# Convolution

EGC 113

Source: MIT OCW Signals and Systems Course

# LTI Systems

$x[n] = \Sigma\, x[k]\, \delta[n-k]$ $\longrightarrow \boxed{\text{LTI}} \longrightarrow$ $y[n] = \Sigma\, x[k]\, h[n-k]$

Index $k = -\infty\ to\ +\infty$

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Here, $y[n] = \Sigma\, x[k]\, h[n-k]$ is called the Convolution Sum

Index $k = -\infty\ to\ +\infty$

Convolution Operation, $x[n] * h[n]$

Operator is "$*$"

Defined as , $x[n] * h[n] = \Sigma\, x[k]\, h[n-k]$ ; Index $k = -\infty\ to\ +\infty$

In Summary, for an LTI System, to determine the O/P of the system to any arbitrary I/P, all we need to know is : Impulse Response, $h[n]$

i.e. LTI System is completely characterized by its Impulse Response

# Convolution Sum

$$x[n] * h[n] = \Sigma\, x[k]\, h[n-k]$$ ; Index $k = -\infty\ to\ +\infty$

Steps Involved :

1) Time Reversal of h : $h[k]$ is time-reversed to obtain $h[-k]$

2) Shift it by "$n$" steps to obtain $h[n-k]$

3) $x[k]$ and $h[n-k]$ are point-wise multiplied for all values of "$k$" for a given value of "$n$"

4) Repeat for all values of "$n$"

# Convolution Integral and Convolution Sum

- ***<u>What you often see:</u>***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

*Also represented as*

$$y(t) = x(t) * h(t)$$

$$y[n] = x[n] * h[n]$$

# Properties

- The convolution operation is ***commutative***. That is, for any two functions $x$ and $h$,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any signals $x$, $h_1$, and $h_2$,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any signals $x$, $h_1$, and $h_2$,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

# Example

- Determine the output y[n], given
  - x[n] = u[n]

$$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$$

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

# Useful Mathematical Relationships

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ which converges only for } |\alpha| < 1, \text{ and}$$

$$\sum_{n-0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ which is a finite sum and hence always converges}$$

# Example

- Compute y[n], given

$$x[n] = 2^n u[-n],$$
$$h[n] = u[n].$$

$$x(t) = e^{2t} u(-t),$$
$$h(t) = u(t-3).$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n],$$
$$h[n] = u[n].$$

# Example

- Are the systems shown earlier:
- 1. Causal
- 2. BIBO Stable
- 3. Invertible
- 4. Memoryless

# Example

- Consider a discrete LTI system whose input and output are related by

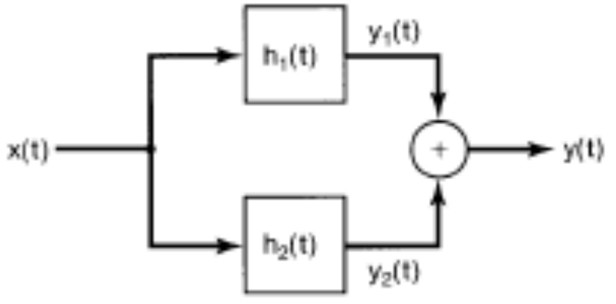
- $y[n] = \sum_{k=-\infty}^{n} 2^{k-n} x[k+1]$

- Determine the impulse response of the system and if it is stable, causal.

# Example

- Consider a system below, where
- $h_1(t) = e^{-2t}u(t)\ h_2(t) = 2e^{-t}u(t)$

- Find the impulse response of the overall system.
- Is the system stable?



(a)

# Example

- Consider a system below, where
- $h_1(t) = u(t)\ h_2(t) = \delta(t-1)$

- Find the impulse response of the overall system.
- Is the system stable?

# Convolution

- LTI systems can be represented as a the convolution of the input with an impulse response.
- Convolution has many useful properties (associative, commutative, etc).

- These carry over to LTI systems
  - Composition of system blocks
  - Order of system blocks

- Useful both practically, and for understanding. While convolution is conceptually simple, it can be practically difficult. It can be tedious to convolve your way through a complex system.
- There has to be a better way ...

