CONTENTS

BAS 303/BAS 403 : Mathematics – IV

1 UNIT

Partial Differential Equations

CONTENTS

PART-1

Origin of Partial Differential Equations, Linear and Non-Linear Partial Differential Equations of First Order, Lagrange's Equations Method to Solve Linear Partial Differential Equations.

Que 1.1. **Form partial differential equations of the equations by eliminating the arbitrary constants :**
$z = ax + by + ab$

Answer

Differentiating z partially w.r.t. x and y,

$$p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq$$

which is a partial differential equation.

Que 1.2. **Form partial differential equations of the equations by eliminating the arbitrary constants :**
$az + b = a^2x + y$

Answer

Differentiating the given relation partially w.r.t. x, we get

$$a \frac{\partial z}{\partial x} = a^2$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = p = a \quad \text{...(1.2.1)}$$

Again differentiating the given relation partially w.r.t. y, we get

$$a \frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = q = \frac{1}{a} \quad \text{...(1.2.2)}$$

Multiplying eq. (1.2.1) and (1.2.2), we get $pq = 1$
which is a partial differential equation.

Que 1.3. **Form the partial differential equation by eliminating the arbitrary function(s) from the following :**
i. $z = f(x^2 - y^2)$
ii. $z = \phi(x) \cdot \psi(y)$
iii. $z = f(x + it) + g(x - it)$

Answer

i. Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) \cdot 2x \quad \text{...(1.3.1)}$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2) \cdot (-2y) \quad \text{...(1.3.2)}$$

Dividing eq. (1.3.1) by eq. (1.3.2), we get

$$\frac{p}{q} = \frac{x}{(-y)} \Rightarrow py + qx = 0$$

which is a partial differential equation.

ii. Differentiating z w.r.t. x, partially, we get

$$\frac{\partial z}{\partial x} = p = \phi'(x)\,\psi(y) \quad \text{...(1.3.3)}$$

Differentiating z w.r.t. y, partially, we get

$$\frac{\partial z}{\partial y} = q = \phi(x)\,\psi'(y) \quad \text{...(1.3.4)}$$

Differentiating eq. (1.3.3) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial y \partial x} = s = \phi'(x)\,\psi'(y) \quad \text{...(1.3.5)}$$

Multiplying eq. (1.3.3) and (1.3.4), we get

$$pq = \phi'(x)\,\psi(y)\,\phi(x)\,\psi'(y) = zs \quad \text{[Using (1.3.5)]}$$

$$\Rightarrow \quad pq - zs = 0$$

which is a partial differential equation.

iii. Given $z = f(x + it) + g(x - it)$
Differentiating z twice partially w.r.t. x and t, we have

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \quad \text{...(1.3}$$

$$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x+it) + i^2 g''(x-it)$$

or $$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it) \quad ...(1.3.7)$$

Adding eq. (1.3.6) and eq. (1.3.7), we obtain $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

which is a partial differential equation of second order.

Que 1.4. **Solve the following differential equations :**
$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$

Answer

Here Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\therefore \quad \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

Taking the first two members, we have $\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$

which on integration gives

$$\log(x-y) = \log(y-z) + \log a$$

or $$\log\left(\frac{x-y}{y-z}\right) = \log a \quad \text{or} \quad \frac{x-y}{y-z} = a \quad ...(1.4.1)$$

Similarly, taking the last two members, we obtain

$$\frac{y-z}{z-x} = b \quad ...(1.4.2)$$

From eq. (1.4.1) and eq. (1.4.2), the general solution is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0.$$

Que 1.5. **Solve $\frac{y^2 z}{x} p + xzq = y^2$.**

Answer

Rewriting the given equation as

$$y^2 zp + x^2 zq = y^2 x$$

The subsidiary equations are

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

The first two fractions give $x^2 dx = y^2 dy$.

Integrating we get $x^3 - y^3 = a$ ...(1.5.1)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(1.5.2)

Hence from eq. (1.5.1) and eq. (1.5.2), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2)$$

Que 1.6. **Solve the partial differential equation**
$x(y^2 + z)\, p - y(x^2 + z)\, q = z(x^2 - y^2)$ **where** $p = \frac{\partial z}{\partial x}$ **and** $q = \frac{\partial z}{\partial y}$.

Answer

Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad ...(1.6.1)$$

Using $x, y, -1$ as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

$$\therefore \quad x\,dx + y\,dy - dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$\Rightarrow \quad x^2 + y^2 - 2z = c_1 \quad ...(1.6.2)$$

Again, using $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y} + \frac{1}{z}dz}{0}$$

$\therefore \quad \frac{1}{x}dz + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\Rightarrow \quad xyz = c_2$$

Hence the general solution is ...(1.6.3)

$$\phi(x^2 + y^2 - 2z, xyz) = 0$$

Que 1.7. **Solve $x^2(y - z)\, p + y^2(z - x)\, q = z^2(x - y)$.**

Answer

Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers 1/x, 1/y and 1/z, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$ which on integration gives

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \qquad \text{...(1.7.1)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$, which on integrating gives

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad \text{...(1.7.2)}$$

Hence from eq. (1.7.1) and eq. (1.7.2), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

Que 1.8. **Solve : $\sqrt{p} + \sqrt{q} = 1$.**

Answer

The equation is of the form, $f(p, q) = 0$ ...(1.8.1)

The complete solution is $z = ax + by + c$

where $\sqrt{a} + \sqrt{b} = 1$ or $b = (1 - \sqrt{a})^2$

$\therefore$ From eq. (1.8.1), the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

Que 1.9. **Solve : $pq = p + q$.**

Answer

The equation is of the form $f(p, q) = 0$ ...(1.9.1)

The complete solution is $z = ax + by + c$

where $ab = a + b$ or $b = \frac{a}{a-1}$

$\therefore$ From eq. (1.9.1), the complete solution is $z = ax + \frac{a}{a-1}y + c$.

Que 1.10. **Solve : $4xyz = pq + 2px^2y + 2qxy^2$.**

Answer

Given : $4xyz = pq + 2px^2y + 2qxy^2$ ...(1.10.1)

Let $x^2 = X$ and $y^2 = Y$

so that $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = 2x\frac{\partial z}{\partial X}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = 2y\frac{\partial z}{\partial Y}$

$\therefore$ After putting the values of p and q in eq. (1.10.1), we get

$$4xyz = 4xy\frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y} + 4x^3y\frac{\partial z}{\partial X} + 4xy^3\frac{\partial z}{\partial Y}$$

or

$$z = x^2\frac{\partial z}{\partial X} + y^2\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}$$

$$= X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}$$

or $z = PX + QY + PQ$,

where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$

It is of the form $z = PX + QY + f(P, Q)$

Its complete solution is $z = aX + bY + ab$ or $z = ax^2 + by^2 + ab$.

Que 1.11. **Solve : $z^2(p^2 + q^2 + 1) = a^2$.**

Answer

The given equation is of the form $f(z, p, q) = 0$

Let $u = x + by$ (note the use of b instead of a, since a is a given constant)

so that $p = \dfrac{dz}{du}$ and $q = b\dfrac{dz}{du}$.

Substituting these values of p and q in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + b^2\left(\frac{dz}{du}\right)^2 + 1\right] = a^2$$

or $$z^2(1 + b^2)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$$

or $$z\sqrt{1 + b^2}\,\frac{dz}{du} = \pm\sqrt{a^2 - z^2}$$

or $$\pm\sqrt{1 + b^2}\,.\,\frac{z}{\sqrt{a^2 - z^2}}\,dz = du$$

Integrating, we have

$$\pm\sqrt{1 + b^2}\sqrt{a^2 - z^2} = u + c$$

or $$(1 + b^2)(a^2 - z^2) = (x + by + c)^2$$

Que 1.12. **Solve : $(x^2 - y^2 - yz)\,p + (x^2 - y^2 - zx)\,q = z\,(x - y)$.**

AKTU 2020-21 (Sem-3), Marks 10

Answer

Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)}$$

Using 1, – 1, – 1 as multipliers, we have

$$\text{Each fraction} = \frac{dx - dy - dz}{x^2 - y^2 - yz - x^2 + y^2 + zx - zx + zy}$$

$$= \frac{dx - dy - dz}{0}$$

$$dx - dy - dz = 0$$

Integrating we get ...(1.12.1)

$$x - y - z = c_1$$

Again from first and second, we have

$$\frac{xdx - ydy}{x^3 - xy^2 - xyz - yx^2 + y^3 + yzx} = \frac{dz}{z(x - y)}$$

$$\Rightarrow \frac{xdx - ydy}{x^3 - xy^2 - yx^2 + y^3} = \frac{dz}{z(x - y)}$$

$$\Rightarrow \frac{xdx - ydy}{(x - y)(x^2 - y^2)} = \frac{dz}{z(x - y)}$$

$$\Rightarrow \frac{xdx - ydy}{x^2 - y^2} = \frac{dz}{z}$$

Integrating, we get

$$\frac{1}{2}\log(x^2 - y^2) = \log z + \frac{1}{2}\log c_2$$

$$\Rightarrow \log\left(\frac{x^2 - y^2}{z^2}\right) = \log c_2$$

$$\Rightarrow \frac{x^2 - y^2}{z^2} = c_2$$

Hence the general solution is

$$\phi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$$

Que 1.13. **Solve $(y + zx)\,p - (x + yz)\,q = x^2 - y^2$**

AKTU 2021-22 (Sem-3), Marks 10

Answer

Given, $(y + xz)\,p - (x + yz)\,q = x^2 - y^2$

This equation is of the form $Pp + Qq = R$ (Lagrange's linear partial differential equation).

Here, $P = y + xz$, $Q = -(x + yz)$, $R = x^2 - y^2$.

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y + xz} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2} \quad ...(1.13.1)$$

Each ratio of (1.13.1) is equal to $\dfrac{xdx + ydy}{(x^2 - y^2)z} = \dfrac{dz + dy}{(1 - z)(y - x)}$

Let us consider,

$$\frac{xdx + ydy}{(x^2 - y^2)z} = \frac{dz}{x^2 - y^2}$$

$xdx + ydy = zdz$

Integrating,

$$\int xdx + \int ydy = \int zdz$$

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_1}{2}$$

$$x^2 + y^2 - z^2 = c_1$$

Consider

$$\frac{dx + dy}{(1 - z)(y - x)} = \frac{dz}{(x + y)(x - y)}$$

$$\frac{dx + dy}{1 - z} = \frac{dz}{-(x + y)}$$

$-(x + y)d(x + y) = (1 - z)\, dz$

Integrating,

$$-\int (x + y)d(x + y) = \int (1 + z)dz$$

$$-\frac{(x + y)^2}{2} = \frac{(1 - z)^2}{-2} + \frac{c_2}{2}$$

$(1 - z)^2 - (x + y)^2 = c_2$

The general solution is,

$$\phi(c_1, c_2) = 0$$

$\phi(x^2 + y^2 - z^2, (1 - z)^2 - (x + y)^2) = 0.$

Que 1.14. **Find the general solution of the partial differential equation $(y + z)p + (z + x)q = (x + y)$.**

AKTU 2022-23 (Sem-4), Marks 10

Answer

$(y + z)p + (z + x)q = (x + y)$

The above equation is of the form $P_p + Q_q = R$

Where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$

Here, auxiliary equations are $\dfrac{dx}{y + z} = \dfrac{dy}{z + x} = \dfrac{dz}{x + y}$

$$\Rightarrow \quad \frac{dx - dy}{y - x} = \frac{dz - dx}{x - z} \Rightarrow \frac{dx - dy}{x - y} = \frac{dz - x}{z + x}$$

Integrating we get

$$\log (x - y) = \log (z - x) + \log a$$

$$x - y = a(z - x) \Rightarrow a = \frac{x - y}{z - x}$$

Again, $\dfrac{dx - dy}{y - x} = \dfrac{dx + dy + dz}{2(x + y - z)}$

$$\Rightarrow \quad \frac{dx + dy + dz}{x + y - z} + \frac{2(dx - dy)}{x - y} = 0$$

Integrating, we get

$\log (x + y + z) + 2 \log (x - y) = \log b$

$\log (x + y + z) + \log (x - y)^2 = \log b$

$\log [(x + y + z)(x - y)^2] = \log b$

$$b = (x + y + z)(x - y)^2$$

Hence, the solution of the given equation is

$$\phi\left[(x + y + z)(x - y)^2, \frac{x - y}{z - x}\right] = 0$$

i.e., $(x + y + z)(x - y)^2 = f\left(\dfrac{x - y}{z - x}\right)$

Que 1.15. **Solve, $(mz - ny)p + (nx - lz)q = ly - mx$, where $p = \dfrac{\partial z}{\partial x}$ & $q = \dfrac{\partial z}{\partial y}$.**

AKTU 2022-23 (Sem-3), Marks 10

Answer

The auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using x, y, z as multipliers, we get

$$\text{Each fraction} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0$

which on integration gives

$$x^2 + y^2 + z^2 = a$$

Again using l, m, n as multipliers, we get

$$\text{Each fraction} = \frac{ldx + mdy + ndz}{0} \quad \text{...(1.15.1)}$$

$\therefore \quad ldx + mdy + ndz = 0$

$\Rightarrow \quad \int lfc + mdy + ndz = c$

$\Rightarrow \quad lx + my + nz = c$ .....

Now, again eq. (1.15.2) $\Rightarrow$ ...(1.15.2)

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

$$= \frac{xdx + ydy + zdz}{mxz - nxy + nxy - lyz + lyz - mxz}$$

$$\Rightarrow \quad = \frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow \quad xdx + ydy + zdz = 0$

$\Rightarrow \quad \int xdx + ydy + zdz = d$

$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = d$

$\Rightarrow \quad x^2 + y^2 + z^2 = d$ ...(1.15.3)

$\therefore$ From eq. (1.15.2) and eq. (1.15.3) the solution of the eq. (1.15.1) is given by

$\varphi(lx + my + nz, x^2 + y^2 + z^2) = 0$

Hence the problem.

PART-2

Charpit's Method to Solve Non-Linear Partial Differential Equations.

Que 1.16. **Solve $(p^2 + q^2)\, y = qz$.**

Answer

Let $\quad f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(1.16.1)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$ ...(1.16.2)

Integrating $\quad p^2 + q^2 = c^2$

Now solving eq. (1.16.1) and eq. (1.16.2),

put $p^2 + q^2 = c^2$ in eq. (1.16.1), so that $q = c^2y/z$

Substituting this value of q in eq. (1.16.2), we get $p = \frac{c\sqrt{(z^2 - c^2y^2)}}{z}$

Hence $\quad dz = pdx + qdy = \frac{c\sqrt{(z^2 - c^2y^2)}}{z}\, dx + \frac{c^2 y}{z}\, dy$

or $\quad z\,dz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\; dx$ or $\frac{\frac{1}{2}\, d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = cdx$

Integrating, we get $c\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Que 1.17. **Solve $2xz - px^2 - 2qxy + pq = 0$**

Answer

Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(1.17.1)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore \quad dq = 0 \quad$ or $\quad q = a$.

Putting $q = a$ in eq. (1.17.1), we get

$$p = \frac{2x(z - ay)}{x^2 - a}$$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}\, dx + ady$$

or $\quad \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}\, dx$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a) \quad$ or $\quad z = ay + b(x^2 - a)$

which is the required complete solution.

Que 1.18. **Solve $2z + p^2 + qy + 2y^2 = 0$.**

Answer

Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$ ...(1.18.1)

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in eq. (1.18.1), we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a-x)^2]$$

$\therefore$

$$dz = pdx + qdy$$

$$= (a-x)\,dx - \frac{1}{y}[2z + 2y^2 + (a-x)^2]\,dy$$

Multiplying both sides by $2y^2$,

$$2y^2\,dz + 4yz\,dy = 2y^2(a-x)\,dx - 4y^3\,dy - 2y(a-x)^2\,dy$$

Integrating

$$2zy^2 = -[y^2(a-x)^2 + y^4] + b$$

or $\quad y^2[(x-a)^2 + 2z + y^2] = b$, which is the desired solution.

Que 1.19. **Solve the PDE $z_x z_y - z = 0$ subject to the condition $z(s, -s) = 1$.**

Answer

Here, we have

$$F(x, y, z, p, q) = pq - z.$$

The characteristics system takes the form

$$\frac{dx}{dt} = F_p = q(t),\ \frac{dy}{dt} = F_q = p(t),\ \frac{dz}{dt} = pF_p + qF_q = 2p(t)q(t)$$

$$\frac{dp}{dt} = -[F_x + p(t)F_z] = p(t),\ \frac{dq}{dt} = -[F_y + q(t)F_z] \ldots$$

Note that

Integrating, we get

$$z = \frac{z^2}{2c}\left(\sin^{-1}\left(\frac{cx}{z}\right)\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{cx}{z}\right)\right) + c_1 + c^2yz + c_2$$

is the required complete integral.

Que 1.21. **Use Cauchy's method of characteristics to solve the first order partial differential equation $u_x + u_y = 1 + \cos y$, $u(0, y) = \sin y$.**

AKTU 2021-22 (Sem-4), Marks 10

Answer

Given : Equation, $u_x + u_y = 1 + \cos y$

Comparing with $au_x + bu_y = f(x, y)$ then

$a = 1, b = 1, f(x, y) = 1 + \cos y$

Then from auxiliary equation

$$\frac{\partial x}{1} = \frac{dy}{1} = \frac{du}{1 + \cos y}$$

Taking 1st and 2nd fraction, $dx = dy$

$$dx - dy = 0$$
$$x - y = c_1$$
$$x = c_1 + y$$

Taking 2nd and 3rd fraction,

$$\frac{dy}{1} = \frac{du}{1 + \cos y}$$
$$\int du = \int (1 + \cos y)\, dy$$
$$u = y + \sin y + c_2 \quad ...(1.21.1)$$

Let $c_2 = g(c_1)$

Equation (1.21.1) becomes

$$u(x, y) = y + \sin y + g(x - y)$$

where $g(x - y)$ is arbitrary function.

Now, given condition

$$u(0, y) = \sin y$$
$$y + \sin y + g(x - y) = \sin y$$
$$g(x - y) = -y$$

so,
$$u(x, y) = y + \sin y - y$$

Solution is given by

$$u(x, y) = \sin y$$

Que 1.22. **Solve the following partial differential equation by Charpit's method : $px + qy = pq$.** **AKTU 2021-22 (Sem-4), Marks 10**

OR

By Charpit's method, find the complete solution of PDE :

$px + qy - pq = 0$. **AKTU 2022-23 (Sem-3), Marks 10**

Answer

$f(x, y, z, p, q) = 0$ is $px + qy - pq = 0$ ...(1.22.1)

$$\frac{\partial f}{\partial x} = p,\ \frac{\partial f}{\partial y} = q,\ \frac{\partial f}{\partial z} = 0,\ \frac{\partial f}{\partial p} = x - q,\ \frac{\partial f}{\partial q} = y - p$$

Charpit's equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

$$\frac{dx}{-(x - q)} = \frac{dy}{-(y - p)} = \frac{dz}{-p(x - q) - q(y - p)} = \frac{dp}{p} = \frac{dq}{q} = \frac{d\phi}{0}$$

We have to choose the simplest integral involving p and q

$\Rightarrow$ $\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a \Rightarrow p = aq$

Putting for p in the given equation (1.22.1), we get

$$q\,(ax + y) = aq^2 \qquad \therefore\ q = \frac{y + ax}{a}$$

$\therefore$ $p = aq = y + ax$ ...(1.22.2)

Now $dz = pdx + qdy$

Putting for p and q in (1.22.2), we get

$$dz = (y + ax)\,dx + \frac{y + ax}{a}\,dy$$
$$adz = (y + ax)a\,dx + (y + ax)\,dy$$
$$adz = (y + ax)\,(adx + dy)$$

Integrating $\quad az = \frac{(y + ax)^2}{a} + b$

PART-3

Solution of Linear Partial Differential Equation of Higher Order with Constant Coefficient.

Que 1.23. **Solve the linear partial differential equation**
$$\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}.$$

Answer

The given equation is

$(D^3 - 3D^2D' + 4D'^3)\,u = e^{x+2y}$ where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

Auxiliary equation is
$$m^3 - 3m^2 + 4 = 0$$
$$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$
$$(m+1)(m^2 - 4m + 4) = 0$$
$$(m-2)^2(m+1) = 0$$
$$\Rightarrow \quad m = 2, 2, -1$$
$$\therefore \quad \text{CF} = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$$
$$\text{PI} = \frac{1}{D^3 - 3D^2D' + 4D'^3}\, e^{x+2y}$$
$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3}\iiint e^u\, du\, du\, du$$
where $x + 2y = u$
$$= \frac{1}{27}e^{x+2y}$$
Hence the complete solution is
$$u = \text{CF} + \text{PI}$$
$$= f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27}e^{x+2y}$$
where f_1, f_2 and f_3 are arbitrary functions.

Que 1.24. **Solve** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$.

Answer

Given equation in symbolic form is $(D^2 + DD' - 6D'^2)\,z = \cos(2x+y)$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Auxiliary equation is $m^2 + m - 6 = 0$ where $m = -3, 2$.
$$\therefore \quad \text{CF} = f_1(y - 3x) + f_2(y + 2x)$$
Since $D^2 + DD' - 6D'^2 = -(2)^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method
$$\text{PI} = \frac{1}{D^2 - DD' - 6D'^2}\cos(2x+y)$$
$$= \frac{1}{(D+3D')(D-2D')}\cos(2x+y)$$
$$= \frac{1}{D+3D'}\left[\int \cos(2x + \overline{c - 2x})\, dx\right]_{c \to y+2x}$$
$$= \frac{1}{D+3D'}\left[\int \cos c\, dx\right]_{c \to y+2x} \quad [\because\ y = c - mx = c - 2x]$$
$$= \frac{1}{D+3D'}\, x\cos(y+2x)$$
$$= \left[\int x\cos(\overline{c+3x} + 2x)\, dx\right]_{c \to y-3x}$$
$$= \left[\int x\cos(5x + c)\, dx\right]_{c \to y-3x}$$
$$= \left[\frac{x\sin(5x+c)}{5} + \frac{\cos(5x+c)}{25}\right]_{c \to y-3x} \quad \text{[Integrating by parts]}$$
$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x})$$
$$= \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$
Hence the complete solution is
$$z = f_1(y-3x) + f_2(y+2x) + \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$

Que 1.25. **Solve** $r - 4s + 4t = e^{2x+y}$.

Answer

Given equation is $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Symbolic form is $(D^2 - 4DD' + 4D'^2)\,z = e^{2x+y}$.

Its auxiliary equation is $(m-2)^2 = 0$, where $m = 2, 2$.
$$\therefore \quad \text{CF} = f_1(y+2x) + xf_2(y+2x)$$
$$\text{PI} = \frac{1}{(D-2D')^2}e^{2x+y}$$
The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ To obtain the PI, we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx$$
$$= \int e^{2x + (c - 2x)}\, dx = xe^c = xe^{2x+y}$$

and from $(D - 2D')\, z = u = xe^{2x+y}$, the solution $\quad [\because\ y = c - mx = c - 2x]$

$$z = \int xe^{2x + (c - 2x)}\, dy = \frac{1}{2} x^2 e^c = \frac{1}{2} x^2 e^{2x+y}$$
$$[\because\ y = c - mx = c - 2x]$$

Hence the complete solution is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2} x^2 e^{2x+y}$.

Que 1.26. **Solve :** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x.$

Answer

The given equation is

$$(D^2 - 2DD' + D'^2)z = \sin x$$

Auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$\therefore$
$$CF = f_1(y + x) + xf_2(y + x)$$
$$PI = \frac{1}{(D - D')^2} \sin (x + 0.y)$$
$$= \frac{1}{(1 - 0)^2} \iint \sin u\, du\, du, \quad \text{where } x = u$$
$$= -\sin u = -\sin x$$

Hence the complete solution is

$$z = CF + PI = f_1(y + x) + xf_2(y + x) - \sin x$$

where f_1 and f_2 are arbitrary functions.

Que 1.27. **Solve the linear partial differential equation**

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4 \sin (2x + y).$$

Answer

The given equation is

$$(D^3 - 4D^2D' + 4DD'^2)z = 4 \sin (2x + y)$$

The auxiliary equation is

$$m^3 - 4m^2 + 4m = 0$$
$$\Rightarrow \quad m(m^2 - 4m + 4) = 0$$

$$\Rightarrow \quad m = 0, 2, 2.$$

$\therefore$
$$CF = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$$
$$PI = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 4 \sin (2x + y)$$
$$= \frac{4}{D}\left[\frac{1}{D^2 - 4DD' + 4D'^2} \sin (2x + y)\right]$$
$$= \frac{4}{D}\left[\frac{1}{(D - 2D')^2} \sin (2x + y)\right]$$
$$= x \cdot \frac{4}{D}\left[\frac{1}{2(D - 2D')} \sin (2x + y)\right]$$
$$= 4x^2 \cdot \frac{1}{D}\left[\frac{1}{2} \sin (2x + y)\right]$$
$$= 2x^2 \frac{1}{D} \sin (2x + y)$$
$$= -2x^2 \frac{\cos (2x + y)}{2} = -x^2 \cos (2x + y).$$

Hence the complete solution is

$$z = CF + PI$$
$$= f_1(y) + f_2(y + 2x) + xf_3(y + 2x) - x^2 \cos (2x + y)$$

where f_1, f_2 and f_3 are arbitrary functions.

Que 1.28. **Solve :** $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x \partial y^2} - 6\dfrac{\partial^3 z}{\partial y^3} = \sin (x + 2y) + e^{3x+y}.$

Answer

The given equation is

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin (x + 2y) + e^{3x+y}$$

Auxiliary equation is

$$m^3 - 7m - 6 = 0$$
$$(m + 1)(m^2 - m - 6) = 0$$
$$\Rightarrow \quad m = -1, -2, 3$$

$\therefore$
$$CF = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$
$$PI = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin (x + 2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{3x+y}.$$

PI corresponding to $\sin(x+2y)$

$$= \frac{1}{(1)^3 - 7(1)(2)^2 - 6(2)^3} \iiint \sin u \, du\, du\, du,$$

where $x + 2y = u$

$$= -\frac{1}{75} \cos u = -\frac{1}{75} \cos(x+2y)$$

PI corresponding to $e^{3x+y} = \dfrac{1}{D^3 - 7DD'^2 - 6D'^3} (e^{3x+y})$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^3 - 7DD'^2 - 6D'^3)} e^{3x+y}$$

$$= x \cdot \frac{1}{3D^2 - 7D'^2} e^{3x+y}$$

$$= x \cdot \frac{1}{3(3)^2 - 7(1)^2} e^{3x+y} = x \cdot \frac{1}{20} e^{3x+y}$$

$\therefore$ Required PI $= -\dfrac{1}{75} \cos(2y+x) + \dfrac{x}{20} e^{3x+y}$

$\therefore$ Complete solution is

$$z = \text{CF} + \text{PI} = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20} e^{3x+y}$$

where f_1, f_2 and f_3 are arbitrary function.

Que 1.29. **Solve $(D^2 - DD')z = \cos x \cos 2y$**

Answer

Auxiliary equation is : $m^2 - m = 0$

$$m(m-1) = 0$$
$$m = 0, 1$$
$$\text{CF} = f_1(y) + f_2(y+x)$$

$$\text{PI} = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} \{\cos(x+2y) + \cos(x-2y)\}$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 - DD'}\cos(x-2y)\right]$$

[Put $D^2 = -1, DD' = -2$ and $D^2 = -1, DD' = 2$]

$$= \frac{1}{2}\left[\frac{1}{-1-(-2)}\cos(x+2y) + \frac{1}{-1-(2)}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right]$$

Thus, the complete solution is

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\left\{\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right\}$$

Que 1.30. **Solve : $r + s - 2t = \sqrt{2x+y}$.**

Answer

The given equation is $r + s - 2t = \sqrt{2x+y}$.

We know that $r = \dfrac{\partial^2 z}{\partial x^2}, s = \dfrac{\partial^2 z}{\partial x \partial y}, t = \dfrac{\partial^2 z}{\partial y^2}$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sqrt{2x+y}$$

$$(D^2 + DD' - 2D'^2) z = \sqrt{2x+y}$$

Auxiliary equation is

$$m^2 + m - 2 = 0$$
$$(m-1)(m+2) = 0$$
$$m = 1, -2$$
$$\text{CF} = f_1(y+x) + f_2(y-2x)$$

$$\text{PI} = \frac{1}{D^2 + DD' - 2D'^2}\sqrt{2x+y}$$

Put $D = 2, D' = 1$, let $2x + y = u$

$$= \frac{1}{(2)^2 + (2)(1) - 2(1)^2} \iint \sqrt{u}\, du\, du$$

$$= \frac{1}{4} \cdot \frac{4}{15} u^{5/2}$$

$$\text{PI} = \frac{1}{15}(2x+y)^{5/2}$$

$\therefore$ Complete solution is $z = \text{CF} + \text{PI}$

$$= f_1(y+x) + f_2(y-2x) + \frac{1}{15}(2x+y)^{5/2}$$

where f_1 and f_2 are arbitrary functions

Que 1.31. **Solve $r + (a+b)s + abt = xy$.**

Answer

Given equation $r + (a+b)s + ab\,t = xy$

We know that $r = \dfrac{\partial^2 z}{\partial x^2}, s = \dfrac{\partial^2 z}{\partial x \partial y}, t = \dfrac{\partial^2 z}{\partial y^2}$

$$\frac{\partial^2 z}{\partial x^2} + (a+b)\frac{\partial^2 z}{\partial x \partial y} + ab\frac{\partial^2 z}{\partial y^2} = xy$$

$$(D^2 + (a+b)\,D\,D' + ab\,D'^2)\,z = xy$$

Put $D = m$ and $D' = 1$

Auxiliary equation is :

$$m^2 + (a+b)\,m + ab = 0$$
$$m^2 + am + bm + ab = 0$$
$$m(m+a) + b(m+a) = 0$$
$$(m+a)\,(m+b) = 0$$
$$m = -a, -b$$
$$\text{CF} = f_1(y - ax) + f_2(y - bx)$$

$$\text{PI} = \frac{1}{(D^2 + (a+b)\,D\,D' + ab\,D'^2)}\,xy$$

$$\text{PI} = \frac{1}{(D^2 + a\,D\,D' + b\,D\,D' + ab\,D'^2)}\,xy$$

$$= \frac{1}{D(D + aD') + bD'(D + aD')}\,xy$$

$$= \frac{1}{(D + aD')(D + bD')}\,xy$$

$$= \frac{1}{D\left[1 + \frac{aD'}{D}\right]D\left[1 + \frac{bD'}{D}\right]}\,xy$$

$$= D^{-1}\left[1\ \frac{aD'}{D}\right]^{-1} D^{-1}\left[1 + \frac{bD'}{D}\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1 + \frac{aD'}{D} + \frac{a^2 D'^2}{D^2} \ldots\right]\left[1 - \frac{bD'}{D} + \frac{b^2 D'^2}{D^2} \ldots\right] xy$$

$$= \frac{1}{D^2}\left[1 + \frac{aD'}{D}\right]\left[1 - \frac{bD'}{D}\right] xy$$

Neglecting higher terms

$$= \frac{1}{D^2}\left[1 - \frac{bD'}{D} + \frac{aD'}{D} - \frac{ab\,D'^2}{D^2}\right] xy$$

$$= \frac{1}{D^2}\left[xy - b\frac{x^2}{2} + \frac{ax^2}{2} - \frac{ab(1)}{D^2}(0)\right] = \frac{1}{D^2}\left[xy + \frac{x^2}{2}(a-b)\right]$$

Integrating twice, we get

$$\text{PI} = \frac{x^3 y}{6} + \frac{x^4}{24}(a-b)$$

$$\text{PI} = \frac{x^3 y}{6} + (a-b)\frac{x^4}{24}$$

Complete solution $z = \text{CF} + \text{PI}$

$$= f_1(y - ax) + f_2(y - bx) + \frac{x^3 y}{6} + \frac{(a-b)}{24}x^4$$

Que 1.32. **Solve the following partial differential equation :**

$$(D^2 - DD' - 2D'^2 + 2D + 2D')\,z = \sin(2x + y)$$

where notations have their usual meaning.

Answer

The given differential equation is

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$$

That can be written as $(D + D')(D - 2D' + 2)z = 0$

Comparing it with $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$

$$m_1 = -1,\ a_1 = 0,\ m_2 = 2,\ a_2 = -2$$

Complementary function

$$\text{CF} = f_1(y - x)e^{0x} + f_2(y + 2x)\,e^{-2x} = f_1(y - x) + f_2(y + 2x)e^{-2x}$$

$$\text{PI} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x + y)$$

Put, $D^2 = -4,\ D'^2 = -1,\ DD' = -2$

$$= \frac{1}{-4 + 2 + 2 + 2D + 2D'}\sin(2x + y) = \frac{1}{2(D + D')}\sin(2x + y)$$

$$= \frac{1}{2}\frac{(D - D')}{(D + D')(D - D')}\sin(2x + y) = \frac{1}{2}\frac{(D - D')}{D^2 - D'^2}\sin(2x + y)$$

$$= \frac{1}{2(-4 + 1)}(D - D')\sin(2x + y)$$

$$= -\frac{1}{6}\{2\cos(2x + y) - \cos(2x + y)\} = -\frac{1}{6}\cos(2x + y)$$

$$z = \text{CF} + \text{PI}$$

$$z = f_1(y - x) + f_2(y + 2x)e^{-2x} - \frac{1}{6}\cos(2x + y)$$

Que 1.33. **Find the solution of the partial differential equation**

$$\left[2D^2 + 5DD' + 3(D')^2\right]z = ye^x\ D' = \frac{\partial}{\partial x},\ D' = \frac{\partial}{\partial y}$$

AKTU 2022-23 (Sem-4), Marks 10

Answer

Auxiliary equation is : $2m^2 + 5m + 3 = 0$

$$2m^2 + 2m + 3m + 3 = 0$$
$$2m\,(m + 1) + 3\,(m + 1) = 0$$
$$(m + 1)\,(2m + 3) = 0$$
$$m = 1, 3/2$$
$$\text{C.F} = f_1(y - x) + f_2\left(y - \frac{3}{2}x\right)$$

$$P.I = \frac{1}{2D^2 + 5DD' + 3D'^2} ye^x$$

$$= \frac{1}{2D^2 + 2DD' + 3DD' + 3D'^2} ye^x = \frac{1}{2D(D + D') + 3D'(D + D')} ye^x$$

$$= \frac{1}{(D + D') + (2D + 3D')} ye^x$$

$$= \frac{1}{D\left[1 + \frac{D'}{D}\right] 2D \left[1 + \frac{3D'}{2D}\right]} ye^x = 2D^{-2}\left[1 + \frac{D'}{D}\right]^{-1}\left[1 + \frac{3D'}{2D}\right]^{-1} ye^x$$

$$= 2D^{-2}\left[1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \ldots\right]\left[1 + \frac{3D'}{2D} + \frac{9D'^2}{4D^2} + \ldots\right]^{-1} ye^x$$

$$= \frac{2}{D^2}\left[1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \ldots\right]\left[1 + \frac{3D'}{2D} + \frac{9D'^2}{4D^2} + \ldots\right] ye^x$$

$$= \frac{2}{D^2}\left[1 + \frac{3D'}{2D} + \frac{9D'^2}{4D^2} + \frac{D'}{D} + \frac{3D'^2}{2D^2} + \frac{9D'^3}{4D^3} + \frac{D'^2}{D^2} + \frac{3D'^3}{2D^3} + \frac{9D'^4}{4D^4}\right] ye^x$$

Neglecting higher power

$$= \frac{2}{D^2}\left[1 + \frac{3D'}{2D} + \frac{D'}{D}\right] ye^x = \frac{2}{D^2}\left[ye^x + \frac{3}{2D} D'(ye^x) + \frac{1}{D} D'(ye^x)\right]$$

$$= \frac{2}{D^2}\left[ye^x + \frac{3}{2D} e^x + \frac{1}{D} e^x\right] = \frac{2}{D^2} ye^x + \frac{6}{2D^3} e^x + \frac{1}{D} e^x$$

$$= 2e^x y + 3e^x + e^x = 2e^x y = 2e^x y + 4e^x$$

Complete solution, $z = \text{C.F} + \text{P.F} = f_1(y - x) + f_2\left(y - \frac{3}{2}x\right) + 2ye^x + 4e^x$

Que 1.34. **Solve the partial differential $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = xy$.**

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Answer

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = xy$$

Let $x = e^X, y = e^Y$ so that $X = \log x$ and $Y = \log y$

and $\quad D = \frac{\partial}{\partial X}$

$D' = \frac{\partial}{\partial Y}$ then the given equation reduces to

$[D(D - 1) - D'(D' - 1)]\, z = e^X \times e^Y$

$[D^2 - D - D'^2 + D]z = e^{X + Y}$

$$[D^2 - D'^2]z = e^{X+Y}$$

$$(D + D')(D - D') = e^{X+Y} \quad \ldots(1.34.1)$$

$$CF = f_1(Y + X) + f_2(Y - X)$$

$$= f_1(\log y + \log x) + f_2(\log y - \log x)$$

$$= f_1(\log yx) + f_2\left(\log \frac{y}{x}\right) \quad \ldots(1.34.2)$$

$$PI = \frac{1}{D^2 - D'^2} e^{X+Y} = \frac{1}{(1)^2 - (1)^2} e^{X+Y} = 0$$

$$z = CF + P.I = f_1(\log xy) + f_2(\log y/x)$$

Que 1.35. **Solve the partial differential equation $(D - D' - 1)(D - D' - 2) = \sin(2x + 3y)$**

AKTU 2021-22 (Sem-3), Marks 10

Answer

Here, C.F. $= e^x \phi_1(y + x) + e^{2x} \phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $\quad P.I = \frac{1}{(D - D' - 1)(D - D' - 2)} \sin(2x + 3y)$

$$= \frac{1}{D^2 - 2DD' + D'2 - 3D + 3D' + 2} \sin(2x + 3y)$$

$$= \frac{1}{-2^2 + 2 \times (2 \times 3) - 3^2 - 3D + 3D' + 2} \sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1} \sin(2x + 3y) = D \frac{1}{-3D^2 + 3DD' + D} \sin(2x + 3y)$$

$$= D \frac{1}{-3 \times (-2^2) + 3 \times (2 \times 3) + D} \sin(2x + 3y)$$

$$= D \frac{1}{D - 6} \sin(2x + 3y)$$

$$= D(D + 6) \frac{1}{D^2 - 36} \sin(2x + 3y) - (D^2 + 6D) \frac{1}{-2^2 - 36} \sin(3x + 2y)$$

$= (1/40) \times [D^2 \sin(2x + 3y) + 6D \sin(2x + 3y)] = (1/40) \times [4 \sin(2x + 3y) = 12 \cos(2x + 3y)]$

Solution is : $z = e^x\phi_1(y + x) + e^2x\, \phi(y + x) + (1/10)\, 5\, [\sin(2x + 3y)\, 3 \cos(2x + 3y)]$

Que 1.36. **Solve $(x^2 D^2 - y^2 D'^2) = xy$ where $D^2 = \frac{\partial^2}{\partial x^2}, D'^2 \frac{\partial^2}{\partial y^2}$**

AKTU 2022-23 (Sem-3), Marks 10

Answer

$$x^2D^2 - y^2D'^2 = xy$$

Let $x = e^X, y = e^Y$, so that $X = \log x$ and $Y = \log y$

and let $D = \frac{\partial}{\partial X}, D' \equiv \frac{\partial}{\partial Y}$ then the given equation reduce to

$$[D(D-1) - D'(D'-1)]z = e^{X+Y}$$
$$[D^2 - D - D'^2 + D]z = e^{X+Y}$$
$$[D^2 - D'^2]z = e^{X+Y}$$
$$[(D+D')(D-D')]z = e^{X+Y}$$
$$CF = f_1(Y-X) + e^X f_2(Y+X)$$

$$P.I = \frac{1}{(D+D')(D-D')}e^{X+Y} = \frac{1}{(1+1)(1-1)}e^{X+Y}$$
$$= \infty$$

Hence complete solution,

$$Z = CF + PI = f_1(Y+X) + e^X f_2(Y-X)$$
$$= f_1(\log y + \log x) + x f_2(\log y - \log x)$$
$$= f_1(\log xy) + x f_2(\log (y/x))$$
$$= g_1(xy) + x g_2(y/x)$$

Where g_1 and g_2 are arbitrary functions.

