one independent variable.
- Partial Differential Equations (PDE) are differential equations having two or more independent variables.

❒ **Order** - An ODE is said to be of order n if the highest derivative of the unknown function in the equation is the n^{th} derivative with respect to the independent variable.

❒ **Linearity** – An ODE is said to be linear only if the function y and all of its derivatives appear by themselves. Thus, it is of the form:

$$\boxed{a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y+b(x)=0}$$

Direction Field Method

❒ **Implicit form** – The implicit form of an ODE is where y' is not separated from the remaining terms of the ODE. It is of the form:

$$\boxed{F(x,y,y')=0}$$

Remark: Sometimes, y' cannot be separated from the other terms and the implicit form is the only one that we can write.

❒ **Explicit form** – The explicit form of an ODE is where y' is separated from the remaining terms of the ODE. It is of the form:

$$\boxed{y'=f(x,y)}$$

❒ **Direction field method** – The direction field method is a graphical representation for the solution of ODE $y'=f(x,y)$ without actually solving for $y(x)$. Here is the procedure:

- Determine the values (x_i,y_i) that form the grid.
- Compute the slope $f(x_i,y_i)$ for each point of the grid.
- Report the associated vector for each point of the grid.

Separation of variables

❒ **Separable** – An ODE is said to be separable if it can be written in the form:

$$\boxed{f(x,y)=g(x)h(y)}$$

❒ **Reduction to separable form** – The following table sums up the variable changes that allow us to change the ODE $y'=f(x,y)$ to $u'=g(x,u)$ that is separable.

Original form	Change of variables	New form
$y'=f\left(\frac{y}{x}\right)$	$u\triangleq\frac{y}{x}$	$u'x+u=f(u)$
$y'=f\left(ax+by+c\right)$	$u\triangleq ax+by+c$	$\frac{u'-a}{b}=f(u)$

Equilibrium

❒ **Characterization** – In order for an ODE to have equilibrium solutions, it must be (1) autonomous and (2) have a value y^* that makes the derivative equal to 0, i.e:

$$(1)\quad\boxed{\frac{dy}{dt}=f(\cancel{x},y)=f(y)}\quad\text{and}\quad(2)\quad\boxed{\exists\ y^*,\ \frac{dy^*}{dt}=f(y^*)=0}$$

❒ **Stability** – Equilibrium solutions can be classified into 3 categories:

- Unstable: solutions run away with any small change to the initial conditions.
- Stable: any small perturbation leads the solutions back to that solution.
- Semi-stable: a small perturbation is stable on one side and unstable on the other.

Linear first-order ODE technique

❒ **Standard form** – The standard form of a first-order linear ODE is expressed with $p(x),r(x)$ known functions of x, such that:

$$\boxed{y'+p(x)y=r(x)}$$

Remark: If $r=0$, then the ODE is homogenous, and if $r\neq 0$, then the ODE is inhomogeneous.

❒ **General solution** – The general solution y of the standard form can be decomposed into a homogenous part y_h and a particular part y_p and is expressed in terms of $p(x),r(x)$ such that:

$$\boxed{y=y_h+y_p}\quad\text{with}\quad\boxed{y_h=Ce^{-\int pdx}}\quad\text{and}\quad\boxed{y_p=e^{-\int pdx}\times\int\left[re^{\int pdx}\right]dx}$$

Remarks: Here, for any function p, the notation $\int pdx$ denotes the primitive of p without additive constant. Also, the term $e^{-\int pdx}$ is called the basis of the ODE and $e^{\int pdx}$ is called the integrating factor.

❒ **Reduction to linear form** – The one-line table below sums up the change of variables that we apply in order to have a linear form:

Name, setting	Original form	Change	New form
Bernoulli, $n\in\mathbb{R}\backslash\{0,1\}$	$y'+p(x)y=q(x)y^n$	$u\triangleq y^{1-n}$	$u'+(1-n)p(x)u=(1-n)q(x)$

Existence and uniqueness of an ODE

Here, we are given an ODE $y' = f(x,y)$ with initial conditions $y(x_0) = y_0$.

❒ **Existence theorem** – If $f(x,y)$ is continuous at all points in a rectangular region containing (x_0,y_0), then $y' = f(x,y)$ has at least one solution $y(x)$ passing through (x_0,y_0).
Remark: If the condition does not apply, then we cannot say anything about existence.

❒ **Uniqueness theorem** – If both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous at all points in a rectangular region containing (x_0,y_0), then $y' = f(x,y)$ has a unique solution $y(x)$ passing through (x_0,y_0).
Remark: If the condition does not apply, then we cannot say anything about uniqueness.