# Example

- Suppose Ram's parents opens a savings account on his 1st birthday and deposit rupees 50 on the first of each month in the account. Determine the money in the account after 1 year and 10 years if the interest rate is 1% per month.

# Example

- The interest calculation can be done easily using linear difference equation. Let y[n] be the amount in his account at n$^{th}$ month.
- $y[n] = y[n-1] + 0.01y[n-1] + x[n]$
- $y[n] = 1.01y[n-1] + x[n]$
- Assuming zero initial conditions, i.e. y[-1] = 0

- The amount after 1 year will be y[12]. Can you calculate that? What about after 2 years?

# Example

- Let's write equation in the form of a geometric series:
- $y[n] = \alpha y[n-1] + \beta x[n]$
- In our case, y[-1] = 0 and x[0]=x[1]=....=D (=50) and $\beta$ =1.
- $y[0] = \beta D$
- $y[1] = \beta D(1+\alpha)$
- $y[2] = \alpha y[1] + \beta D = \beta D(\alpha^2 + \alpha + 1)$
- ….
- $y[N] = \beta D(\alpha^N + \alpha^{N-1} + \cdots + \alpha + 1) = \beta D \sum_{m=0}^{N} \alpha^m = \beta D \frac{1-\alpha^{N+1}}{1-\alpha}$

# Example

- In our case, $\alpha$ = 1.01, b = 1, D =50, N = 12 and 24,

- The amount will be,
- $y[12] = 50\,\frac{1-1.01^{13}}{1-1.01}$= 690.47

- $y[24] = 50\,\frac{1-1.01^{25}}{1-1.01} = 1412.16$
- $y[121] = 50\,\frac{1-1.01^{121}}{1-1.01} = 11666.95$

- How will you calculations change if the interest was computed quarterly or annually?

## Multiple Representations of Discrete-Time Systems

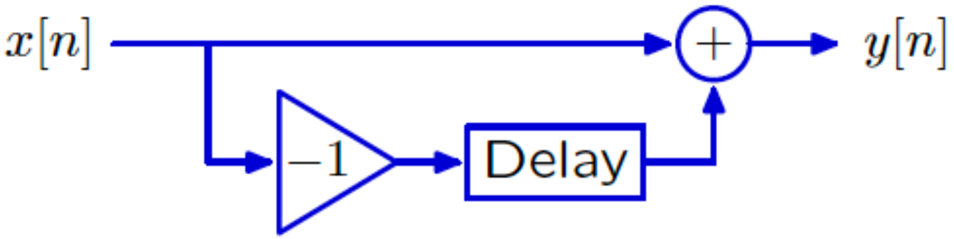
Systems can be represented in different ways to more easily address different types of issues.

**Verbal description:** 'To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.'

**Difference equation:**

$$y[n] = x[n] - x[n-1]$$

**Block diagram:**




We will exploit particular strengths of each of these representations.

# Difference Equations

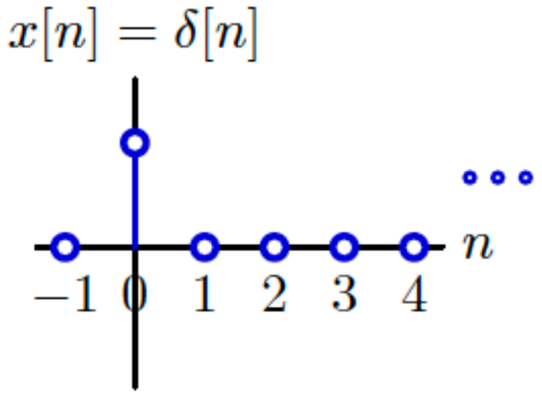
Difference equations are mathematically precise and compact.

Example:

$$y[n] = x[n] - x[n-1]$$

Let $x[n]$ equal the "unit sample" signal $\delta[n]$,

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$




We will use the unit sample as a "primitive" (building-block signal) to construct more complex signals.

## Check Yourself

Solve

$$y[n] = x[n] - x[n-1]$$

given

$$x[n] = \delta[n]$$

How many of the following are true?