Que 1.37. **Solve the partial differential equation :**

$$D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2.$$

AKTU 2020-21 (Sem-3), Marks 10

Answer

$D(D + D' - 1)(D + 3D' - 1)z = x^2 - 4xy + 2y^2$

This can be written as :

$D(D + D' - 1)(D + 3D' - 2)z = 0$

Comparing with,

$(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$

$m_1 = -1, a_1 = 1, m_2 = -3, a_2 = 2$

Complementary function,

$CF = f_1(y) + f_2(y - x)e^x + f_2(y - 3x)e^{2x}$

$$PI = \frac{1}{D(D+D'-1)(D+3D'-2)}(x^2 - 4xy + 2y^2)$$

$$\frac{1}{2D}[1-(D+D')]^{-1}\left\{1-\frac{D+3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}(1 + D + D' + (D+D')^2 + ...)\left\{1 + \frac{D+3D'}{2} + \left(\frac{D+3D'}{2}\right)^2 + ...\right\}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left[1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D'^2}{4} + \frac{19D'^2}{4} + ...\right](x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left[x^2 - 4xy + 2y^2 + 3(x - 2y) + 5(2y - 2x) + \frac{7}{2} + 19 - 22\right]$$

$$= \frac{1}{2D}\left[x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right]$$

Hence, complete solution is, $z = C.F + P.I$

$$= f_1(y) + e^x f_2(y - x) + e^{2x} f_3(y - 3x) + \frac{1}{6}x^3 - x^2y + xy^2 - \frac{7}{4}x^2 + 2xy + \frac{x}{4}$$

PART-4

Equations Reducible to Linear Partial Differential Equations with Constant Coefficients.

Que 1.38. **Solve the linear partial differential equation**

$$x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4.$$

AKTU 2021-22 (Sem-3), Marks 10

Answer

Put $x = e^X, y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D \equiv \frac{\partial}{\partial X}, D' \equiv \frac{\partial}{\partial Y}$ and

$DD' = \frac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{3X+4Y}$

$\Rightarrow$ $[(D^2 - 4DD' + 4D'^2) - (D - 2D')]z = e^{3X+4Y}$

$\Rightarrow$ $(D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$

Its

$$CF = f_1(Y + 2X) + e^X f_2(Y + 2X)$$
$$= f_1(\log y + 2\log x) + x f_2(\log y + 2\log x)$$
$$= f_1(\log yx^2) + x f_2(\log yx^2) = g_1(yx^2)) + xg_2(yx)^2$$

$$PI = \frac{1}{D - 2D' - 1}\left[\frac{1}{D - 2D'}e^{3X+4Y}\right]$$

$$= \frac{1}{D - 2D' - 1}\left[\frac{1}{3-8}\int e^u\, du\right] \text{ where } 3X + 4Y = u$$

$$= \frac{1}{D - 2D' - 1}\left[-\frac{1}{5}e^{3X+4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{D-2D'-1}e^{3X+4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{3-8-1}e^{3X+4Y}\right] = \frac{1}{30}e^{3X+4Y} = \frac{1}{30}x^3y^4$$

Hence the complete solution is

$$z = \text{CF} + \text{PI} = g_1(yx^2) + xg_2(yx^2) + \frac{1}{30}x^3y^4$$

where g_1 and g_2 are arbitrary functions.

Que 1.39. **Solve : $(x^2D^2 + 2xy\,DD' + y^2D'^2)z = x^my^n$.**

AKTU 2020-21 (Sem-3), Marks 10

Answer

Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and let $D \equiv \frac{\partial}{\partial X}$, $D' \equiv \frac{\partial}{\partial Y}$ and $DD' = \frac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$[D(D-1) + 2DD' + D'(D'-1)]z = e^{mX+nY}$

$\Rightarrow \quad (D^2 + 2DD' + D'^2 - D - D')z = e^{mX+nY}$

$\Rightarrow \quad [(D+D')^2 - (D+D')]z = e^{mX+nY}$

$\Rightarrow \quad (D+D')(D+D'-1)z = e^{mX+nY}$

$\therefore$
$$\text{CF} = f_1(Y-X) + e^X f_2(Y-X)$$
$$= f_1(\log y - \log x) + xf_2(\log y - \log x)$$
$$= f_1\left(\log\frac{y}{x}\right) + xf_2\left(\log\frac{y}{x}\right) = g_1\left(\frac{y}{x}\right) + xg_2\left(\frac{y}{x}\right)$$

$$\text{PI} = \frac{1}{(D+D')(D+D'-1)}e^{mX+nY}$$
$$= \frac{1}{(m+n)(m+n-1)}e^{mX+nY}$$
$$= \frac{x^my^n}{(m+n)(m+n-1)}$$

Hence complete solution is

$$z = \text{CF} + \text{PI}$$
$$= g_1(y/x) + xg_2(y/x) + \frac{x^my^n}{(m+n)(m+n-1)}$$

where g_1 and g_2 are arbitrary functions.

Que 1.40. **Solve : $x^2r - y^2t + px - qy = \log x$.**

Answer

Let $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D = \frac{\partial}{\partial X}$ and $D' = \frac{\partial}{\partial X}$,

then the given equation reduces to

$$[D(D-1) - D'(D'-1) + D - D']z = X \quad \text{...(1.40.1)}$$

$\Rightarrow$
$$(D^2 - D'^2)z = X$$

which is a homogeneous linear partial differential equation with constant coefficients.

$$\text{CF} = \phi_1(Y+X) + \phi_2(Y-X)$$

$\therefore$
$$\text{PI} = \frac{1}{D^2 - D'^2}(X) = \frac{1}{(1)^2 - (0)^2}\iint u\,du\,du$$

and
$$X = u$$

where
$$= \int \frac{u^2}{2}du = \frac{u^3}{6} = \frac{X^3}{6}$$

Hence solution to eq. (1.40.1) is

$$z = \phi_1(Y+X) + \phi_2(Y-X) + \frac{X^3}{6}$$
$$= \phi_1(\log y + \log x) + \phi_2(\log y - \log x) + \frac{(\log x)^3}{6}$$

Therefore the complete solution to the given differential equation is

$$z = \phi_1(\log xy) + \phi_2\left(\log\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

$$z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

where f_1 and f_2 are arbitrary functions.

☺☺☺

2 UNIT

Application of Partial Differential Equations and Fourier Transform

CONTENTS

PART-1

Method of Separation.

Que 2.1. **Classify the following partial differential equation** $(1-x^2)\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x \partial y}+(1-y^2)\frac{\partial^2 z}{\partial y^2}-2z=0.$

Answer

$$(1-x^2)\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x \partial y}+(1-y^2)\frac{\partial^2 z}{\partial y^2}-2z=0$$

On comparing above equation with ideal form,

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0$$

$$A = (1-x^2)$$
$$B = -2xy$$
$$C = (1-y^2)$$

$$B^2-4AC = (-2xy)^2-4(1-x^2)(1-y^2)$$
$$= 4x^2y^2-4(1-y^2-x^2+x^2y^2)$$
$$= 4x^2y^2-4+4y^2+4x^2-4x^2y^2)$$
$$= 4(x^2+y^2)-4$$

For hyperbolic : $B^2-4AC>0$, for $x \ge 1$ or $y \ge 1$ or both $x, y \ge 1$

For elliptical : $B^2-4AC<0$, for x and $y \le 0$

For parabolic : $B^2-4AC=0$, for any of x and $y = 1$ and 0

Que 2.2. **Apply method of separation of variables to solve** $\frac{\partial^2 z}{\partial x \partial y}=e^{-x}\cos y$**, given that** $z=0$ **when** $x=0$ **and** $\frac{\partial z}{\partial x}=0$ **when** $y=0.$

Answer

$$\frac{\partial^2 z}{\partial x \partial y}=e^{-x}\cos y \quad ...(2.2.1)$$

Let $z = X(x).\ Y(y)$

where X is a function of x only and Y is a function of y only.

$$\frac{\partial z}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}$$

From given eq. (2.2.1),

$$\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} = e^{-x}\cos y$$

$$e^x\frac{\partial X}{\partial x} = \frac{\cos y}{\frac{\partial Y}{\partial y}} = k \text{ (say)}$$

Now $\quad e^x\frac{\partial X}{\partial x} = k$

$$\partial X = ke^{-x}\partial x$$

$$X = -ke^{-x} + C_1$$

and $\quad k\frac{\partial Y}{\partial y} = \cos y$

$$\partial Y = \frac{1}{k}\cos y\,\partial y$$

$$Y = \frac{1}{k}\sin y + C_2$$

Thus $\quad z = XY$

$$z = (-ke^{-x} + C_1)\left(\frac{1}{k}\sin y + C_2\right) \quad ...(2.2.2)$$

Putting $\quad z = 0$ when $x = 0$

$$0 = -k + C_1$$

$$C_1 = k$$

From eq. (2.2.2),

$$z = (-ke^{-x} + k)\left(\frac{1}{k}\sin y + C_2\right) \quad ...(2.2.3)$$

$$\frac{\partial z}{\partial x} = ke^{-x}\left(\frac{1}{k}\sin y + C_2\right)$$

Putting $\quad \frac{\partial z}{\partial x} = 0,\ y = 0$

$$0 = ke^{-x}(0 + C_2)$$

$$C_2 = 0$$

From eq. (2.2.3), $\quad z = k(1 - e^{-x})\left(\frac{1}{k}\sin y\right) = (1 - e^{-x})\sin y$

Que 2.3. **Solve the Laplace equation $u_{xx} + u_{yy} = 0\ x \in (0, 1), y \in (0, 1)$ with the conditions $u(x, 0) = u(x, 1) = 0$ and $u(0, y) = 0, u(1, y) = f(y)$ by using the method of separation of variables.**

AKTU 2022-23 (Sem-3), Marks 10

Answer

Given Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(2.3.1)$$

Let $u = XY$, where X is a function of x only and Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

and $\quad \frac{\partial^2 u}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$

From eq. (2.3.1),

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i : $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$ (say)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

and $\quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$

$$X = C_1 x + C_2,\ Y = C_3 y + C_4$$

At $\quad y = 0,\ Y = 0 \Rightarrow C_4 = 0$

Also, $\quad y = 1,\ Y = 0 \Rightarrow C_3 = 0$

$\therefore \quad Y = 0$

Thus, $\quad u = XY = X(0)$

$u = 0$ (not possible)

Case ii : $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2$ (say)

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

and $$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

If $$y = 0,\ Y = 0$$

$$C_3 + C_4 = 0$$

$$C_4 = -C_3$$

and $$Y = 0 \text{ at } y = 1$$

$$0 = C_3 e^{k} - C_3 e^{-k}$$

$$C_3 (e^{k} - e^{-k}) = 0$$

$$C_3 = 0,\ C_4 = 0,\ Y = 0$$

(Not possible)

Case iii : $$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 e^{kx} + C_2 e^{-kx},$$

$$Y = C_3 \cos ky + C_4 \sin ky$$

At $$y = 0,\ Y = 0,\ C_3 = 0$$

$$Y = C_4 \sin ky$$

At $$y = 1,\ Y = 0$$

$$0 = C_4 \sin k$$

$$\sin k1 = 0$$

$$k1 = n\pi$$

$$k = \frac{n\pi}{1}$$

Thus, $$u = (C_1 e^{kx} + C_2 e^{-kx}) C_4 \sin \frac{n\pi y}{1} \quad \text{...(2.3.2)}$$

At $$x = 0,\ u = 0$$

$$0 = (C_1 + C_2) C_4 \sin \frac{n\pi y}{1}$$

$$C_1 + C_2 = 0$$

$$C_2 = -C_1$$

From eq. (2.3.2),

$$u = \frac{2}{2} C_4 C_1 \left(e^{kx} - e^{-kx}\right) \sin \frac{n\pi y}{1}$$

$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{1}} - e^{\frac{-n\pi x}{1}}}{2}\right) \sin\left(\frac{n\pi y}{1}\right) \quad \text{...(2.3.3)}$$

Let $$b_n = 2C_1 C_4$$

At $$x = 1,\ u = f(y)$$

From eq. (2.3.3),

$$f(y) = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi a}{1}} - e^{\frac{-n\pi a}{1}}}{2}\right) \sin\left(\frac{n\pi y}{1}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{1}\right) \sin\left(\frac{n\pi y}{1}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{1}\right) = \frac{2}{1}\int_0^1 f(y) \sin\left(\frac{n\pi y}{1}\right) dy$$

$$b_n = \frac{2}{1 \sinh\left(\frac{n\pi a}{1}\right)} \int_0^1 f(y) \sin\left(\frac{n\pi y}{1}\right) dy \quad \text{...(2.3.4)}$$

Thus, $$u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{1}\right) \sin\left(\frac{n\pi y}{1}\right)$$

where b_n is given by eq. (2.3.4)

Que 2.4. **Solve by method of separation of variable for PDE**

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,\ u(x, 0) = 4e^{-x}.$$

Answer

Given, $$x\frac{\partial u}{\partial x} + \frac{2\partial u}{\partial y} = 0 \quad \text{...(2.4.1)}$$

Assuming $x = 3$ in eq. (2.4.1), we get

$$3\frac{\partial u}{\partial x} + \frac{2\partial u}{\partial y} = 0$$

Let $u = XY$, where X is a function x and Y is a function of y only.

$$\frac{\partial u}{\partial x} = X'Y \quad \text{...(2.4.2)}$$

and $$\frac{\partial u}{\partial y} = XY' \quad \text{...(2.4.3)}$$

Putting $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in eq. (2.4.1), we get

$$3X'Y + 2XY' = 0$$

Dividing by XY, we get

$$3\frac{X'}{X} + 2\frac{Y'}{Y} = 0$$

or $$\frac{3X'}{X} = -\frac{2Y'}{Y} = k \text{ (say)}$$

Taking $3\dfrac{X'}{X} = k$

$$\frac{dX}{Xdx} = \frac{k}{3}$$

$$\frac{dX}{X} = \frac{k}{3}\,dx$$

On integrating, we get

$$\log X = \frac{k}{3}x + \log C_1$$

$$X = C_1 e^{\frac{k}{3}x}$$

Similarly, $\dfrac{Y'}{Y} = -\dfrac{k}{2}$

$$\frac{dY}{Y} = -\frac{k}{2}\,dy$$

On integrating, we get,

$$\log Y = -\frac{k}{2}y + \log C_2$$

$$Y = C_2 e^{-\frac{k}{2}y}$$

Therefore the complete solution

$$u = XY$$

$$u = C_1C_2 e^{\frac{k}{3}x} \cdot e^{-\frac{k}{2}y}$$

$$u = C_1C_2 e^{\frac{k}{3}x-\frac{k}{2}y} \quad ...(2.4.4)$$

Now, $u(x, 0) = C_1C_2 e^{\frac{k}{3}x}$

$$4e^{-x} = C_1C_2 e^{\frac{k}{3}x}$$

On comparing the coefficients, we get

$$C_1C_2 = 4 \text{ and } \frac{k}{3} = -1 \quad \therefore \quad k = -3$$

Putting the value of C_1C_2 and k in eq. (2.4.4), we get

$$u(x, y) = 4e^{-x+\frac{3}{2}y}$$

Que 2.5. **Solve the *P.D.E.* by separation of variables method,**

$$u_{xx} = u_y + 2u,\ u(0, y) = 0,\ \frac{\partial}{\partial x}u(0, y) = 1 + e^{-3y}.$$

Answer

Let

$$u = XY \quad ...(2.5.1)$$

where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy} = XY'$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} = YX''$$

From the given equation,

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = k(\text{say}) \quad ...(2.5.2)$$

Taking

$$\frac{X''}{X} = k$$

$$\Rightarrow \quad X'' - kX = 0$$

$\Rightarrow$ Auxiliary equation is

$$m^2 - k = 0$$

$$\Rightarrow \quad m = \pm\sqrt{k}$$

$$\therefore \quad \text{C.F.} = C_1e^{\sqrt{k}x} + C_2e^{-\sqrt{k}x}$$

$$\text{P.I.} = 0$$

$$\therefore \quad X = C_1e^{\sqrt{k}x} + C_2e^{-\sqrt{k}x} \quad ...(2.5.3)$$

Taking

$$\frac{Y'}{Y} + 2 = k$$

$$\Rightarrow \quad \frac{Y'}{Y} = k - 2$$

$$\Rightarrow \quad \frac{dY}{Y} = (k-2)\,dy$$

Integration yields,

$$\log Y = (k-2)y + \log C_3 \quad ...(2.5.4)$$

$$\Rightarrow \quad Y = C_3e^{(k-2)y}$$

Hence from eq. (2.5.1)

$$u(x, y) = (C_1e^{\sqrt{k}x} + C_2e^{-\sqrt{k}x})\,C_3e^{(k-2)y} \quad ...(2.5.5)$$

Applying the condition $u(0, y) = 0$ in eq. (2.5.5), we get

$$u(0, y) = 0 = (C_1 + C_2)\,C_3e^{(k-2)y}$$

$$\Rightarrow \quad C_1 + C_2 = 0 \Rightarrow C_2 = -C_1 \quad ...(2.5.6)$$

From eq. (2.5.5), most general solution is

$$u(x, y) = \Sigma\, C_1C_3\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)e^{(k-2)y} \quad ...(2.5.7)$$

$$\frac{\partial u}{\partial x} = \Sigma\, C_1C_3\sqrt{k}\left(e^{\sqrt{k}x} + e^{-\sqrt{k}x}\right)e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 1 + e^{-3y} = \Sigma C_1C_3\sqrt{k}(2)\,e^{(k-2)y} = \sum_{n=1}^{\infty} b_n\, e^{(k-2)y}$$

Comparing the coefficients, we get

$$b_1 = 1, \quad k - 2 = 0$$

$$2C_1C_3\sqrt{k} = 1, k = 2$$

$$\therefore \quad C_1C_3 = \frac{1}{2\sqrt{2}}$$

$$b_3 = -1, \quad k - 2 = -3$$

$$2C_1C_3\sqrt{k} = 1, k = -1$$

$$\therefore \quad C_1C_3 = \frac{1}{2i}$$

Hence from eq. (2.5.1), the particular solution is

$$u(x, y) = \frac{1}{2\sqrt{2}}(e^{\sqrt{2}x} - e^{-\sqrt{2}x}) + \frac{1}{2i}(e^{ix} - e^{-ix})e^{-3y}$$

$$\Rightarrow \quad u(x, y) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,x + e^{-3y}\sin x$$

Que 2.6. **Solve by the method of separation of variables, the heat equation $u_t = u_{xx}$, $0 < x < 1, t > 0$ subject to the initial and boundary conditions $u(x, 0) = x - x^2$, $u(0, t) = u(1, t) = 0$.**

AKTU 2022-23 (Sem-3), Marks 10

Answer

Given, the heat equation

$$u_t = u_{xx}, \ 0 < x < 1, t > 0 \quad ...(2.6.1)$$

Subject to the initial and boundary conditions

$$u(x, 0) = x - x^2, \ u(0, t) = u(1, t) = 0 \quad ...(2.6.2)$$

Assuming separable solutions

$$u(x, t) = X(x)T(t), \quad ...(2.6.3)$$

Shows that the heat eq. (2.6.1) becomes

$$XT' = X''T,$$

which, after dividing by XT and expanding gives

$$\frac{T'}{T} = \frac{X''}{X}, \quad ...(2.6.4)$$

implying that $\quad T' = \lambda T, \quad X'' = \lambda X, \quad ...(2.6.5)$

where λ is a constant, From (2) and (3), the boundary conditions becomes

$$X(0) = X(1) = 0 \quad ...(2.6.6)$$

Integrating the X equation in (2.6.5) gives rise to three cases depending on the sign of λ where $\lambda = -k^2$ for some constant k is applicable which we have as the solution.

$$X(x) = c_1 \sin kx + c_2 \cos kx \quad ...(2.6.7)$$

Imposing the boundary conditions (6) shows that $c_1 \sin 0 + c_2 \cos 0 = 0$, $c_1 \sin k + c_2 \cos k = 0$, ...(2.6.8)

we get

$$c_2 = 0, \ c_1 \sin k = 0 \Rightarrow k = 0, \pi, 2\pi, \ldots n\pi, \quad ...(2.6.9)$$

where n is an integer. From (5), we further deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t}$$

giving the solution

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where we have set $c_1c_3 = b_n$. Using the initial condition gives

$$u(x, 0) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

At this point, we recognize that we have a Fourier sine series and that the coefficients b_n are chosen such that

$$b_n = 2\int_0^1 (x - x^2)\sin n\pi x \, dx$$

$$= 2\left[\frac{1 - 2x}{n^2\pi^2}\cos n\pi x + \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right)\cos n\pi x\right]_0^1$$

$$= \frac{4}{n^3\pi^3}(1 - (-1)^n).$$

Thus, the solution of the PDF as

$$u(x, t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x$$

Que 2.7. **Solve the following partial differential equation by using method of separation of variables :**

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial^2 y} = 0; \ z(x, 0) = 0, \ z(x, \pi) = 0, \ z(0, y) = 4\sin 3y.$$

AKTU 2021-22 (Sem-3), Marks 10

Answer

Applying the method of separation of variables, let $z = X(x)\,Y(y)$. Putting this with given equation, we get

$$X'Y + XY'' = 0 \text{ or } \frac{X'}{X'} = -\frac{Y''}{Y} = K \text{ (a constant).}$$

$$\frac{X'}{X'} - K = 0 \text{ and } \frac{Y''}{Y} + K = 0$$

Case I : $K = 0$ then we get $X = c_2$ and $Y = c_2y + c_3$

Hence $z = c_1(c_2y + c_3)$ which is not suitable here.

Case II : $K > 0$, let $K = p^2$ then we have

$$\frac{X'}{X'} = p^2 \text{ and } \frac{Y''}{Y} + p^2 = 0 \text{ which give}$$

$$X = c_1 e^{p^2 x} \text{ and } Y = c_2 \cos py + c_2 \sin py$$

Hence $$z = c_1 e^{p^2 x} (c_1 \cos py + c_3 \sin py)$$

which suits the given boundary conditions.

The condition $z(x, 0) = 0$ gives $c_2 = 0$ hence

$$z = ce^{p^2 x} \sin py$$

Now, $z(x, u) = 0$ gives $\sin py = 0$ $[p = n, n, \in I]$

Thus, we can write $z = ce^{p^2 x} \sin ny$

Further, $z(0, y) = 4\sin 3y$ gives $c = 4$ and $n = 3$

The solution is $z = 4e^{2x} \sin 3y$.

Que 2.8. **Solve the following partial differential equation by method of separation of variables :**

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + 2u = 0. \; u(x, 0) = 10e^{-x} - 6e^{-4x}.$$

AKTU 2021-22 (Sem-4), Marks 10

Answer

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + 2u = 0 \quad ...(2.8.1)$$

Let the solution of equation (2.8.1) be

$$u(x, t) = X(x)\, T(t)$$

$$X'T - XT' + XT = 0$$

$$(X' - X)T = XT'$$

$$\frac{X' - X}{X} = \frac{T'}{T} = k \text{ (say)}$$

Now, $$\frac{X' - X}{X} = k \Rightarrow \frac{X'}{X} - 1 = k \Rightarrow \frac{X'}{X} = k + 1$$

On integrating both side

$$\int \frac{X'}{X} = \int (k+1)$$

$$\log X = (k+1)x + \log C_1$$

$$\log X = (k+1)x \log e + \log C_1$$

$$\log X = \log e^{(k+1)x} + \log C_1$$

$$X = C_1 e^{(k+1)x}$$

Now, $$\frac{T'}{T} = k$$

On integrating both side

$$\int \frac{T'}{T} = \int k$$

$$\log T = kt + \log C_2$$

$$T = C_2 e^{kt}$$

$$u(x, t) = X.\, T = C_1 e^{(k+1)x} C_2 e^{kt}$$

$$\therefore \quad = C_1 C_2 e^{(k+1)x + kt} \quad ...(2.8.2)$$

Given : $$u(x, 0) = 10\, e^{-x} - 6\, e^{-4x}$$

when $$t = 0$$

$$10\, e^{-x} - 6\, e^{-4x} = C_1 C_2 e^{(k+1)x + kt}$$

$$e^{-x}(10 - 6\, e^{-3x}) = C_1 C_2 e^{(k+1)x}$$

On comparing both side

$$C_1 C_2 = 10 - 6\, e^{-3x} \qquad k + 1 = -1 \Rightarrow k = -2$$

Put value of $C_1\, C_2$ and k in eq. 2

$$u\,(x, t) = (10 - 6\, e^{-3x})\, e^{-x-2t}$$

Que 2.9. **Solve the partial differential equation by the method of separation of variables $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, given that $u = 5e^{-y} - e^{-5y}$, when $x = 0$.**

AKTU 2022-23 (Sem-4), Marks 10

Answer

Given : $$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$$

Let the solution be $u(x, y) = X(x) = Y(y)$

$$4X'Y + XY' = 3XY$$

$$4X'Y = (3Y - Y')X$$

$$\frac{4X'}{X} = \frac{3Y - Y'}{Y} = K \text{ (say)}$$

Solving $$\frac{X'}{X} = \frac{K}{4}$$

On integrating, $\log X = \dfrac{Kx}{4} + \log C_1 \Rightarrow X = C_1 e^{Kx/4}$

Solving $3 - \frac{Y'}{Y} = K \Rightarrow \frac{Y'}{Y} = 3 - K$

On integrating, $\log Y = (3 - K)y + \log C_2 \Rightarrow Y = C_2 e^{(3-K)}$

$$u(x, y) = C_1 C_2 e^{Kx/4 + (3-K)y}$$

When $x = 0$, $u(0, y) = C_1 C_2 e^{(3-K)y}$

$$5e^{-y} - e^{-5y} = C_1 C_2 e^{(3-K)y}$$

On comparing, $C_1 C_2 e^{(3-K)y} = 5e^{-y}$

$$C_1C_2 = 5 \text{ and } 3 - K = -1 \Rightarrow = 4$$

$$u(x_1, y) = 5e^{x-y} \quad ...(2.9.1)$$

On comparing, $C_1C_2 e^{(3-K)y} = -e^{-5y}$

$$C_1C_2 = -1 \text{ and } 3 - K = -5 \Rightarrow K = 8$$

$$u(x, y) = -e^{2x-5y} \quad ...(2.9.2)$$

From equation (2.9.1) and (2.9.2)

$$u(x, y) = 5e^{x-y} - e^{2x-5y}$$

PART-2

Solution of One Dimensional Heat Equation, Wave Equation.

Que 2.10. Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x),\ 0 \le x \le 1$$

AKTU 2020-21 (Sem-3), Marks 10

Answer

Given wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad ...(2.10.1)$$

Let $y = XT$, where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 y}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2}$$

and

$$\frac{\partial^2 y}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

Substituting these values in eq. (2.10.1)

$$X\frac{\partial^2 T}{\partial t^2} = a^2 T\frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{a^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \text{(say)}$$

$$(D^2 + k^2)X = 0 \text{ and } (D^2 + a^2k^2)T = 0$$

$$m = \pm ki \text{ and } m = \pm aki$$

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 \cos akt + C_4 + \sin akt$$

Thus $y = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos akt + C_4 \sin akt)$ ...(2.10.2)

The boundary conditions are,

$$y(0, t) = 0$$

$$y(1, t) = 0$$

Initial conditions are,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x, 0) = \sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x),\ 0 \le x \le 1$$

Put $x = 0, y = 0$ in eq. (2.10.2),

$$0 = C_1(C_3 \cos akt + C_4 \sin akt)$$

$$C_1 = 0$$

From eq. (2.10.2),

$$y = C_2 \sin kx \cdot (C_3 \cos akt + C_4 \sin akt) \quad ...(2.10.3)$$

Now

$$\frac{\partial y}{\partial t} = C_2 \sin kx \cdot (-ak C_3 \sin akt + akC_4 \cos akt)$$

$$= ak\, C_2 \sin kx \cdot (-C_3 \sin akt + C_4 \cos akt)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$0 = ak\, C_2 \sin kx(C_4)$$

$$C_4 = 0$$

From eq. (2.10.3),

$$y = C_2 \sin kx \cdot C_3 \cos akt$$

$$y = A_n \sin kx \cdot \cos akt \quad [\because C_2C_3 = A_n] \quad ...(2.10.4)$$

Put $x = 1, y = 0$ in eq. (2.10.4)

$$0 = A_n \sin k \cdot \cos akt$$

$$\sin k = 0$$

$$k = n\pi$$

From eq. (2.10.4),

$$y = A_n \sin(n\pi x)\cos(an\pi t) \quad ...(2.10.5)$$

Now, $t = 0 \quad y = \sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x)$

$\sum A_n \sin(n\pi x) = \sin \pi x + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x)$

which will be satisfied by taking

$$A_n = \frac{1}{n} \text{ and } n = 1, 3, 5$$

Hence the required solution is from eq. (2.10.5)

$$y = \frac{1}{n}\sin(n\pi x)\cos(an\pi t) \text{ for } n = 1, 3, 5$$

Que 2.11. Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$ representing the vibrations of the string of length l fixed at both ends. Given that $y(0, t) = 0$; $y(l, t) = 0$, $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t}(x, 0) = 0$; $0 < x < l$.

Answer

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} \quad ...(2.11.1)$$

Let

$$y = XT \quad ...(2.11.2)$$

Where X is a function of x only and T is a function of t only, be a solution of eq. (2.11.1)

Then, $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Putting these values in eq. (2.11.1), we get

$$\frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = k \text{ (say)} \quad ...(2.11.3)$$

$$X'' = kX$$

$$X'' - kX = 0 \quad ...(2.11.4)$$

When k is negative and $k = -p^2$, say

$$X = C_1 \cos px + C_2 \sin px$$

$$T = C_3 \cos Cpt + C_4 \sin Cpt$$

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt) \quad ...(2.11.5)$$

Due to vibrations problem, y must be periodic function of x and t.

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt)$$

Now applying boundary conditions that

$y = 0$ when $x = 0$ and $y = 0$ when $x = l$, we get

$$0 = C_1 (C_3 \cos cpt + C_4 \sin Cpt) \quad ...(2.11.6)$$

$$0 = (C_1 \cos pl + C_2 \sin pl)(C_3 \cos Cpt + C_4 \sin Cpt) \quad ...(2.11.7)$$

From eq. (2.11.6), we have $C_1 = 0$ and eq. (2.11.7) reduces to

$$C_2 \sin pl (C_3 \cos Cpt + C_4 \sin Cpt) = 0$$

$$\sin pl = 0$$

$$pl = n\pi \text{ or } p = \frac{n\pi}{l} \text{ where } n = 1,2,3,....$$

A solution of wave equation

$$y = C_2\left(C_3 \cos \frac{n\pi Ct}{l} + C_4 \sin \frac{n\pi Ct}{l}\right)\sin \frac{n\pi x}{l}$$

$$= \left(a_n \cos \frac{n\pi Ct}{l} + b_n \sin \frac{n\pi Ct}{l}\right)\sin \frac{n\pi x}{l}$$

where $C_2 C_3 = a_n$ and $C_2 C_4 = b_n$

$$y = \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi Ct}{l} + b_n \sin \frac{n\pi Ct}{l}\right)\sin \frac{n\pi x}{l} \quad ...(2.11.8)$$

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$, where $t = 0$ ...(2.11.9)

we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad ...(2.11.10)$$

and

$$0 = \sum_{n=1}^{\infty} \frac{n\pi C}{l} b_n \sin \frac{n\pi x}{l} \quad ...(2.11.11)$$

Since eq. (2.11.10) represents Fourier series for $f(x)$, we have

$$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad ...(2.11.12)$$

From eq. (2.11.11), $b_n = 0$, for all n

Hence eq. (2.11.8) reduces to

$$y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi Ct}{l} \sin \frac{n\pi x}{l} \quad ...(2.11.13)$$

where a_n is given by eq. (2.11.12) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

Que 2.12. Find the displacement of a finite string of length L that is fixed at both ends and is released from rest with an initial displacement $f(x)$.

Answer

Let the equation of the string be

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = XT$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$$

It will satisfy the given differential equation

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 \cos kCt + C_4 \sin kCt$$

$$u = (C_1 \cos kx + C_2 \sin kx)\,(C_3 \cos kct + C_4 \sin kCt)$$

According to given conditions,

$$u(0, t) = u(L, t) = 0 \quad \text{...(2.12.1)}$$

$$u(x, 0) = f(x)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$$

Applying $u(0, t) = 0$ in eq. (2.12.1), $C_1 = 0$

$$u = \sin kx\,(A_n \cos kCt + B_n \sin kCt) \quad \text{...(2.12.2)}$$

where, $C_2 C_3 = A_n$

$C_2 C_4 = B_n$

Applying $u(L, t) = 0$

$$k = \frac{n\pi}{L}$$

$$u = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi Ct}{L}\right) + B_n \sin\left(\frac{n\pi Ct}{L}\right)\right]$$

$$\left(\frac{\partial u}{\partial t}\right) = \frac{n\pi C}{L}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n \sin\frac{n\pi Ct}{L} + B_n \cos\frac{n\pi Ct}{L}\right] \quad \text{...(2.12.3)}$$

Applying $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0, B_n = 0$

From eq. (2.12.3)

$$u = \sum A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi Ct}{L}\right) \quad \text{...(2.12.4)}$$

Applying $u(x, 0) = f(x)$ in eq. (2.12.4)

$$f(x) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{...(2.12.5)}$$

Thus complete solution is given by eq. (2.12.4) where A_n is given by eq. (2.12.5).

Que 2.13. **Write the solution of two dimensional wave equation.**

Answer

Equation of two dimensional wave is given by

$$\frac{\partial^2 u}{\partial t^2} = C^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{...(2.13.1)}$$

The boundary conditions are $u(0, y, t) = 0$ ...(i)

$u(a, y, t) = 0$ ...(ii)

$u(x, 0, t) = 0$ ...(iii)

$u(x, b, t) = 0$ ...(iv)

The initial conditions are

$u(x, y, 0) = f(x, y)$ ...(v)

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y)$ ...(iv)

Let $u = X(x)\,Y(y)\,T(t)$ is a solution of eq. (2.13.1). Differentiating partially w.r.t. x, y, and t and putting the values in eq. (2.13.1),

$$XY\frac{\partial^2 T}{\partial t^2} = C^2\left(YT\frac{\partial^2 X}{\partial x^2} + XT\frac{\partial^2 Y}{\partial y^2}\right)$$

Dividing both sides by XYT,

$$\frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_1^2,\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_2^2,\ \frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$

Where $k^2 = k_1^2 + k_2^2$

$X = C_1 \cos k_1 x + C_2 \sin k_1 x,$

$Y = C_3 \cos k_2 y + C_4 \sin k_2 y$

and $T = C_5 \cos kCt + C_6 \sin kCt$

Thus $u = (C_1 \cos k_1 x + C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt)$ ...(2.13.2)

Case ii : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_1^2,\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_2^2$ and $\frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$

Where, $k^2 = k_1^2 + k_2^2$

Its solution is given by

$$u = (C_1 e^{k_1 x} + C_2 e^{-k_1 x})(C_3 e^{k_2 y} + C_4 e^{-k_2 y})(C_5 e^{kCt} + C_6 e^{-kCt}) \quad ...(2.13.3)$$

Case iii : When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$ and $\frac{1}{C^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

Its solution is

$$u = (C_1 x + C_2)(C_3 y + C_4)(C_5 t + C_6) \quad ...(2.13.4)$$

Solution of two dimensional wave equation is given by eq. (2.13.2). Apply boundary condition (i) to eq. (2.13.2),

$\Rightarrow$ $0 = C_1$

From eq. (2.13.2),

$$u = C_2 \sin k_1 x(C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.5)$$

At $x = a, u = 0$

$$0 = C_2 \sin k_1 a(C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt)$$

$$\sin k_1 a = 0 = \sin m\pi$$

$$k_1 = \frac{m\pi}{a}$$

From eq. (2.13.5),

$$u = C_2 \sin \frac{m\pi x}{a}(C_3 \cos k_2 y + C_4 \sin k_2 y)(C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.6)$$

Now at $y = 0, u = 0$

From eq. (2.13.6),

$\Rightarrow$ $0 = C_3$

$$u = C_2 \sin \frac{m\pi x}{a} C_4 \sin k_2 y(C_5 \cos kCt + C_6 \sin kCt) \quad ...(2.13.7)$$

At $y = b, u = 0$

$$\sin k_2 b = 0 = \sin n\pi$$

$$k_2 = \frac{n\pi}{b}$$

From eq. (2.13.7)

$$u = C_2 C_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}(C_5 \cos kCt + C_6 \sin kCt)$$

or $$u = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)(A_{mn} \cos kCt + B_{mn} \sin kCt) \quad ...(2.13.8)$$

Now apply initial condition (v),

$$f(x, y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)A_{mn}$$

$$A_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \quad ...(2.13.9)$$

Differentiate eq. (2.13.8) w.r.t. t,

$$\frac{\partial u}{\partial t} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)[-kCA_{mn} \sin kCt + kCB_{mn} \cos kCt]$$

At $t = 0, \frac{\partial u}{\partial t} = g(x, y)$

$$g(x, y) = kC \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)B_{mn}$$

$$kCB_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b g(x, y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy \quad ...(2.13.10)$$

Solution of two dimensional wave equation is given by eq. (2.13.8) and the values of A_{mn} and B_{mn} are given by eq. (2.13.9) and eq. (2.13.10).

Que 2.14. **Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$.**

Answer

Equation of heat in one dimension is given by

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = X(x)\, T(t)$$

$$X\frac{\partial T}{\partial t} = C^2 T\frac{\partial^2 X}{\partial x^2} \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2 T}\frac{\partial T}{\partial t} = -k^2 \text{(let)}$$

$$X = C_1 \cos kx + C_2 \sin kx$$

and $$T = C_3 e^{-k^2 C^2 t}$$

Thus, $$u = (C_1 \cos kx + C_2 \sin kx)\, C_3 e^{-k^2 C^2 t} \quad ...(2.14.1)$$

Given boundary conditions are $u(0, t) = u(2, t) = 0$

Put $u(0, t) = 0$ in eq. (2.14.1),

$$0 = C_1 C_3 e^{-k^2 C^2 t}$$

From eq. (2.14.1), $C_1 = 0$

$$u = C_2 C_3 \sin kx\, e^{-k^2C^2t}$$

$$u = A_n \sin kx\, e^{-k^2C^2t}$$

Apply $u(2, t) = 0$ ...(2.14.2)

$$0 = A_n \sin 2k\, e^{-k^2C^2t}$$

$$\sin 2k = 0$$

$$k = \frac{n\pi}{2}$$

From eq. (2.14.2),

$$u = \sum A_n \sin \frac{n\pi x}{2} e^{-n^2\pi^2C^2t/4}$$

...(2.14.3)

Apply initial condition $u(x, 0) = 100\,x$

$$100\,x = \sum A_n \sin \frac{n\pi x}{2}$$

$$A_n = \frac{2}{2}\int_0^2 100x \cdot \sin \frac{n\pi x}{2} dx$$

$$A_n = 100\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= 100\left[-\frac{4}{n\pi}\cos n\pi\right] = -\frac{400}{n\pi}(-1)^n = \frac{400}{n\pi}(-1)^{n+1}$$

Thus from eq. (2.14.3), $u = \sum_{n=1}^{\infty} \frac{400}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{2} e^{-n^2\pi^2C^2t/4}$

Que 2.15. **Write two dimensional heat conduction equation with solution.**

Answer

The partial differential equation of two dimensional heat conduction problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{C^2}\frac{\partial u}{\partial y} \quad \text{... (2.15.1)}$$

The boundary conditions are

$$u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0$$

and the initial condition is

$$u(x, y, 0) = f(x, y)$$

Let the solution be $u = X\,Y\,T$

Where X is a function of x, Y is a function of y and T is a function of t.