Numerical methods for ODE - Initial value problems

In this section, we would like to find $y(t)$ for the interval $[0,t_f]$ that we divide into $N+1$ equally-spaced points $t_0 < t_1 < ... < t_N = t_f$, such that:

$$\frac{dy}{dt} = f(t,y) \quad \textrm{with} \quad y(0) = y_0$$

❒ **Error** – In order to assess the accuracy of a numerical method, we define its local and global errors $\epsilon_{\textrm{local}}, \epsilon_{\textrm{global}}$ as follows:

$$\boxed{\epsilon_{\textrm{local}} = |y^{\textrm{exact}}(t_n) - y^{\textrm{numerical}}(t_n)|} \quad \textrm{and} \quad \boxed{\epsilon_{\textrm{global}} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}|y^{\textrm{exact}}(t_n) - y^{\textrm{numerical}}(t_n)|^2}}$$

Remarks: If $\epsilon_{local} = O(h^k)$, then $\epsilon_{global} = O(h^{k-1})$. Also, when we talk about the 'error' of a method, we refer to its global error.

❒ **Taylor series** – The Taylor series giving the exact expression of y_{n+1} in terms of y_n and its derivatives is:

$$\boxed{y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + ... = \sum_{k=0}^{+\infty}\frac{h^k}{k!}y_n^{(k)}}$$

We can also have an expression of y_n in terms of y_{n+1} and its derivatives:

$$\boxed{y_n = y_{n+1} - hy_{n+1}' + \frac{h^2}{2}y_{n+1}'' - \frac{h^3}{6}y_{n+1}''' + ... = \sum_{k=0}^{+\infty}\frac{(-h)^k}{k!}y_{n+1}^{(k)}}$$

❒ **Stability** – The stability analysis of any ODE solver algorithm is performed on the model problem, defined by:

$$\boxed{y' = \lambda y \quad \textrm{with} \quad y(0) = y_0 \quad \textrm{and} \quad \lambda < 0}$$

which gives $y_n = y_0\sigma^n$, for which h verifies the condition $|\sigma(h)| < 1$.

❒ **Euler methods** – The Euler methods are numerical methods that aim at estimating the solution of an ODE:

Type	Update formula	Error	Stability condition
Forward Euler	$y_{n+1} = y_n + hf(t_n,y_n)$	$O(h)$	$h < \frac{2}{\lvert\lambda\rvert}$
Backward Euler	$y_{n+1} = y_n + hf(t_{n+1},y_{n+1})$	$O(h)$	None

❒ **Runge-Kutta methods** – The table below sums up the most commonly used Runge-Kutta methods:

Type	Method	Update formula	Error	Stability condition
RK1	Euler's	$y_{n+1} = y_n + hk_1$ where $k_1 = f(t_n,y_n)$	$O(h)$	$h < \frac{2}{\lvert\lambda\rvert}$
RK2	Heun's	$y_{n+1} = y_n + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$ where $k_1 = f(t_n,y_n)$ and $k_2 = f(t_n+h,y_n+hk_1)$	$O(h^2)$	$h < \frac{2}{\lvert\lambda\rvert}$

System of linear ODEs

❒ **Definition** – A system of n first order linear ODEs

$$\begin{cases} y_1' = a_{11}y_1 + ... + a_{1n}y_n \\ \vdots \\ y_n' = a_{n1}y_1 + ... + a_{nn}y_n \end{cases}$$

can be written in matrix form as:

$$\boxed{\vec{y}' = A\vec{y}}$$

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

❒ **System of homogeneous ODEs** – The resolution of the system of 2 homogeneous linear ODEs $\vec{y}' = A\vec{y}$ is detailed in the following table:

Case	Eigenvalues $\leftrightarrow$ Eigenvectors	Solution
Real distinct eigenvalues	$\lambda_1 \leftrightarrow \vec{\eta}_{\lambda_1}$ $\lambda_2 \leftrightarrow \vec{\eta}_{\lambda_2}$	$\vec{y} = C_1\vec{\eta}_{\lambda_1}e^{\lambda_1 t} + C_2\vec{\eta}_{\lambda_2}e^{\lambda_2 t}$
Double root eigenvalues	$\lambda \leftrightarrow \vec{\eta}$ $\vec{\rho}$ s.t. $(A-\lambda I)\vec{\rho} = \vec{\eta}$	$\vec{y} = [(C_1+C_2t)\vec{\eta} + C_2\vec{\rho}]e^{\lambda t}$
Complex conjugate eigenvalues	$\alpha + i\beta \leftrightarrow \vec{\eta}_R + i\vec{\eta}_I$ $\alpha - i\beta \leftrightarrow \vec{\eta}_R - i\vec{\eta}_I$	$\vec{y} = C_1(\cos(\beta t)\vec{\eta}_R - \sin(\beta t)\vec{\eta}_I)e^{\alpha t}$ $+C_2(\cos(\beta t)\vec{\eta}_I + \sin(\beta t)\vec{\eta}_R)e^{\alpha t}$