1. $y[2] > y[1]$
2. $y[3] > y[2]$
3. $y[2] = 0$
4. $y[n] - y[n-1] = x[n] - 2x[n-1] + x[n-2]$
5. $y[119] = 0$

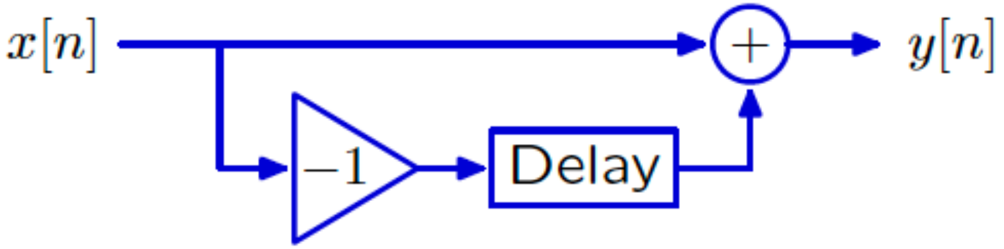
In what ways are difference equations different from block diagrams?

**Difference equation:**

$$y[n] = x[n] - x[n-1]$$

Difference equations are "declarative."
They tell you rules that the system obeys.

**Block diagram:**

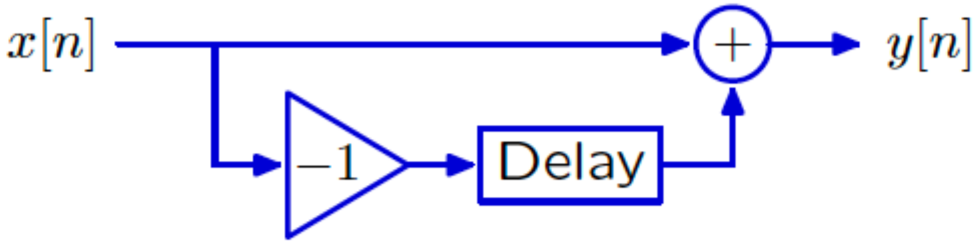



Block diagrams are "imperative."
They tell you what to do.

Block diagrams contain **more** information than the corresponding difference equation (e.g., what is the input? what is the output?)

## From Samples to Signals

**Operators** manipulate signals rather than individual samples.




Nodes represent whole signals (e.g., $X$ and $Y$).

The boxes **operate** on those signals:

- Delay = shift whole signal to right 1 time step
- Add = sum two signals
- $-1$: multiply by $-1$

**Signals** are the primitives.

**Operators** are the means of combination.

# Operator Notation

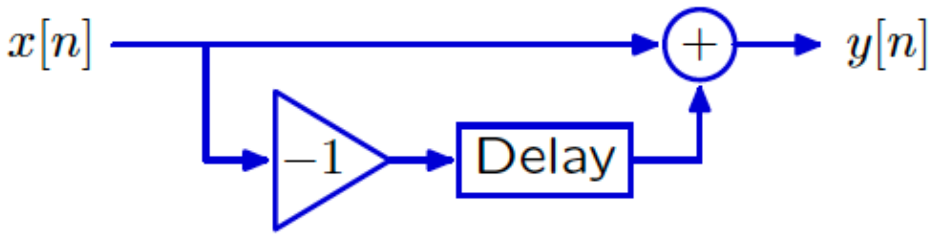
Symbols can now compactly represent diagrams.

Let $\mathcal{R}$ represent the right-shift **operator**:

$$Y = \mathcal{R}\{X\} \equiv \mathcal{R}X$$

where $X$ represents the whole input signal ($x[n]$ for all $n$) and $Y$ represents the whole output signal ($y[n]$ for all $n$)

Representing the difference machine




with $\mathcal{R}$ leads to the equivalent representation

$$Y = X - \mathcal{R}X = (1 - \mathcal{R})\,X$$

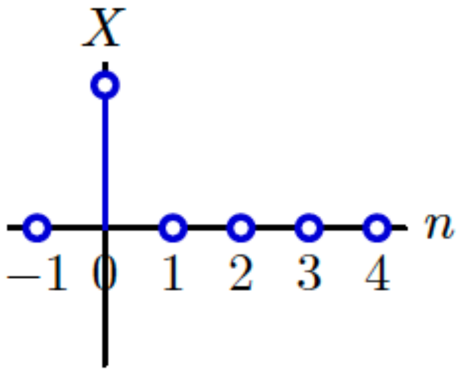
# Operator Notation: Check Yourself

Let $Y = \mathcal{R}X$. Which of the following is/are true:

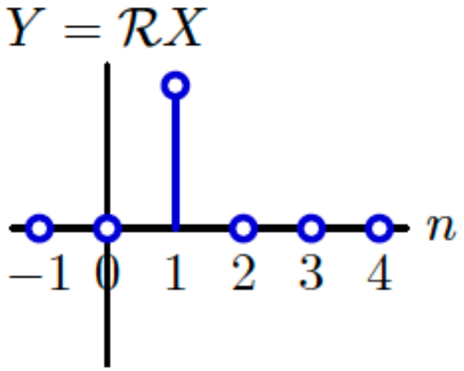
1. $y[n] = x[n]$ for all $n$
2. $y[n+1] = x[n]$ for all $n$
3. $y[n] = x[n+1]$ for all $n$
4. $y[n-1] = x[n]$ for all $n$
5. none of the above

## Check Yourself

Consider a simple signal:



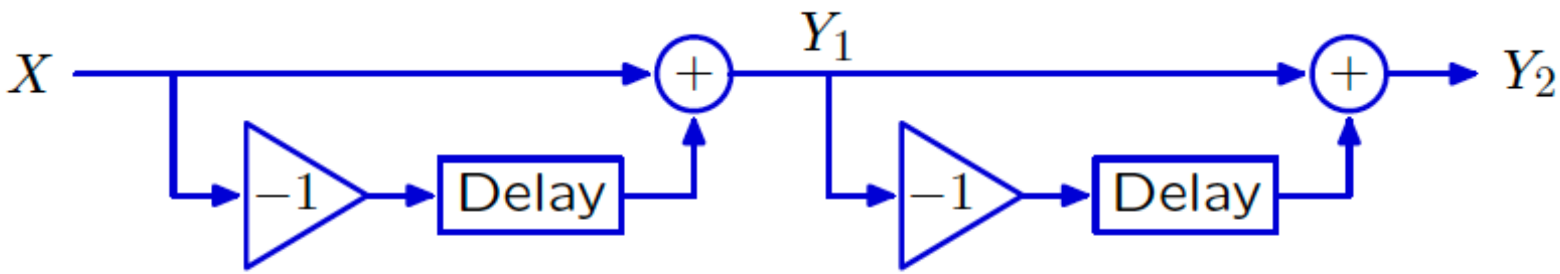

Then



# Operator Representation of a Cascaded System

System operations have simple operator representations.

Cascade systems $\rightarrow$ multiply operator expressions.




Using operator notation:

$$Y_1 = (1 - \mathcal{R})\, X$$

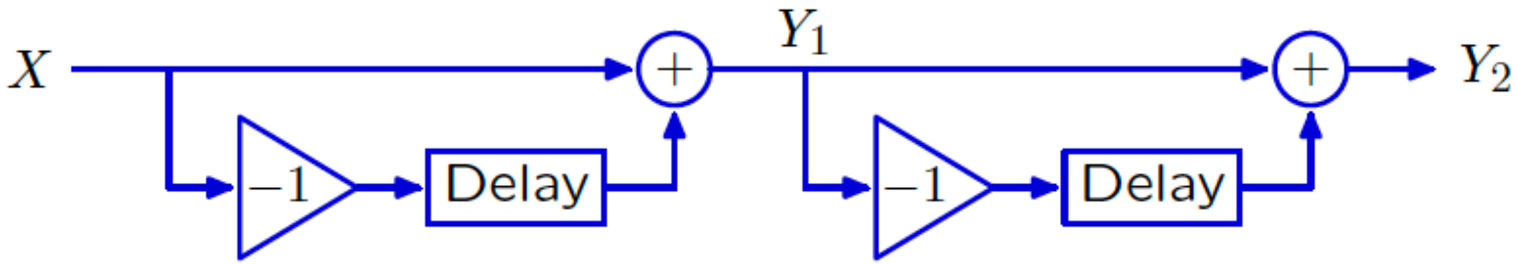
$$Y_2 = (1 - \mathcal{R})\, Y_1$$

Substituting for $Y_1$:

$$Y_2 = (1 - \mathcal{R})(1 - \mathcal{R})\, X$$

# Operator Algebra

Operator expressions can be manipulated as polynomials.