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = YT\frac{d^2X}{dx^2}, \frac{\partial^2 u}{\partial y^2} = XT\frac{d^2Y}{dy^2}, \frac{\partial u}{C^2\partial t} = \frac{XY}{C^2}\frac{dT}{dt}$$

Now equation from (2.15.1)

$$YT\frac{d^2X}{dx^2} + XT\frac{d^2Y}{dy^2} = \frac{XY}{C^2}\frac{dT}{dt}$$

or $$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{C^2T}\frac{dT}{dt}$$

(Dividing by XYT on both sides)

Since X is a function of independent variable x, Y of y and T of t, there are three possibilities :

i. $$\frac{1}{X}\frac{d^2X}{dx^2} = 0, \frac{1}{Y}\frac{d^2Y}{dy^2} = 0, \frac{1}{C^2T}\frac{dT}{dt} = 0$$

ii. $$\frac{1}{X}\frac{d^2X}{dx^2} = k_1^2, \frac{1}{Y}\frac{d^2Y}{dy^2} = k_2^2, \frac{1}{C^2T}\frac{dT}{dt} = k^2$$

iii. $$\frac{1}{X}\frac{d^2X}{dx^2} = -k_1^2, \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_2^2, \frac{1}{C^2T}\frac{dT}{dt} = -k^2$$

Case i : $X = C_1x + C_2$, $Y = C_3y + C_4$, $T = C_5$

Thus $u = XYT$

$\Rightarrow$ $u = (C_1x + C_2)(C_3y + C_4)C_5$

Case ii : $X = C_1 e^{k_1x} + C_2 e^{-k_1x}$

$Y = C_3 . e^{k_2y} + C_4 e^{-k_2y}$, $T = C_5 e^{C^2k^2t}$

Thus $u = X\,Y\,T$

$\Rightarrow$ $u = (C_1 e^{k_1x} + C_2 e^{-k_1x})(C_3 . e^{k_2y} + C_4 e^{-k_2y}) C_5 e^{C^2k^2t}$

Case iii : $X = C_1 \cos k_1x + C_2 \sin k_1 x$,

$Y = C_3 \cos k_2y + C_4 \sin k_2y$, $T = C_5 e^{-C^2k^2t}$

$\therefore$ $u = (C_1 \cos k_1x + C_2 \sin k_1x)(C_3 \cos k_2y + C_4 \sin k_2y) C_5 e^{-C^2k^2t}$

Out of these three solutions, we have to choose that solution which satisfies the heat equation. Accordingly, case (iii) is accepted here.

$\therefore$ $u = (C_1 \cos k_1x + C_2 \sin k_1x)(C_3 \cos k_2y + C_4 \sin k_2y) C_5 e^{-C^2k^2t}$ ...(2.15.2)

Now we apply boundary conditions on putting $u = 0$ and $x = 0$ in eq. (2.15.2), we get

$$0 = C_1(C_3 \cos k_2y + C_4 \sin k_2y) C_5 e^{-C^2k^2t}$$

$\Rightarrow \quad C_1 = 0$

$\therefore$ The eq. (2.20.2) reduces to

$$u = (C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y) C_5 e^{-C^2 k^2 t}$$

or

$$u = \sin k_1 x (A_{mn} \cos k_2 y + B_{mn} \sin k_2 y) e^{-k^2 C^2 t} \quad ...(2.15.3)$$

Now at $x = a, u = 0$

$$0 = \sin k_1 a (A_{mn} \cos k_2 y + B_{mn} \sin k_2 y) e^{-k^2 C^2 t}$$

$$\sin k_1 a = 0 = \sin m\pi$$

$$k_1 = \frac{m\pi}{a}$$

From eq. (2.15.3),

$$u = \sin\left(\frac{m\pi x}{a}\right)[A_{mn} \cos k_2 y + B_{mn} \sin k_2 y]\, e^{-k^2 C^2 t}$$

Now at $y = 0, u = 0$

$\Rightarrow \quad A_{mn} = 0 \quad ...(2.15.4)$

and at $y = b, u = 0$

$$\sin k_2 b = 0 = \sin n\pi$$

$$k_2 = \frac{n\pi}{b}$$

Thus from eq. (2.15.4),

$$u = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-k^2 C^2 t} \quad ...(2.15.5)$$

where

$$k^2 = k_1^2 + k_2^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$k = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Now the initial condition $u(x, y, 0) = f(x, y)$

From eq. (2.15.5),

$$f(x, y) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where

$$B_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx\, dy$$

Solution is given by eq. (2.15.5) and value of B_{mn} is given by the above equation.

Que 2.16. **A square plate is bounded by lines $x = 0, y = 0; x = 20, y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while the upper three edges are kept at 0 °C. Find the steady state temperature.**

Answer

The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Its solution is

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$$u(0, y) = 0$$

$$0 = C_1(C_3 e^{py} + C_4 e^{-py})$$

$\therefore \quad C_1 = 0$

Then,

$$u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py})$$

$$u(20, y) = 0$$

$$0 = C_2 \sin 20p\,(C_3 e^{py} + C_4 e^{-py})$$

$$\sin 20p = \sin n\pi = 0$$

$\therefore \quad p = \dfrac{n\pi}{20}$

$$u(x, y) = \sin\frac{n\pi x}{20}\left(C_2 C_3 e^{\frac{n\pi}{20}y} + C_2 C_4 e^{-\frac{n\pi}{20}y}\right)$$

$$= \sin\frac{n\pi x}{20}\left(Ae^{\frac{n\pi}{20}y} + Be^{-\frac{n\pi}{20}y}\right)$$

where, $A = C_2 C_3$ and $B = C_2 C_4$

$$u(x, 0) = 0$$

$$0 = \sin\frac{n\pi x}{20}(A + B)$$

$$A = -B$$

$\therefore \quad u(x, y) = A \sin\dfrac{n\pi}{20}x\left[e^{\frac{n\pi}{20}y} - e^{-\frac{n\pi}{20}y}\right]$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{20}x \sinh\frac{n\pi}{20}y$$

$$u(x, 20) = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin\frac{n\pi x}{20}$$

$$= x(20 - x)$$

where, $$b_n = \frac{2}{20 \times \sinh n\pi} \int_0^{20} x(20 - x) \sin \frac{n\pi}{20} x \, dx$$

$$= \frac{2}{20 \times \sinh n\pi} \left[\left\{ (20x - x^2) \left(\frac{-\cos \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) \right\}_0^{20} - \int_0^{20} (20 - 2x) \left(\frac{-\cos n\pi \frac{}{20} x}{\frac{n\pi}{20}} \right) dx \right]$$

$$= \frac{1}{10 \sinh n\pi} \times \frac{20}{n\pi} \int_0^{20} (20 - 2x) \cos \frac{n\pi}{20} x \, dx$$

$$= \frac{2}{n\pi \sinh n\pi} \left[\left\{ (20 - 2x) \frac{\sin \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right\}_0^{20} - \int_0^{20} (-2) \left(\frac{\sin \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) dx \right]$$

$$= \frac{4}{n\pi \sinh n\pi} \times \frac{20}{n\pi} \left(\frac{-\cos \frac{n\pi}{20} x}{\frac{n\pi}{20}} \right) = \frac{4 \times 20^2}{n^3 \pi^3 \sinh n\pi} \left(1 - \cos \frac{n\pi}{20} \right)$$

$$= \begin{cases} \dfrac{3200}{n^3 \pi^3 \sinh n\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$\therefore \quad u(x, y) = \frac{3200}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{\sin \frac{n\pi}{20} x \sinh \frac{n\pi}{20} y}{n^3 \sinh n\pi}$$

$$\therefore \quad u(x, y) = \frac{3200}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin \frac{(2n+1)}{20} x \sinh \frac{(2n+1)}{20} \pi y}{(2n+1)^3 \sinh (2n+1)\pi}$$

Que 2.17. **A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = a\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement.**

AKTU 2022-23 (Sem-4), Marks 10

Answer

The displacement $y(x, t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(2.17.1)}$$

The boundary conditions are

i. $y(0, t) = 0, \forall t \geq 0.$

ii. $y(\forall tl^3, t)0. = 0.$

iii. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, for $0 < x < l$

iv. $y(x, 0) = y^0 \sin^3 (\pi x / l)$, for $0 < x < l$.

The suitable solution of (2.17.1) is given by

$y(x, t) = (A \cos\lambda x + B \sin \lambda x) (C \cos \lambda \text{ at} + D \sin \lambda \text{ at})$ ...(2.17.2)

Using (i) and (ii) in (2.17.2), we get

$$A = 0 \text{ and } \lambda = \frac{n\pi}{l}$$

$$\therefore \quad y(x, t) = \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right) \quad \text{...(2.17.3)}$$

Now, $$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l} \left(-C \sin \frac{n\pi at}{l} \cdot \frac{n\pi a}{l} + D \cos \frac{n\pi at}{l} \cdot \frac{n\pi a}{l} \right)$$

Using (iii) in the above equation, we get

$$0 = B \sin \frac{n\pi x}{l} D \frac{n\pi a}{l}$$

Here, B cannot be zero. Therefore $D = 0$.

Hence eq. (2.17.3) becomes

$$y(x, t) = B \sin \frac{n\pi x}{l} C \cos \frac{n\pi at}{l}$$

$$= B_1 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}, \text{ where } B_1 = BC$$

The most general solution is,

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad \text{...(2.17.4)}$$

Using (2.17.4), wet get

$$y_0 \sin^3 \frac{n\pi}{l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

i.e., $$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = y_0 \left\{ \frac{3}{4} \sin \frac{\pi x}{l} - \frac{1}{4} \sin \frac{3\pi x}{l} \right\}$$

i.e., $$B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \ldots$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

Equating the like coefficients on both sides, we get

$$B_1 = \frac{3y_0}{4},\ B_3 = \frac{-y_0}{4},\ B_2 = B_4 = \ldots = 0$$

Substituting in (2.17.4), we get

$$y(x, t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

Que 2.18. **A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0 °C and are kept at that temperature. Calculate the temperature function $u(x, t)$.**

AKTU 2020-21 (Sem-3), Marks 10

Answer

The temperature function $u(x, t)$ satisfies the differential equation,

$$\frac{\partial u}{\partial t} = c^2\frac{\partial u}{\partial x^2}$$

The boundary conditions associated with the problem are :

$$u(0, t) = 0,\ u(l, t) = 0$$

The initial condition is $u(x, 0) = u_0$

Its solution is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} u_n \sin\left(\frac{n\pi x}{l}\right)e^{-\lambda_n t}$$

$$\lambda_n = \frac{n\pi c}{t}$$

Since, $\mu(x, 0) = \mu_0$

we have,

where, $$a_n = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi nx}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l \mu_0 \sin\left(\frac{n\pi x}{l}\right)dx$$

Hence, $$u(x, t) = \frac{4\mu_0}{\pi}\sum_{n=1,3,5,\ldots\ldots} \frac{1}{n}\sin\frac{n\pi x}{l}e\frac{-c^2n^2\pi^2t}{l^2}$$

Que 2.19. **A laterally insulated bar of length has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If the temperature at B is suddenly reduced to 0°C and kept s while that of A is maintained at 0°C. Find the temperature at a distance x from A at any time t.**

AKTU 2021-22 (Sem-3), Marks 10

OR

An insulated rod of length 1 has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced at 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.

AKTU 2022-23 (Sem-4), Marks 10

Answer

The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \ldots(2.19.1)$$

The boundary conditions associated with the problem are

$$u(0, 1) = 0,\ u(l, t) = 0$$

when $t = 0$, the heat flow is independent of time (steady state condition) and so eq. (2.19.1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \ldots(2.19.2)$$

In general solution is given by $u = ax + b$

where a and b are arbitrary.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ we get from eq. (2.19.2), $b = 0$

and $a - \dfrac{100}{t}$

Thus the initial condition is expressed by $u(x, 0) = \dfrac{100}{l}$ v.

The solution of (2.19.1) with the respective boundary condition is given by

$$ux, l = \sum_{a=4}^{\infty} a_n\, x, l$$

$$= \sum_{a=4}^{\infty} a_n \sin\left(\frac{a\pi x}{L}\right)e\pi^2,\ \lambda = \frac{n\pi c}{L}$$

Since $$u(x, 0) = \sum_{a=4}^{\infty} a_n \sin\left(\frac{a\pi x}{L}\right)$$

where $$a_n = \frac{2}{I}\int_0^I \frac{100}{I}x\sin\left(\frac{n\pi x}{I}\right)dx$$

$$= \frac{200}{l}\left[\left[x\left[\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (1)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos \pi x\right] = -\frac{200}{n\pi}(-1) = \frac{200}{n\pi}(-1)$$

Hence $\quad u(x, t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)}{n}\sin\frac{n\pi x}{l}$

PART-3

Two Dimensional Heat Heat Equation (Only Laplace Equation), and Their Application.

Que 2.20. **Find the possible general solutions of two dimensional Laplace equation using method of separation of variables.**

Answer

Laplace equation in two dimension is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(2.20.1)$$

Let $\quad u = XY \quad ...(2.20.2)$

Then, $\quad \dfrac{\partial^2 u}{\partial x^2} = X''Y$

$$\frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting in eq. (2.20.1),

$$X''Y + XY'' = 0$$

or $\quad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = k \text{ (say)} \quad ...(2.20.3)$

Now, $\quad \dfrac{d^2X}{dx^2} - kX = 0 \quad ...(2.20.4)$

$$\frac{d^2Y}{dy^2} + kY = 0$$

Solving eq. (2.20.4), we get

i. When k is positive and $k = p^2$

$$X = C_1 e^{px} + C_2 e^{-px}$$
$$Y = C_3 \cos py + C_4 \sin py$$

ii. When k is negative and $k = -p^2$

$$X = C_1 \cos px + C_2 \sin px,\ Y = C_3 e^{py} + C_4 e^{-py}$$

iii. When $\quad k = 0$

$$X = C_1 x + C_2,\ Y = C_3 y + C_4$$

Thus, the various possible solutions of Laplace equation (2.20.2) are

$$u = (C_1 e^{px} + C_1 e^{-px})(C_3 \cos py + C_4 \sin py) \quad ...(2.20.5)$$
$$u = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(2.20.6)$$
$$u = (C_1 x + C_2)(C_3 y + C_4) \quad ...(2.20.7)$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions.

Que 2.21. **Solve : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to boundary conditions $u(0, y) = 0 = u(\pi, y)$, and $u(x, 0) = u_0$, $\lim\limits_{y \to \infty} u(x, y) = 0$, $0 < x < \pi$.**

Answer

Given, $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Let, $\quad u = XY$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$u = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad ...(2.21.1)$$

$$u(0, y) = 0$$
$$0 = C_1 (C_3 e^{ky} + C_4 e^{-ky})$$
$$C_1 = 0 \quad ...(2.21.2)$$

From eq. (2.21.1),

$$u = \sin kx\,(A_n e^{ky} + B_n e^{-ky})$$
$$u(\pi, y) = 0$$
$$\sin k\pi = 0 \Rightarrow k = n \quad ...(2.21.3)$$
$$u = \sin nx\,(A_n e^{ny} + B_n e^{-ny})$$

$\lim\limits_{y \to \infty} u(x, y) = 0$, it satisfies only when $A_n = 0$.

From eq. (2.21.3),

$$u = \sum B_n e^{-ny} \sin nx$$

Now $\quad u(x, 0) = u_0$ ...(2.21.4)

$$u_0 = \sum B_n \sin nx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{-2u_0}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-2u_0}{\pi}\left[\frac{(-1)^n - 1}{n}\right]$$

Thus from eq. (2.21.4),

$$u = \sum \frac{-2u_0}{\pi n}[(-1)^n - 1]e^{-ny} \sin nx$$

Que 2.22. **Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy-plane, $0 \le x \le a$ and $0 \le y \le b$ satisfying the following boundary conditions $u(x, 0) = 0, u(x, b) = 0$ and $u(0, y) = 0, u(a, y) = f(y)$.**

Answer

Given Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(2.22.1)$$

Let $u = XY$, where X is a function of x only and Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

From eq. (2.22.1),

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i :

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \text{ (say)}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

and

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$X = C_1 x + C_2,\ Y = C_3 y + C_4$$

At $\quad y = 0,\ Y = 0 \Rightarrow C_4 = 0$

Also, $\quad y = b,\ Y = 0 \Rightarrow C_3 = 0$

$$Y = 0$$

$\therefore \quad u = XY = X(0)$

Thus, $\quad u = 0$ (not possible)

Case ii :

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (say)}$$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

and

$$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$$X = C_1 \cos kx + C_2 \sin kx,\ Y = C_3 e^{ky} + C_4 e^{-ky}$$

If $\quad y = 0,\ Y = 0$

$$C_3 + C_4 = 0$$

$$C_4 = -C_3$$

and $\quad Y = 0$ at $y = b$

$$0 = C_3 e^{kb} - C_3 e^{-kb}$$

$$C_3(e^{kb} - e^{-kb}) = 0$$

$$C_3 = 0,\ C_4 = 0,\ Y = 0$$ (not possible)

Case iii :

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 e^{kx} + C_2 e^{-kx},$$

$$Y = C_3 \cos ky + C_4 \sin ky$$

At $\quad y = 0,\ Y = 0,\ C_3 = 0$

$$Y = C_4 \sin ky$$

At $\quad y = b,\ Y = 0$

$$0 = C_4 \sin kb$$

$$\sin kb = 0$$

$$kb = n\pi$$

$$k = \frac{n\pi}{b}$$

$$u = (C_1 e^{kx} + C_2 e^{-kx})C_4 \sin\frac{n\pi y}{b} \quad ...(2.22.2)$$

Thus,

At $\quad x = 0,\ u = 0$

$$0 = (C_1 + C_2)C_4 \sin\frac{n\pi y}{b}$$

$$C_1 + C_2 = 0$$

$$C_2 = -C_1$$

From eq. (2.22.2),

$$u = \frac{2}{2}C_4 C_1\left(e^{kx} - e^{-kx}\right)\sin\frac{n\pi y}{b}$$

Let
$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right) \quad ...(2.22.3)$$

At
$$b_n = 2C_1 C_4$$
$$x = a, \; u = f(y)$$

From eq. (2.22.3),

$$f(y) = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi a}{b}} - e^{\frac{-n\pi a}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

$$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad ...(2.22.4)$$

Thus,
$$u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where b_n is given by eq. (2.22.4)

Que 2.23. **In a telephone of wire of length l, a steady voltage distribution of 20 volts at the source end and 12 volts at the terminal end is maintained. At time $t = 0$, the terminal is grounded. Assuming $L = 0$, $G = 0$, determine the voltage and current where symbols have their usual meanings.**

Answer

The telegraph line equation is

$$\frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial t}$$

$$\frac{\partial V}{\partial t} = \frac{1}{RC} \frac{\partial^2 V}{\partial x^2} \quad ...(2.23.1)$$

Here, V_s = Initial steady voltage satisfying

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$V_s = 20 + \frac{(12-20)}{l} x$$

$$V_s = 20 - \frac{8}{l} x = V(x, 0)$$

$$V(x, 0) = 20 - \frac{8}{l} x \quad ...(2.23.2)$$

And let V_s' = steady voltage after grounding the terminal end (terminal voltage = 0)

$$V_s' = 20 - \frac{20x}{l} \quad ...(2.23.3)$$

$$V(x, t) = V_s' + V_t(x, t)$$

$$V(x, t) = 20 - \frac{20x}{l} + \sum b_n e^{-n^2\pi^2 t / l^2 RC} \sin\left(\frac{n\pi x}{l}\right) \quad ...(2.23.4)$$

Putting $t = 0$, $V(x, 0)$ is given by eq. (2.23.2)

From eq. (2.23.4),

$$20 - \frac{8}{l} x = 20 - \frac{20x}{l} + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{12x}{l} = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l \frac{12}{l} x \sin\frac{n\pi x}{l} dx$$

$$b_n = \frac{24}{l^2}\left[x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} \right]_0^l$$

$$= \frac{24}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi \right] = -\frac{24}{n\pi}(-1)^n$$

$$b_n = \frac{24}{n\pi}(-1)^{n+1}$$

Thus from eq. (2.23.4),

$$V(x, t) = 20 - \frac{20x}{l} + \sum \frac{24}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t / l^2 RC} \sin\left(\frac{n\pi x}{l}\right)$$

Also,

$$\frac{\partial V}{\partial x} = -\frac{20}{l} + \sum \frac{24}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t / l^2 RC} \left(\frac{n\pi}{l}\right) \cos\frac{n\pi x}{l} = -L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{20}{lL} - \sum \frac{24}{Ll}(-1)^{n+1} e^{-n^2\pi^2 t / l^2 RC} \cos\left(\frac{n\pi x}{l}\right)$$

On integrating

$$i = \frac{20\,t}{lL} + \frac{24\, l^2\, RC}{n^2\, \pi^2 Ll} \sum (-1)^{n+1} e^{\frac{-n^2\pi^2 t}{l^2 RC}} \cos\left(\frac{n\pi x}{l}\right) + A$$

At $t = 0$, $i = 0$, $A = 0$

$\therefore$
$$i = \frac{20\,t}{lL} + \frac{24l\,RC}{n^2\pi^2 L}\sum (-1)^{n+1} e^{\frac{-n^2\pi^2 t}{l^2 RC}} \cos\left(\frac{n\pi x}{l}\right)$$

Que 2.24. **Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that**

$$i(x, 0) = i_0,\ e(x, 0) = e_0 \sin \frac{\pi x}{l}$$

Also R and G are negligible.

Answer

Since R and G are negligible, transmission line equations becomes

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} \quad ....(2.24.1)$$

and
$$\frac{\partial i}{\partial x} = -C\frac{\partial e}{\partial t} \quad ....(2.24.2)$$

For elimination of i, differentiating eq. (2.24.1) partially w.r.t x and eq. (2.24.2) partially w.r.t t, we have

$$\frac{\partial^2 e}{\partial x^2} = -L\frac{\partial^2 i}{\partial x \partial t} \text{ and } \frac{\partial^2 i}{\partial t \partial x} = C\frac{\partial^2 e}{\partial t^2}$$

Hence,
$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \quad ....(2.24.3)$$

The initial conditions are $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin \frac{\pi x}{l}$ ....(2.24.4)

Since, the ends are suddenly grounded, the boundary conditions are

$$e(0, t) = e(l, t) = 0$$

Also $i = i_0$ (constant) when $t = 0$ ....(2.24.5)

$\therefore$
$$\frac{\partial i}{\partial x} = 0 \text{ which gives } \frac{\partial e}{\partial t} = 0 \text{ when } t = 0 \quad ....(2.24.6)$$

Now let $e = XT$ be a solution of eq. (2.24.3) where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 e}{\partial x^2} = X''T \text{ and } \frac{\partial^2 e}{\partial t^2} = XT''$$

$\therefore$ From eq. (2.24.3) $X''T = LCXT''$

Separating the variables $\dfrac{X''}{X} = LC\dfrac{T''}{T} = -p^2$ (say)

This leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} + p^2X = 0 \text{ and } \frac{d^2T}{dt^2} + \frac{p^2}{LC}T = 0$$

$\therefore$
$$X = C_1 \cos px + C_2 \sin px$$

$$T = C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}$$

$$e = XT = (C_1 \cos px + C_2 \sin px)$$

$\Rightarrow$
$$\left(C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}\right) \quad ...(2.24.7)$$

Applying the boundary conditions eq. (2.24.5) in eq. (2.24.7), we get

$$C_1 = 0 \text{ and } p = \frac{n\pi}{l}, n \text{ being an integer}$$

$\therefore$ Eq. (2.24.7) becomes

$$e = C_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi t}{l\sqrt{LC}} + C_4 \sin\frac{n\pi t}{l\sqrt{LC}}\right)$$

or
$$e = \sin\frac{n\pi x}{l}\left(A \cos\frac{n\pi t}{l\sqrt{LC}} + B \sin\frac{n\pi t}{l\sqrt{LC}}\right) \quad ...(2.24.8)$$

where
$$A = C_2C_3 \text{ and } B = C_2C_4$$

$$\frac{\partial e}{\partial t} = \sin\frac{n\pi x}{l}\left(-\frac{An\pi}{l\sqrt{LC}}\sin\frac{n\pi t}{l\sqrt{LC}} + \frac{Bn\pi}{l\sqrt{LC}}\cos\frac{n\pi t}{l\sqrt{LC}}\right)$$

Since $\dfrac{\partial e}{\partial t} = 0$ when $t = 0$, we get

$$B = 0$$

$\therefore$ From eq. (2.24.8)
$$e = A \sin\frac{n\pi x}{l}\cos\frac{n\pi t}{l\sqrt{LC}}$$

By superposition,
$$e = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi t}{l\sqrt{LC}} \text{ is also a solution}$$

But
$$e = e_0 \sin\frac{\pi x}{l} \text{ when } t = 0$$

$\therefore$
$$e_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

$\Rightarrow$
$$A_1 = e_0 \text{ and } A_2 = A_3 = ... = 0$$

Hence,
$$e = e_0 \sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}}$$

Now,
$$-L\frac{\partial i}{\partial t} = \frac{\partial e}{\partial x}$$

$\therefore$
$$\frac{\partial i}{\partial t} = -\frac{1}{L}\cdot\frac{e_0\pi}{l}\cos\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}}$$

Integrating w.r.t. t, regarding x as constant

$$i = -\frac{e_0\pi}{Ll}\cos\frac{\pi x}{l}\cdot\frac{l\sqrt{LC}}{\pi}\sin\frac{\pi t}{l\sqrt{LC}} + f(x) \quad ...(2.24.9)$$

where $f(x)$ is an arbitrary constant function.

Since $i = i_0$ when $t = 0$, we have $i_0 = 0 + f(x)$ or $f(x) = i_0$

$\therefore$ From eq. (2.24.9), we have

$$i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}}$$

Que 2.25. **Solve $\dfrac{\partial^2 V}{\partial x^2} = LC \dfrac{\partial^2 V}{\partial t^2}$ assuming that the initial voltage is $V_0 \sin \dfrac{\pi x}{l}$; $V_t(x_0) = 0$ and $V = 0$ at the ends $x = 0$ and $x = l$ for all t.**

Answer

Let
$$V = XT \qquad \text{...(2.25.1)}$$
where X is a function of x only and T is a function of t only.

$$\frac{\partial^2 V}{\partial x^2} = TX'' \text{ and } \frac{\partial^2 V}{\partial t^2} = TX''$$

Substituting in the given equations, we get

$$TX'' = LCTX''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -p^2 \text{ (say)}$$

i. $$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

$\therefore$ $$X = C_1 \cos px + C_2 \sin px$$

ii. $$LC\frac{T''}{T} = -p^2 \Rightarrow \quad T'' + \frac{p^2}{LC}T = 0$$

$\therefore$ $$T = C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}$$

Hence,
$$V = XT = (C_1 \cos px + C_2 \sin px)\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right) \qquad \text{...(2.25.2)}$$

Boundary conditions are

$$V(0, t) = 0 = V(l, t) \text{ and } \frac{\partial V}{\partial t} = 0 \text{ when } t = 0$$

Applying conditions on eq. (2.25.2), we get

$$0 = C_1\left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$$

$\Rightarrow$ $$C_1 = 0$$

From eq. (2.25.2),
$$V = C_2 \sin px \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right) \qquad \text{...(2.25.3)}$$

$$V(l, t) = 0 = C_2 \sin pl \left(C_3 \cos \frac{pt}{\sqrt{LC}} + C_4 \sin \frac{pt}{\sqrt{LC}}\right)$$

$$\sin pl = 0 = \sin n\pi \, (n \in I)$$

$\Rightarrow$ $$p = \frac{n\pi}{l}$$

$\therefore$ Hence, from eq. (2.25.3)

$$V = C_2 \sin \left[C_3 \cos \frac{n\pi t}{l\sqrt{LC}} + C_4 \sin \frac{n\pi t}{l\sqrt{LC}}\right] \qquad \text{...(2.25.4)}$$

$$\frac{\partial V}{\partial t} = C_2 \frac{n\pi}{l\sqrt{LC}} \sin \frac{n\pi x}{l}\left[-C_3 \sin \frac{n\pi t}{l\sqrt{LC}} + C_4 \sin \frac{n\pi t}{l\sqrt{LC}}\right]$$

At $t = 0$,
$$0 = C_2 \frac{n\pi}{l\sqrt{LC}} \sin \frac{n\pi x}{l} . C_4$$

$$C_4 = 0$$

$\Rightarrow$ Hence from eq. (2.25.4),

$$V = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}} \qquad \text{...(2.25.5)}$$

Now,
$$V(x, 0) = V_0 \sin \frac{\pi x}{l} = C_2 C_3 \sin \frac{n\pi x}{l}$$

Comparing, we get

$$C_2 C_3 = V_0 \text{ and } n = 1$$

Hence, the required solution is

$$V(x, t) = V_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Que 2.26. **Determine the solution of one dimensional heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$, $(t > 0)$ and the initial condition $u(x, 0) = 3 \sin \dfrac{\pi x}{l}$: l being the length of the bar.**

AKTU 2021-22 (Sem-4), Marks 10

Answer

Equation of heat is one dimension is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u = X(x)\, T(t)$$

$$X\frac{\partial T}{\partial t} = T\frac{\partial^2 X}{\partial x^2} \Rightarrow \frac{1}{X}\frac{\partial^2 x}{\partial x^2} = \frac{1}{T}\frac{\partial T}{\partial t} = -k^2 \text{ (let)}$$

$$X = C_1 \cos kx + C_2 \sin kx$$

and $\quad T = C_3 e^{-k^2 t}$

$$u = (C_1 \cos kx + C_2 \sin kx)\, C_3 e^{-k^2 t} \quad \text{...(2.26.1)}$$

Given boundary conditions are $u(0, t) = a$

Put $\quad u(0, t) = 0$ in eq. (2.26.1)

$$0 = C_1 C_3 e^{-k^2 t}$$

$$C_1 = 0$$

From eq. (2.26.1)

$$u = C_2 C_3 \sin kx\, e^{-k^2 t}$$

$$u = A_n \sin kx\, e^{-k^2 t} \quad \text{...(2.26.2)}$$

Apply $\quad u(l, t) = 0$

$$0 = A_n \sin lk\, e^{-k^2 t}$$

$$\sin lk = 0$$

$$k = \frac{n\pi}{l}$$

From eq. (2.26.2)

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 t}{l^2}} \quad \text{...(2.26.3)}$$

Put $\quad t = 0$ in eq. (2.26.3)

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

or $\quad f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$

Multiply both side by $\sin\left(\frac{m\pi x}{l}\right)$ and then integrating w.r.t x between the limits $x = 0$ to $x = l$

$$\int_0^l f(x) \sin\left(\frac{m\pi x}{l}\right) dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx$$

$$\left[\because \int_0^l \sin\left(\frac{mnx}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} \frac{l}{2}, & m = n \\ 0, & m \neq n \end{cases}\right]$$

$$\int_0^l f(x) \sin\left(\frac{m\pi x}{l}\right) dx = A_n \frac{l}{2}$$

$$A_n = \frac{2^l}{l} \int_0^l f(x) \sin\left(\frac{m\pi x}{l}\right) dx$$

Given, $\quad f(x) = 3 \sin \frac{\pi x}{l}$

$$A_n = \frac{2}{l}\int_0^l 3 \sin\frac{\pi x}{l} \sin\frac{m\pi x}{l}\, dx$$

$$= \frac{6}{l}\int_0^l \left[\left(\cos\frac{\pi x}{l} - \frac{m\pi x}{l}\right) - \cos\left(\frac{\pi x}{l} + \frac{m\pi x}{l}\right)\right] dx$$

$$= \frac{6}{l}\int_0^l \left[\cos\left(\frac{\pi}{l} - \frac{m\pi}{l}\right)x - \cos\left(\frac{\pi}{l} - \frac{m\pi}{l}\right)x\right] dx$$

$$= \frac{6}{l}\left[\int_0^l \frac{1}{\left(\frac{\pi}{l} - \frac{m\pi}{l}\right)} \cos\left(\frac{\pi}{l} - \frac{m\pi}{l}\right)x\left(\frac{\pi}{l} - \frac{m\pi}{l}\right) dx - \int_0^l \frac{1}{\left(\frac{\pi}{l} + \frac{m\pi}{l}\right)} \cos\left(\frac{\pi}{l} + \frac{m\pi}{l}\right)x\left(\frac{\pi}{l} - \frac{m\pi}{l}\right) dx\right]$$

$$\Rightarrow \frac{6}{l}\left[\frac{l}{\pi - m\pi}\left[\sin\left(\frac{\pi}{l} - \frac{m\pi}{l}\right)x\right]_0^l - \left(\frac{l}{m\pi + \pi}\right)\left[\sin\left(\frac{\pi}{l} + \frac{m\pi}{l}\right)\right]_0^l\right]$$

$$\Rightarrow \frac{6}{l}\left[\frac{l}{\pi - m\pi}\left[\sin\left(\frac{\pi - mx}{l}\right) l - \sin 0\right] - \frac{l}{m\pi + \pi}\left[\sin\frac{\pi + m\pi}{l} \times l - \sin 0\right]\right]$$

$$\Rightarrow \frac{6}{l}\left[\frac{l}{\pi - m\pi}(-1) - \frac{l}{m\pi + \pi}(-1)\right] \quad [\because \sin n\pi = 0]$$

$$\Rightarrow \frac{6}{l}\left[\frac{l}{\pi + m\pi} - \frac{l}{\pi - m\pi}\right]$$

$$= \frac{6}{l} \times \frac{l(\pi - m\pi) - (\pi + m\pi)l}{\pi^2 - m^2\pi^2}$$

$$= 6\frac{\pi - m\pi - \pi - m\pi}{\pi^2 - m^2\pi^2}$$

$$= -\frac{12\,m\pi}{\pi^2 - m^2\pi^2} = \frac{-12\,m}{\pi - m^2\pi}$$

Hence the general solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} A_n\, e^{-\frac{n^2\pi^2 t}{l^2}} . \sin\frac{n\pi x}{l}$$

where $\quad A_n = \dfrac{-12\,m}{\pi - m^2\pi}$

Que 2.27. **Determine the solution of Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = f(x)$.**

AKTU 2021-22 (Sem-4), Marks 10

Answer

Laplace equation in two dimension is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...(2.27.1)}$$

Let $u(x, y)$ be solution of (2.27.1)

Suppose $u(x, y) = X(x)\, Y(y)$ ...(2.27.2)

then $\quad \dfrac{\partial^2 u}{\partial x^2} = X''Y \quad$ and $\quad \dfrac{\partial^2 u}{\partial y^2} = XY''$

So, eq. (2.27.1) become $X''Y + XY'' = 0$

$$\Rightarrow \quad \frac{X''}{X} = \frac{-Y''}{Y} = K \quad \text{...(2.27.3)}$$

For $k < 0$ *i.e.*, $k = -\lambda^2$

then eq. (2.27.3) $X'' = -\lambda^2 X$ and $Y'' = +\lambda^2 Y$

$\Rightarrow \quad X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

$\Rightarrow \quad X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ and $Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$

So, form eq. (2.27.2)

$$u(x, y) = (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_3 e^{\lambda y} + C_4 e^{-\lambda y}) \quad \text{...(2.27.4)}$$

The nature of solution will depend on the boundary condition.

Put boundary condition in eq. (2.27.4)

$u(x, 0) = 0 = (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_3 + C_4)$

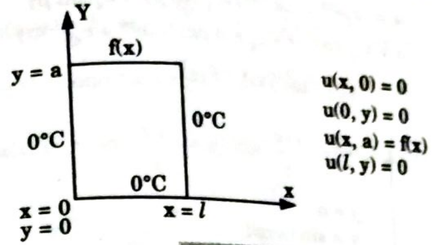

Fig. 2.27.1.

We get $C_3 = -C_4$

Then $u(x, y) = [C_1 \cos\lambda x + C_2 \sin\lambda x]\,[C_3 e^{\lambda y} + C_4 e^{-\lambda y}]$ ...(2.27.5)

Again $u(0, y) = 0$, then we get $C_1 = 0$

Again $u(l, y) = 0$

$\Rightarrow C_2 C_3 \sin\lambda l\,[e^{\lambda y} - e^{-\lambda y}]$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = n\pi/l$

All this value put in eq. (2.27.5), then we get

$$u(x, y) = C_2 C_3 \sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{a}}\right)$$

$$= 2C_n \sin\frac{n\pi x}{l}\left(\sinh\frac{n\pi y}{l}\right)$$

$$\Rightarrow \quad u(x, y) = \sum_{n \ge 1} D_n \sin\frac{n\pi x}{l} \sinh\left(\frac{n\pi y}{l}\right)$$

Que 2.28. **Solve the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary conditions, $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\dfrac{n\pi x}{l}$.**

AKTU 2020-21 (Sem-3), Marks 10

Answer

Given equation $= \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ...(2.28.1)

The three possible solutions of eq. (2.28.1) are :

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(2.28.2)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(2.28.3)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(2.28.4)$$

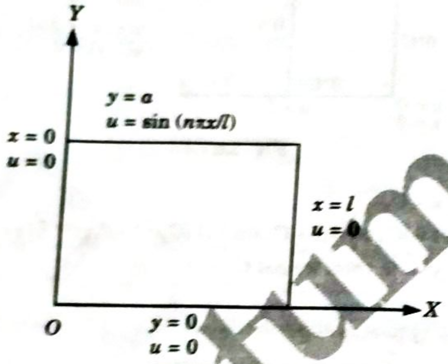

Fig. 2.28.1.

We have to solve eq. (2.28.1) satisfying the following boundary conditions

$$u(0, y) = 0 \quad ...(2.28.5)$$

$$u(l, y) = 0 \quad ...(2.28.6)$$

$$u(x, 0) = 0 \quad ...(2.28.7)$$

$$u(x, a) = \sin n\pi x/l \quad ...(2.28.8)$$

Using eq. (2.28.5) and eq. (2.28.6) in eq. (2.28.2), we get

$$c_1 + c_2 = 0, \text{ and } c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using eq. (2.28.5) and eq. (2.28.6) in eq. (2.28.4). Hence the suitable solution for the present problem is solution eq. (2.28.3). Using eq. (2.28.5) in eq. (2.28.3), we have $c_5(c_7 e^{py} + c_8 e^{-py}) = 0$ i.e., $c_5 = 0$

$\therefore$ eq. (2.28.3) becomes $u = c_6 \sin px(c_7 e^{py} + c_8 e^{-py})$ ...(2.28.9)

Using eq. (2.28.6), we have $c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$

$\therefore$ Either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ when $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, ...$

$\therefore$ Eq. (2.28.9) becomes $u = c_6 \sin (n\pi/l)(c_7 e^{n\pi y/l} + c_8 e^{-n\pi y/l})$ ...(2.28.10)

Using eq. (2.28.7), we have

$$0 = c_6 \sin n\pi x/l \,.\, (c_7 + c_8) \text{ i.e., } c_8 = -c_7.$$

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin \frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \text{ where } b_n = c_6 c_7$$

Now using the condition eq. (2.28.8), we have

$$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get $\quad b_n = \dfrac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin \frac{n\pi x}{l} = \frac{\sinh (n\pi y/l)}{\sinh (n\pi a/l)} \sin \frac{n\pi x}{l}$$

Que 2.29. **A string is stretched and fastened to two points l m apart. Motion is started by displacing the string in the form $u(x, 0) = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by $u(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$.**

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Answer

The equation of string is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad ...(2.29.1)$$

Since, the string is stretched between the two points (0, 0) and (l, 0), hence the displacement of the string at these point will be zero.

$\therefore \quad y(0, t) = 0 \quad ...(2.29.2)$

and $\quad y(l, t) = 0 \quad ...(2.29.3)$

Since the string is released from rest hence its initial velocity will be zero.

$\therefore \quad \dfrac{\partial y}{\partial t} = 0$ at $t = 0 \quad ...(2.29.4)$

Since, the string is displaced from the initial position at time $t = 0$ hence the initial displacement is given is

$$y(x, 0) = K \sin \frac{\pi x}{l} \quad ...(2.29.5)$$

Conditions (2), (3), (4) (5) are the boundary conditions let us now proceed to solve equation (2.29.1),

Let $\quad y = XT \quad ...(2.29.6)$

where X is a function of x only and T is a function of t only

$$\frac{\partial y}{\partial t} = \frac{\partial (XT)}{\partial t} = X\frac{dT}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial\left(X\frac{dT}{dt}\right)}{\partial t} = X\frac{d^2T}{dt^2}$$

Similarly, $$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2T}{dx^2} \quad ...(2.29.7)$$

Substituting eq. (2.29.7) in eq. (2.29.1), we get

$$X\frac{d^2T}{dt^2} = a^2T\frac{d^2X}{dx^2} \Rightarrow XT'' = a^2TX''$$

Now making some cases, we get

Case 1 : $\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = -p^2$

i. $$\frac{1}{a^2}\frac{T''}{T} = -p^2$$

$$\frac{d^2T}{dt^2} + a^2p^2T = 0$$

Auxiliary equation is given by

$$m^2 + a^2p^2 = 0$$

$$m = \pm api$$

$\therefore$ $$CF = c_1 \cos apt + c_2 \sin apt$$

$$PI = 0$$

$\therefore$ $$CF = c_1 \cos apt + c_2 \sin apt$$

$$T = CF + PI$$

$$= c_1 \cos apt + c_2 \sin apt \quad ...(2.29.8)$$

ii. $$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

$$\frac{X''}{X} = -p^2$$

Auxiliary equation is

$$m^2 + p^2 = 0$$

$$m = \pm pi$$

$\therefore$ $$C.F = c_3 \cos (px) + c_4 \sin (px) = X \quad ...(2.29.9)$$

Hence, $$y(x, t) = (c_1 \cos apt + c_2 \sin apt)\,(c_3 \cos (px) + c_4 \sin (px)) \quad ...(2.29.10)$$

Case 2 : $\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = p^2$

i. $$\frac{1}{a^2}\frac{T''}{T} = p^2 \Rightarrow \frac{d^2T}{dt^2} - a^2p^2T = 0$$

Auxiliary equation is $m^2 - p^2a^2 = 0 \Rightarrow m = \pm pa$

$$C.F = c_5e^{apt} + c_6e^{-apt}$$

$\therefore$ $$P.I = 0$$

$$T = c_5e^{apt} + c_6e^{-apt}$$

$\therefore$

ii. $$\frac{X''}{X} = p^2$$

$$\frac{d^2X}{dx^2} - p^2X = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ $$C.F = c_7e^{px} + c_8e^{-px}$$

$$P.I = 0$$

$\therefore$ $$X = c_7e^{px} + c_8e^{-px}$$

hence, $$y(x, t) = (c_5 \cos e^{apt} + c_6e^{-apt})(c_7e^{px} + c_8e^{-px}) \quad ...(2.29.11)$$

Case 3 : $\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = 0 \text{ (say)}$

i. $$\frac{1}{a^2}\frac{T''}{T} = 0 \Rightarrow T'' = 0 \text{ or } \frac{d^2T}{dt^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$C.F = c_9 + c_{10}t$$

$$P.I = 0$$

$\therefore$ $$T = c_9 + c_{10}t$$

ii. $$\frac{X''}{X} = 0 \text{ (say)} \Rightarrow \frac{d^2X}{dx^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0,0$$

$\therefore$ $$C.F = c_{11} + c_{12}x$$

$$P.I = 0$$

$\therefore$ $$X = C_{11} + C_{12}x \quad ...(2.29.12)$$

hence, $y(x, t) = (c_9 + c_{10}x)(c_{11} + c_{12}x)$

Hence, solution (10) is the general solution of one-dimension wave equation given by the equation (2.29.1),

$y(x, t) = (c_1 \cos apt + c_2 \sin apt)(c_3 \cos px + c_4 \sin px)$

Applying the boundary condition,

$y(0, t) = 0 = (c_1 \cos apt + c^2 \sin apt)\, c_3$

$\Rightarrow$ $$c_3 = 0$$

$\therefore$ From (10), $y(x, t) = (c_1\cos apt + c_2 \sin apt)\, c_4 \sin px \quad ...(2.29.13)$

Again, now using the boundary conditions,

$$y(x, t) = 0 = (c_1 \cos apt + c_2 \sin apt)\, c_4 \sin pl$$

$$\sin pl = 0 = \sin n\pi \; (n \in I)$$

$\Rightarrow$

$\therefore$ $$\pi = \frac{n\pi}{l}$$

Hence from (13),

$$y(x,t) = \left(c_1 \cos \frac{n\pi a t}{l} + c_2 \sin \frac{n\pi a t}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(2.29.14)$$

now $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi a}{l}\left[-c_1 \sin \frac{n\pi a t}{l} + c_2 \sin \frac{n\pi a t}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

Now again using boundary conditions,

At $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi a}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $c_2 = 0$

$\therefore$ From (14), $y(x, t) = c_1 c_4 \cos \frac{n\pi a t}{l} \sin \frac{n\pi x}{l}$ ...(2.29.15)

$$y(x, 0) = k \sin \frac{n\pi}{l} = c_1 c_4 \sin \frac{n\pi x}{l}$$

where $c_1 c_4 = k,\ n = 1$

Hence from (15), $y(x, t) = k \cos \frac{\pi a x}{l} \sin \frac{\pi x}{l}$ which is required solution.

PART-4

Complex Fourier Transform, Fourier Sine Transform, Fourier Cosine Transform, Inverse Transform, Convolution Theorem.

Que 2.30. **Find the Fourier transform of the following function $f(x) = 1 - x^2$, if $|x| \le 1$ and $f(x) = 0$, if $|x| > 1$.**

Answer

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx = \int_{-1}^{1} e^{isx}(1 - x^2) dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{-s^2} + (-2)\frac{e^{isx}}{-is^3}\right]_{-1}^{1}$$

$$= \frac{2e^{is}}{-s^2} + \frac{2e^{-is}}{-s^2} + \frac{2}{is^3}(e^{is} - e^{-is}) = -\frac{2}{s^2}(e^{is} - e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is})$$

$$= -\frac{4\cos s}{s^2} + \frac{4 \sin s}{s^3}$$

$$F(s) = \frac{4}{s^3}(\sin s - s \cos s)$$

Now using inverse Fourier transform

$$F^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(\sin s - s \cos s)\, e^{-isx} dx$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin s - s \cos s)}{s^3}(\cos sx - i \sin x) ds$$

$$f(x) = \frac{4}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s \cos s}{s^3}\right) \cos sx\, ds$$

Put $x = \frac{1}{2}$

$$1 - \left(\frac{1}{2}\right)^2 = \frac{4}{\pi}\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds$$

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx = -3\pi/16$$

Que 2.31. **Find the Fourier transform of $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$**

Answer

The Fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Substituting the value of $f(x)$, we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1.e^{isx} dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{1}{(is)}[e^{ias} - e^{-ias}]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{s} \cdot \frac{e^{ias} - e^{-ias}}{2i} = \frac{1}{\sqrt{2\pi}} \frac{2 \sin sa}{s} = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

Que 2.32. **Find the Fourier transform of the following function defined for $a > 0$ by $f(t) = e^{-at^2}$**

Answer

Given, $f(t) = e^{-at^2}$

Now first we need to find the Fourier transform of e^{-t^2}.

Now, $$F(s) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t). e^{-ist} dt$$

$$\therefore \quad F(s) = \int_{-\infty}^{\infty} e^{-t^2}.e^{-ist}\, dt = \int_{-\infty}^{\infty} e^{-(t^2 + ist)}\, dt = \int_{-\infty}^{\infty} e^{-\left\{\left(t + \frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}} dt$$

$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left\{t + \frac{is}{2}\right\}^2} dt$$

$$F(s) = F\{f(x)\} = e^{-s^2/4}\int_{-\infty}^{\infty} e^{-x^2} dx \quad \left(\begin{array}{l}\because (t + is/2) = x \\ \Rightarrow dt = dx\end{array}\right)$$

$$F(s) = e^{-s^2/4}\sqrt{\pi} \quad ...(2.32.1) \quad \left(\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right)$$

Now from the Change of scale property of Fourier transform *i.e.*,

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

So for Fourier transform of $f(x) = e^{-ax^2}$, we get

$$F(e^{-ax^2}) = F\{f(\sqrt{a}x)\} = \frac{1}{\sqrt{a}}.F\left(\frac{s}{\sqrt{a}}\right)$$

$$F(e^{-ax^2}) = \frac{1}{\sqrt{a}}.\sqrt{\pi}\, e^{-\frac{\left(\frac{s}{\sqrt{a}}\right)^2}{4}} \quad \text{(Using eq. (2.32.1))}$$

$$F(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}.\, e^{\frac{-s^2}{4a}}$$

or, by changing the variable of function, we get

$$F(e^{-at^2}) = \sqrt{\frac{\pi}{a}}.\, e^{\frac{-s^2}{4a}}$$

Que 2.33. **Find the finite Fourier sine transform of**

$$f(x) = x(\pi - x) \text{ in } 0 < x < \pi$$

Answer

Finite Fourier sine transform of $f(x)$ is

$$F_s(n) = \int_0^L f(x)\sin\frac{n\pi x}{L}dx = \int_0^{\pi} x(\pi - x).\sin nx dx$$

$$= x(\pi - x).\left[\frac{-\cos nx}{n}\Big|_0^{\pi} - (\pi - 2x).\left(\frac{-\sin nx}{n^2}\right)\Big|_0^{\pi} + (-2).\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$F_s(n) = 0 + 0 + \frac{2}{n^3}(1 - \cos n\pi) = \frac{2}{n^3}[1 - (-1)^n]$$

Que 2.34. **Find the Fourier transform of block function $f(t)$ of height 1 and duration defined by**

$$f(t) = \begin{cases} 1 & \text{for } |t| \le \dfrac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

Answer

Given :
$$f(t) = \begin{cases} 1 & \text{for } |t| \le \dfrac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-ist}dt = \int_{-a/2}^{a/2} e^{-ist}dt$$

($\because$ $f(t) = 0$ outside this limit)

$$= \frac{1}{-is}[e^{-ist}]_{-a/2}^{a/2} = \frac{1}{-is}\left[e^{-\frac{ias}{2}} - e^{\frac{ias}{2}}\right]$$

On multiplying and dividing by $2i$,

$$= \frac{2i}{-is}\left[\frac{e^{-\frac{ias}{2}} - e^{\frac{ias}{2}}}{2i}\right] = \frac{2}{s}\sin\frac{as}{2}$$

Que 2.35. **Find the Fourier cosine transform of $\dfrac{1}{1+x^2}$ and hence find Fourier sine transform of $\dfrac{x}{1+x^2}$.**

Answer

$$f(x) = \frac{1}{1+x^2}$$

Fourier cosine transform of $f(x)$,

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$$

$$I = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{1+x^2}\cos sx\, dx \quad \text{...(2.35.1)}$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{-x}{1+x^2}\sin sx\, dx = -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{(1+x^2-1)\sin sx\, dx}{x(1+x^2)}$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx}{x}dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx\, dx}{x(1+x^2)}$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\frac{\pi}{2} + \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx\, dx}{(1+x^2)x} \quad \text{...(2.35.2)}$$

$$\frac{d^2I}{ds^2} = 0 + \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\cos sx\, dx}{1+x^2}$$

$$\frac{d^2I}{ds^2} = I \Rightarrow m = \pm 1$$

C.F. $= C_1e^s + C_2e^{-s}$

P. I. $= 0$

$$I = C_1e^s + C_2e^{-s} \quad \text{...(2.35.3)}$$

$$\frac{dI}{ds} = C_1e^s - C_2e^{-s} \quad \text{...(2.35.4)}$$

$$I = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{1+x^2}dx = \sqrt{\frac{\pi}{2}}$$

From eq. (2.35.2),

$$\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}}$$

Putting $s = 0$ in eq. (2.35.3) and eq. (2.35.4), we get

$$I = C_1 + C_2 \quad ...(2.35.5)$$

$$\frac{dI}{ds} = -\sqrt{\frac{\pi}{2}} = C_1 - C_2 \quad ...(2.35.6)$$

From eq. (2.35.5) and eq. (2.35.6), we get

$$C_1 = 0,\ C_2 = \sqrt{\frac{\pi}{2}}$$

Thus $\quad I = \sqrt{\frac{\pi}{2}}\, e^{-s}$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\cos sx\, dx}{1+x^2} = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

On differentiating, we get

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{-x\sin sx}{1+x^2}\, dx = -\sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{x\sin sx\, dx}{1+x^2} = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$\int_0^{\infty} \frac{x\sin sx}{1+x^2}\, dx = \frac{\pi}{2} e^{-s}$$

Que 2.36. **Using the Fourier integral transformation, show that**

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos sx}{s^2+a^2}\, ds,\ a > 0,\ x \ge 0$$

Answer

Using Fourier cosine integral representation

$$F(x) = \frac{2}{\pi}\int_0^{\infty} \cos sx \int_0^{\infty} e^{-at} \cos st\, dt\, ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^{\infty} \cos sx \left[\frac{e^{-at}}{a^2+s^2}(-a\cos st + s\sin st)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos sx \left[\frac{-1}{a^2+s^2}(-a)\right] ds$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos sx}{a^2+s^2}\, ds$$

Que 2.37. **State the Convolution theorem for Fourier transform. Prove that the Fourier transform of the convolution of the two functions equal to the product of their Fourier transforms.**

Answer

Convolution Theorem for Fourier Transform : The convolution of two functions $F(x)$ and $G(x)$ over the interval $(-\infty, \infty)$ is defined as

$$F * G = \int_{-\infty}^{\infty} F(u)\, G(x-u)\, du$$

$$F\{F(x) * G(x)\} = F\int_{-\infty}^{\infty} F(u)\, G(x-u)\, du$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(u)\, G(x-u)\, du\right] e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(x-u)\, e^{isx}\, dx\right] F(u)\, du,$$

Put $x - u = t$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} G(t)\, e^{ist}\, dt\right] F(u)\, e^{isu}\, du$$

$$= \int_{-\infty}^{\infty} e^{isu} F(u).\, F\{G(t)\} = \int_{-\infty}^{\infty} e^{isx} F(x).\, F\{G(t)\}$$

$$F\{F(x) * G(x)\} = F\{F(x)\}\, F\{G(x)\}$$

Hence proved.

Que 2.38. **Find the inverse Fourier sine transform of $\frac{1}{x}e^{-ax}$.**

Answer

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx$$

Let, $\quad I = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx$

$$\frac{dI}{ds} = \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^{\infty} = \frac{a}{s^2+a^2}$$

$$I = \tan^{-1}\frac{s}{a} + A$$

at $\quad s = 0, I = 0,$

Putting, $\quad 0 = \tan^{-1} 0 + A$

$\therefore \quad A = 0$

Thus, $I = \tan^{-1}\left(\frac{s}{a}\right)$

PART-5

Application of Fourier Transform to Solve Partial Differential Equation.

Que 2.39. **Determine the distribution of temperature in the semi infinite medium $x \ge 0$ when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.**

Answer

Let $u(x, t)$ be the temperature at point x at any time t. Heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, (x > 0, t > 0) \quad ...(2.39.1)$$

Given initial condition $u(x, 0) = F(x)$ ...(2.39.2)

The boundary condition $u(0, t) = 0$ ...(2.39.3)

Taking Fourier sine transform of eq. (2.39.1), we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px \, dx = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

$$\frac{d\bar{u}}{dt} = c^2[p(u)_{x=0} - p^2\bar{u}] = -c^2p^2\bar{u} \quad \{\text{using } (u)_{x=0} = 0\}$$

$$\frac{d\bar{u}}{dt} + c^2p^2\bar{u} = 0 \quad ...(2.39.4)$$

Solution to eq. (2.39.2) is $\bar{u} = C_1 e^{-c^2p^2t}$ ....(2.39.5)

Taking Fourier sine transform of eq. (2.39.2), we get

$$(\bar{u})_{t=0} = \int_0^\infty F(x) \sin px \, dx = f_s(p)$$

From eq. (2.39.5), $(\bar{u})_{t=0} = C_1 \Rightarrow C_1 = f_s(p)$

From eq. (2.39.5), $\bar{u} = f_s(p)e^{-c^2p^2t}$

Now taking its inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty f_s(p)e^{-c^2p^2t} \sin px \, dp$$

Que 2.40. **Solve one dimensional wave equation given by $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.**

Answer

We know that

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{isx}dx = \left[e^{isx}f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} is\, e^{isx}f(x)dx$$

Assuming that $f(x) \to 0$ as $x \to \pm\infty$

$$F\{f'(x)\} = -is\int_{-\infty}^{\infty} e^{isx}f(x)\,dx$$

$$F\{f'(x)\} = -is\, F\{f(x)\}$$

$$\therefore \quad F\left\{\frac{\partial u}{\partial x}\right\} = -isF\{u\} \quad \left[f'(x) = \frac{\partial u}{\partial x}\right]$$

$$F\left\{\frac{\partial u}{\partial x}\right\} = -is\, u(s)$$

Similarly, $F\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right] = F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}e^{isx}dx = -s^2u(s)$.

and $F\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2}\{u(s)\}$ ...(2.40.1)

Thus taking Fourier transform both sides of wave equation, we have

$$D^2\{u(s)\} = -s^2c^2 u(s) \quad ...(2.40.2)$$

Eq. (2.40.2) is a second order ordinary differential equation for $u(s)$.

Solving eq. (2.40.2), we have

$$m^2 = -s^2c^2$$

$$m = \pm isc$$

Thus $u(s) = C_1 e^{isct} + C_2 e^{-isct}$

General solution of eq. (2.40.2) is

$$u(s) = f(s)\, e^{isct} + g(s)\, e^{-isct} \quad ...(2.40.3)$$

$\because$ Corresponding to different values of s, eq. (2.40.2) has different values of C_1 and C_2.

To find $u(x)$, we now take inverse Fourier sine transform of eq. (2.40.3),

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(s)e^{-isx}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)e^{isct} + g(s)e^{-isct}]e^{-isx}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(s)e^{-is(x-ct)} + g(s)e^{-is(x+ct)}]ds$$

$$u(x, t) = F(x - ct) + G(x + ct)$$

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3 UNIT

Statistical Techniques–I

CONTENTS

PART-1

Overview of Measure of Central Tendency, Moments, Skewness, Kurtosis.

Que 3.1. **The first four moments of a distribution about the value 4 of the variables are –1.5, 17, – 30 and 80. Find moments $\mu_1, \mu_2, \mu_3, \mu_4$ about mean. Also find β_1 and β_2.**

AKTU 2022-23 (Sem-3), Marks 10

Answer

Given : $\mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 80$

$A = 4$

Moment about the mean,

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 17 - (-1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu_1' \mu_2' + 2(\mu_1')^3$$

$$= (-30) - 3(-1.5)(17) + 2(-1.5)^3 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_1' \mu_3 + 6\mu_1'^2 \mu_2' - 3(\mu_1')^4$$

$$= 80 - 4(-1.5)(-30) + 6(-1.5)^2 (17) - 3(-1.5)^4$$

$$= 80 - 34.31$$

$$= 45.68$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1580.06}{3209.05} = 0.5$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{45.68}{217.56} = 0.21$$

Que 3.2. **Define skewness and kurtosis of a distribution. The first four moments of a distribution are 0, 2.5, 0.7, and 18.71. Find the coefficient of skewness and kurtosis.**

Answer

Skewness : The term skewness means lack of symmetry *i.e.*, when a distribution is not symmetric then it is called a skewed distribution and this distribution may be positively skewed or negatively skewed.

Kurtosis : It tells whether the distribution, if plotted on a graph would give us a normal curve, a curve more flat than the normal curve, or more peaked than the normal curve.

Numerical :

$$\mu_1 = 0,\ \mu_2 = 2.5,\ \mu_3 = 0.7,\ \mu_4 = 18.71$$

Coefficient of skewness $(\beta_1) = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{(0.7)^2}{(2.5)^3} = 0.03136$ (+ ve)

The distribution is positively skewed.

$$\text{Kurtosis } (\beta_2) = \frac{\mu_4}{\mu_2^{\,2}} = \frac{18.71}{(2.5)^2} = 2.9936 < 3$$

The distribution is platykurtic.

Que 3.3. **Find the M.G.F. of the random variable X having the following probability density function**

$$F(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Also find mean and variance of X.

Answer

Moment generating function is given by

$$M_x(t) = E(e^{tx})$$

$$= \int_0^1 x e^{tx}\, dx + \int_1^2 (2-x) e^{tx}\, dx + \int_2^\infty 0\, e^{tx}\, dx$$

$$= \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2}\,(e^{2t} + 1 - 2e^t)$$

$$= \frac{(e^t - 1)^2}{t^2} = \left(\frac{e^t - 1}{t}\right)^2$$

$$= \frac{1}{t^2}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right]^2$$

$$M_x(t) = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots$$

Now

$$V_1 = \frac{d}{dt} M_x(t) = \frac{1}{2!} + 0 = \frac{1}{2}$$

$$V_2 = \frac{d^2}{dt^2} M_x(t) = \frac{d}{dt}\left(\frac{1}{2!} + \frac{2t}{3!} + \ldots\right) = \frac{1}{3}$$

$$\text{Mean} = \bar{x} = V_1 = \frac{1}{2}$$

$$\text{Variance} = u_2 = V_2 - \bar{x}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Que 3.4. **The first four moments of a distribution about $x = 4$ are 1, 4, 10 and 45. Calculate the moments about the mean and comment upon the skewness and kurtosis of the distribution.**

OR

Compute skewness and Kurtosis, if the first four moments of a frequency distribution about the value 4 of the variable are 1, 4, 10 and 45.

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Answer

Given : $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$ and $\mu_4' = 45$, $x = 4$

Moments about mean :

$$\mu_1 = 0,\ \mu_2 = \mu_2' - (\mu_1')^2 = 4 - (1)^2 = 4 - 1 = 3$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= 10 - 3(1)(4) + 2(1)^3 = 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4$$

$$= 45 - 4(1)(10) + 6(1)^2(4) - 3(1)^4$$

$$= 45 - 40 + 24 - 3 = 26$$

Coefficient of skewness :

$$\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} = 0$$

Coefficient of kurtosis :

$$\frac{\mu_4}{\mu_2^{\,2}} = \frac{26}{3^2} = 2.88 < 3,$$ *i.e.*, curve is platykurtic.

Que 3.5. **Find all four central moments and discuss skewness and kurtosis and also Karl Pearson skewness for the frequency distribution given below :**

Range of Expend in ₹ (100)/month	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
No. of Families	38	292	389	212	69

Answer

Moments about mean are given by

Range	f	x	$(x-A)$	$f(x-A)$	$(x-A)^2$	$f(x-A)^2$	$(x-A)^3$	$f(x-A)^3$	$(x-A)^4$	$f(x-A)^4$	Cumulative Frequency
2 – 4	38	3	– 4	– 152	16	608	– 64	– 2432	256	9728	38
4 – 6	292	5	– 2	– 584	4	1168	– 8	– 2336	16	4672	330
6 – 8	389	7(A)	0	0	0	0	0	0	0	0	719
8 – 10	212	9	2	424	4	848	8	1696	16	3392	931
10 – 12	69	11	4	276	16	1104	64	4416	256	17664	1000
	Σf = 1000			$\Sigma f(x-A)$ = – 36		$\Sigma f(x-A)^2$ = 3728		$\Sigma f(x-A)^3$ = 1344		$\Sigma f(x-A)^4$ = 35,456	

$$\mu'_1 = \frac{\Sigma f(x-A)}{\Sigma f} = \frac{-36}{1000} = -0.036$$

$$\mu'_2 = \frac{\Sigma f(x-A)^2}{\Sigma f} = \frac{3728}{1000} = 3.728$$

$$\mu'_3 = \frac{\Sigma f(x-A)^3}{\Sigma f} = \frac{1344}{1000} = 1.344,\ \mu'_4 = \frac{\Sigma f(x-A)^4}{\Sigma f} = \frac{35456}{1000} = 35.456$$

Central moments are given by

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 3.728 - (-0.036)^2 = 3.7267$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$$

$$= 1.344 - 3(3.728)(-0.036) + 2(-0.036)^3$$

$$= 1.344 + 0.402624 - 0.000093312$$

$$\mu_3 = 1.7465$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$

$$= 35.456 - 4(1.344)(-0.036) + 6(3.728)(-0.036)^2 - 3(-0.036)^4$$

$$= 35.456 + 0.193536 + 0.028988 - 5.0388 \times 10^{-6}$$

$$\mu_4 = 35.6785$$

Coefficient of skewness, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

$$\beta_1 = \frac{(1.7465)^2}{(3.7267)^3}$$

$$\beta_1 = 0.0589 \text{ (positive)}$$

The curve is positively skewed.

Coefficient of kurtosis, $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{35.6785}{(3.7267)^2}$

$\beta_2 = 2.569\ (<3)$, *i.e.*, curve is platykurtic.

Measure of Karl Pearson's skewness is given by

$$\text{Mean} = A + \frac{\Sigma f(x-A)}{\Sigma f}$$

$$= 7 + \frac{(-36)}{1000} = 6.964$$

$$\text{Median} = l + \frac{\frac{N}{2} - c.f}{f}.i = 6 + \frac{\frac{1000}{2} - 330}{389} \times 2$$

$$= 6 + 0.437 \times 2 = 6.874$$

Standard Deviation (S.D.) = $\sqrt{\dfrac{\Sigma f(x-A)^2}{\Sigma f} - \left(\dfrac{\Sigma f(x-A)}{\Sigma f}\right)^2}$

$$= \sqrt{\frac{3728}{1000} - \left(\frac{-36}{1000}\right)^2}$$

$$= \sqrt{3.728 - 0.001296} = \sqrt{3.726} = 1.930$$

Karl Pearson's coefficient of skewness = $\dfrac{3(\text{Mean} - \text{Median})}{S.D}$

$$S_k = \frac{3(6.964 - 6.874)}{1.930}$$

$$S_k = \frac{0.27}{1.930} = 0.1398$$

Since $S_k > 0$

∴ Distribution is positively skewed.

Que 3.6. **The following table represents the height of a batch of 100 students. Calculate skewness and kurtosis :**

Height (in cm)	59	61	63	65	67	69	71	73	75
No. of students	0	2	6	20	40	20	8	2	2

Answer

Height (cm) x	No. of student f	$u = \dfrac{x-67}{2}$	fu	fu^2	fu^3	fu^4
59	0	– 4	0	0	0	0
61	2	– 3	– 6	18	– 54	162
63	6	– 2	– 12	24	– 48	96
65	20	– 1	– 20	20	– 20	20
67	40	0	0	0	0	0
69	20	1	20	20	20	20
71	8	2	16	32	64	128
73	2	3	6	18	54	162
75	2	4	8	32	128	512
	$N = \Sigma f = 100$		$\Sigma fu = 12$	$\Sigma fu^2 = 164$	$\Sigma fu^3 = 144$	$\Sigma fu^4 = 1100$

Moments about 67 :

$$\mu_1' = \left(\frac{\sum fu}{N}\right) h = \left(\frac{12}{100}\right)(2) = 0.24$$

$$\mu_2' = \left(\frac{\sum fu^2}{N}\right) h^2 = \left(\frac{164}{100}\right)(4) = 6.56$$

$$\mu_3' = \left(\frac{\sum fu^3}{N}\right) h^3 = \frac{144}{100} \times 8 = 11.52$$

$$\mu_4' = \left(\frac{\sum fu^4}{N}\right) h^4 = \frac{1100}{100} \times 16 = 176$$

Moments about mean :

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 6.56 - (.24)2 = 6.5024$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= 11.52 - 3(6.56)(0.24) + 2(0.24)^3 = 6.824448$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 176 - 4(11.52)(0.24) + 6(6.56)(0.24)^2 - 3(0.24)^4$$

$$= 167.19798$$

Coefficient of skewness, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

$$= \frac{(6.824448)^2}{(6.5024)^3} = 0.1694 \text{ (positive)}$$

Hence, the curve is positively skewed.

Coefficient of kurtosis, $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3.9544 > 3$

Hence the distribution is leptokurtic.

Que 3.7. **Calculate the first four central moments about the mean of the following data**

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

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Answer

x	f	$u = \frac{x-4}{2}$	fu	fu^2	fu^3	fu^4
0	1	–2	–2	4	–8	16
1	8	–1.5	–12	18	–27	40.5
2	28	–1	–28	28	–28	28
3	56	–0.5	–28	14	–7	3.5
4	70	0	0	0	0	0
5	56	0.5	28	14	7	3.5
6.	28	1	28	28	28	28
7	8	1.5	12	18	27	40.5
8	1	2	2	4	8	18
	$N = 256$		$\Sigma fu = 0$	$\Sigma fu^2 = 128$	$\Sigma fu^3 = 0$	$\Sigma fu^4 = 176$

Moments about 4 :

$$\mu_1' = \frac{\sum fu}{N} \times h = 0$$

$$\mu_2' = \frac{\sum fu^2}{N} \times h^2 = \frac{128}{256} \times 4 = 2$$

$$\mu_3' = \frac{\sum fu^3}{N} \times h^3 = 0$$

$$\mu_4' = \frac{\sum fu^4}{N} \times h^4 = \frac{176}{265} \times 16 = 11$$

Moments about mean :

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_2'^2 = 2$$

$$\mu_3 = \mu_3' - 3\mu_2'^2 \mu_1' + 2(\mu_2') = 0$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1' + 3\mu_1'^4 = 0$$

$$= 11$$

Que 3.8. **First four moments about 2 are 1, 2.5, 5.5 and 16 respectively. Find the first four central moments, moments about origin and coefficient of skewness.**

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Answer

Given : $\mu_1' = 1$, $\mu_2' = 2.5$, $\mu_3' = 5.5$ and $\mu_4' = 16$, $A = 2$

Central moments are given by

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2.5 - 1 = 1.5$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3 = 5.5 - 3 \times 2.5 \times 1 + 2 \times (1)^3$$

$$= 5.5 - 6.5 + 2 = 1.5$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^3 - 3\mu_1'^4$$

$$= 16 - 4 \times 5.5 \times 1 + 6 \times 2.5 (1)^2 - 3 \times (1)^4$$

$$= 16 - 22 + 15 - 3 = 6$$

Moments about, origin,

$$v_1 = A + \mu_1' = 2 + 1 = 3$$

$$v_2 = \mu_2 + (v_1)^2 = 1.5 + 9 = 10.5$$

$$v_3 = \mu_3 + 3v_1 v_2 - 2v_1^3$$

$$= 1.5 + 3 \times 3 \times 10.5 - 2 \times 27 = 15$$

$$v_4 = \mu_4 + 4v_1 v_3 - 6v_1^2 v_2^3 + 3v_1^4$$

$$= 6 + 4 \times 3 \times 15 - 6 \times (3)^2 \times 10.5 + 3 \times (3)^4$$

$$= -138$$

Coefficient of skewness :

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(1.5)^3}{(1.5)^3} = 1$$

PART-2

Curve Fitting, Method of Least Square, Fitting of Straight Lines, Fitting of Second Degree Parabola Exponential Curves.

Que 3.9. **Use the method of least squares to obtain the normal equations and fit the curve for $y = \frac{c_0}{x} + c_1\sqrt{x}$ to the following table of values :**

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

Answer

Normal equations to the curve $y = \frac{c_0}{x} + c_1\sqrt{x}$ are

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}}$$

$$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sum x$$

Table of values is :

x	y	y/x	$y\sqrt{x}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{x^2}$
0.1	21	210	6.64078	3.16228	100
0.2	11	55	4.91935	2.23607	25
0.4	7	17.5	4.42719	1.58114	6.25
0.5	6	12	4.24264	1.41421	4
1	5	5	5	1	1
2	6	3	8.48528	0.70711	0.25
$\Sigma x = 4.2$		$\Sigma(y/x) = 302.5$	$\Sigma y\sqrt{x} = 33.71524$	$\Sigma \frac{1}{\sqrt{x}} = 10.10081$	$\Sigma \frac{1}{x^2} = 136.5$

Substituting the values in normal equations, we get

$$302.5 = 136.5\, c_0 + 10.10081\, c_1 \quad ...(3.9.1)$$

and
$$33.71524 = 10.10081\, c_0 + 4.2\, c_1 \quad ...(3.9.2)$$

Solving eq. (3.9.1) and eq. (3.9.2), we get

$$c_0 = 1.97327 \text{ and } c_1 = 3.28182$$

Hence the required equation of curve is

$$y = \frac{1.97327}{x} + 3.28182\sqrt{x}$$

Que 3.10. **Using the least square method fit a second degree polynomial from the following data :**

x	0	1	2	3	4	5	6	7	8
y	12.0	10.5	10.0	8.0	7.0	8.0	7.5	8.5	9.0

Also, estimate y at $x = 6.5$.

Answer

Let a second degree polynomial, $y = ax^2 + bx + c$

The normal equations for the given polynomial are given as follows :

$$\Sigma x^2 y = c\Sigma x^2 + b\Sigma x^3 + a\Sigma x^4$$

$$\Sigma y = nc + b\Sigma x + a\Sigma x^2$$

$$\Sigma yx = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

x	y	x^2	x^3	x^4	xy	x^2y
0	12.0	0	0	0	0	0
1	10.5	1	1	1	10.5	10.5
2	10.0	4	8	16	20.0	40.0
3	8.0	9	27	81	24.0	72.0
4	7.0	16	64	256	28.0	112.0
5	8	25	125	625	40.0	200.0
6	7.5	36	216	1296	45.0	270
7	8.5	49	343	2401	59.5	416.5
8	9.0	64	512	4096	72.0	576
$\Sigma x=36$	$\Sigma y=80.5$	$\Sigma x^2=204$	$\Sigma x^3=1296$	$\Sigma x^4=8772$	$\Sigma xy=299$	$\Sigma x^2y=1697$

$$n = 9$$

Putting value in normal equations, we have

$$204a + 36b + 9c = 80.5 \quad ...(3.10.1)$$

$$1296a + 204b + 36c = 299 \quad ...(3.10.2)$$

$$8772a + 1296b + 204c = 1697 \quad ...(3.10.3)$$

On solving eq. (3.10.1), eq. (3.10.2) and eq. (3.10.3),

$$a = 0.18, \quad b = -1.85, \quad c = 12.18$$

Then,
$$y = 0.18x^2 - 1.85x + 12.18$$

At
$$x = 6.5$$

$$y = 7.76$$

Que 3.11. **Fit the curve $pv^\gamma = K$ to the following data :**

p (kg / cm^2)	0.5	1	1.5	2	2.5	3
v (litres)	1620	1000	750	650	520	460

Answer

$$pv^\gamma = K$$

$$v = \left(\frac{K}{p}\right)^{1/\gamma} = K^{1/\gamma}\, p^{-1/\gamma}$$

Taking log,
$$\log v = \frac{1}{\gamma}\log K - \frac{1}{\gamma}\log p$$

which is the form
$$Y = A + BX$$

Where $Y = \log v$, $X = \log p$, $A = \frac{1}{\gamma}\log K$ and $B = -\frac{1}{\gamma}$

p	v	X	Y	XY	X^2
0.5	1620	– 0.30103	3.20952	– 0.96616	0.09062
1	1000	0	3	0	0
1.5	750	0.17609	2.87506	0.50627	0.03101
2	620	0.30103	2.79239	0.84059	0.09062
2.5	520	0.39794	2.716	1.08080	0.15836
3	460	0.47712	2.66276	1.27046	0.22764
Total		ΣX = 1.05115	ΣY = 17.25573	ΣXY = 2.73196	ΣX^2 = 0.59825

Here, $m = 6$

Substituting the values in normal equations, we get

$$17.25573 = 6A + 1.05115\,B$$

and

$$2.73196 = 1.05115\,A + 0.598825\,B$$

On solving, we get

$$A = 2.99911 \quad \text{and} \quad B = -\,0.70298$$

$$\therefore \qquad \gamma = -\frac{1}{B} = \frac{1}{0.70298} = 1.42252$$

Again, $\log K = \gamma A = 4.26629$

$\therefore$ $K = \text{antilog}\,(4.26629) = 18462.48$

Hence required curve is $pv^{1.42252} = 18462.48$.