Using difference equations:

$$\begin{aligned} y_2[n] &= y_1[n] - y_1[n-1] \\ &= (x[n] - x[n-1]) - (x[n-1] - x[n-2]) \\ &= x[n] - 2x[n-1] + x[n-2] \end{aligned}$$

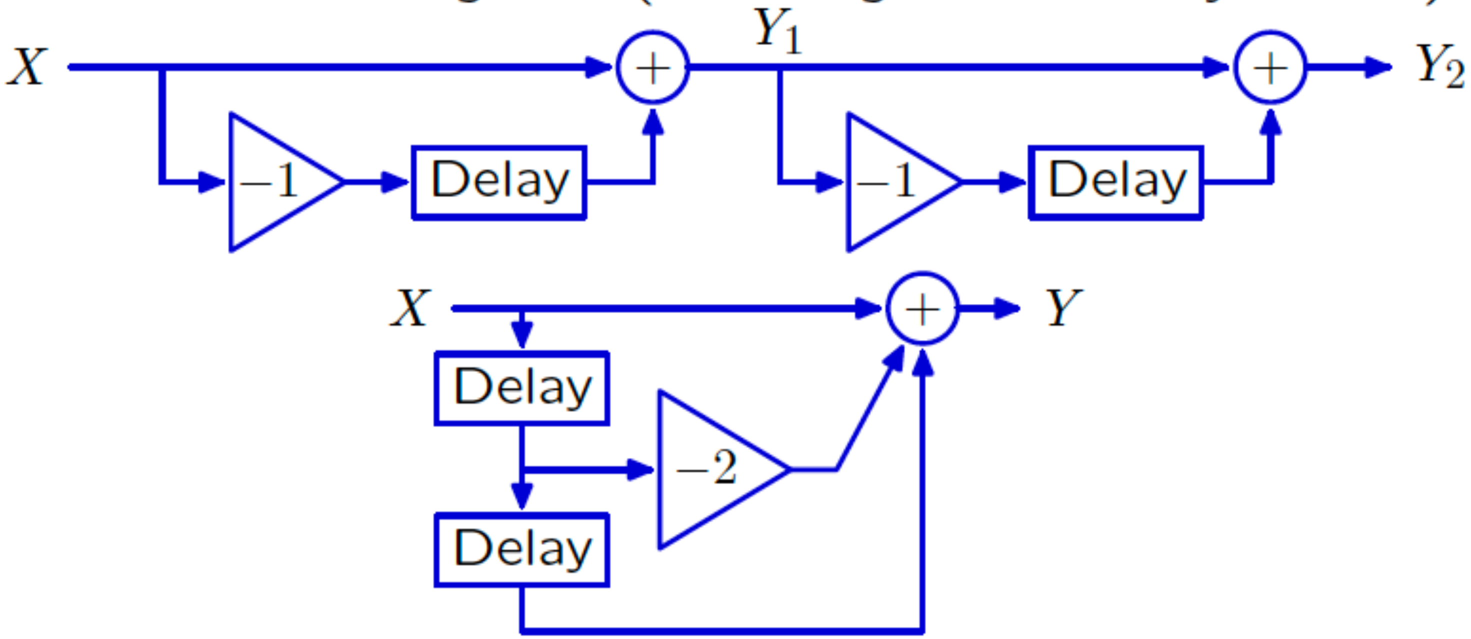
Using operator notation:

$$\begin{aligned} Y_2 &= (1-\mathcal{R})\,Y_1 = (1-\mathcal{R})(1-\mathcal{R})\,X \\ &= (1-\mathcal{R})^2 X \\ &= (1-2\mathcal{R}+\mathcal{R}^2)\,X \end{aligned}$$

## Operator Algebra

Operator notation facilitates seeing relations among systems.