Que 3.12. **Determine the least square approximation of the type $ax^2 + bx + c$ to the function 2^x, at points $x_i = 0, 1, 2, 3, 4$.**

Answer

Here
$$y = 2^x = ax^2 + bx + c$$

Normal equations for the given curve are,

$$\Sigma yx^2 = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$
$$\Sigma yx = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$
$$\Sigma y = a\Sigma x^2 + b\Sigma x + mc$$

Table of values is, (Here $m = 5$)

x	y	xy	x^2	yx^2	x^3	x^4
0	1	0	0	0	0	0
1	2	2	1	2	1	1
2	4	8	4	16	8	16
3	8	24	9	72	27	81
4	16	64	16	256	64	256
$\Sigma x = 10$	$\Sigma y = 31$	$\Sigma xy = 98$	$\Sigma x^2 = 30$	$\Sigma yx^2 = 346$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

Substituting the values in normal equations, we get

$$346 = 354a + 100b + 30c$$
$$98 = 100a + 30b + 10c$$
$$31 = 30a + 10b + 5c$$

On solving, $a = 1.143$, $b = -\,0.971$ and $c = 1.286$

$\therefore$ $y = 1.143x^2 - 0.971x + 1.286$

Que 3.13. **Fit a parabolic curve of second degree to the following data :**

X :	**0**	**1**	**2**	**3**	**4**
Y :	**1**	**1.8**	**1.3**	**2.5**	**6.3**

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Answer

Let the parabola of fit be $y = a + b\,x + c\,x^2$ ...(3.13.1)

Normal equations are
$$\Sigma y = 5a + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\Sigma x = 10$	$\Sigma y = 12.9$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 37.1$	$\Sigma x^2y = 130.3$

Substituting the values in normal equations, we get

$$12.9 = 5a + 10b + 30c$$
$$37.1 = 10a + 30b + 100c$$
$$130.3 = 30a + 100b + 354c$$

On solving, we get $a = 1.42$, $b = -\,1.07$, $c = 0.55$

Substituting the value of ab and c in Equation (1) we get

$$y = 1.42 - 1.07x + 0.55x^2$$

Que 3.14. **Using the method of least square fit a curve of the form $y = ab^x$ to the following data :**

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

AKTU 2020-21 (Sem-3), Marks 10

Answer

$$y = ab^x$$
$$\log y = \log a + x \log b$$

Let $Y = \log_e y,\ A = \log_e a,\ B = \log_e b$

$$Y = A + Bx$$

Normal equations are,

$$\sum_{i=1}^{5} Y_i = nA + B\sum_{i=1}^{5} x$$

$$\sum_{i=1}^{5} x_i Y_i = A\sum_{i=1}^{5} x_i + B\sum_{i=1}^{5} x_i^2 \qquad \text{(Here } n = 5\text{)}$$

x	y	$Y = \log y$	x^2	$x_i Y_i$
2	8.3	0.91907	4	1.82814
3	15.4	1.18752	9	3.5625
4	33.1	1.51982	16	6.0792
5	65.2	1.81424	25	9.0712
6	127.4	2.10516	36	12.6309
$\sum_{i=1}^{5} x = 20$		$\sum_{i=1}^{5} Y = 7.5458$	$\sum_{i=1}^{5} x^2 = 90$	$\sum_{i=1}^{5} x_i Y_i = 33.18194$

Substituting the values in normal equations, we get

$$7.5458 = 5A + 20B$$
$$33.18194 = 20A + 90B$$

On solving, we get

$$A = 9677/31250 = 0.309664$$

and $B = 149937/500000 = 0.299874$

$\therefore$ $0.309664 = \log_e a$

$\Rightarrow$ $a = e^{0.309664} = 1.36$ (approx)

and $0.299874 = \log_e b$

$\Rightarrow$ $b = e^{0.299874} = 1.34$ (approx)

$\therefore$ $y = ab^x \Rightarrow y = 1.36(1.34)^x$

Que 3.15. **Fit a second degree parabola to the following data :**

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

OR

Fit a parabolic curve of regression of y on x to the following data :

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
f	1.1	1.3	1.6	2.0	2.7	3.4	4.1

AKTU 2021-22 (Sem-3), Marks 10

Answer

We shift the origin to (2.5, 0) and take 0.5 as the new unit. This changes the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. Normal equations are :

$$\Sigma y = am + b\Sigma X + c\Sigma X^2$$
$$\Sigma Xy = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2 y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	–3	1.1	–3.3	9	9.9	–27	81
1.5	–2	1.3	–2.6	4	5.2	–8	16
2.1	–1	1.6	–1.6	1	1.6	–1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total =	$\Sigma X = 0$	$\Sigma y = 16.2$	$\Sigma Xy = 14.3$	$\Sigma X^2 = 28$	$\Sigma X^2y = 69.9$	$\Sigma X^3 = 0$	$\Sigma X^4 = 196$

Substituting the values in normal equations, we get

$$7a + 28c = 16.2\ ; \quad 28b = 14.3\ ; \quad 28a + 196c = 69.9$$

On solving, we get

$$a = 2.07, \quad b = 0.511, \quad c = 0.061$$
$$y = 2.07 + 0.511X + 0.061X^2$$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511(2x - 5) + 0.061(2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$.

This is the required parabola of best fit.

Que 3.16. **Use least the method of squares to the curve $y = c_0 x + \dfrac{c_1}{\sqrt{x}}$ for the following data :**

x	0.2	0.3	0.5	1	2
y	16	14	11	6	3

AKTU 2021-22 (Sem-4), Marks 10

Answer

Normal equation to the curve $y = c_0 x + \dfrac{C_1}{\sqrt{x}}$

$$\Sigma y/x = C_0 + C_1 \sum \frac{1}{x\sqrt{x}}$$

$$\sum y\sqrt{x} = C_0 \sum x\sqrt{x} + C_1$$

Table of value is

x	y	$\sqrt{x}$	$x\sqrt{x}$	$1/x\sqrt{x}$	y/x	$y\sqrt{x}$
0.2	16	0.447	0.089	11.236	80	7.152
0.3	14	0.548	0.164	6.098	46.667	7.672
0.5	11	0.707	0.354	2.825	22	7.777
1	6	1	1	1	6	6
2	3	1.414	2.828	0.354	1.5	4.242
$\Sigma x = 4$			$\Sigma x\sqrt{x}$ = 4.435	$\Sigma 1/x\sqrt{x}$ = 21.513	$\Sigma y/x$ = 156.167	$\Sigma y\sqrt{x}$ = 32.843

Substituting the value in normal equation we get

$$80 = C_0 + 21.513\, C_1 \quad \text{...(3.16.1)}$$

$$32.843 = 4.435\, C_0 + C_2 \quad \text{...(3.16.2)}$$

Solving eq. (3.16.1) and (3.16.2)

$$C_0 = 6.636 \text{ and } C_1 = 3.410$$

Hence the required equation of curve is

$$y = 6.636\,x + \frac{3.410}{\sqrt{x}}$$

Que 3.17. **Use the method of least squares to fit the curve $y = ab^x$ for the following data**

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

AKTU 2022-23 (Sem-4), Marks 10

Answer

$$y = ab^x$$

$$\log y = \log a + \log b$$

$$y = \log_e y,\ A = \log_e a,\ B = \log_e b$$

$$y = A + Bx$$

Normal equations are

x	y	$\gamma = \log y$	x^2	$x_i\gamma_i$
2	144	2.15836	4	4.31672
3	172.8	2.23754	9	6.71262
4	207.4	2.31659	16	9.26636
5	248.6	2.39585	25	11.97925
6	298.5	2.47421	36	14.84526
$\sum_{i=1}^{5} x = 20$		$\sum_{i=1}^{5} \gamma = 11.58255$	$\sum_{i=1}^{5} x^2 = 90$	$\sum_{i=1}^{5} x_i\gamma_i = 47.12021$

Substituting the value in normal equation, we get

$$11.58255 = 5A + 20B \quad \text{...(3.17.1)}$$

$$47.12021 = 20A + 90B \quad \text{...(3.17.2)}$$

Multiply equation (3.17.1) by 4 and substracting eq. (3.17.1) from eq. (3.17.2)

$$47.12021 - 46.33020 = 70B$$

$$B = \frac{0.79001}{70} = 0.011285$$

$$\log_e b = 0.011285 \Rightarrow b = e^{0.011285} \Rightarrow 1.12 \text{ (approx)}$$

From eq. (3.17.1), put the value of B

$$11.58255 = 5A + 20 \times 0.011285$$

$$5A = 11.35685$$

$$A = 2.27137$$

$$\log_e a = 2.27137$$

$$a = e^{2.27137} = 9.69 \text{ (approx)}$$

$$y = 9.69(1.12)^x$$

PART-3

Correlation and Rank Correlation.

Que 3.18. **Calculate the rank coefficient from the sales and expenses of 10 firms as given below :**

Sales X	45	56	39	54	45	40	56	60	30	36
Expenses Y	40	36	30	44	36	32	45	42	20	36

Answer

X	Y	R_1	R_2	$d = R_1 - R_2$	d^2
45	40	5.5	4	1.5	2.25
56	36	2.5	6	– 3.5	12.25
39	30	8	9	– 1	1
54	44	4	2	2	4
45	36	5.5	6	– 0.5	0.25
40	32	7	8	– 1	1
56	45	2.5	1	1.5	2.25
60	42	1	3	– 2	4
30	20	10	10	0	0
36	36	9	6	3	9
					$\Sigma d^2 = 36$

Repeated Rank of X column :

$45 = 2$ times $= m_1$

$56 = 2$ times $= m_2$

Repeated Rank of Y column :

$36 = 3$ times $= m_3$

Rank correlation coefficient,

$$r = 1 - \frac{6\left[\Sigma d^2 + \frac{1}{12}\Sigma(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right]}{N(N^2-1)}$$

$$= 1 - \frac{6\left[36 + \frac{1}{12}(2^3 - 2 + 2^3 - 2 + 3^3 - 3)\right]}{10(99)}$$

$$= 1 - \frac{6[36 + 3]}{990} = 1 - \frac{234}{990}$$

$$r = 0.7636$$

Que 3.19. **Calculate the correlation coefficient for the following heights (in inches) of fathers (X) and their sons (Y):**

X :	**65**	**66**	**67**	**67**	**68**	**69**	**70**	**72**
Y :	**67**	**68**	**65**	**68**	**72**	**72**	**69**	**71**

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Answer

$$\bar{X} = \frac{65 + 66 + 67 + 67 + 68 + 69 + 70 + 72}{8} = \frac{544}{8} = 68$$

$$\bar{Y} = \frac{67 + 68 + 65 + 68 + 72 + 72 + 69 + 71}{8} = \frac{552}{8} = 69$$

X	Y	$(X - \bar{X})$	$(Y - \bar{Y})$	$(X - \bar{X})^2$	$(Y - \bar{Y})^2$	$(X - Y)(Y - \bar{Y})$
65	67	– 3	– 2	9	4	6
66	68	– 2	– 1	4	1	2
67	65	– 1	– 4	1	16	4
67	68	– 1	– 1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
				$\Sigma(X - \bar{X})^2 = 36$	$\Sigma(Y - \bar{Y})^2 = 44$	$\Sigma(X - \bar{X})(Y - \bar{Y}) = 24$

Now, the coefficient of correlation is given as,

$$\gamma_{XY} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\sqrt{\Sigma(X - \bar{X})^2 \times \Sigma(Y - \bar{Y})^2}} = \frac{24}{\sqrt{36 \times 44}} = \frac{24}{39.79} = 0.603$$

Que 3.20. **Ten students got the following percentage of marks in principles of Economics and Statistics :**

Roll Nos.	1	2	3	4	5	6	7	8	9	10
Marks in Economics	78	36	98	25	75	82	90	62	65	39
Marks in Statistics	84	51	91	60	68	62	86	58	53	47

Calculate the coefficient of correlation.

Answer

Let the marks in the two subjects be denoted by x and y respectively.

x	y	$u = x - 65$	$v = y - 66$	u^2	v^2	uv
78	84	13	18	169	324	234
36	51	– 29	– 15	841	225	435
98	91	33	25	1089	625	825
25	60	– 40	– 6	1600	36	240
75	68	10	2	100	4	20
82	62	17	– 4	289	16	– 68
90	86	25	20	625	400	500
62	58	– 3	– 8	9	64	24
65	53	0	– 13	0	169	0
39	47	– 26	– 19	676	361	494
Total		$\Sigma u = 0$	$\Sigma v = 0$	$\Sigma u^2 = 5398$	$\Sigma v^2 = 2224$	$\Sigma uv = 2704$

Here, $n = 10,\ \bar{u} = \frac{1}{n}\Sigma u_i = 0,\ \bar{v} = \frac{1}{n}\Sigma v_i = 0$

$$r_{uv} = \frac{n\Sigma uv - \Sigma u\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$$

$$= \frac{(10 \times 2704) - (0 \times 0)}{\sqrt{(10 \times 5398) - (0)^2}\sqrt{(10 \times 2224) - (0)^2}} = 0.780$$

Hence, $r_{xy} = r_{uv} = 0.780.$

PART-4

Regression Analysis : Regression Lines of y on x and x an y.

Que 3.21. **If $4x - 5y + 33 = 0$ and $20x - 9y = 107$ are two lines of regression. Find the mean values of x and y, the coefficient of correlation and the standard deviation of y if the variance of x is 9.**

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Answer

Since both the lines of regression pass through the point $(\bar{x}, \bar{y})$ therefore we have,

$$4\bar{x} - 5\bar{y} + 33 = 0 \quad \text{...(3.21.1)}$$

$$20\bar{x} - 9\bar{y} + 107 = 0 \quad \text{...(3.21.2)}$$

Multiply equation (3.21.1) by (3.21.5), we get

$$20\bar{x} - 5\bar{y} + 165 = 0 \quad \text{...(3.21.3)}$$

Substracting eq. (3.21.3) from eq. (3.21.2), we get

$$16\bar{x} - 272 = 0$$

$$\bar{y} = 17$$

From eq. (3.21.1)

$$4\bar{x} - 5 \times 17 + 33 = 0$$

$$4\bar{x} = 52$$

$$\bar{x} = 13$$

Hence $\bar{x} = 13,\ \bar{y} = 17$

Now, variance of $x \Rightarrow \sigma_2^{\,2} = 9$

$\sigma_x = 3$

The equation of lines of regression can be written as

$y = 0.8x + 6.6$ and $x = 0.45y + 5.35$

The regression coefficient of y on x is

$$\frac{r\sigma_y}{\sigma_x} = 0.8 \quad \text{...(3.21.4)}$$

The regression coefficient of x on y is

$$\frac{r\sigma_x}{\sigma_y} = 0.45 \quad \text{...(3.21.5)}$$

Multiplying eq. (3.21.4) and eq. (3.21.5), we get

$$r^2 = 0.8 \times 0.45 = 0.36$$

$$r = 0.6$$

From eq. (3.21.4), we get standard deviation of y

$$\sigma_y = \frac{0.8\sigma_x}{r} = \frac{0.8 \times 3}{0.6} = 4$$

Que 3.22. **If the θ is the acute angle between the two regression lines in the case of two variables x and y, show that $\tan\theta = \frac{1-r^2}{r}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^{\,2}+\sigma_y^{\,2}}$, where r, σ_x, σ_y have their usual meanings. Explain the significance of the formula when $r = 0$ and $r = \pm 1$.**

Answer

Coefficient of correlation is given by,

$$r = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\sqrt{n \Sigma\, dx^2 - (\Sigma\, dx)^2}\,\sqrt{\eta \Sigma\, dy^2 - (\Sigma\, dy)^2}}$$

where, $dx = x - \bar{x}$

$dy = y - \bar{y}$

Coefficients of regression are given by,

$$b_{xy} = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\eta \Sigma\, dy^2 - (\Sigma\, dy)^2}$$

$$b_{yx} = \frac{\eta \Sigma\, dxdy - \Sigma\, dx\, \Sigma\, dy}{\eta \Sigma\, dx^2 - (\Sigma\, dx)^2}$$

Equations to the lines of regression of y on x and x on y are

$$y - \bar{y} = \frac{r\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = \frac{r\sigma_x}{\sigma_y}(y - \bar{y})$$

Their slopes are $m_1 = \frac{r\sigma_y}{\sigma_x}$ and $m_2 = \frac{\sigma_y}{r\sigma_x}$

$$\therefore \quad \tan \theta = \pm \frac{m_2 - m_1}{1 + m_2 m_1} = \pm \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$= \pm \frac{1 - r^2}{r} \cdot \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2} = \pm \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Since $r^2 \leq 1$ and σ_x, σ_y are positive.

$\therefore$ +ve sign gives the acute angle between the lines.

Hence, $$\tan \theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\theta = \frac{\pi}{2}$

$\therefore$ The two lines of regression are perpendicular to each other.

Hence the estimated value of y is the same for all values of x and vice-versa.

When $r = \pm 1$, $\tan \theta = 0$ so that $\theta = 0$ or π.

The lines of regression will coincide and there is a perfect correlation between the two variables x and y.

Que 3.23. **In a partially destroyed laboratory record of an analysis of correlation data, the following results are legible variance of $x = 9$. Regression equations are**

$$8x - 10y = -66$$
$$40x - 18y = 214$$

Find

1. **Mean values of x and y**
2. **Standard deviation of y**
3. **Correlation coefficient between x & y.**

Answer

Since both the lines of regression pass through the point $(\bar{x}, \bar{y})$ therefore, we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \quad \text{...(3.23.1)}$$

$$40\bar{x} - 18\bar{y} - 214 = 0 \quad \text{...(3.23.2)}$$

Multiplying eq. (3.23.1) by 5, we get

$$40\bar{x} - 50\bar{y} + 330 = 0 \quad \text{...(3.23.3)}$$

Subtracting eq. (3.23.3) from eq. (3.23.2), we get

$$32\bar{y} - 544 = 0 \;\therefore\; \bar{y} = 17$$

$\therefore$ From eq. (3.23.1), $8\bar{x} - 170 + 66 = 0$

$$8\bar{x} = 104$$
$$\bar{x} = 13$$

Hence $\bar{x} = 13,\ \bar{y} = 17$

Now, variance of $x = \sigma_x^2 = 9$

$$\sigma_x = 3$$

The equations of lines of regression can be written as

$$y = 0.8x + 6.6 \text{ and } x = 0.45y + 5.35$$

$\therefore$ The regression coefficient of y on x is $\frac{r\sigma_y}{\sigma_x} = 0.8$ ...(3.23.4)

The regression coefficient of x on y is $\frac{r\sigma_x}{\sigma_y} = 0.45$ ...(3.23.5)

Multiplying eq. (3.23.4) and eq. (3.23.5), we get

$$r^2 = 0.8 \times 0.45 = 0.36$$
$$r = 0.6$$

From eq. (3.23.4), we get standard deviation of y,

$$\sigma_y = \frac{0.8\sigma_x}{r} = \frac{0.8 \times 3}{0.6} = 4.$$

Que 3.24. **Find the coefficient of correlation (r) and obtain the equation to the lines of regression for the following data :**

x	6	2	10	4	8
y	9	11	5	8	7

OR

From the following data, determine the equations of line of regression of y on x and x on y.

x	6	2	10	4	8
y	9	11	5	8	7

AKTU 2021-22 (Sem-4), Marks 10

Answer

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	XY	X^2	Y^2
6	9	0	1	0	0	1
2	11	– 4	3	– 12	16	9
10	5	4	– 3	– 12	16	9
4	8	– 2	0	0	4	0
8	7	2	– 1	– 2	4	1
$\Sigma x = 30$	$\Sigma y = 40$	$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma XY = -26$	$\Sigma X^2 = 40$	$\Sigma Y^2 = 20$

$$\bar{x} = \frac{\Sigma x}{N} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{40}{5} = 8$$

Regression coefficient of y on x,

$$b_{yx} = \frac{\Sigma XY}{\Sigma X^2} = \frac{-26}{40} = -0.65$$

Regression coefficient of x on y,

$$b_{xy} = \frac{\Sigma XY}{\Sigma Y^2} = \frac{-26}{20} = -1.3$$

Equation of regression line (y on x) is

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$

$$y - 8 = \frac{-26}{40}(x - 6)$$

or, $y - 8 = -0.65\,(x - 6)$

or, $y = -0.65x + 11.9$

Regression equation (x on y) is

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$

$$x - 6 = -1.3\,(y - 8)$$

$$x = -1.3y + 16.4$$

Correlation coefficient,

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$$r^2 = -0.65 \times (-1.3) = 0.845$$

$$r = -\sqrt{0.845} \quad \text{(As both } b_{yx}, b_{xy} \text{ are negative)}$$

$$r = -0.919$$

Que 3.25. **If for two random variables, x and y with same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$, then show that $\dfrac{b}{\beta} = \dfrac{1-a}{1-\alpha}$**

Also find the common mean.

Answer

Here, $b_{yx} = a, b_{xy} = \alpha$

Let the common mean be m, then regression lines are

$$y - m = a(x - m)$$

$$y = ax + m(1 - a) \quad ...(3.25.1)$$

and

$$x - m = \alpha\,(y - m)$$

$$x = \alpha y + m\,(1 - \alpha) \quad ...(3.25.2)$$

Comparing eq. (3.25.1) and eq. (3.25.2) with the given equations,

$$b = m\,(1 - a),\ \beta = m\,(1 - \alpha)$$

$$\therefore \quad \frac{b}{\beta} = \frac{1-a}{1-\alpha}$$

Que 3.26. **For 10 observations on price (x) and supply (y) the following data were obtained**

$$\Sigma x = 130, \Sigma y = 220, \Sigma x^2 = 2288$$
$$\Sigma y^2 = 5506 \text{ and } \Sigma xy = 3467$$

Obtain the two lines of regression.

Answer

$$\bar{x} = \frac{\Sigma x}{N}$$

$$\bar{x} = \frac{130}{10} = 13$$

$$\bar{y} = \frac{\Sigma y}{N}$$

$$\bar{y} = \frac{220}{10} = 22$$

Regression coefficient of y on x, $b_{yx} = \frac{\Sigma XY}{\Sigma X^2}$

$$b_{yx} = \frac{3467}{2288} = 1.52$$

Regression coefficient of x on y, $b_{xy} = \frac{\Sigma XY}{\Sigma Y^2} = \frac{3467}{5506} = 0.63$

Equation of regression line (y on x) is,

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$

$$y - 22 = 1.52(x - 13)$$

$$y = 1.52x + 2.24$$

Regression equation (x on y) is,

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$

$$x - 13 = 0.63\,(y - 22)$$

$$x = 0.63y - 0.86$$

Que 3.27. **The following table gives age (x) in years of cars and annual maintenance cost (y) in hundred rupees**

x	1	3	5	7	9
y	15	18	21	23	22

Calculate the maintenance cost for a 4-year-old car after finding the regression equation. **AKTU 2020-21 (Sem-3), Marks 10**

Answer

x	1	3	5	7	9
y	15	18	21	23	22

x	y	x^2	y^2	xy
1	15	1	225	15
3	18	9	324	54
5	21	25	441	105
7	23	49	529	161
9	22	81	484	198
$\Sigma x = 25$	$\Sigma y = 99$	$\Sigma x^2 = 165$	$\Sigma y^2 = 2003$	$\Sigma xy = 533$

$$\bar{x} = \frac{\Sigma x}{N} = \frac{25}{5} = 5$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{99}{5} = 19.8$$

$$b_{yx} = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{5 \times 533 - 25 \times 99}{5 \times 165 - 625} = \frac{2665 - 2475}{200}$$

$$= \frac{190}{200} = 0.95$$

Equation of regression line (y on x) is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - \bar{y} = 0.95\,(x - \bar{x})$$

$$y = 19.8 + 0.95\,(x - 5)$$

$$y = 19.8 + 0.95\,x - 4.75$$

$$y = 0.95\,x + 15.05 \quad ...(3.27.1)$$

Substituting $x = 4$ in eq. (3.27.1),

$$y = 0.95 \times 4 + 15.05$$

$$= 18.85 \text{ (in hundred rupees)}$$

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4 UNIT

Statistical Techniques–II

CONTENTS

PART-1

Overview of Probability Random Variables (Discrete and Continuous Random Variable) Probability Mass Function and Probability Density Function, Expectation and Variance.

Que 4.1. From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction, (ii) two particular engineers must be included, (iii) one particular architect must be excluded.

Answer

i. Number of committees $= {}^6C_3 \times {}^5C_2 = \dfrac{6.5.4}{3.2.1} \times \dfrac{5.4}{2.1} = 200.$

ii. Here we have to choose one engineer from the remaining four engineers.

$\therefore$ Number of committees $= {}^4C_1 \times {}^5C_2 = 4 \times \dfrac{5.4}{2.1} = 40.$

iii. Here we have to choose two architects from the remaining four architects.

$\therefore$ Number of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6.5.4}{3.2.1} \times \dfrac{4.3}{2.1} = 120.$

Que 4.2. A five figure number is formed by the digit 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.

Answer

The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ Total number of 5-figure number formed = 5! – 4! = 96.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, number ending in 04, 12, 20, 24, 32, 40.

Number ending in 04 = 3! = 6, Number ending in 12 = 3! – 2! = 4.

Number ending in 20 = 3! = 6, Number ending in 24 = 3! – 2! = 4.

Number ending in 32 = 3! = 2! = 4, and Number ending in 40 = 3! = 6.

The number having 12, 24, 32 in the extreme right are (3! – 2!) since the number having zero on the extreme left are excluded.

Que 4.3. **A has one share in a lottery in which there is 1 prize and 2 blanks : *B* has three shares in a lottery in which there are 3 prizes and 6 blanks. Compare the probability of *A*'s success to that of *B*'s success.**

Answer

A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which *A* can get a prize is 1.

$\therefore$ The probability of *A*'s success $= \dfrac{1}{3}$.

Again *B* can draw a ticket in ${}^9C_3 = \dfrac{9.8.7}{3.2.1} = 84$ ways.

The number of ways in which *B* gets all blanks $= {}^6C_3 = \dfrac{6.5.4}{3.2.1} = 20$

$\therefore$ The number of ways of getting a prize = 84 – 20 = 64

Thus the probability of *B*'s success = 64/84 = 16/21

Hence A's probability of success : *B*'s probability of success $= \dfrac{1}{3} : \dfrac{16}{21}$

$= 7 : 16$

Que 4.4. **A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.**

Answer

Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in ${}^8C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^8C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly, 2 red balls out of 6 can be drawn in ${}^6C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^6C_2}{{}^{14}C_2} = \frac{15}{91}$$

Hence, the probability of drawing 2 balls of the same colour (either both white or both red).

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}$$

Que 4.5. **A bag contains 10 white and 15 black balls. If two balls are drawn in succession without replacement, then find the probability that the first ball is white and the second ball is black.**

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Answer

Let A : be the event that first ball is white

And B : be the event that second ball is black

$$P(A) = \frac{\text{Number of white balls}}{\text{Total number of balls}} = \frac{10}{25} = \frac{2}{5}$$

$$P(B) = \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{15}{24} = \frac{5}{8}$$

{24 because 1 ball was taken out and not replaced}

Hence the probability that first is white and second is black

$$= \frac{2}{5} \times \frac{5}{8} = \frac{1}{4}$$

Que 4.6. **Three machines I, II and III are manufacture respectively 0.4, 0.5 and 0.1 of the total production. The percentage of defective items produced by I, II and III is 2, 4 and 1 per cent respectively. For an item chosen at random, what is the probability it is defective ?**

Answer

The defective item produced by machine I $= \dfrac{0.4 \times 2}{100} = \dfrac{0.8}{100}$

The defective item produced by machine II $= \dfrac{0.5 \times 4}{100} = \dfrac{2}{100}$

The defective item produced by machine III $= \dfrac{0.1 \times 1}{100} = \dfrac{0.1}{100}$

The total defective items produced by machines, I, II or III

$$= \frac{0.8}{100} + \frac{2}{100} + \frac{0.1}{100} = \frac{2.9}{100} = 0.029$$

Que 4.7. ***A* can hit a target 3 times in 5 shots, *B* can hit a target 2 times in 5 shots and *C* can hit a target 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) at least two shots hit ?**

Answer

Probability of A hitting the target = 3/5

Probability of B hitting the target = 2/5

Probability of C hitting the target = 3/4

i. In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5}\times\frac{2}{3}\times\left(1-\frac{3}{4}\right)=\frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5}\times\frac{3}{4}\times\left(1-\frac{3}{5}\right)=\frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4}\times\frac{3}{5}\times\left(1-\frac{2}{5}\right)=\frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100}+\frac{12}{100}+\frac{27}{100} = 0.45$$

iii. In order that at least two shots may hit the target, we must also consider the cases of all A, B, C hitting the target in addition to the three cases of (i) for which

$$p_4 = \text{Chance that } A, B, C \text{ all hit} = \frac{3}{5}\times\frac{2}{5}\times\frac{3}{4}=\frac{18}{100}$$

Since all these are mutually exclusive events, the probability of at least two shots hit.

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100}+\frac{12}{100}+\frac{27}{100}+\frac{18}{100} = 0.45$$

Que 4.8. **State Baye's Theorem. The contents of urns I, II and III are as follows : 1 white, 2 black and 3 red balls; 2 white, 1 black and 1 red balls; 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urn I ?**

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Answer

Statement : Let $E_1, E_2, ..., E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2, ..., E_n$ have non-zero probability of occurrence and they form a partition of S.

Let A be any event associated with S, then according to Baye's theorem,

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$

for any $k = 1, 2, 3 ...., n$

Numerical :

Let B_1, B_2 and B_3 denote the event of selecting urn I, urn II and urn III. Let A be the event that two balls drawn are white and red.

$$\therefore \quad P(B_1) = \frac{1}{3}$$

$$P(B_2) = \frac{1}{3}$$

$$P(B_3) = \frac{1}{3}$$

Now, $$P(A/B_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

$$P(A/B_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$$

$$P(A/B_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

Using Baye's Theorem,

Required probability = $P(B_1/A)$

$$= \frac{P(B_1)P(A/B_1)}{P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}} = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{3}+\frac{2}{11}} = \frac{33}{118}$$

Que 4.9. **Two urns contain 4 white, 6 blue and 4 white, 5 blue balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is white. What is the probability that it was drawn from the (i) first urn (ii) second urn.**

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Answer

Let U_1 : the ball is drawn from U_1

U_2 : the balls drawn from U_2

W : the ball is white

i. We have to find $P(U_1/W)$
By Baye's theorem

$$P(U_1/W) = \frac{P(U_1)P(W/U_1)}{P(U_1)P(W/U_1) + P(U_2)P(W/U_2)} \quad ...(4.9.1)$$

Since the two urns are equally likely to be selected

$$P(U_1) = P(U_2) = \frac{1}{2}$$

Also, $P(W/U_1) = P$ (a white ball is drawn from U_1)

$$= \frac{4}{10}$$

Also, $P(W/U_2) = P$ (a white ball is drawn from U_2)

$$= \frac{4}{10}$$

From the eq. (4.9.1)

$$P(U_1/W) = \frac{\frac{1}{2}\times\frac{4}{10}}{\frac{1}{2}\times\frac{4}{10}+\frac{1}{2}\times\frac{4}{10}} = \frac{4}{8} = \frac{1}{2}$$

ii.

$$P(U_2/W) = \frac{P(U_2)P(W/U_2)}{P(U_1)P(W/U_1) + P(U_2)P(W/U_2)}$$

$$P(U_2/W) = \frac{\frac{1}{2}\times\frac{4}{10}}{\frac{1}{2}\times\frac{4}{10}+\frac{1}{2}\times\frac{4}{10}} = \frac{1}{2}$$

Que 4.10.

i. Is the function defined as follows a density function ?

$$f(x) = e^{-x}, \quad x \ge 0$$
$$= 0, \quad x < 0,$$

ii. If so, determine the probability that the variate having this density will fall in the interval (1, 2) ?

iii. Also find the cumulative probability function $F(2)$?

Answer

i. $f(x) \ge 0$ for every x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

ii. Required probability $= P(1 \le x \le 2) = \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2}$

$$= 0.368 - 0.135 = 0.233$$

This probability is equal to the shaded area in Fig. 4.17.1(a).

iii. Cumulative probability function $F(2)$:

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^2 e^{-x}\,dx = 1 - e^{-2}$$
$$= 1 - 0.135 = 0.865$$

which is shown in Fig. 4.10.1(b).

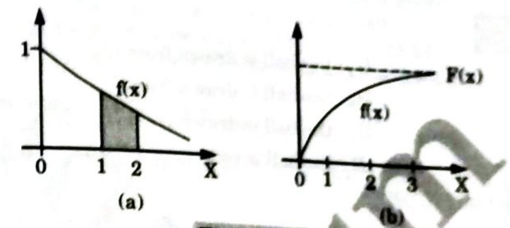

Fig. 4.10.1.

Que 4.11. There are three bags : first containing 1 white, 2 red, 3 green balls : second containing 2 white, 3 red, 1 green balls and third containing 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.

Answer

Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two bags are white and red.

Now $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

$P(A/B_1) = P$ (a white and a red ball are down from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \frac{2}{5}$

$$P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \frac{1}{5}$$

By Baye's theorem,

$$P(B_2/A) = \frac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$$

$$= \frac{\frac{1}{3}\times\frac{2}{5}}{\frac{1}{3}\times\frac{2}{15}+\frac{1}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{1}{5}} = \frac{6}{11}$$

Que 4.12. **Three urns contain 6 red, 4 black; 4 red, 6 black; 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from the first urn.**

Answer

Let U_1 : the ball is drawn from U_1.
U_2 : the ball is drawn from U_2.
U_3 : the ball is drawn from U_3.
R : the ball is red.

We have to find $P(U_1/R)$.

By Baye's theorem,

$$P(U_1/R) = \frac{P(U_1)P(R/U_1)}{P(U_1)P(R/U_1) + P(U_2)P(R/U_2) + P(U_3)P(R/U_3)} \quad ...(4.12.1)$$

Since the three urns are equally likely to be selected

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

Alos $P(R/U_1) = P$(a red ball is drawn from urn I) $= \frac{6}{10}$

$P(R/U_2) = P$(a red ball is drawn from urn II) $= \frac{4}{10}$

$P(R/U_3) = P$(a red ball is drawn from urn III) $= \frac{5}{10}$

$\therefore$ From eq. (4.12.1), we have $P(U_1/R) = \dfrac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{4} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{2}{5}$

Que 4.13. **The probability density function of a variate X is**

X	0	1	2	3	4	5	6
$p(X)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

i. **Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$**
ii. **What will be the minimum value of k so that $P(X \leq 2) > 3$.**

Answer

i. If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1$$

$$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$
$$k = 1/49$$

$\therefore$
$$P(X < 4) = k + 3k + 5k + 7k = 16k = 16/49$$
$$P(X \geq 5) = 11k + 13k = 24k = 24/49$$
$$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49$$

ii. $$P(X \leq 2) = k + 3k + 5k = 9k = 0.3 \text{ or } k > 1/30$$

Thus, minimum value of $k = \frac{1}{30}$

Que 4.14. **A random variable X has the following probability function :**

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

i. **Find the value of k**
ii. **Evaluate $P(X < 6)$, $(P \geq 6)$**
iii. **$P(0 < X < 5)$**

OR

A random variable X has the following probability distribution values of X :

X :	0	1	2	3	4	5	6	7
$P(X)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Then, evaluate $P(X \geq 6)$. **AKTU 2022-23 (Sem-3), Marks 10**

Answer

i. If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

i.e., $$10k^2 + k - 1 = 0$$
$$10k^2 + 10k - k - 1 = 0$$
$$10k(k + 1) - 1(k + 1) = 0$$
$$k = \frac{1}{10}, k = -1$$

ii. $$P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$
$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2$$
$$= k[8 + k] = \frac{1}{10}\left[8 + \frac{1}{10}\right] = \frac{9}{10} \times \frac{8}{10} = \frac{81}{100}$$

$$P(X \geq 6) = 2k^2 + 7k^2 + k$$

$$= \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

iii. $$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= k + 2k + 2k + 3k = 8k$$

$$= 8 \times \frac{1}{10} = \frac{8}{10}$$

Que 4.15. **A variate X has the probability distribution**

x	–3	6	9
$P(X = x)$	1/6	1/2	1/3

Find $E(X)$ and $E(X^2)$. Hence evaluate $E(2X + 1)^2$.

Answer

$$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2$$

$$E(X)^2 = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$\therefore$ $$E(2X + 1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$

$$= 4(93/2) + 4(11/2) + 1 = 209$$

Que 4.16. **The frequency distribution of a measurable characteristic varying between 0 and 2 is as under**

$$f(x) = x^3, \quad 0 \leq x \leq 1$$

$$= (2 - x)^3, 1 \leq x \leq 2$$

Calculate the standard deviation and also the mean deviation about the mean.

Answer

Total frequency $$N = \int_0^1 x^3\, dx + \int_1^2 (2 - x)^3\, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$\therefore$ $$\mu_1' \text{(about the origin)} = \frac{1}{N}\left[\int_0^1 x\, x^3\, dx + \int_1^2 x(2 - x)^3\, dx\right]$$

$$= 2\left\{\left[\frac{x^5}{5}\right]_0^1 + \left[-x\frac{(2 - x)^4}{4}\right]_1^2 - \left[\frac{(2 - x)^5}{20}\right]_1^2\right\} = 2\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{20}\right) = 1$$

$$\mu_2' \text{(about the origin)} = \frac{1}{N}\left[\int_0^1 x^2\, x^3\, dx + \int_1^2 x^2 (2 - x)^3\, dx\right]$$

$$= 2\left\{\left[\frac{x^6}{6}\right]_0^1 + \left[-x^2\frac{(2 - x)^4}{4}\right]_1^2 + \frac{1}{2}\int_1^2 x(2 - x)^4\, dx\right\}$$

$$= 2\left\{\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\left[\frac{1}{5} + \frac{1}{30}\right]\right\} = \frac{16}{15}$$

Hence $$\sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{16}{15} - 1 = \frac{1}{15}$$

i.e., Standard deviation $\sigma = \frac{1}{\sqrt{15}}$

Mean deviation about the mean :

$$= \frac{1}{N}\left\{\int |x - 1|\, x^3\, dx + \int_1^2 |x - 1|(2 - x)^3\, dx\right\}$$

$$= 2\left\{\int_0^1 (1 - x)x^3\, dx + \int_1^2 (x - 1)(2 - x)^3\, dx\right\}$$

$$= 2\left\{\left(\frac{1}{4} - \frac{1}{5}\right) + \left(0 + \frac{1}{20}\right)\right\} = \frac{1}{5}$$

Que 4.17. **A bag A contains 8 white and 4 black balls. A second bag B contains 5 white and 6 black balls. One ball is drawn at random from bag A and is placed in bag B. Now, a ball is drawn at random from bag B. It is found that this ball is white. Find the probability that a black ball has been transferred from bag A.**

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Answer

Let B_1 : transfer of white ball to bag B.

B_1 : transfer of black ball to bag B.

$P(B_1) - \frac{8}{12}$; $P(B_1) = \frac{4}{12}$

Let E be the event of drawing a white ball from bag B after transfer.

$P(E/B_1)$ = Probability of drawing white ball if black ball is transferred to bag $B - \frac{6}{12}$

$P(E/B_2)$ = Probability of drawing a white ball if white ball is transferred to bag $B = \frac{7}{12}$

$\therefore$ $P(E) = P(B_1) \cdot P(E/B_1) \cdot P(B_2) \cdot P(E/B_2)$

$$\frac{8}{12} \cdot \frac{6}{12} + \frac{4}{12} \cdot \frac{7}{12} = \frac{48}{194} + \frac{28}{144} = \frac{76}{144}$$

$\therefore$ The required probability $P(B/E)$

$$= \frac{P(B_1).P(E/B_1)}{P(E)} = \frac{\frac{8}{12}\cdot\frac{6}{12}}{\frac{76}{144}} = \frac{48}{76} \text{ Choice } (B)$$

PART-2

Discrete and Continuous Probability Distribution : Binomial, Poisson and Normal Distributions.

Que 4.18. **Let the random variable X assume the value 'r' with the probability law $p(X = r) = q^{r-1}p$; $r = 1, 2, 3$. Find the m.g.f of X and hence it's mean and variance.** **AKTU 2021-22 (Sem-3), Marks 10**

Answer

By the definition of m.g.f.

$$M_X(t) = E[e^{tX}] = \sum_{-1}^{\infty} e^{tn}\, pq^{n-1} = \frac{p}{q}\sum_{n=1}^{\infty}(qe^t)^n$$

$$= \frac{p}{q}(qe^t)\sum_{n=1}^{\infty}(qe^t)^{n-1}$$

$$= pe^t[1 + qe^t + (qe^t)^2 + ...] = pe^t\left[\frac{1}{1-qe^t}\right]$$

Now $\quad \frac{d}{dt}[M_X(t)] = \frac{pe^t}{(1-qe^t)^2}$

and $\quad \frac{d^2}{dt^2}[M_X(t)] = pe^t \cdot \frac{(1+qe^t)}{(1-qe^t)^2}$ (Verify it)

$\therefore \quad \mu_1'$ (about the origin) $= \left\{\frac{d}{dt}[M_X(t)]\right\}_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p}$ $\quad (\because \; p + q = 1)$

μ_2' (about the origin) $= \left\{\frac{d^2}{dt^2}[M_X(t)]\right\}_{t=0} = \frac{p(1+q)}{(1-q)^2} = \frac{1+q}{p^2}$

Hence, $\quad$ Mean $= \mu_1' = \frac{1}{p}$

and $\quad$ variance $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$.

Que 4.19. **Define a binomial distribution. Prove that the Poisson distribution is the limiting case of binomial distribution.**

OR

Show that Poisson distribution is a particular limiting form of the binomial distribution when p or q is very small, and n is large enough. **AKTU 2020-21 (Sem-3), Marks 10**

Answer

A. Binomial distribution : A random variable x is said to have a binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$p(x = r) = \begin{cases} {}^nC_r p^r q^{n-r}, & r = 0, 1, 2, ... n \\ 0, & \text{otherwise} \end{cases}$$

Also $\quad p + q = 1$

B. Poisson distribution as a limiting case of binomial distribution : If the parameters n and p of a binomial distribution are known, we can find the distribution. But in situations where n is very large and p is very small, application of binomial distribution is very labourious. However, if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite, say λ, we get the Poisson approximation to the binomial distribution. Now, for a binomial distribution

$$P(X = r) = {}^nC_r q^{n-r} p^r = \frac{n(n-1)(n-2)...(n-r+1)}{r!} \times (1-p)^{n-r} \times p^r$$

$$= \frac{n(n-1)(n-2)...(n-r+1)}{r!} \times \left(1 - \frac{\lambda}{n}\right)^{n-r} \times \left(\frac{\lambda}{n}\right)^r \quad \left(\text{Since } np = \lambda \;\therefore\; p = \frac{\lambda}{n}\right)$$

$$= \frac{\lambda^r}{r!} \times \frac{n(n-1)(n-2)...(n-r+1)}{n^r} \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)....\left(\frac{n-r+1}{n}\right) \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{r-1}{n}\right) \times \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

As $n \to \infty$, each of the $(r-1)$ factors,

$\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), ... \left(1-\frac{r-1}{n}\right)$ tends to 1. Also $\left(1-\frac{\lambda}{n}\right)^r$ tends to 1.

Since $\underset{x \to \pm\infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$, the Naperian base.

$$\therefore \quad \left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda} \to e^{-1} \text{ as } n \to \infty$$

Hence in the limiting case when $n \to \infty$, we have

$$P(X=r) = \frac{e^{-\lambda}.\lambda^r}{r!} \quad (r = 0, 1, 2, 3, ...)$$

Here P is called the Poisson probability distribution.

Que 4.20. **Find the mean and variance of binomial distribution.**

Answer

Mean of binomial distribution :

For the binomial distribution, $P(r) = {}^nC_r\, q^{n-r} p^r$

Mean $\quad \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r\ {}^nC_r q^{n-r}\ p^r$

$$= 0 + 1.\ {}^nC_1\, q^{n-1} p + 2.\ {}^nC_2\, q^{n-2} p^2 + 3.\ {}^nC_3\, q^{n-3} p^3 + ... + n.\ {}^nC_n p^n$$

$$= nq^{n-1} p + 2.\frac{n(n-1)}{2.1} q^{n-2} p^2 + 3.\frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + ... + np^n$$

$$= np^{n-1} p + n(n-1)\, q^{n-2} p^2 + \frac{n(n-1)(n-2)}{2.1} q^{n-3} p^3 + ... + np^n$$

$$= np[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2} p + {}^{n-1}C_2 q^{n-3} p^2 + ... + {}^{n-1}C_{n-1} p^{n-1}]$$

$$= np(q+p)^{n-1} = np \qquad (\because p + q = 1)$$

Hence, the mean of binomial distribution is np.

Variance of binomial distribution :

$$\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2 = \sum_{r=0}^{n} [r + r(r-1)]\, P(r) - \mu^2$$

$$= \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1)\, P(r) - \mu^2$$

$$= \mu + \sum_{r=2}^{n} r(r-1)\ {}^nC_r q^{n-r} p^r - \mu^2$$

(Since the contribution due to $r = 0$ and $r = 1$ is zero)

$$= \mu + [2.1.\ {}^nC_2 q^{n-2} p^2 + 3.2.\ {}^nC_3 q^{n-3} p^3 + ... + n(n-1)\ {}^nC_n p^n] - \mu^2$$

$$= \mu + \left[2.1.\frac{n(n-1)}{2.1} q^{n-2} p^2 + 3.2.\frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + ... + n(n-1) p^n\right] - \mu^2$$

$$= \mu + [n(n-1) q^{n-2} p^2 + n(n-1)(n-2) q^{n-3} p^3 + ... + n(n-1) p^n] - \mu^2$$

$$= \mu + n(n-1) p^2 [q^{n-2} + (n-2) q^{n-3} p + ... + p^{n-2}] - \mu^2$$

$$= \mu + n(n-1) p^2 [{}^{n-2}C_0 q^{n-2} + {}^{n-2}C_1 q^{n-3} p + ... + {}^{n-2}C_{n-2} p^{n-2}] - \mu^2$$

$$= \mu + n(n-1) p^2 (q+p)^{n-2} - \mu^2 = \mu + n(n-1) p^2 - \mu^2 \qquad [\because q + p = 1]$$

$$= np + n(n-1) p^2 - n^2 p^2 = np[1-p] = npq. \qquad [\because \mu = np]$$

Hence, the variance of binomial distribution is npq.

Que 4.21. **Out of 800 families with four children each, how many families would be expected to have (i) 2 boys and 2 girls, (ii) at least one boy, (iii) no girl, (iv) at most two girls. Assume equal probabilities for boys and girls.**

Answer

Probability of having boy $= P = \frac{1}{2}$

Probability of having girl $= Q = \frac{1}{2}$

Number of children $= n$

i. Probability of getting 2 boy and 2 girl $= {}^nC_r P^r Q^{n-r}$

$$= {}^4C_2 P^2 Q^{4-2}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2! \times 2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= 6 \times \frac{1}{4} \times \frac{1}{4} = 0.375$$

ii. Probability of getting at least one boy $= 1 - {}^4C_0 P^0 Q^4$

$$= 1 - \frac{4!}{0! \times 4!} \times \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16} = 0.937$$

iii. Probability of getting no girl $= {}^4C_4 P^4 Q^0$

$$= {}^4C_4 \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^0$$

$$= \frac{4!}{0!4!}\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625$$

iv. Probability of getting at most two girls

$$= {}^4C_4 P^4 Q^0 + {}^4C_3 P^3 Q^1 + {}^4C_2 P^2 Q^2$$

$$= 0.0625 + \frac{4 \times 3!}{1 \times 3!} \times \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + 0.375$$

$$= 0.0625 + \frac{1}{4} + 0.375$$

$$= 0.625 + 0.250 + 0.375$$

$$= 0.6875$$

Que 4.22. **Find the mean and variance of Poisson distribution.**

Answer

For the Poisson distribution, $P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$

Mean of Poisson distribution :

$$\text{Mean, } \mu = \sum_{r=0}^{\infty} rP(r) = \sum_{r=0}^{\infty} r . \frac{e^{-\lambda} \lambda^r}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{r\lambda^r}{r!} = e^{-\lambda}\left(0 + \frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + ...\right)$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + ...\right) = \lambda e^{-\lambda} . e^{\lambda} = \lambda$$

Thus, the mean of the Poisson distribution is equal to the parameter λ.

Variance of Poisson distribution :

$$\text{Variance,} \qquad \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 . \frac{\lambda^r e^{-\lambda}}{r!} - \lambda^2 = e^{-\lambda} \sum_{r=0}^{\infty} \frac{r^2 \lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\left[\frac{1^2 \lambda}{1!} + \frac{2^2 \lambda^2}{2!} + \frac{3^2 \lambda^3}{3!} + \frac{4^2 \lambda^4}{4!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{2\lambda^2}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{(1+1)\lambda}{1!} + \frac{(1+2)\lambda^2}{2!} + \frac{(1+3)\lambda^3}{3!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + ...\right) + \left(\frac{\lambda}{1!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + ...\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + ...\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda} [e^{\lambda} + \lambda e^{\lambda}] - \lambda^2$$

$$= \lambda e^{-\lambda} . e^{\lambda}(1 + \lambda) - \lambda^2$$

$$= \lambda(1 + \lambda) - \lambda^2 = \lambda.$$

Hence, the variance of the Poisson distribution is also λ.

Que 4.23. **The distribution of the number of road accidents per day in a city is Poisson with mean 4. Find the number of days out of 100 days when there will be**

i. no accident
ii. at least 2 accidents
iii. at most 3 accidents
iv. between 2 and 5 accidents

Answer

Mean, $\lambda = 4$, Number of days, $N = 100$

i. $$P(r = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-4} = 0.01831$$

$\therefore$ Required number of days $= N . P(r = 0)$

$$= 100 \times 0.01831 = 1.831 \approx 2$$

ii. $P(r \geq 2) = 1 - P(r < 2) = 1 - [P(r = 0) + P(r = 1)]$

$$= 1 - \left[e^{-4} + \frac{e^{-4}(4)}{1!}\right] = 1 - 5e^{-4} = 0.90842$$

$\therefore$ Required number of days $= N . P(r \geq 2)$

$$= 100 \times 0.90842 = 90.842 \approx 91$$

iii. $P(r \leq 3) = P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

$$= \frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!}$$

$$= e^{-4} + 4e^{-4} + 8e^{-4} + \frac{64}{6}e^{-4} = 0.43347$$

$\therefore$ Required number of days $= N . P(r \leq 3)$

$$= 100 \times 0.43347 = 43.347 \approx 43$$

iv. $P(2 < r < 5) = P(r = 3) + P(r = 4) = \frac{e^{-4}(4)^3}{3!} + \frac{e^{-4}(4)^4}{4!}$

$$= \left(\frac{64}{6} + \frac{256}{24}\right)e^{-4} = 0.3907$$

$\therefore$ Required number of days

$$= N \cdot P(2 < r < 5) = 100 \times 0.3907 = 39.07 \approx 39$$

Que 4.24. **For continuous random variable X if**

$$f(x) = \frac{3}{4}(x^2 + 1), 0 \le x \le 1.$$

Then,

i. Verify that $f(x)$ is a probability distribution function.

ii. Find λ such that $P(X \le \lambda) = P(X > \lambda)$.

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Answer

$$f(x) = \frac{3}{4}(x^2 + 1) \quad 0 \le x \le 1$$

$$P(X \le \lambda) = P(X > \lambda)$$

i. $f(x) \ge 0 \ \forall \ x \in [0, 1]$

and $\int_0^1 f(x)\,dx = \int_0^1 \frac{3}{4}(x^2 + 1)\,dx = \frac{3}{4}\left[\frac{1}{3} + 1\right]$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

$\therefore$ $f(x)$ in a valid pdf.

ii. $P(X \le \lambda) = P(X > \lambda) = 1 - P(X \le \lambda)$

$\Rightarrow$ $P(X \le \lambda) = \frac{1}{2}$

$\Rightarrow$ $\int_0^\lambda f(x)\,dx = \frac{1}{2} \Rightarrow \int_0^\lambda \frac{3}{4}(x^2 + 1)\,dx = \frac{1}{2}$

$\Rightarrow$ $\int_0^\lambda (x^2 + 1)\,dx = \frac{2}{3} \Rightarrow \frac{\lambda^3}{3} + \lambda = \frac{2}{3}$

$\Rightarrow$ $\lambda^3 + 3\lambda - 2 = 0$

$\Rightarrow$ $\lambda \ne$ not an integer

$= 0.5961$

Que 4.25. **In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.**

Answer

Mean number of defectives $= 2 = np = 20p$

$\therefore$ The probability of a defective part is $p = 2/20 = 0.1$

The probability of a non-defective part $= 0.9$

$\therefore$ The probability of at least three defectives in a sample of 20

$= 1 -$ (Probability that either none, or one, or two are non-defective parts)

$= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}]$

$= 1 - (0.9)^{18} \times 4.51 = 0.323$

Thus the number of samples having at least three defective parts out of 1000 samples

$= 1000 \times 0.323 = 323$

Que 4.26. **Fit a Poisson distribution to the set of observation :**

x	0	1	2	3	4
f	122	60	15	2	1

Answer

Mean $= \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{60 + 36 + 6 + 4}{200} = 0.5$

$\therefore$ Mean of Poisson distribution *i.e.*, $m = 0.5$

Hence, the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

$\therefore$ The theoretical frequencies are

x	0	1	2	3	4
f	121	61	15	2	0

$[\because \ e^{-0.5} = 0.61]$

Que 4.27. **If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.**

Answer

It follows a Poisson distribution as the probability of occurrence is very small.

Mean $(m) = np = 2000(0.001) = 2$

Probability that more than 2 will get a bad reaction

= 1 – [Probability that no one gets a bad reaction + Probability that one gets a bad reaction + Probability that two gets bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] \quad [\because m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32 \quad [\because e = 2.718]$$

Que 4.28. **In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and SD of 60 hours. Estimate the number of bulbs likely to burn for**

a. More than 2150 hours

b. Less than 1950 hours and

c. More than 1920 hours and but less than 2160 hours.

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Answer

Here $\mu = 2040$ hours and $\sigma = 60$ hours

a. For $x = 2150$, $z = \frac{x - \mu}{\sigma} = 1.833$

$\therefore$ Area against $z = 1.83$ is 0.4664

(According to normal distribution table)

The area required in this case is to the right of the ordinate at $z = 1.83$.

i.e., Area $= 0.5 - 0.4664 = 0.0336$

Thus the number of bulbs to burns for more than 2150 hours

$= 0.0336 \times 2000 = 67$ approximately

b. For $x = 1950$, $z = \frac{x - \mu}{\sigma} = -1.5$

The area required in this case is to the left of $z = -1.33$

i.e., Area $= 0.5 - 0.4082$(table value for $z = 1.33$) $= 0.0918$

$\therefore$ The number of bulbs expected to burn for less than 1950 hours.

$= 0.0918 \times 2000 = 184$ approximately.

c. When $x = 1920$, $z = \frac{1920 - 2040}{60} = -2$

When $x = 2160$, $z = \frac{2160 - 2040}{60} = 2$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area for $z = 2$, *i.e.*, $= 2 \times 0.4772 = 0.9544$.

Thus, the required number of bulbs $= 0.9544 \times 2000 = 1909$ nearly.

Que 4.29. **Calculate the moment generating function of the discrete Binomial distribution given by, $P(x) = {}^nC_x p^x q^{n-x}$ where $(q = 1 - p)$. Also find the first and second moments about the mean.**

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Answer

Moment Generating Function of Binomial Distribution :

The probability mass function is given by,

$$f(x) = C(n, x)p^x(1-p)^{n-x}$$

Here the term $C(n, x)$denotes the number of combinations of n elements taken x at a time, and x can take the values 0, 1, 2, 3, . . ., n.

Use the probability mass function to obtain the moment generating function of X :

$$M(t) = \Sigma_{x=0}^{n} (e^{tx}\, C(n, x) >) p^x (1-p)^{n-x}$$

It becomes clear that you can combine the terms with exponent of x :

$$M(t) = \Sigma_{x=0}^{n} (pe^t)^x\, C(n, x) >)(1-p)^{n-x}$$

Furthermore, by use of the binomial formula, the above expression is simply :

$$M(t) = [(1-p) + pe^t]^n$$

The first moment is the mean and the second moment about the mean is the sample variance.

Calculation of the Mean :

In order to find the mean and variance, you'll need to know both $M'(0)$ and $M'(0)$. Begin by calculating your derivatives, and then evaluate each of them at $t = 0$.

You will see that the first derivative of the moment generating function is :

$$M'(t) = n(pe^t)[(1-p) + pe^t]^{n-1}$$

From this, you can calculate the mean of the probability distribution. $M(0) = n(pe^0)[(1-p) + pe^0]^{n-1} = np$. This matches the expression that we obtained directly from the definition of the mean.

Calculation of the Variance :

The calculation of the variance is performed in a similar manner. First, differentiate the moment generating function again, and then we evaluate this derivative at $t = 0$.

$M''(t) = n(n-1)(pe^t)^2[(1-p) + pe^t]^{n-2} + n(pe^t)[(1-p) + pe^t]^{n-1}$.

To calculate the variance of this random variable you need to find $M''(t)$. Here you have $M''(0) = n(n-1)p^2 + np$. The variance σ^2 of your distribution is

$$\sigma^2 = M''(0) - [M'(0)]^2 = n(n-1)p^2 + np - (np)^2 = np(1-p).$$

Que 4.30. **A sample of 100 dry battery cells tested to find the length of life produced the following results : $\bar{x} = 12$ hours, $\sigma = 3$ hours. Assuming the data to be normally distributed, what percentage of battery cells are expected to have life (i) more than 15 hours (ii) less than 6 hours (iii) between 10 and 14 hours.**

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Answer

Let x denotes the length of life of dry battery cells,

Also, $Z = \dfrac{x-\bar{x}}{\sigma} = \dfrac{x-12}{3}$

i. When $x = 15$, $Z = \dfrac{15-12}{3} = \dfrac{3}{3} = 1$

$P(x > 15) = P(z > 1)$
$= P(0 < Z < \infty) - P(0 < Z < 1)$
$= 0.5 - 0.3413 = 0.1587$
$= 15.87\ \%$

ii. When $x = 6$, $Z = \dfrac{6-12}{3} = -2$

$\therefore$ $P(x < 6) = P(Z < -2)$
$= P(\infty < Z < -2) - P(0 < Z < -2)$
$= 0.5 - 0.4772 = 0.0228$
$= 2.28\ \%$

iii. When $x = 10$, $Z = \dfrac{10-12}{3} = -\dfrac{2}{3} = -0.67$

When $x = 14$, $Z = \dfrac{2}{3} = 0.67$

$P(10 < x < 14) = P(-0.67 < Z < 0.67)$
$= P(-0.67 < Z < 0) + P(0 < Z\ 0.67)$
$= 2P(0 < Z < 0.67)$
$= 2 \times 0.2486$
$= 0.4972$
$= 49.72\ \%$

Que 4.31. **In a sample of 1000 cases, the mean of a certain test is 14 and S.D is 2.5. Assuming the distribution to be normal, find**

i. How many students score between 12 and 15 ?
ii. How many score above 18 ?
iii. How many score below 8 ?

Given $f(0.8) = 0.2881, f(0.4) = 0.1554, f(1.6) = 0.4452, f(2.4) = 0.4918$.

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Answer

Here, $\mu = 14$ and $\sigma = 2.5$

i. For $x = 12$, $z = \dfrac{12-14}{2.5} = \dfrac{-2}{2.5} = -0.8$

For $x = 15$, $z = \dfrac{15-14}{2.5} = \dfrac{1}{2.5} = 0.4$

The nuclear of students score more than 12 and less 15 will be represented by the area between $z = -0.8$ and $z = 0.4$. The area for $z = 0.8$ is 0.2881 and for $z = 0.4 = 0.1554$.
Total area $= 0.2881 \times 0.1554 = 0.0448$
Thus, the required number of students $= 0.0448 \times 1000 = 44.8 \approx 45$

ii. For $x = 18$, $z = \dfrac{12-18}{2.5} = \dfrac{-6}{2.5} = -2.4$

$\therefore$ Area against $z = 2.4$ is 0.4918
(According to normal distribution table)
The area required in this case is to the right of the ordinate at $z = 2.4$

Area $= 0.5 - 0.4918 = 0.0082$

Thus, the number of students scoring more than 18
$= 0.0082 \times 1000 = 8.2 \approx 9$

iii. For $x = 8$, $z = \dfrac{12-8}{2.5} = \dfrac{4}{2.5} = 1.6$

$\therefore$ Area against $z = 1.6$ is 0.4452
(According to normal distribution table)
Thus, the number of students scoring less than 8
$= 0.0548 \times 1000 = 54.8 \approx 55$

Que 4.32. **Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones**

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

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Answer

$n = 5$ $\quad N = 2 + 14 + 20 + 34 + 22 + 8 = 100$

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i}$$
$$= \frac{0 \times 2 + 1 \times 14 + 2 \times 20 + 3 \times 34 + 4 \times 22 + 5 \times 8}{100}$$
$$= \frac{14 + 40 + 102 + 88 + 40}{100} = \frac{284}{100} = 2.84$$

We know that, Mean $= np$
$2.84 = 5 \times p$
$p = \dfrac{2.84}{5} = 0.568$
$q = 1 - 0.568 = 0.432$

Expected Binomial distribution,

$$P(r) = {}^nc_r\, p^r\, q^{n-r}$$
$$= {}^5c_r\, (0.568)^r\, (0.432)^{5-r} \text{ where } r = 0, 1, 2, 3, 4, 5$$

For $r = 0$,
$$P(0) = {}^5c_0\, (0.568)^0\, (0.432)^5 = 0.015$$
$$\text{Expected} = 100 \times 0.015 = 1.5 = 2.$$

For $r = 1$,
$$P(1) = {}^5c_1\, (0.568)^1\, (0.432)^4 = 0.098$$
$$\text{Expected} = 100 \times 0.098 = 9.8 \approx 10$$

For $r = 2$,
$$P(2) = {}^5c_2\, (0.568)^2\, (0.432)^3 = 0.260$$
$$\text{Expected} = 100 \times 0.260 = 26$$

For $r = 3$,
$$P(3) = {}^5c_3\, (0.568)^3\, (0.432)^2 = 0.342$$
$$\text{Expected} = 100 \times 0.342 = 34.2 \approx 34$$

For $r = 4$,
$$P(4) = {}^5c_4\, (0.568)^4\, (0.432) = 0.225$$
$$\text{Expected} = 100 \times 0.224 = 22.5 \approx 23$$

For $r = 5$,
$$P(5) = {}^5c_5\, (0.568)^5\, (0.432)^0 = 0.059$$
$$\text{Expected} = 100 \times 0.059 = 5.9 \approx 6$$

Hence, fitted Binomial distribution,

x	0	1	2	3	4	5
f	2	10	26	34	26	6

Que 4.33. **The number of accidents in a year involving taxi drivers in a city follows a Poisson distribution with mean equal to 3. Out of 1000 taxi drivers, find approximately the number of drivers such that**

i. No accident in a year

ii. More than three accidents in a year.

(given, $e^{-3} = 0.04979$). **AKTU 2021-22 (Sem-3), Marks 10**

$$= 1 - e^{-3}\left[1 + 3 + \frac{9}{2}\right] = 1 - 8.5 \times e^{-3}$$
$$= 1 - 8.5 \times 0.4979 = 1 - 0.42322 = 0.37678$$

Required number of drives $= 1000 \times 0.57678 = 576.78 \approx 577.$

Que 4.34. **In a normal distribution, 12 % of the items are under 30 and 85 % items are under 60. Find the mean and standard deviation.**

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Answer

Let is and σ be the mean and standard deviation respectively 12% of the items are under 30, 85% of items are under 60.

$P(x < 30) = 0.12$ $\qquad$ $P(x < 60) = 0.85$

$P(x > 30) = 1 - 0.12 = 0.88$ $\qquad$ $P(x > 60) = 1 - 0.85 = 0.15$

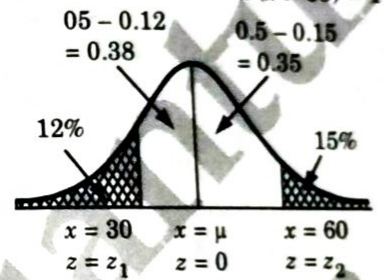

Fig. 4.34.1.

For $\qquad x = 30,\ z = \dfrac{x - \mu}{\sigma}$

$$\frac{30 - \mu}{\sigma} = z_1 \text{ (say)} \qquad \text{...(4.34.1)}$$

For $x = 60$, $\qquad z = \dfrac{x - \mu}{\sigma}$

$$\frac{60 - \mu}{\sigma} = z_2 \text{ (say)} \qquad \text{...(4.34.2)}$$

$$P(0 < z < z_2) = 0.35$$

From normal distribution table, $z_2 = 1.04$

$$\sigma = \frac{30}{2.22} = 13.513$$

From eq. (4.34.3),

$$30 - \mu = -1.18 \times 13.513$$
$$\mu = 30 + 15.945 = 45.945$$

Que 4.35. **If X variable follow the Poisson distribution such that $P(X = 2) = 9\ P\ (X = 4) + 90P\ (X = 6)$. Find mean, variance and distribution.**

AKTU 2022-23 (Sem-4), Marks 10

Answer

$$P(X = 2) = 9P(X = 4) + 90P(X = 6)$$

$$\frac{e^{-\lambda}\lambda^2}{21} = \frac{9e^{-\lambda}\lambda^4}{41} + 90P\frac{e^{-\lambda}\lambda^6}{61}$$

$$\frac{\lambda^2 e^{-\lambda}}{2} = \lambda^2 e^{-\lambda}\left(\frac{9\lambda^2}{41} + \frac{90\lambda^4}{61}\right)$$

$$\frac{1}{2} = \frac{9}{41}\lambda^2 + \frac{90}{61}\lambda^4$$

$$\frac{1}{2} = \frac{9\lambda^2}{24} + \frac{90}{360}\lambda^4$$

$$1 = \frac{9\lambda^2}{12} + \frac{90}{180}\lambda^4$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{2}$$

$$4 = 3\lambda^2 + 2\lambda^4$$

Let $\lambda^2 = X$

$$3X + 2X^2 - 4 = 0$$
$$2X^2 + 3X - 4 = 0$$

$$X = \frac{-3 \pm \sqrt{41}}{4}$$

$$\lambda^2 = \frac{-3 + \sqrt{41}}{4}$$

$$\lambda = \frac{\sqrt{-3 + \sqrt{41}}}{2}$$

Que 4.36. **The following table gives the number of days in a 50 day period during which automobile accidents occurred in a city.**

No. of accidents	0	1	2	3	4
No. of days	21	18	7	3	1

Fit a Poisson distribution to the data and calculate the theoretical frequencies.

AKTU 2021-22 (Sem-4), Marks 10

Answer

$$\text{Mean} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{0 + 18 + 14 + 9 + 4}{50} = \frac{45}{50} = 0.9$$

$\therefore$ Mean of Poisson distribution *i.e.*, $m = 0.9$

Hence, the theoretical frequency for r success is

$$= \frac{Ne^{-m}(m)^r}{r!}$$

$$= \frac{50}{r!}e^{-0.9}(0.9)^r \quad \text{where } r = 0, 1, 2, 3, 4$$

$\therefore$ The theoretical frequency are

No. of accidents	0	1	2	3	4
No. of days	21	19	9	3	1

$[\because e^{-0.9} = 0.41]$

☺☺☺

Quantum Series

5
UNIT

Statistical Techniques–III

CONTENTS

PART-1

Introduction of Sampling Theory, Hypothesis, Null Hypothesis, Alternative Hypothesis, Testing a Hypothesis, Level of Significance, Confidence Limits.

Que 5.1. **A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5 % level of significance.**

Answer

Suppose the coin is unbiased.
Then the probability of getting the head in a toss = 1/2
∴ Expected number of successes = 1/2 × 400 = 200
The observed value of successes = 216
Thus the excess of observed value over expected value = 216 – 200 = 16

Also SD of simple sampling = $\sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $$z = \frac{x - np}{\sqrt{(npq)}} = \frac{16}{10} = 1.6$$

As $z < 1.96$, the hypothesis is accepted at 5 % level of significance i.e., we conclude that the coin is unbiased at 5 % level of significance.

Que 5.2. **In a city A, 20 % of a random sample of 900 school boys had a certain slight physical defect. In another city *B*, 18.5 % of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?**

Answer

We have $$n_1 = 900, n_2 = 1600$$

and $$p_1 = \frac{20}{100} = \frac{1}{5}, p_2 = \frac{18.5}{100}$$

∴ $$p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$$

and $$q = 1 - 0.19 = 0.81$$

Thus $$e^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$= 0.19 \times 0.81\left(\frac{1}{900} + \frac{1}{1600}\right) = 0.0017$$

giving $$e = 0.04 \text{ (approx.)}$$

Also $p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore z = \frac{p_1 - p_2}{e} = \frac{0.015}{0.04} = 0.37$

As $z < 1$, the difference between the proportions is not significant.

Que 5.3. **The mean of a certain normal population is equal to the Standard Error (SE) of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative ?**

Answer

If μ be the mean and σ the SD of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

Also for a sample of size 25, we have

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{25}} = \frac{\bar{x} - \sigma/10}{\sigma/5}$$

$$= \frac{10\bar{x} - \sigma}{10} \times \frac{5}{\sigma} = \frac{10\bar{x} - \sigma}{2\sigma} = \frac{5\bar{x}}{\sigma} - \frac{1}{2}$$

Since $\bar{x}$ is negative, $z < -\frac{1}{2}$

$\therefore$ The probability that a normal variate $z < -\frac{1}{2}$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= 0.5 - 0.915 = 0.3085, \text{ from normal table.}$$

Que 5.4. **An unbiased coin is thrown *n* times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of *n* that will ensure this result with 90% confidence.**

Answer

SE of the proportion of heads $= \sqrt{\frac{pq}{n}} = \sqrt{\frac{1}{2}\frac{1}{2}\frac{1}{n}} = \frac{1}{2\sqrt{n}}$

90 % of confidence = 45 % of the total area under the normal curve on each side of the mean.

$\therefore$ The corresponding value of $z = 1.645$, from normal table.

Thus $\quad p \mp 1.645\,\sigma = 0.49 \text{ or } 0.51$

i.e., $\quad 0.5 - 1.645 \frac{1}{2\sqrt{n}} = 0.49 \text{ and } 0.5 + 1.645 \frac{1}{2\sqrt{n}} = 0.51$

Hence $\quad \frac{1.645}{2\sqrt{n}} = 0.01 \text{ or } \sqrt{n} = \frac{329}{4} \text{ or } n = 6765 \text{ approximately}$

PART-2

Test of Significance of Difference of Means.

Que 5.5. **Explain the test of significance of difference of means.**

Answer

Given two independent samples $x_1, x_2, x_3, ..., x_{n1}$ and $y_1, y_2, y_3, ..., y_{n2}$, with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y from a normal populations with the same variance, we have to test the hypothesis that the population mean μ_1 and μ_2 are the same.

For this, we calculate $\quad t = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ ...(5.5.1)

where $\quad \bar{x} = \frac{1}{n_1}\sum_1^{n_1} x_i, \; \bar{y} = \frac{1}{n_2}\sum_1^{n_2} y_i$

and

$$\sigma_s^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2] = \frac{1}{n_1 + n_2 - 2}\left\{\sum_1^{n_1}(x_i - \bar{x})^2 + \sum_1^{n_2}(y_i - \bar{y})^2\right\}$$

It can be shown that the variate t defined by eq. (5.5.1) follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

It the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1 % level of significant.

If $t < t_{0.05}$, the data is said to be consistent with hypothesis, that $\mu_1 = \mu_2$.

Que 5.6. **Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?**

Boys	1	2	3	4	5	6	7	8	9	10	11
Marks I test	23	20	19	21	18	20	18	17	23	16	19
Marks II test	24	19	22	18	20	22	20	20	23	20	17

Answer

We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1;$$

$$\sigma_s^2 = \frac{\Sigma(d-\bar{d})^2}{n-1} = \frac{50}{10} = 5 \quad i.e., \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching it implies that the mean of the difference between the marks of the two test is zero *i.e.*, $\mu = 0$.

Then $$t = \frac{\bar{d}-\mu}{\sigma_s}\sqrt{n} = \frac{1-0}{2.24}\sqrt{11} = 1.48 \text{ nearly and}$$

$$df\ \nu = 11 - 1 = 10.$$

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	–1	–2	4
3	19	22	3	2	4
4	21	18	–3	–4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	–	–1	1
10	16	20	4	3	9
11	19	17	–2	–3	9
			$\Sigma d = 11$		$\Sigma(d-\bar{d})^2 = 50$

We know that $t_{0.05}$ (for $\nu = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5 % level of significance *i.e.*, the test provides no evidence that the students have benefited by extra coaching.

Que 5.7. **Samples of sizes 10 and 14 were taken from two normal populations with SD 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5 % level.**

Answer

We have, $\bar{x}_1 = 20.3$, $\bar{x}_2 = 18.6$, $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$, $s_2 = 5.2$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = 22.775$$

$\therefore$ $$S = 4.772$$

Null hypothesis, H_0 : $\mu_1 = \mu_2$, *i.e.*, the means of the two populations are the same.

Alternative hypothesis, H_1 : $\mu_1 \neq \mu_2$

Test statistic : Under H_0, the test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{20.3-18.6}{4.772\sqrt{\frac{1}{10}+\frac{1}{14}}} = 0.8604$$

The tabulated value of t at 5 % level of significance for 22 df is $t_{0.05} = 2.0739$

Conclusion :

Since t = 0.8604 < $t_{0.05}$, the null hypothesis H_0 is accepted; *i.e.*, there is no significant difference between their means.

Que 5.8. **The heights of 6 randomly chosen sailors in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.**

Answer

Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

$$n_1 = 6, n_2 = 9$$

Null hypothesis, H_0 : $\mu_1 = \mu_2$. *i.e.*, the mean of both the population are the same.

Alternative hypothesis, H_1 : $\mu_1 > \mu_2$

Calculation of two sample means :

X_1	63	65	68	69	71	72
$X_1 - \bar{X}_1$	–5	–3	0	1	3	4
$(X_1 - \bar{X}_1)^2$	25	9	0	1	9	16

$$\bar{X}_1 = \frac{\Sigma X_1}{n_1} = 68 \,;\, \Sigma(X_1 - \bar{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \bar{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \bar{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\bar{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66 \,;\, \Sigma(X_2 - \bar{X}_2)^2 = 152.0002$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \bar{X}_1)^2 + \Sigma(X_2 - \bar{X}_2)^2]$$

$$= 16.3077$$

$\therefore$ $$S = 4.038$$

Test statistic :

Under H_0, $$t = \frac{\bar{X}_1 - \bar{X}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.038\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.1569$$

The value of t at 5 % level of significance for 13 df is 1.77 $(df = n_1 + n_2 - 2)$

Conclusion : Since $t_{\text{calculated}} < t_{0.05} = 1.77$, the null hypothesis H_0 is accepted. *i.e.*, there is no significant difference between their averages. *i.e.*, the sailors are not on the average taller than the soldiers.

Que 5.9. **The 9 items of a sample have the following values : 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5 ?**
[The tabulated value of $t_{0.05}$ = 2.31 for 8 d.f]

AKTU 2021-22 (Sem-4), Marks 10

Answer

$$\bar{X} = \frac{(45 + 47 + 50 + 48 + 47 + 40 + 53 + 51 + 52)}{9}$$

$$= 49.11$$

X	45	47	50	52	48	47	49	53	51	SUM
$X - \bar{X}$	– 4.11	– 2.11	0.89	2.89	– 1.11	– 2.11	– 0.11	3.89	1.89	
$(X - \bar{X})^2$	16.89	4.45	0.79	8.35	1.23	4.45	1.21	15.13	3.57	56.07

$$s^2 = \sum\frac{(X - \bar{X})^2}{9} = \frac{56.07}{9} = 6.23$$

i. **The null hypothesis H_0 : μ = 47.5.** Alternative hypothesis $H_1 : \mu \neq 47.5$.

ii. **Calculation of test statistic :** Since the sample size is small, we use t-distribution;

$$t = \frac{(\bar{X} - \mu)}{\left(\frac{s}{\sqrt{9-1}}\right)} = \frac{(49.11 - 47.5)}{\left(\frac{\sqrt{6.33}}{\sqrt{(9-1)}}\right)} = 1.82$$

iii. **Level of significance :** $\alpha = 0.05$.

iv. **Critical value :** The value of t_α at 5 % level of significance for $v = 9 - 1 = 8$ degrees of freedom is 2.31.

v. **Decision :** Since the calculated value of $|t| = 1.82$ is less than the table value $t_\alpha = 2.31$. Therefore, the null hypothesis is accepted.

Que 5.10. **The annual rainfall in Lucknow city is normally distributed with mean 45 cm. The rainfall during the last five years are 48 cm, 42 cm, 40 cm, 44 cm and 43 cm respectively. Can we conclude that the average rainfall during the last five years is less than the normal rainfall? Test at 5 % level of significance. [The tabulated value of $t_{0.05}$ = 2.776 and $t_{0.1}$ = 2.132 for 4 degree of freedom.**

AKTU 2022-23 (Sem-4), Marks 10

Answer

We have the mean and standard deviation of the small sample as

$$\bar{x} = \frac{1}{n}\sum_i x_i = \frac{1}{5}(48 + 42 + 40 + 44 + 43) = 43.4$$

$$s^2 = \left(\frac{1}{n}\sum_i x_i^2\right) - \bar{x}^2 = \frac{1}{5}(9453) - 43.4)^2 = 7.04$$

Null hypothesis, H_0 : $\bar{x} = \mu$ (there is no significant difference in the rain fall).

Alternate hypothesis, H_1 : $\bar{x} < \mu$.

We use the left tailed test with 5 % level of significance. Now t (0.05) for one tailed test = $t(0.1)$ for two tailed test with $n = 5$. The value of t for $P = 0.05$ and $\upsilon = 4$ is 2.132. The test statistic is given by

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n = 1}} = \frac{43.4 - 45}{1.3266} = -1.206$$

Since, $|t| = 1.206 < 2.132$, we accept the null hypothesis. There is no significance difference in the rainfall.

PART-3

T-Test, Z-Test, and Chi-Square Test.

Que 5.11. The annual rainfall at a certain place is normally distributed with mean 45 cm. The rainfall during the last five years are 48 cm, 42 cm, 40 cm, 44 cm and 43 cm. Can we conclude that the average rainfall during the last five years is less than the normal rainfall ? Test at 5% level of significance.

Answer

We have the mean and standard deviation of the small sample as

$$\bar{x} = \frac{1}{n}\sum_i x_i = \frac{1}{5}(48 + 42 + 40 + 44 + 43) = 43.4$$

$$s^2 = \left(\frac{1}{n}\sum_i x_i^2\right) - \bar{x}^2 = \frac{1}{5}(9453) - 43.4)^2 = 7.04$$

Null hypothesis, $H_0 : \bar{x} = \mu$ (there is no significant difference in the rain fall).

Alternate hypothesis, $H_1 : \bar{x} < \mu$.

We use the left tailed test with 5 % level of significance. Now t (0.05) for one tailed test = t(0.1) for two tailed test with $n = 5$. The value of t for $P = 0.05$ and $\upsilon = 4$ is 2.132. The test statistic is given by

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n = 1}} = \frac{43.4 - 45}{1.3266} = -1.206$$

Since, $|t| = 1.206 < 2.132$, we accept the null hypothesis. There is no significance difference in the rainfall.

Que 5.12. The height of 8 males participating in an athletic championship are found to be 175 cm, 168 cm, 165 cm, 167 cm, 160 cm, 173 cm and 168 cm. Can we conclude that the average height is greater than 165 cm ? Test at 5 % level of significance.

Answer

Null hypothesis, H_0 : $\mu = 165$ cm.

Alternate hypothesis, H_1 : $\mu < 165$ cm

We use the right tailed test with 5 % level of significance. We have $n = 8$.