"Equivalent" block diagrams (assuming both initially at rest):



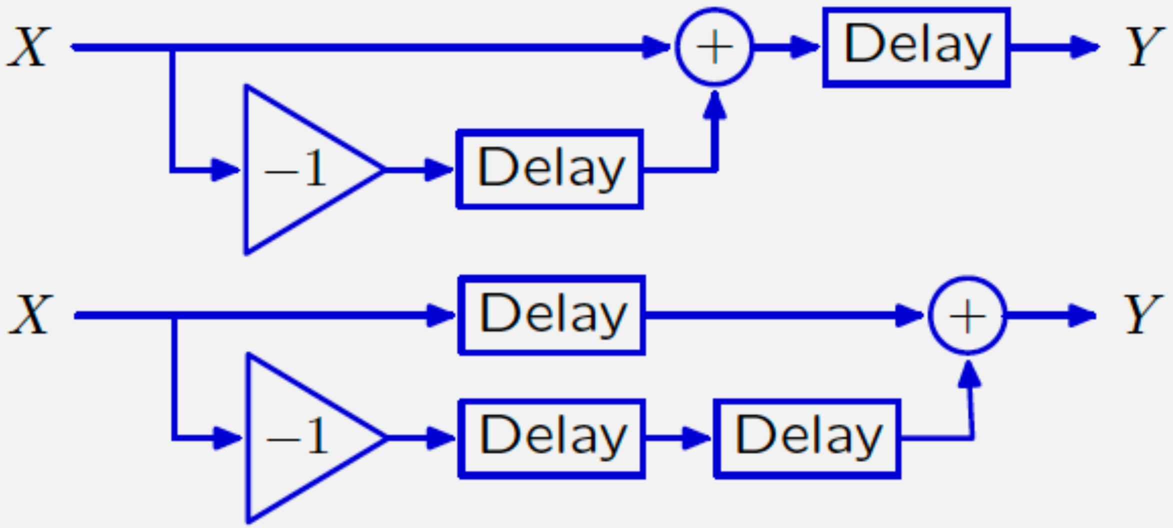

Equivalent operator expressions:

$$(1-\mathcal{R})(1-\mathcal{R}) = 1 - 2\mathcal{R} + \mathcal{R}^2$$

The operator equivalence is much easier to see.

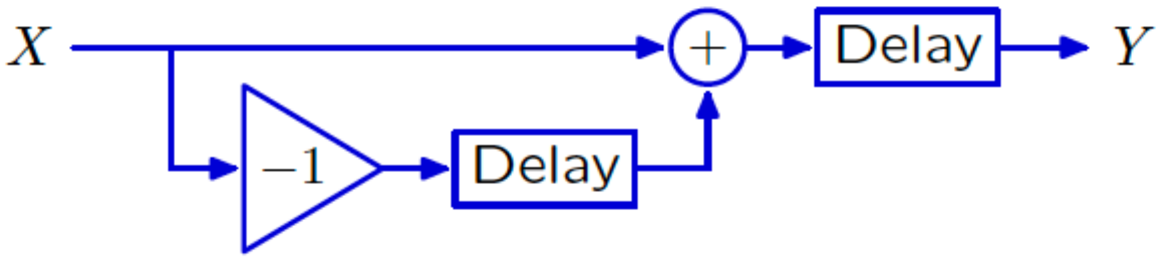
## Check Yourself

Operator expressions for these "equivalent" systems (if started "at rest") obey what mathematical property?