Since, t (0.05) for one tailed test = t (0.1) for two tailed test, we have for 7 degrees of freedom and $P = 0.05$, $t = 1.895$. We compute the sample mean and standard deviation. We have

$$\bar{x} = \frac{1}{8}(175 + 168 + 170 + 167 + 160 + 173 + 168)$$
$$= 168.25$$

$$s^2 = \frac{1}{8}\left(\frac{1}{n}\sum_i x_i^2\right) - \bar{x}^2 = \frac{1}{8}(226616) - (168.25)^2$$
$$= 18.9375.$$

The test statistic is given by

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{168.25 - 165}{4.3517/2.6458} = 1.976$$

Since, $|t| = 1.976 < 1.895$, we reject the null hypothesis and accept the alternative hypothesis. The average height is greater than 165 cm.

Que 5.13. The scores of 10 candidates obtained in tests before and after attending some coaching classes are given below :

Before	54	76	92	65	75	78	66	82	80	78
After	60	80	86	72	80	72	66	88	82	73

Is the coaching for the test effective ? Test at 5 % level of significance.

AKTU 2022-23 (Sem-3), Marks 10

Answer

The data relates to the marks obtained by the same set of students. Hence, we can regard that the marks are correlated.

If x_i, y_i denote the marks obtained in the two tests, we obtain the values of $d_i = x_i - y_i$ as –6, –4, 6, –7, –5, 6, 0, –6, –2, 5.

We find
$$\bar{d} = \frac{1}{n}\Sigma d_i = -\frac{13}{10} = -1.3,$$

$$s_d^2 = \frac{1}{n}\Sigma d_i^2 - \bar{d}^2 = \frac{1}{10}(263) - 1.69 = 24.61.$$

We define

Null hypothesis, H_0 : $\bar{d} = 0$ (the students have not benefited from coaching).

Alternate hypothesis, H_1 : $\bar{d} < 0$ (the students have benefited from coaching).

We shall use the one tailed test. Now, t (0.05) for one tailed test = t(0.1) for two tailed test, with the degrees of freedom = $n - 1 = 9$. The value of t for $P = 0.05$ and $\upsilon = 9$ is 1.833.

$$t = \frac{\bar{d}}{s_d/\sqrt{n-1}} = \frac{-1.3}{4.9608/3} = -0.786.$$

We find $|t| = 0.786 < 1.833$. Hence, we accept the null hypothesis that the students have not benefited from coaching.

Que 5.14. **Two random samples of sizes 9 and 7 gave the sum of square of deviations from their respective means as 175 and 95 respectively. Can they be regarded as drawn from normal populations with the same variance ?**

Answer

We have

$$n_1 = 9,\ \Sigma(x_i - \bar{x})^2 = n_1 s_1^2 = 175,$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{175}{8} = 21.875.$$

$$n_2 = 7,\ \Sigma(y_i - \bar{y})^2 = n_2 s_2^2 = 95,$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{95}{6} = 15.8333.$$

Now, $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$. Hence, we take $\nu_1 = n_1 - 1 = 8$, and $\nu_2 = n_2 - 1 = 6$. We define

Null hypothesis, H_0 : $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$.

Alternate hypothesis, H_1 : $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

At 5 % level of significance, we have, $F_{0.05}(8, 6) = 4.15$

Now, the F-statistic is given by

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{21.875}{15.8333} = 1.381 < 4.15$$

Therefore, we accept the null hypothesis H_0. The two random samples might have come from two normal populations with the same variance.

Que 5.15. **The values in two random samples are given below :**

Sample 1 :	15	25	16	20	22	24	21	17	19	23		
Sample 2 :	35	31	25	38	26	29	32	34	33	27	29	31

Can we conclude that the two samples are drawn from the same population ? Test at 5% level of significance.

AKTU 2022-23 (Sem-3), Marks 10

Answer

We have,

$$n_1 = 10$$

$$\bar{x} = \frac{15 + 25\ 16 + 20 + 22 + 24 + 21 + 17 + 19 + 23}{10}$$

$$\frac{202}{10} = 20.2$$

$$S_1^2 = \frac{1}{10}\left(\sum x_i^2\right) - \bar{x}^2 = \left[\frac{1}{10}(4186)\right] - (20.2)^2$$

$$= 418.6 - 408.04$$

$$S_1^2 = 10.56$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10}{9}(10.56)^2 = 123.90$$

$$n_2 = 12$$

$$\bar{x} = \frac{35 + 31 + 25 + 38 + 26 + 29 + 32 + 34 + 33 + 27 + 29 + 31}{12}$$

$$= \frac{370}{12} = 30.83$$

$$s_2^2 = \frac{1}{12}\left(\sum x_i^2\right) - \bar{x}^2 = \frac{1}{12} \times 11572 - (30.83)^2$$

$$= 964.33 - 950.49$$

$$s_2 = 13.84$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{12}{11} \times (13.84)^2 = 208.96$$

Now, $\hat{\sigma}_2^2 > \hat{\sigma}_1^2$, we take $\upsilon_1 = n_1 - 1 = 9$ and $\upsilon_2 = n_2 - 1 = 11$

We define

Null hypothesis, H_0 : $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$

Alternate hypothesis, $H_1 = \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$

At 5% level of significance, we have $F_{0.05}(9, 11) = 2.90$

Now, the F-statistic is given by

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{123.90}{208.96} = 0.59 < 2.90$$

Thus we accept the null hypothesis. There is no significant difference between the two sample.

We can conclude that the two samples see drawn from the same population.

Que 5.16. **A survey of 240 families with 4 children shows the following distribution :**

Number of boys	4	3	2	1	0
Number of families	10	55	105	58	12

Test the hypothesis that male and female births are equal probable. (Given : $\chi^2_{0.05}$ = 9.49 and 11.1 for 4 *d.f.* and 5 *d.f.* respectively)

Answer

Null hypothesis, H_0 : Male and female are equally probable.

Number of boys	4	3	2	1	0
Number of girls	0	1	2	3	4
Number of families	10	55	105	58	12

Alternate hypothesis, H_1 : Male and female birth are not equally probable.

Calculation of expected frequencies $(q+p)^n$,

Probability of female birth $= p = \frac{1}{2}$

Probability of male birth $= q = \frac{1}{2}$

$$(q+p)^n = q^n + {}^nc_1 pq^{n-1} + {}^nc_2 p^2 q^{n-2} + {}^nc_3 p^3 q^{n-3} \ldots p^n$$

$$= \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^4$$

$$\text{Number of girls} = 240\left[\frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right]$$

$$= 240 \times \frac{1}{16} + 240 \times \frac{4}{16} + 240 \times \frac{6}{16} + 240 \times \frac{4}{16} + 240 \times \frac{1}{16}$$

$$= 15 + 60 + 90 + 60 + 15$$

These are the expected frequencies of female births.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
10	15	–5	25	1.67
55	60	–5	25	0.41
105	90	15	225	2.5
58	60	–2	4	0.067
12	15	–3	9	0.6
			Total	5.247

Given, $\chi^2_{0.05}$ = 9.49 and 11.1 for 4 *d.f.* and 5 *d.f.*

Since the calculated value of χ^2 (5.247) < χ^2 value at 4 *d.f.* and 5 *d.f.*

Hence, the null +hypothesis is accepted *i.e.*, the male and female birth is equally probable.

Que 5.17. **From the following table regarding the color of eyes of father and son, test if the color of son's eye is associated with that of father.**

		Eye color of son	
		Light	Not Light
Eye color of father	Light	471	51
	Not Light	148	230

Given $\chi^2_{0.05}$ (1) = 3.841.

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Answer

Given table can be written as :

		Eye color of son		
		Light	Not Light	Total
Eye color of father	Light	471	51	522
	Not Light	148	230	378
	Total	619	281	900

Let us suppose the null hypothesis is that there is no association between color of son's eye and father's eye.

The expected frequencies are,

$$E(471) = \frac{619 \times 522}{900} = 359.02$$

$$E(51) = \frac{281 \times 522}{900} = 162.98$$

$$E(148) = \frac{619 \times 378}{900} = 259.98$$

$$E(230) = \frac{281 \times 378}{900} = 118.02$$

f_o	f_e	$f_o - f_e$	$(f_o - f_e)^2$	$(f_o - f_e)^2/f_e$
471	359.02	111.98	12539.5204	34.927
51	162.98	– 111.98	12539.5204	76.939
148	259.98	– 111.98	12539.5204	48.233
230	118.02	111.98	12539.5204	106.249
				$\sum \frac{(f_o - f_e)^2}{f_e} = 266.348$

$$x^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 266.348$$

Table value of x^2 at 5 % level of significance for 5 degree of freedom is 3.841. Since the calculated value is greater than table value therefore null hypothesis is rejected. Thus, there is an association between color of son's eye and color of father's eye.

Que 5.18. **In an experiment on immunization of cattle from tuberculosis the following results were obtained :**

	Affected	Unaffected
Inoculated	12	28
Not inoculated	13	7

Examine the effect of vaccine in controlling the incidence of the disease. [Given $\chi^2_{0.05,1} = 3.84$]

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Answer

Affected	Unaffected	Total	
Inoculated	12	28	40
Not inoculated	13	7	20
Total	25	35	60

Let us suppose the null hypothesis is that there is no effect of vaccine in controlling the incidence of the diseases.

$$E(12) = \frac{25 \times 40}{60} = 16.67 \qquad E(13) = \frac{25 \times 20}{60} = 8.33$$

$$E(28) = \frac{35 \times 40}{60} = 23.33 \qquad E(7) = \frac{35 \times 20}{60} = 11.67$$

f_o	f_e	$f_o - f_e$	$(f_o - f_e)^2$	$(f_o - f_e)^2/f_e$
12	16.67	– 4.67	21.81	1.31
28	23.33	4.67	21.81	0.93
13	8.33	4.67	21.81	2.62
7	11.67	– 4.67	21.81	1.87
				$\sum \frac{(f_o - f_e)^2}{f_e} = 6.73$

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 6.73$$

Table value χ^2 at 5 % level of significance for 5 degree of freedom is 3.84. Since the calculated value is greater than table value therefore null hypothesis is rejected. Thus, there is significant effect of vaccine in controlling the incidence of diseases.

Que 5.19. **In two independent sample of size 8 and 10, the sum of square of deviations of the sample values from the respective means were 84.4 and 102.6. Test whether the difference of variances of populations is segment or not. Use a 5% level of significance. [$F_{005\,(7,\,9)} = 3.29$]**

AKTU 2021-22 (Sem-3), Marks 10

Answer

We have $\quad n_1 = 8\ (x_i - \bar{x})_2 = n_1 s_1^2 = 84.4$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{84.4}{7} = 12.057$$

$$n_2 = 10,\ \Sigma(y_i - \bar{y})^2\ n_2 s_2^2 = 102.6$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{102.6}{9} = 11.4$$

Now, $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$. Hence, we take $\upsilon_1 = n_1 - 1 = 7$, and $\upsilon_2 = n_2 - 1 = 9$.

We define

Null hypothesis, H_0 : $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$.

Alternate hypothesis, H_1 : $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

At 5 % level of significance, we have, $F_{0.05}(7, 9) = 3.29$
Now, the F-statistic is given by

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{12.057}{11.4} = 1.057 < 3.29$$

Therefore, we accept the null hypothesis. There is no significant difference between the population variance.

Que 5.20. **The demand for a particular spare part in a factory was found to vary from day-to-day. In a sample study the following information was obtained**

Days	Mon	Tue	Wed	Thurs	Fri	Sat
No. of parts demanded	1124	1125	1110	1120	1126	1115

Use χ^2-test to test the hypothesis that the number of parts demanded does not depend on the day of the week.
[The value of $\chi^2_{0.05} = 11.07$ for 5 *d. f*].

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Answer

Null hypothesis H_0 = Number of parts demanded does not depend on all days of week.
Alternative Hypothesis H_1 = Number of parts demanded depends on the day of the week.
Test statistics

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

O = Observed frequency
E = Expected frequency
Total number of parts demanded = 6720

Under the null hypothesis, $E_i = \frac{6720}{6} = 1120$

O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
1124	1120	4	16	0.014
1125	1120	5	25	0.022
1110	1120	– 10	100	0.089
1120	1120	0	0	0
1126	1120	6	36	0.032
1115	1120	– 5	25	0.022
			Total = 0.179	

Given $\chi^2_{0.05} = 11.07$
Since, the calculate value of χ^2 (0.179) < χ^2 value at 5 d.f.
Hence, null hypothesis is accepted *i.e.*, the number of parts demanded does not depend on the day of the week.

Que 5.21. **In an experiment on pea breading the following frequency of seeds were obtained :**

Red and Yellow	Wrinkled and Yellow	Round and Green	Wrinkled and Green	Total
315	101	108	32	556

Theory predicts the frequencies should be in the proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment. Test at 5 % level of significance. [The tabulated value of $\chi^2_{0.05} = 7.815$ for 3 degree of freedom.]

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Answer

Observed value (f_0)	Expected values (f_e)
315 Red and Yellow	$\left(\frac{9}{16}\right) \times 566 = 312.75$
101 Wrinkled and yellow	$\frac{3}{16} \times 566 = 104.25$
108 Round and green	$\frac{3}{16} \times 566 = 104.25$
32 Wrinkled and green	$\frac{1}{16} \times 566 = 34.75$
566 Total seeds	566 Total seeds

f_0	f_e	$f_0 - f_e$	$(f_0 - f_e)^2$	$(f_0 - f_e)^2 / f_e$
315	312.75	2.25	5.0625	0.0162
101	104.25	– 3.25	10.5625	0.1013
108	104.25	– 3.25	14.0625	0.1349
32	34.75	– 2.75	7.5625	0.2176
				$\sum \frac{(f_0 - f_e)^2}{f_e} = 0.47$

$$x^2 = \sum \frac{(f_0 - f_e)^2}{f_e} = 0.47$$

Table value x^2 at 5% level of significance for 3 degree of freedom is 1.815.
Since the calculated value of $x^2(0.47) < 7.815$
Hence the null hypothesis is accepted.

PART-4

Statistical Quality Control (SQC), Control Charts, Control Charts for Variables ($\bar{X}$, R Charts).

Que 5.22. **If number of samples = 20, size of each sample = 5, $\bar{R} = 2.32\,\bar{\sigma}$, $\bar{\bar{x}} = 99.6$, $\bar{R} = 7.0$. Find the values of control limit for drawing a mean chart. [$n = 5$, mean range = 2.32 (population S.D.)]**

Answer

Here, we have

$$\bar{\bar{x}} = 99.6$$
$$\bar{R} = 7.0$$
$$\bar{R} = 2.32\,\bar{\sigma} \quad \Rightarrow \quad \bar{\sigma} = \frac{\bar{R}}{232} = \frac{7}{2.32} = 3.0172$$
$$n = 5$$

UCL
$$= \bar{X} + 3\left(\frac{\bar{\sigma}}{\sqrt{n}}\right) = 99.6 + \left(3 \times \frac{3.0172}{\sqrt{5}}\right)$$
$$= 99.6 + \frac{9.0516}{2.2361} = 99.6 + 4.0479 = 103.6479$$

LCL
$$= \bar{X} - 3\left(\frac{\bar{\sigma}}{\sqrt{n}}\right) = 99.6 - 4.0479 = 95.5521$$

CL
$$= 99.6$$

$\bar{X}$ - Chart :

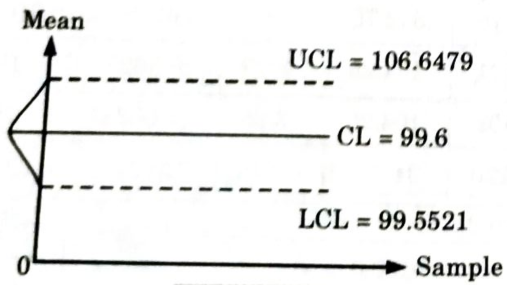

Fig. 5.22.1.

Que 5.23. **Control on measurements of pitch diameter of thread in aircraft fittings is checked with 5 samples each containing 5 times at equal intervals of time. The measurements are given below. Constant $\bar{X}$ and R charts and state your inference from the charts.**

Sample No.	1	2	3	4	5
Measurements x	46	41	40	42	43
	45	41	40	43	44
	44	44	42	43	47
	43	42	40	42	47
	42	40	42	45	45

Answer

Here, we have

Sample No.	1	2	3	4	5
Measurements x	46	41	40	42	43
	45	41	40	43	44
	44	44	42	43	47
	43	42	40	42	47
	42	40	42	45	45
Total	$\Sigma x = 220$	$\Sigma x = 208$	$\Sigma x = 204$	$\Sigma x = 215$	$\Sigma x = 226$
$\bar{X} = \frac{\Sigma x}{n}$	$\frac{220}{5} = 44$	$\frac{208}{5} = 41.6$	$\frac{204}{5} = 40.8$	$\frac{215}{5} = 43$	$\frac{226}{5} = 43$
R	$46 - 42 = 4$	$44 - 40 = 4$	$42 - 40 = 2$	$45 - 42 = 3$	$47 - 43 = 4$

$$\bar{X} = \frac{\Sigma \bar{X}}{5} = \frac{44 + 41.6 + 40.8 + 43.0 + 45.2}{5}$$

From the table of control chart, for sample size of 5 items, $A_2 = 0.577$,

Limits for $\bar{X}$-Chart :

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R} = 42.92 + 0.577 \times 3.4 = 44.88$$
$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2\,\bar{R} = 42.92 - 0.577 \times 3.4 = 40.96$$
$$\bar{X}_3 = 40.8 < LCL = 40.96$$
$$\bar{X}_5 = 45.2 > UCL = 44.88$$

All sample points do not lie between control limits. Hence, the process is out of control.

$\bar{\bar{X}}$ -Chart :

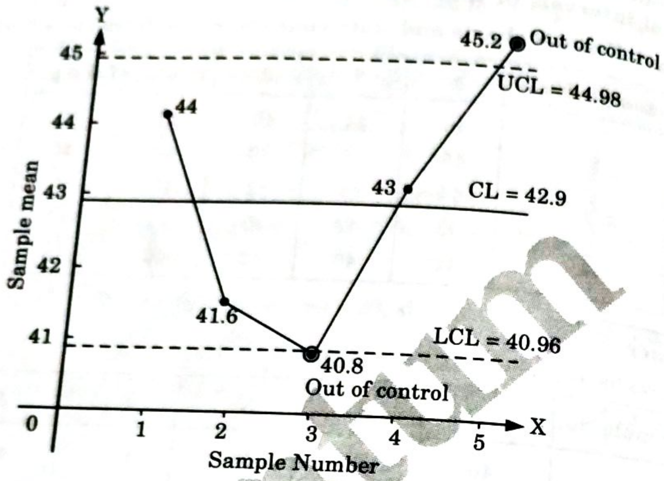

Fig. 5.23.1.

From the control chart table, $D_3 = 0, D_4 = 2.115$

***R*-Chart :**

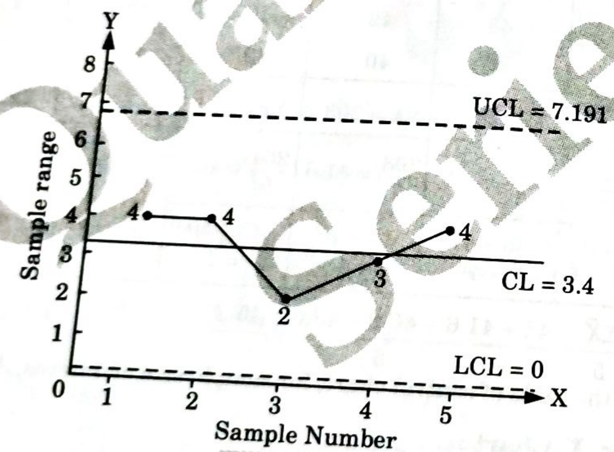

Fig. 5.23.2.

Limits for *R*-Chart :

$$UCL_R = D_4 \bar{R}$$
$$= 2.115 \times 3.4 = 7.191$$
$$LCL_R = D_3 \bar{R} = 0$$
$$CL_R = \bar{R} = 3.4$$

All sample points lie between control limits. Hence, the variability is under control. But process is out of control due to $\bar{X}$ - chart.

Que 5.24. **The given table shows that the value of sample mean $\bar{X}$ and the range *R* for 10 samples of size 5 each. Draw mean and range chart and also comment on the state of control of the process. (Given $A_2 = 0.58, D_3 = 0, D_4 = 2.115$).**

Sample No.	1	2	3	4	5	6	7	8	9	10
$\bar{X}$	45	46	48	52	53	37	51	46	47	38
R	4	5	6	7	4	5	7	6	6	4

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Answer

Mean of means $\bar{\bar{x}} = \dfrac{\Sigma x}{n}$

$$= \frac{45 + 46 + 48 + 52 + 53 + 37 + 51 + 46 + 47 + 38}{10}$$

$$= \frac{463}{10} = 46.3$$

From the table of control chart for sample size of 5
$A_2 = 0.58, D_3 = 0, D_4 = 3.115$
Control limits of $\bar{x}$ – chart

$$ULC = \bar{\bar{X}} + A_2\bar{R}\ ,\ LCL = \bar{\bar{X}} - A_2\bar{R}$$

Mean of range $\bar{R} = \dfrac{4+5+6+7+4+5+7+6+6+4}{10} = 5.4$

$ULC = 46.3 + 0.58 \times 5.4 = 49.432$
$LCL = 463 - 0.58 \times 5.4 = 43.168$

$\bar{X}$ - **Chart :**

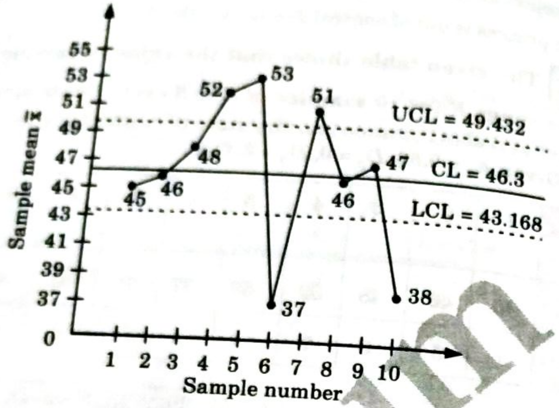

Fig. 5.24.1.

Hence we have the means of the sample number 4, 5, 7 which are greater than upper control limit (UCL) and sample number 6, 10 are less than lower control limit. Five of ten samples points falls outside the control limits.

Hence, the process is very much out of control.

***R*-Chart :**

From the table of control chart $D_3 = 0, D_4 = 2.115, \bar{R} = 5.4$

From the control chart table for sample size of 5 $A_2 = 0.58$

Control units for *R*-Chart :

$$UCL = D_4\bar{R} = 2.115 \times 5.4 = 11.4$$

$$LCL = D_3\bar{R} = 0$$

$$CL = \bar{R} = 5.4$$

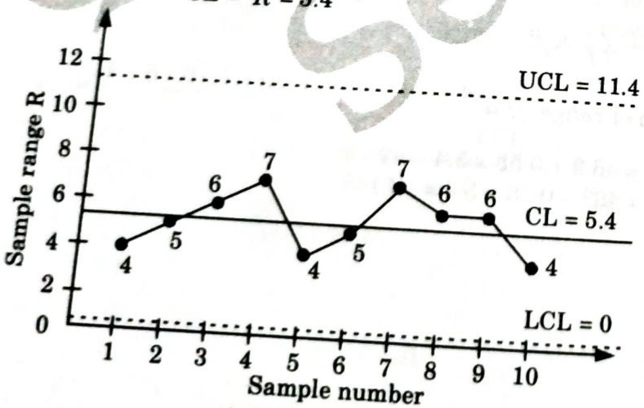

Fig. 5.24.2.

All values of R lie between the control limits 11.4 and 0. Hence, the variability is under control. Still the process is out of control due to $\bar{X}$ - chart.

PART-5

Control Charts for Variables (p, np and C-Charts).

Que 5.25. **15 sample with size 200 each taken at an interval of 45 minutes from a manufacturing process the average fraction defective was 0.068. Calculate the values of central line, upper and lower control line.**

Answer

Here, we have

Average fraction defective = 0.068

$\therefore$ Central line CL = $\bar{p} = 0.068$

We know that

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.068 + 3\sqrt{\frac{0.068(1-0.068)}{200}}$$

$$= 0.068 + 0.053 = 0.121$$

$$LCL_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.068 - 0.053 = 0.15$$

Que 5.26. **Construct a *p*-chart for the following data :**

Number of samples (each of 100 items)	1	2	3	4	5	6	7	8	9	10
Number of defectives	12	10	6	8	9	9	7	10	11	8

Answer

Number of Sample	Number of units in a sample (*n*)	Number of defectives *d*	Fraction defective $p = d/n$
1	100	12	0.12
2	100	10	0.10
3	100	6	0.06
4	100	8	0.08
5	100	9	0.09
6	100	9	0.09
7	100	7	0.07

8	100	10	0.10
9	100	11	0.11
10	100	8	0.08
Total	$N = 100$	$\Sigma d = 90$	

Average fraction defective = $\bar{p}$

$$= \frac{\text{Total no. of defective in all samples combined}}{\text{Total no. of items in all samples}}$$

$$= \frac{\Sigma d}{N} = \frac{90}{1000} = 0.09$$

Standard limits :

Upper Control Limit, $UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

$$= 0.09 + 3\sqrt{\frac{0.09\,(1-0.09)}{100}}$$

$$= 0.09 + 3\sqrt{0.000819}$$

$$= 0.09 + 3 \times 0.0286 = 0.09 + 0.0858 = 0.1758$$

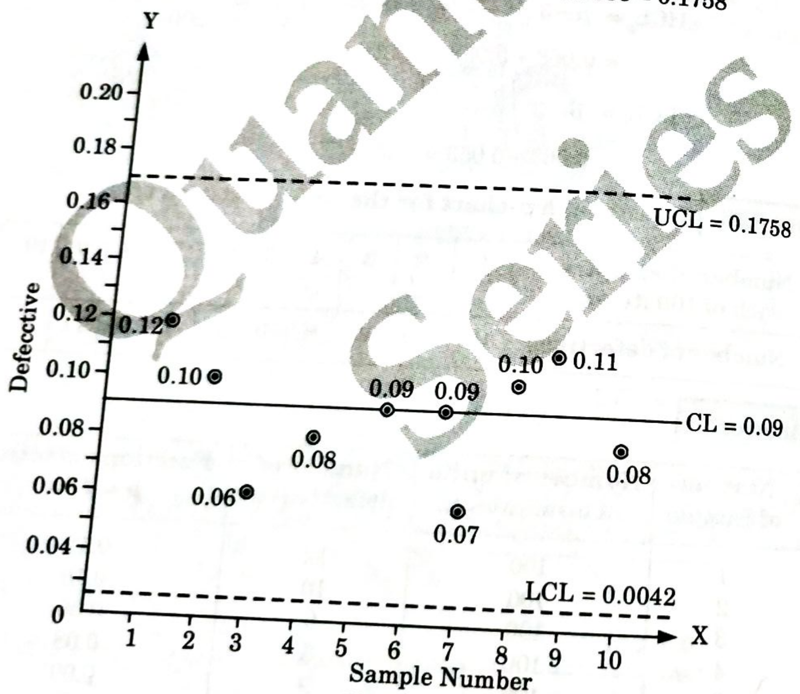

Fig. 5.26.1

Lower Control Limit, $LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

$$= 0.09 - 0.0858 = 0.0042$$

All the values lie between control limits.

Hence, the variability is under control.

Que 5.27. **The following set of data covering 15 consecutive production days on the number of defectives found in daily production from a sample of 200 units. Draw a *p*-chart and test whether the production process was in control.**

Production day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of defectives	10	5	10	12	11	9	22	4	12	24	21	15	8	14	4

Number of Sample	Number of units in a sample (n)	Number of defectives d	Fraction defective $p = d/n$
1	200	10	0.05
2	200	5	0.025
3	200	10	0.05
4	200	12	0.06
5	200	11	0.055
6	200	9	0.045
7	200	22	0.011
8	200	4	0.02
9	200	12	0.06
10	200	24	0.012
11	200	21	0.105
12	200	15	0.075
13	200	8	0.04
14	200	14	0.07
15	200	4	0.02
Total	3000	181	

i. Average fraction defective

$$= \bar{p} = \frac{\Sigma d}{N}$$

$$= \frac{\text{Total no. of defective in all samples combined}}{\text{Total number of items in all samples}}$$

$$= \frac{181}{30,00} = 0.0603$$

ii. Standard limits

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.0603 + 3\sqrt{\frac{0.0603\,(1-0.0603)}{200}}$$

$$= 0.0603 + 0.0505 = 0.1108$$

$$LCL_p = 0.0603 - 0.0505 = 0.0098$$

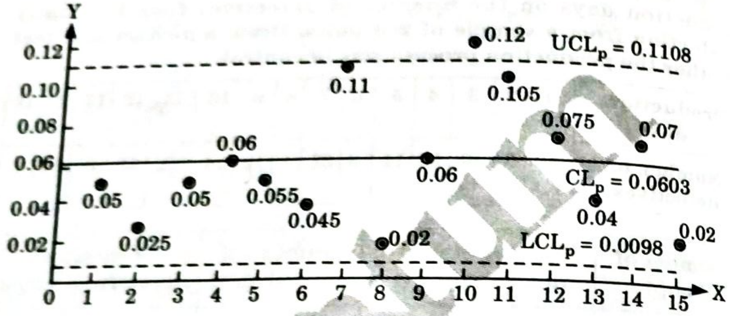

Fig. 5.27.1.

One sample point is above the UCL line, so the production process is to be corrected to make it under control.

Que 5.28. **A drilling machine bores holes with a mean diameter of 0.5230 cm and a standard deviation of 0.0032 cm. Calculate the 2-sigma and 3-sigma upper and lower control limits for means of sample of 4.**

Answer

Mean diameter $\bar{\bar{x}} = 0.5230$ cm

S.D. $\sigma = 0.0032$ cm, $n = 4$

i. 2-sigma limits are as follows :

$$CL = \bar{\bar{x}} = 0.5230 \text{ cm}$$

$$UCL = \bar{\bar{x}} + 2\frac{\sigma}{\sqrt{n}} = 0.5230 + 2 \times \frac{0.0032}{\sqrt{4}} = 0.5262 \text{ cm}$$

$$LCL = \bar{\bar{x}} - 2\frac{\sigma}{\sqrt{n}} = 0.5230 - 2 \times \frac{0.0032}{\sqrt{4}} = 0.5198 \text{ cm.}$$

ii. 3-sigma limits are as follows :

$$CL = \bar{\bar{x}} = 0.5230 \text{ cm}$$

$$UCL = \bar{\bar{x}} + 3\frac{\sigma}{\sqrt{n}} = 0.5230 + 3 \times \frac{0.0032}{\sqrt{4}} = 0.5278 \text{ cm}$$

$$LCL = \bar{\bar{x}} - 3\frac{\sigma}{\sqrt{n}} = 0.5230 - 3 \times \frac{0.0032}{\sqrt{4}} = 0.5182 \text{ cm.}$$

Que 5.29. **In a blade manufacturing factory, 1000 blades are examined daily. Draw the *np*-chart for the following table and examine whether the process is under control ?**

Date number of defective	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Blades	9	10	12	8	7	15	10	12	10	8	7	13	14	15	16

Answer

Here, $n = 1000$

Σnp = total number of defectives = 166

Σn = total number inspected = 1000 × 15

$\therefore$ $$\bar{p} = \frac{\Sigma\, np}{\Sigma n} = \frac{166}{1000 \times 15} = 0.011$$

$$n\bar{p} = 1000 \times 0.011 = 11$$

$\therefore$ Control limits are $CL = n\bar{p} = 11$

$$UCL_{np} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} = 11 + 3\sqrt{11(1-0.011)} = 20.894$$

$$LCL_{np} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} = 11 - 3\sqrt{11(1-0.011)} = 1.106$$

***np*-Chart :**

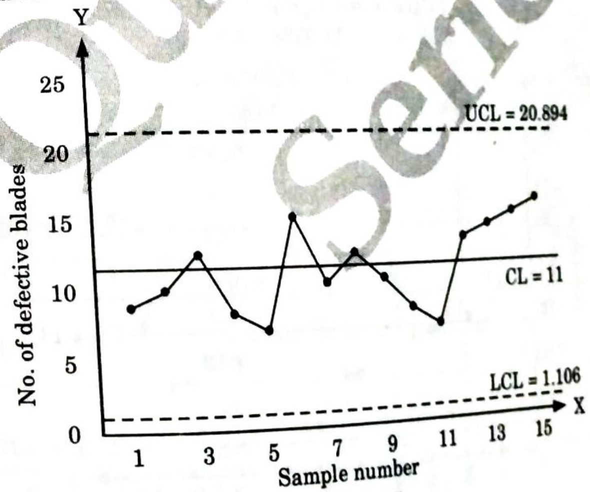

Fig. 5.29.1.

Since all the points lie within the control limits, the process is under control.

Que 5.30. **An inspection of 10 samples of size 400 each from 10 lots revealed the following number of defective units 17, 15, 14, 26, 9, 4, 19, 12, 9, 15.**
Calculate control limits for the number of defective units and state whether the process is under control or not.

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Answer

Here, we have

Total number of items in 10 samples = $N = 10 \times 400 = 4000$

Total number of defectives in 10 samples = Σd

$= 17 + 15 + 14 + 26 + 9 + 4 + 19 + 12 + 9 + 15$

$= 140$

Average fraction defective = $\bar{p} = \dfrac{\Sigma d}{N}$

$= \dfrac{140}{4000} = 0.035$

$n\bar{p} = 400 \times 0.035 = 14$

$CL = n\bar{p} = 14$

$UCL = n\,\bar{p} + 3\sqrt{n\,\bar{p}\,(1-\bar{p})} = 14 + 3\sqrt{14\,(1-0.035)}$

$= 14 + 3\sqrt{14 \times 0.965} = 14 + 3\sqrt{13.51}$

$= 14 + 3 \times 3.676 = 14 + 11.028 = 25.028$

$LCL = n\,\bar{p} - 3\sqrt{n\,\bar{p}\,(1-\bar{p})}$

$= 14 - 11.0288 = 2.972$

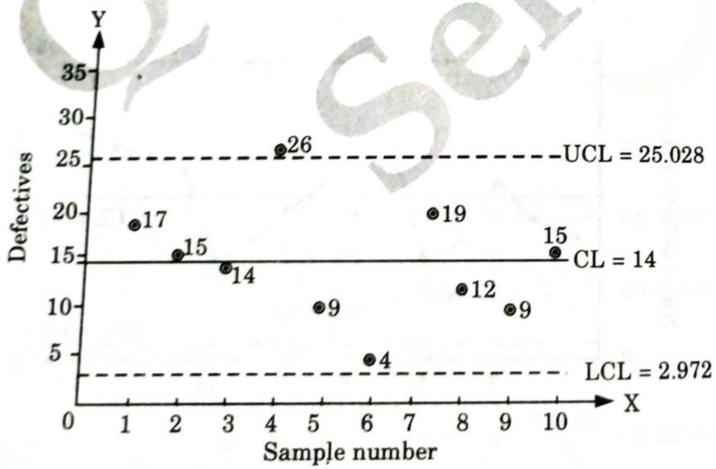

Fig. 5.30.1.

The number of defective units 26 in the 4th sample is greater than the value of the upper control limits of 25. Hence, the sample point falls outside the upper control limit, so, the process is not under control.

Que 5.31. **Distinguish between the *np*-chart and *p*-chart. Following is the data of defective of 10 samples of size 100 each. Construct *np*-chart and examine whether the process is in statistical control ?**

Sample No.	1	2	3	4	5	6	7	8	9	10
No. of defectives	6	9	12	5	12	8	8	16	13	7

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Answer

Difference :

S. No.	*p*-chart	*np*-chart
1.	*p*-charts show the proportion of nonconforming units on the *y*-axis.	*np*-charts show the whole number of nonconforming units on the *y*-axis.
2.	When the subgroup sizes are different the center line on the *p*-chart is straight.	When the subgroup sizes are different the center line on the *np*-chart varies.

Numerical :

Number of sample	Number of unit in sample (n)	Number of defectives(d)	Fraction defective $p = d/n$
1	100	6	0.06
2	100	9	0.09
3	100	12	0.12
4	100	5	0.5
5	100	12	0.12
6	100	8	0.08
7	100	8	0.08
8	100	16	0.16
9	100	13	0.13
10	100	7	0.07
Total	1000	96	

$n = 100$

Σd = total number of defectives = 96

Σn = total number of items of all combined $= 100 \times 10 = 1000$

$$\bar{p} = \frac{\Sigma d}{\Sigma n} = \frac{96}{1000} = 0.096$$

Control limits are, $CL = n\bar{p} = 100 \times 0.096 = 9.6$

$$UCL = n\bar{p} + 3 \times \sqrt{n\bar{p}\bar{q}} = 9.6 + 3 \times \sqrt{100 \times 0.096 \times (1 - 0.096)}$$

$$= 9.6 + 3 \times \sqrt{8.6784} = 9.6 + 3 \times 2.946$$

$$= 9.6 + 8.838 = 18.438$$

$$LCL = 9.6 - 8.838 = 0.762$$

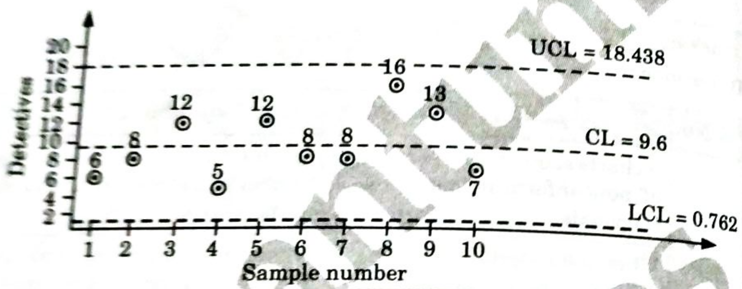

Fig. 5.31.1.

Since all the sample points are inside the control limits the processes is in a state of statistical control.

Que 5.32. Following is the data of defectives of 10 samples of size 100 each.

Sample no.	1	2	3	4	5	6	7	8	9	10
No. of defectives	15	11	9	6	5	4	3	2	7	1

Construct *p*-chart and state whether the process is in statistical control.

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Answer

Number of sample	Number of units in a sample (n)	Number of deflective	Fraction defective $p = d/n$
1	100	15	0.15
2	100	11	0.11
3	100	9	0.09
4	100	6	0.06
5	100	5	0.05
6	100	4	0.04
7	100	3	0.03
8	100	2	0.02
9	100	7	0.07
10	100	1	0.01
Total	$N = 1000$	$\Sigma d = 63$	

1. Average fraction defective

$$\bar{p} = \frac{\Sigma d}{N} = \frac{63}{1000} = 0.063$$

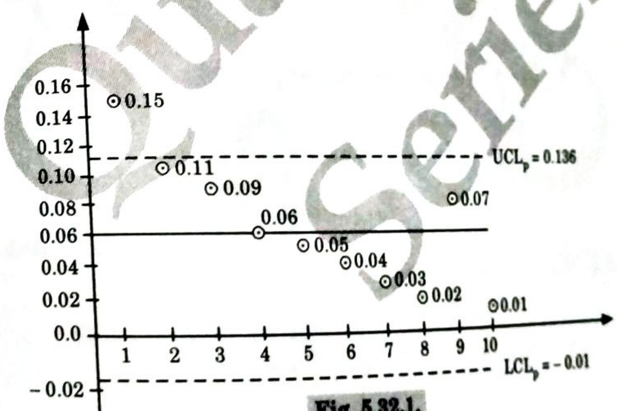

Fig. 5.32.1.

2. Standard limits

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$= 0.063 + 3\sqrt{\frac{0.063 \times 0.937}{100}}$$
$$= 0.063 + 0.073$$
$$= 0.136$$
$$LCL_p = 0.063 - 0.073$$
$$= -0.01$$

One sample point is above the UCL line so the process is not in statistical control.