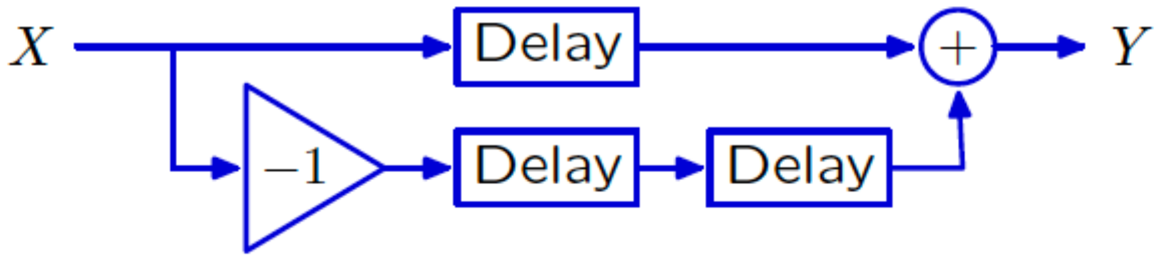



1. commutate
2. associative
3. distributive
4. transitive
5. none of the above

## Check Yourself




$$Y = \mathcal{R}(1 - \mathcal{R})X$$



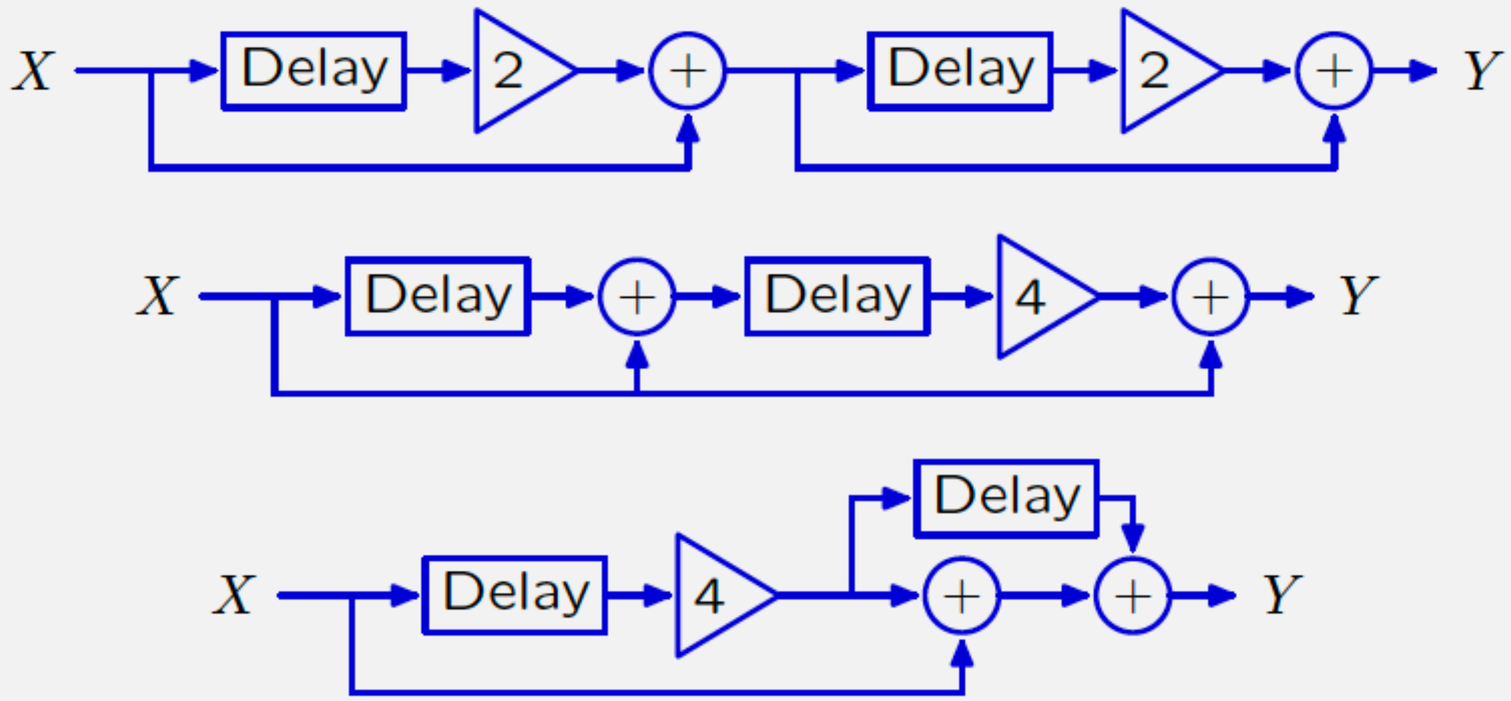

$$Y = (\mathcal{R} - \mathcal{R}^2)X$$

Multiplication by $\mathcal{R}$ distributes over addition.