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1 UNIT Partial Differential Equations (2 Marks Questions)

1.1. Find the particular integral of the following partial differential equation $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Ans.
$$PI = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y)$$

Replace D^2 by -2^2, DD' by -2.1 and D'^2 by -1^2.

It will be a case of failure. Thus

$$PI = x\frac{1}{2D + D'}\cos(2x + y) = \frac{x(2D - D')}{4D^2 - D'^2}\cos(2x + y)$$
$$= x\frac{1}{4(-2^2) - (-1^2)}(2D - D')\cos(2x + y)$$
$$= \frac{x}{-16 + 1}\{2D\cos(2x + y) - D'\cos(2x + y)\}$$
$$= \frac{x}{-15}\{-2.2\sin(2x + y) + \sin(2x + y)\}$$
$$= \frac{x}{15}\{4\sin(2x + y) - \sin(2x + y)\} = \frac{x}{5}\sin(2x + y)$$

1.2. Solve : $(D - 5D' + 4)^3 z = 0$.

Ans. $(D - 5D' + 4)^3 z = 0$

$$CF = e^{-4x}[f_1(y + 5x) + x f_2(y + 5x) + x^2 f_3(y + 5x)]$$
$$PI = 0$$

Thus
$$z = CF + PI$$
$$z = e^{-4x}[f_1(y + 5x) + x f_2(y + 5x) + x^2 f_3(y + 5x)]$$

1.3. Solve : $(D^2 - 2D')z = 0$.

Ans. There is no linear factor. Let us assume that $z = \sum Ae^{hx+ky}$ be the solution corresponding to

$$(D^2 - 2D')z = 0$$

$$(D^2 - 2D')z = Ah^2 \sum e^{hx+ky} - 2Ak \sum e^{hx+ky} = 0$$

$$= \sum A e^{hx+ky}(h^2 - 2k) = 0$$

$\because \quad \sum A e^{hx+ky} \neq 0$

$\therefore \quad h^2 - 2k = 0$

or $\quad k = h^2/2$

Hence, the general solution is, $z = \sum A e^{hx+(h^2/2)y}$

1.4. Solve : $(D^3 D'^2 + D^2 D'^3) z = 0$.

Ans. $D^2 D'^2 (D + D')z = 0$

Putting $D = m$ and $D' = 1$, we get auxiliary equation as

$m^2(m + 1) = 0$

$m = 0, 0, -1$

Thus, $\quad CF = \phi_1(y + 0x) + x\phi_2(y + 0x) + \phi_3(y - x)$

$PI = 0$

$\therefore \quad z = \phi_1(y + 0x) + x\phi_2(y + 0x) + \phi_3(y - x)$

$z = \phi_1(y) + x\phi_2(y) + \phi_3(y - x)$

1.5. Solve the PDE $(D - D' - 1)(D + D' - 2)\, z = e^{2x-y}$.

Ans. $(D - D' - 1)(D + D' - 2)\, z = e^{2x-y}$

$CF = e^x f_1(y + x) + e^{2x} f_2(y - x)$

$$PI = \frac{1}{(D - D' - 1)(D + D' - 2)} e^{2x-y}$$

$$= \frac{1}{2(-1)} e^{2x-y} = -\frac{1}{2} e^{2x-y}$$

Thus $\quad z = e^x f_1(y + x) + e^{2x} f_2(y - x) - \frac{1}{2} e^{2x-y}$

1.6. Formulate the PDE by eliminating the arbitrary function from $\phi(x^2 + y^2, y^2 + z^2) = 0$

Ans. Given, $\phi(x^2 + y^2, y^2 + z^2) = 0$ ...(1.6.1)

Let, $\quad u(x, y, z) = x^2 + y^2$ ...(1.6.2)

$v(x, y, z) = y^2 + z^2$ ...(1.6.3)

Differentiating eq. (1.6.1) partially w.r.t. to x, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$\phi_u[2x + 0 \times p] + \phi_v[0 + 2zp] = 0 \qquad \left[\because \frac{\partial z}{\partial x} = p\right]$$

$\phi_u(2x) + \phi_v(2zp) = 0$

Similarly differentiating eq. (1.6.1) partially w.r.t. to y, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0$$

$$\phi_u[2y + 0 \times q] + \phi_v[2y + 2zq] = 0 \qquad \left(\because \frac{\partial z}{\partial y} = q\right)$$

Eliminating ϕ_u and ϕ_v (*i.e.*, the coefficient matrix should be singular),

$$\begin{vmatrix} 2x & 2zp \\ 2y & 2y + 2zq \end{vmatrix} = 0$$

$x(y + zq) - yzp = 0$

$xy + xzq - yzp = 0$

$xy + z(xq - yp) = 0$

1.7. Solve $p - q = 1$.

Ans. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + (a - 1)\,y + c$ is the desired solution.

1.8. Solve $z = px + qy + \sqrt{(1 + p^2 + q^2)}$.

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Ans. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{1 + p^2 + q^2}$

$\therefore \quad$ Its complete solution is $z = ax + by + \sqrt{1 + a^2 + b^2}$.

1.9. Find the particular integral of

$$\frac{2\partial^2 z}{\partial x^2} + \frac{3\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x - y$$

Ans. We have $[2D^2 + 3DD' + (D')^2 + D + D']\, z = x - y$.

We write $[2D^2 + 3DD' + (D')^2 + D + D'] = D[1 + D^{-1}D' + 3D' + 2D + D^{-1}(D')^2]$

The particular integral is given by

$z = D^{-1}[1 + \{D^{-1}D' + 3D' + 2D + D^{-1}(D')^2\}]^{-1}(x - y)$

$= D^{-1}[1 - \{D^{-1}D' + 3D' + 2D + D^{-1}(D')^2\} + ...](x - y)$

$= D^{-1}[x - y - \{D^{-1}(-1) + 3(-1) + 2(1)\}]$

$= D^{-1}[x - y + x + 1] = x^2 + (1 - y)x.$

1.10. What is the auxiliary equation of Charpit Method ?

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Ans. Auxiliary equation of Charpit Method :

$$\frac{dp}{a} = \frac{dq}{b} = \frac{dz}{3q^2 p} = \frac{dx}{q^2} = \frac{dy}{2pq}$$

1.11. Solve the following partial differential equation $(D^2 + DD')\, z = 0$.

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Ans.
$$(D^2 + DD')\, z = 0$$
$$D\,(D + D')\, z = 0$$
Corresponding to the factor D, part of C.F $= f_1(y)$
Corresponding to the factor $(D + D')$ part of C.F $= f_2(y - x)$
Hence complete solution is
$$Z = \text{C.F} + \text{P.I}$$
$$\text{CF} = f_1(y) + f_2(y - x)$$
$$PI = 0$$
$$z = f_1(y) + f_2(y - x)$$

1.12. Derive a partial differential equation by eliminating the constants a and b from $z = ax + a^2y^2 + b$.

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Ans. Differentiating z partially w.r.t x and y
$$p = \frac{\partial z}{\partial x} = a,\; q = \frac{\partial z}{\partial x} = 2ya^2$$
Substituting for a and b in the given equation we get
$$z = px + \frac{q}{2}y + b$$
Constant b cannot be eliminated.

1.13. Solve the partial differential equation $p + q = 1$

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Ans.
$$p + q = 1 \quad \text{...(1.13.1)}$$
The complete integral of equation (1.13.1)
$$z = ax + by + c \quad \text{...(1.13.2)}$$
$$p = \frac{\partial z}{\partial x} = a,\, q = \frac{\partial z}{\partial y} = b$$
$$p + q = a + b = 1$$
$$b = 1 - a$$
Solution is given by $z = ax + (1 - a)\, y + c$

1.14. Calculate particular Integral (P. I.) of $(D - 3D' + 2)z = e^{x+2y}$.

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Ans.
$$\text{PI} = \frac{1}{D - 3D' + 2} e^{x+2y}$$
$$= \frac{1}{1 - 3 \times 2 + 2} e^{x+2y} = -\frac{1}{3} e^{x+2y}$$

1.15. Obtain a partial differential equation that governs the family of surfaces $z = (x - \alpha)^2 + (y - \beta)^2$.

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Ans. The given equation,
$$z = (x - \alpha)^2 + (y - \beta)^2 \quad \text{...(1.15.1)}$$
Now, differentiate partially w.r.t x, we get
$$\frac{\partial z}{\partial x} = 2(x - \alpha)$$
$$p = 2(x - \alpha)$$
Again differential equation (1.15.1) w.r.t y ...(1.15.2)
$$\frac{\partial z}{\partial y} = 2(y - \beta)$$
$$q = 2(y - \beta)$$
Equation (1.15.1) becomes ...(1.15.3)
$$z = \frac{p^2}{4} + \frac{q^2}{4}$$
This is required P.D.E.

1.16. Find the complete integral of the partial differential equation
$$z = px + qy + \frac{p}{p+q},\; p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

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Ans. The given equation is
$$z = px + qy + \frac{p}{q+p} \quad \text{...(1.16.1)}$$
It is of the form of $z = pz + qy + f(p, q)$
The complete integral is
$$z = ax + by + f(a, b)$$
i.e.,
$$z = ax + by + \frac{a}{a+b} \quad \text{...(1.16.2)}$$
$$(a + b)\, z = a(a + b)\, x + (a + b)\, by + a$$
$$az + bz = a^2x + abx + b^2y + aby + a \quad \text{...(1.16.3)}$$
Differentiate eq. (1.16.3) w.r.t a
$$z = 2ax + bx + by + 1 \quad \text{...(1.16.4)}$$
Differentiate eq. (1.16.3) w.r.t b
$$z = ax + 2by + ay \quad \text{...(1.16.5)}$$
From eq. (1.16.4) and eq. (1.16.5)
$$2ax + bx + by + 1 = ax + 2by + ay$$
$$ax + bx = by + ay - 1$$
$$ax - ay = by - bx - 1$$
$$a(x - y) = by - bx - 1$$
$$a = -\frac{(x - y)b}{(x - y)} - \frac{1}{x - y}$$
$$a = -b - \frac{1}{x - y} \Rightarrow a = \frac{1}{y - x} - b$$

Put value of a in equation (1.16.5)

$$z = \left(\frac{1}{y-x} - b\right)x + 2by + \left(\frac{1}{y-x} - b\right)y$$

$$z(y-x) = 1 - b(y-x)x + 2by(y-x) + 1 - b(y-x)y$$
$$z(y-x) = b(y-x)(-x+2y-y)$$
$$z(y-x) = b(y-x)(y-x)$$

$$b = \frac{z}{(y-x)}$$

Put the value of a and b in equation (1.16.2)

$$z = \left(\frac{1}{y-x} - b\right)x + \frac{z}{y-x}y + \frac{1-b(y-x)}{z+1-b(y-x)}$$

1.17. Find partial differential equation (PDE) by eliminating a and b from $z = ax + by + a^2 + b^2$.

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Ans. $z = ax + by + a^2 + b^2$

Differentiating partially with respect to x, y we get

$$\frac{\partial z}{\partial x} = p = a$$

$$\frac{\partial z}{\partial y} = q = b$$

Then the partial differential equation is

$px + qy + p^2 + q^2 = z$

1.18. Solve the PDE, $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$.

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Ans. $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

The given equation is $D^2 - D'^2 = 0$

Where $D \equiv \frac{\partial}{\partial x}$ and $D \equiv \frac{\partial}{\partial y}$

Auxiliary equation $m^2 - 1 = 0$

$(m+1)(m-1) = 0 \Rightarrow m = 1, -1$

C.F. $= f_1(y-x) + f_2(y+x)$

PI = 0

Hence, the complete solution is

$$z = \text{C.F} + \text{PI} = f_1(y-x) + f_2(y+x)$$

Where f_1 and f_2 are arbitrary functions.

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2 UNIT

Application of Partial Differential Equations and Fourier Transform (2 Marks Questions)

2.1. Classify the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ans. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

On comparing above equation with standard form,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$A = 1, B = 0, C = 1$

Now $B^2 - 4AC = 0 - 4 \times 1 \times 1 = -4 < 0$

So, it will represent elliptic equation.

2.2. Classify the equation $u_{xx} + 3u_{xy} + u_{yy} = 0$.

OR

Classify the partial differential equation $u_{xx} + 3u_{xy} + u_{yy} = 0$

AKTU 2021-22 (Sem-3), Marks 02

Ans. Let $u_{xx} + 3u_{xy} + u_{yy} = 0$

$A = 1, B = 3, C = 1$

$B^2 - 4AC = 9 - 4(1)(1) = 5 > 0$

Hence, the given partial differential equation is hyperbolic.

2.3. Solve $\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial t}$ using method of separation of variables.

Ans. Let $u = XT$, where X is a function of x and T is a function of t only.

$$\frac{\partial u}{\partial x} = T\frac{\partial X}{\partial x}, \frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

$$T\frac{\partial X}{\partial x} = 3X\frac{\partial T}{\partial t}$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = \frac{3}{T}\frac{\partial T}{\partial t} = k \text{ (let)}$$

$$\frac{\partial X}{X} = k\partial x$$

$$\ln X = kx + C_1$$
$$X = e^{kx + C1}$$
$$\frac{3}{T}\frac{\partial T}{\partial y} = k$$
$$\ln T = \frac{k}{3}t + C_2$$
$$T = e^{\frac{k}{3}t + C_2}$$
$$u = XT$$
$$u = e^{(kx + C_1)}e^{\frac{k}{3}t + C_2} = Ae^{(x + t/3)k}$$

2.4. Write down the solution for the PDE $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Ans. On comparing the given PDE $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ to one dimensional heat equation we have

$$C^2 = 1$$

So, the solution of this PDE is

$$u = (C_1 \cos kx + C_2 \sin kx)\, C_3\, e^{-k^2 t}$$

2.5. Find the steady state temperature distribution in a rod of 2m whose ends are kept at 30 °C and 70 °C respectively.

Ans. Steady state temperature distribution is given by

$$\frac{\partial^2 u}{\partial x^2} = 0$$
$$u = C_1 x + C_2 \quad ...(2.5.1)$$

at $x = 0, u = 30$

$$C_2 = 30$$

and at $x = 2, u = 70$

$$70 = 2C_1 + C_2 \Rightarrow C_1 = 10$$

Hence $u = 30x + 10$

2.6. Write two dimensional heat equation.

OR

Write down the two dimensional heat equation.

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Ans. Two dimensional heat equation is $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$

2.7. Write down transmission line equations.

OR

Explain the radio equations.

AKTU 2020-21 (Sem-3), Marks 02

OR

Write radio wave equations.

AKTU 2021-22 (Sem-3), Marks 02

Ans. i. **Telegraph equations :**

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \quad ...(2.7.1)$$
$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \quad ...(2.7.2)$$

Both eq. (2.7.1) and eq. (2.7.2) are known as telegraph equations.

ii. **Radio equations :**

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \quad ...(2.7.3)$$
$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \quad ...(2.7.4)$$

Eq. (2.7.3) and eq. (2.7.4) are known as radio equations.

2.8. Write down the case (or equation) for submarine cable.

Ans. Transmission line equation is

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV \quad ...(2.8.1)$$
$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \quad ...(2.8.2)$$

For submarine cables, $L = C = 0$ hence the eq. (2.8.1) and eq. (2.8.2) will reduce to

$$\frac{\partial^2 V}{\partial x^2} = RGV$$

and

$$\frac{\partial^2 i}{\partial x^2} = RGi$$

2.9. Specify with suitable example the clarification of Partial Differential Equation (PDE) for elliptic, parabolic and hyperbolic differential equation.

Ans. Let, $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad ...(2.9.1)$

where A, B, C are constants or continuous functions of x and y possessing continuous partial derivatives and A is positive.

Now eq. (2.9.1) is

i. Elliptic, if $B^2 - 4AC < 0$

ii. Hyperbolic, if $B^2 - 4AC > 0$
iii. Parabolic, if $B^2 - 4AC = 0$

Example : Consider partial differential equation as follows :

$$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

Here $A = 4, B = 4, C = 1$

$\therefore$ $B^2 - 4AC = (4)^2 - 4(4)(1) = 16 - 16 = 0$

Hence, the given equation is parabolic.

2.10. Classify the following partial differential equation

$$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

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Ans. $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$

On comparing above equation with ideal form,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + f\left(x, t, u\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

$A = 4, B = 4, C = 1$

$B^2 - 4AC = 16 - 4 \times 4 \times 1 = 16 - 16 = 0$

Given partial differential equation is parabolic.

2.11. Tell the classification of the following partial differential equation

$$5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial t^2} = 0$$

AKTU 2021-22 (Sem-4), Marks 02

Ans. $5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial t^2} = 0$

On comparing above equation with ideal form,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

$A = 5, \quad B = -9, \quad C = 4$

$B^2 - 4A\,C = 81 - 80 = 1$

$\because B^2 - 4A\,C > 0$

So, given partial equation is hyperbolic equation.

2.12. Write down the two-dimensional wave equation.

AKTU 2021-22 (Sem-4), Marks 02

OR

Write the wave equation in two dimensions.

AKTU 2022-23 (Sem-3), Marks 02

Ans.

$$\frac{\partial^2 u}{\partial f^2} = c^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Where $c^2 = T/m$

m is mass of the membrane per unit area.

2.13. Classify the partial differential equation

$$r + 2s + (\sin^2 x)\,t + q = 0,\ r = \frac{\partial^2 z}{\partial x^2}\ s = \frac{\partial^2 z}{\partial y^2} \text{ and } t = \frac{\partial^2 z}{\partial x \partial y}$$

AKTU 2022-23 (Sem-4), Marks 02

Ans. $r + 2s + (\sin^2 x)\,t + q = 0$

$$\frac{\partial^2 z}{dx^2} + (\sin^2 x)\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} + q = 0$$

On comparing above equation with ideal form,

$$A + \frac{\partial^2 u}{dx^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$A = 1 \quad B = \sin^2 x \quad C = 2$

$B^2 - 4AC = \sin^4 x - 4x^2$

$= \sin^4 x - 8$

For elliptical, $B^2 - 4AC < 0$, for $x \geq 0$

2.14. Classify the PDE, $u_{xx} + u_{yy} - u_{xy} = 0$.

AKTU 2022-23 (Sem-3), Marks 02

Ans. Let $u_{xx} + u_{yy} - u_{xy} = 0$

On comparing with standard form

$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + F(u) = 0$

$A = 1, B = -1, C = 1$

$B^2 - 4AC = 1 - 4 \times 1 \times 1 = -3 < 0$

Hence the given partial differential equation is elliptic.

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3 UNIT Statistical Techniques–I (2 Marks Questions)

3.1. What is the meaning of skewness ?

Ans. The term skewness means lack of symmetry *i.e.*, when a distribution is not symmetric then it is called a skewed distribution and this distribution may be positively skewed or negatively skewed.

3.2. What do you understand by measure of kurtosis ? Discuss in brief.

Ans. Measure of kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \quad \mu_2 = \frac{\Sigma(x-\bar{x})^2}{N}, \quad \mu_4 = \frac{\Sigma(x-\bar{x})^4}{N}$$

If $\beta_2 = 3$, the curve is normal or mesokurtic.
If $\beta_2 > 3$, the curve is peaked or leptokurtic.
If $\beta_2 < 3$, the curve is flat topped or platykurtic.

3.3. Discuss in brief the types of correlation.

Ans. **Types of correlation :**

1. **Positive correlation :** If a decrease in the value of one variable X results in a corresponding decrease in value of other variable Y on an average, the correlation is said to be positive.
2. **Negative correlation :** If the decrease in the values of one variable X results in the increase to a corresponding values of Y, the correlation between X and Y is said to be negative.
3. **Linear correlation :** When all the plotted point lies approximately on a straight line, then the correlation is said to be linear correlation.
4. **Perfect correlation :** If the deviation of one variable X is proportional to the deviation in other variable Y, then the correlation is said to be perfect correlation.

3.4. Define regression lines.

Ans. A line of regression is the straight line which gives the best fit in the least square to the given frequency.

3.5. If covariance between x and y variable is 10 and the variance of x and y are respectively 16 and 9, find the coefficient of correlation.

Ans. Coefficient of correlation,

$$= \frac{\text{Covariance }(x, y)}{\sqrt{\text{Variance } x} \times \sqrt{\text{Variance } y}}$$

$$= \frac{10}{\sqrt{16}\,.\sqrt{9}} = 0.833$$

3.6. Write the normal equations to fit a curve $y = ax^2 + b$ by least square method.

Ans. Normal equations for the given curve are :

$$\Sigma y = a\Sigma x^2 + bx$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3$$

3.7. The first two moments of a distribution about the value '2' of the variable are 1, 16. Show that mean is 3, variance is 15.

AKTU 2020-21 (Sem-3), Marks 02

Ans. Let us denote the n-moment about the value 2 as :

$$M_{n,2} = E[(x-2)^n]$$

$$M_1 = E[(x-2)] = 1 \quad \text{...(3.7.1)}$$

$$M_2 = E[(x-2)^2] = 16 \quad \text{...(3.72)}$$

From eq. (1),

$$E[x] - E[2] = 1 \Rightarrow E[x] - 2 = 1$$

Mean $= \mu_s = E[x] = 3$

From eq. (2),

$$E[(x-2)^2] = 16$$

$$E[(x-3+1)^2] = E[(x-3)^2 + 2(x-3) + 1]$$

$$\Rightarrow E[(x-3)^2] + 2E[x-3] + E(1) = 16$$

$$\text{Variance} = \mu^2 \Rightarrow E[(x-3)^2]$$

$$= 16 - 2E[x-3] - E(1)$$

$$= 16 - 2(E[x] - 3) - 1$$

$$= 16 - 2(3-3) - 1$$

$$= 16 - 1 = 15$$

3.8. If the regression coefficient is 0.8 and 0.2. What will be the value of coefficient of correlation ?

AKTU 2020-21 (Sem-3), Marks 02

Ans. Correlation coefficient,

$$R = \sqrt{0.8 \times 0.2} = \sqrt{0.16} = 0.40$$

3.9. In an asymmetrical distribution mean is 16 and median is 20. Calculate the mode of the distribution.

AKTU 2021-22 (Sem-3), Marks 02

Ans.

$$\text{Mode} = 3\,\text{Median} - 2\,\text{Mean}$$
$$= 3 \times 20 - 2 \times 16 = 60 - 32 = 28$$

3.10. The lines of regression of y on x and x on y are respectively $y = x + 5$ and $16x - 9y = 94$. Find the correlation coefficient.

AKTU 2021-22 (Sem-3), Marks 02

Ans. Given equation of regression lines are,

$$y = x + 5$$
$$y - x = 4 \quad \ldots(3.10.1)$$
$$16x - 9y = 94 \quad \ldots(3.10.2)$$

Multiply equation (3.10.1) by 9 and add equation (3.10.2)

$$9y - 9x = 45$$
$$16x - 9y = 94$$
$$7x = 139$$
$$x = \frac{139}{7}$$

Substituting $x = \frac{139}{7}$ in equation (3.10.1)

$$y = \frac{139}{7} + 5 = \frac{139 + 35}{7} = \frac{174}{7}$$

Since point of intersection of two regression lines is $(\bar{x}, \bar{y})$

$$\therefore \quad \bar{x} = \frac{139}{7} = 19.86 \qquad \bar{y} = \frac{174}{7} = 24.86$$

$y - x = 5$ is regression equation Y on X. So, equation becomes

$Y - X = 5, \quad Y = X + 5$

Comparing with $Y = b_{YX} X + a$ we get

$$b_{YX} = 1$$

$16x - 9y = 94$ is regression equation of X on Y. So, equation becomes

$16X - 9Y = 94$

$$X = \frac{9Y}{16} + \frac{94}{16}$$

Comparing with $X = b_{XY} Y + a$

$$b_{XY} = \frac{9}{16}$$

$$r = \pm\sqrt{b_{XY} \cdot b_{YX}} = \pm\sqrt{\frac{9}{16} \times 1} = \frac{3}{4}$$

Correlation coefficient is $= \frac{3}{4}$

3.11. Calculate the moment generating function of the negative exponential function $f(x) = \lambda e^{-\lambda x}$; $x, \lambda > 0$

AKTU 2021-22 (Sem-4), Marks 02

Ans. The moment generating function for X ~ exponential (λ) is

$$M(t) = E\,[e^{tX}]$$
$$= \int_0^\infty e^{tx}\,\lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx$$
$$= \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{\lambda - t}$$

when $t - \lambda < 0$, or equivalently, when $t < \lambda$.

3.12. Regression coefficients are 0.8 and 0.8, what would be the value of coefficient of correlation ?

AKTU 2021-22 (Sem-4), Marks 02

Ans. Regression coefficients are, $r_{xy} = r_{yx} = 0.8$

Coefficient of correlation $= \sqrt{(r_{xy} \times r_{yx})} = \sqrt{0.8 \times 0.8} = 0.8$

3.13. What is the relation between the regression coefficients and the coefficient of correlation ?

AKTU 2022-23 (Sem-4), Marks 02

Ans.

$$r = \sqrt{b_{yx} + b_{xy}}$$

3.14. The fourth central moment is 48. What must be its standard deviation in order that the distribution be mesokurtic.

AKTU 2022-23 (Sem-4), Marks 02

Ans. For mesokurtic, $\beta_2 = 3$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$
$$3 = \frac{48}{\mu_2^2}$$
$$\mu_2^2 = \frac{48}{3}$$
$$\mu_2^2 = 16$$
$$\mu_2 = 4$$
$$\mu_2 = \text{Variance} = \sigma^2$$
$$\sigma^2 = 4, \quad \sigma = 2$$

3.15. Write the formula of Karl Pearson correlation coefficient and write the range of correlation coefficient.

AKTU 2022-23 (Sem-3), Marks 02

Ans. Karl Pearson's coefficient correlation (r),

$$= \frac{\text{Sum of products of devations from their respective means}}{\text{Numer of pairs} \times \text{Standard devations of both series}}$$

$$r = \frac{\Sigma xy}{N \times \sigma_x \times \sigma_y}$$

Where, N = Number of pair of observations

x = Deviation of X series from mean $(X - \bar{X})$

y = Deviation of Y series from mean $(Y - \bar{Y})$

σ_x = Standard deviation of X series $\left(\sqrt{\frac{\Sigma x^2}{N}}\right)$

σ_y = Standard deviation of Y series $\left(\sqrt{\frac{\Sigma y^2}{N}}\right)$

r = Coefficient of correlation

Correlation coefficient ranges between – 1 and + 1.

3.16. Find the arithmetic mean of the following frequency distribution :

x	1	2	3	4	5	5	7
f	5	9	12	17	14	10	6

AKTU 2022-23 (Sem-3), Marks 02

Ans.

x_i	f_i	$x_i f_i$
1	5	5
2	9	18
3	12	36
4	17	68
5	14	70
5	10	50
7	6	42
	$\Sigma f_i = 73$	$\Sigma x_i f_i = 289$

$$\text{Arithmetic mean} = \frac{\sum x_i f_i}{\sum f_i} = \frac{289}{73} = 3.96$$

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4 UNIT Statistical Techniques–II (2 Marks Questions)

4.1. Given $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, evaluate $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$

Ans. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad \frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B)$ or $P(A \cap B) = \frac{1}{12}$

Thus $\quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}$$

$$P(A \cup B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}$$

4.2. Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.

Ans.

a. There are six possible ways in which the die can fall out of which there is only one ways of getting 4.

Thus, the required chance $= \frac{1}{6}$.

b. There are six possible ways in which the die can fall out of which there are only 3 ways of getting 2, 4 or 6.

Thus, the required chance $= 3/6 = \frac{1}{2}$.

4.3. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace ?

Ans. A = Event of drawing a spade

and B = Event of drawing an ace

A and B are not mutually exclusive.

AB = Event of drawing the ace of spades

$$P(A) = \frac{13}{52},\ P(B) = \frac{4}{52},\ P(AB) = \frac{1}{52}$$

$$\therefore \quad P(A + B) = P(A) + P(B) - P(AB)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

4.4. If the function $f(x)$ is defined by $f(x) = ce^{-x}$, $0 < x < \infty$ calculate the value of c which changes $f(x)$ to a probability density function.

AKTU 2020-21 (Sem-3), Marks 02

Ans. Since $f(x)$ is a probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} c\,e^{-x}\,dx = 1$$

$$c\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\frac{c}{-1}[0-1] = 1$$

$$\therefore \quad c = 1$$

4.5. Identify the following statement is true or false "For a Binomial distribution, mean is 6 and variance is 9.

AKTU 2020-21 (Sem-3), Marks 02

Ans. Let p, q = probabilities of certain events

$$np = 6 \quad npq = 9$$

$$q = \frac{npq}{np} = \frac{9}{6} = \frac{3}{2}$$

But q cannot be greater than 1 $(\because$ Probability $\le 1)$

So, given statement is false.

4.6. Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Prove that the chance that exactly two of them will be children is 10/21.

AKTU 2021-22 (Sem-3), Marks 02

Ans. Total number of ways = ${}^9C_4$ = 126 ways

There are 4 children, and we have to select exactly 2 children.

Hence, we have ${}^4C_2$ ways = 6 ways.

Also, choose other two people from men and women. So we have, ${}^5C_2$ ways = 10 ways.

Hence, the required probability = (6 × 10)/126 = 10/21

4.7. If the probability density functions

$$f(x) = \begin{cases} kx^3, & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$ **find the value of 'k'. Also find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.**

AKTU 2021-22 (Sem-3), Marks 02

Ans.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_0^3 kx^3\,dx + \int_3^{\infty} 0\,dx = 1$$

$$0 + k\left[\frac{x^4}{4}\right]_0^3 + 0 = 1$$

$$k\left[\frac{81}{4} - 0\right] = 1$$

$$k = \frac{81}{4}$$

$$p\left(x > \frac{1}{2} \text{ and } x < \frac{3}{2}\right) = \int_{-\infty}^{1/2} f(x)\,dx + \int_{1/2}^{3/2} f(x)\,dx$$

$$= \int_{-\infty}^{1/2} \frac{4}{81}x^3\,dx + \int_{1/2}^{3/2} \frac{4}{81}x^3\,dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{-\infty}^{1/2} + \frac{4}{81}\left[\frac{x^4}{4}\right]_{1/2}^{3/2}$$

$$= \frac{4}{81\times 4}\left[\left(\frac{1}{2}\right)^4 - 0\right] + \frac{4}{81\times 4}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{81}\left[\left(\frac{1}{16}\right) + \left(\frac{80}{16}\right)\right] = \frac{1}{81} \times \frac{81}{16} = \frac{1}{16}$$

4.8 A die is tossed twice. A success is getting 2 or 3 on a toss. Calculate mean.

AKTU 2021-22 (Sem-4), Marks 02

Ans. Let x be the random variable denoting the number of times 2 or 3 comes (the number of successes) when a die is tossed twice.

Then x takes the values 0, 1, 2

Let $P(X = 0)$ be probability of not getting 2 or 3.

$P(X = 0) = 16/36 = 4/9$

Let $P(X = 1)$ be probability of getting 2 or 3.

$P(X = 1) = 16/36 = 4/9$

Let $P(X = 1)$ be probability of getting two times 2 or two times 3 or getting 2 and 3 both.

$P(X = 1) = 4/36 = 1/9$

Thus the probability distribution of X is given by

$X = x$	$X = 0$	$X = 1$	$X = 2$
$P(X = x)$	4/9	4/9	1/9

We know that mean, $E(X) = \Sigma X_i P_i = 0 \times 4/9 + 1 \times 4/9 + 2 \times 1/9$

$E(X) = 0 + 4/9 + 2/9 = 6/9$

Thus mean $E(X) = 6/9 = 2/3$

4.9. Write Statement of Baye's theorem.

AKTU 2021-22 (Sem-4), Marks 02

Ans. Baye's Theorem Statement :

Let $E_1, E_2, ..., E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2 ..., E_n$ have non-zero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Baye's theorem,

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k)P(A \mid E_k)}$$

for any $k = 1, 2, 3, ..., n$

4.10. A and B are any two independent events such that $P(A) = 0.4$, $P(A \cup B^c) = 0.7$. Find the $P(B)$, where B^c is the complementary event of event B.

AKTU 2022-23 (Sem-4), Marks 02

Ans.

$P(A) = 0.4$

$P(A \cup B^c) = 0.7$

$P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$

$P(A \cup B^c) = P(A) + P(B^c) - P(A) \, . \, P(B^c)$

[$\because$ A, B are independent $P(A \cap B) = P(A) \, . \, P(B)$]

$0.7 = 0.4 + P(B^c) - 0.4 \, P(B^c)$

$0.3 = 0.6 \, P(B^c)$

$$P(B^c) = \frac{0.3}{0.6} = \frac{1}{2}$$

$P(B^c) = 1 - P(B^c)$

$$= 1 - \frac{1}{2} = 1 - \frac{1}{2}$$

4.11. The random variable X is said to follow the Normal distribution with mean 9 and standard deviation 3, find x^* such that $P(X > x^*) = 0.16$.

AKTU 2022-23 (Sem-4), Marks 02

Ans.

$P(X > x) = 0.16$

$$P\left(\frac{X - \mu_x}{\sigma_x} > \frac{x - 9}{3}\right) = 0.16$$

$$P\left(z > \frac{x - 9}{3}\right) = 0.16$$

The corresponding value of z-score from standard table at probability value of 0.16 is 0.0636.

$$\frac{x - 9}{3} = 0.0636$$

$x = 9 + 0.0636*3 = 9.1908$

4.12. If $P(A) = \frac{1}{4}, P(B) = \frac{1}{2}, P(A \cup B) = \frac{5}{8}$, find the value of $P(A \cap B)$.

AKTU 2022-23 (Sem-3), Marks 02

Ans.

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{2}, \quad P(A \cup B) = \frac{5}{8}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{5}{8} = \frac{1}{4} + \frac{1}{2} - P(A \cap B)$$

$$P(A \cap B) = \frac{1}{4} + \frac{1}{2} - \frac{5}{8} = \frac{2 + 4 - 1}{8} = \frac{5}{8}$$

4.13. Write probability mass function of binomial distribution with mean and variance of the distribution.

AKTU 2022-23 (Sem-3), Marks 02

Ans. Probability mass function of binomial distribution :

The formula the binomial probability mass function is

$$P(x; p, n) = \binom{n}{x} (p)^x (1 - p)^{(n - x)} \quad \text{for } x = 0, 1, 2, ..., n$$

where $\binom{n}{x} = \frac{n!}{x!(n - z)!}$

Mean and variance of binomial distribution : Refer Q. 4.21, Page 4–15G, Unit-4.

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5 UNIT Statistical Techniques–III (2 Marks Questions)

5.1. In two large populations there are 30 % and 25 % respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations ?

Ans. Here $p_1 = 0.3, p_2 = 0.25$ so that $p_1 - p_2 = 0.05$

$$\therefore \quad e^2 = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2} = \frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}$$

so that $\quad e = 0.0195$

$$\therefore \quad z = \frac{p_1 - p_2}{e} = \frac{0.05}{0.0195} = 2.5 \text{ nearly}$$

Hence, it is unlikely that the real difference will be hidden.

5.2. Explain sampling and its objectives.

Ans. A part selected from the population is called a sample. The process of selection of a sample is called sampling. A random sample is one in which each member of population has an equal chance of being included in it. There are ${}^NC_n$ different samples of size n that can be picked up from a population of size N.

Objective of sampling :

1. Gathering the maximum information about the population with the minimum effort, cost and time.
2. To obtain the best possible values of the parameters under specific conditions.
3. The logic of the sampling theory is the logic of induction in which we pass from a particular (sample) to general population.

5.3. Define statistical hypothesis.

Ans. Statistical hypothesis is an assumption on conjecture or guess about the parameters of population distribution. When more than one population is considered, statistical hypothesis consists of relationship between the parameter of the populations.

5.4. Define null hypothesis.

AKTU 2022-23 (Sem-3), Marks 02

OR

Write down the definition of the null hypothesis.

AKTU 2022-23 (Sem-4), Marks 02

Ans. Null hypothesis is denoted by H_0 is the statistical hypothesis which is to be actually tested for acceptance or rejection.

5.5. What is upper control limit ?

Ans. Upper Control Limit (UCL) the highest value that a quality characteristic can take before the process becomes out of control.

5.6. When was the test statistic $F = \frac{S_1^2}{S_2^2}$ is used ?

AKTU 2020-21, 2021-22 (Sem-3); Marks 02

Ans. If the null hypothesis is true, then the F test-statistic can be simplified. Then the ratio of sample variances ($F = S_1^2/S_2^2$) will be used.

5.7. Explain the t-test for small samples.

AKTU 2020-21, 2021-22 (Sem-3); Marks 02

Ans. The t-distribution is used to test the significance of :

i. The mean of a small sample.
ii. The difference between the means of two small samples or to compare two small samples.
iii. The correlation coefficient.

Let $x_1, x_2, ..., x_n$, be the members of random sample drawn from a normal population with mean μ. If $\bar{x}$ be the mean of the sample then

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \quad \text{where } s^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1}$$

5.8. What do you mean by statistical quality control (SQC)?

AKTU 2021-22 (Sem-3), Marks 02

OR

What is Statistical Quality Control (SQC) ? Define in brief.

AKTU 2022-23 (Sem-4), Marks 02

Ans. Statistical quality control can be simply defined as an economic and effective system of maintaining and improving the quality of outputs throughout the whole operating process of specification, production and inspection based on continuous testing with random samples.

5.9. When we use F-test. **AKTU 2021-22 (Sem-4), Marks 02**

Ans. The F-test is used when we want to carry out the test for the equality of the two population variances. If anyone wants to test whether or not two independent samples have been drawn from a normal population with the same variability, then we generally employs the F-test.

5.10. Discuss (in brief) "Control Charts".

AKTU 2022-23 (Sem-3), Marks 02

Ans.

1. The control chart is a graph used to study how a process changes over time with data plotted in time order.
2. Control charts is a graph used in production control to determine whether quality and manufacturing processes are being controlled under stable conditions.

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B.Tech.
(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2020-21
MATHEMATIC-IV

Time : 3 Hours **Max. Marks : 100**

Note : 1. Attempt all sections. If require any missing data; then choose suitably.

SECTION-A

1. Attempt all questions in brief : **(2 × 10 = 20)**

a. **What is the auxiliary equation of Charpit Method ?**

Ans. Refer Q. 1.10, Page SQ–3U, Unit-1, Two Marks Questions.

b. **Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.**

Ans. Refer Q. 1.8, Page SQ–3U, Unit-1, Two Marks Questions.

c. **Classify the following partial differential equation**

$$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

Ans. Refer Q. 2.10, Page SQ–10U, Unit-2, Two Marks Questions.

d. **Explain the radio equations.**

Ans. Refer Q. 2.7, Page SQ–8U, Unit-2, Two Marks Questions.

e. **The first two moments of a distribution about the value '2' of the variable are 1, 16. Show that mean is 3, variance is 15.**

Ans. Refer Q. 3.7, Page SQ–13U, Unit-3, Two Marks Questions.

f. **If the regression coefficient is 0.8 and 0.2. What will be the value of coefficient of correlation ?**

Ans. Refer Q. 3.8, Page SQ–13U, Unit-3, Two Marks Questions.

g. **If the function $f(x)$ is defined by $f(x) = ce^{-x}$, $0 < x < \infty$ calculate the value of c which changes $f(x)$ to a probability density function.**

Refer Q. 4.4, Page SQ–18U, Unit-4, Two Marks Questions.

dentify the following statement is true or false "For a Binomial distribution, mean is 6 and variance is 9.