## Check Yourself

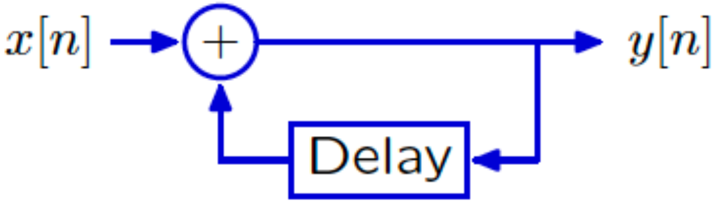
How many of the following systems are equivalent to

$$Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\,X \quad ?$$

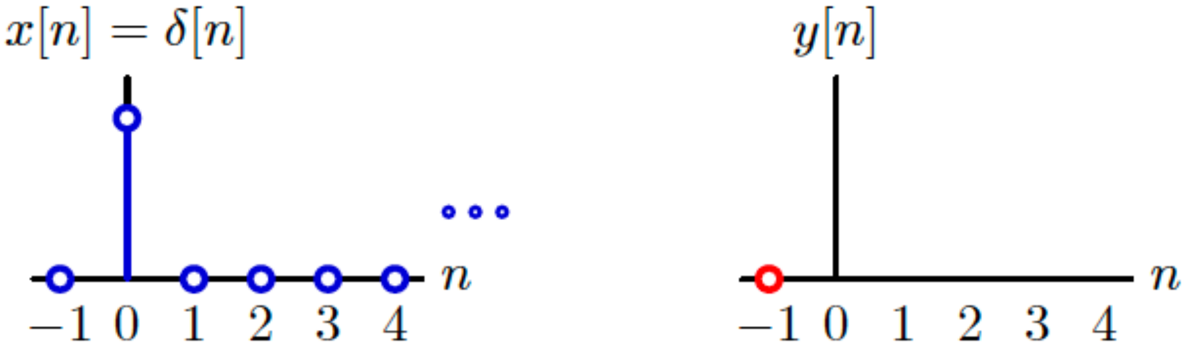


# Example: Accumulator

Try step-by-step analysis: it always works. Start "at rest."




Find $y[n]$ given $x[n] = \delta[n]$: $\quad y[n] = x[n] + y[n-1]$



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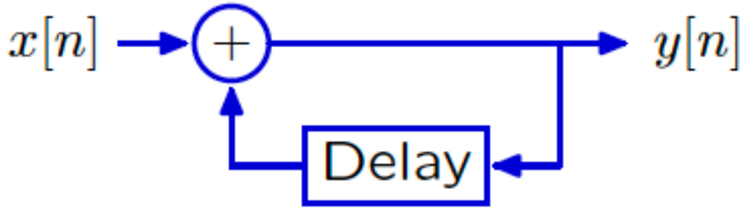
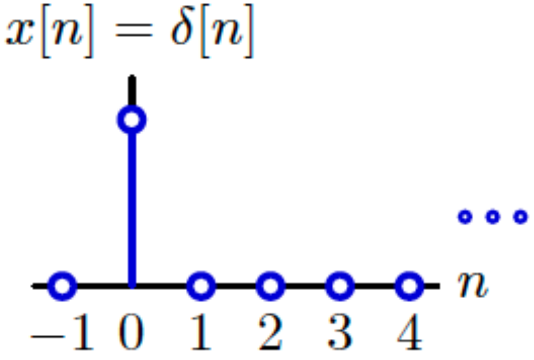
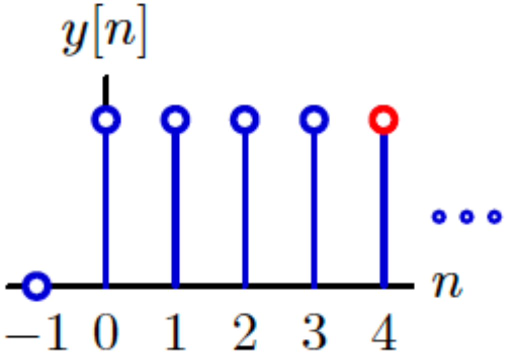



Find $y[n]$ given $x[n] = \delta[n]$: $\quad y[n] = x[n] + y[n-1]$

$$
\begin{aligned}
y[0] &= x[0] + y[-1] &&= 1 + 0 = 1 \\
y[1] &= x[1] + y[0] &&= 0 + 1 = 1 \\
y[2] &= x[2] + y[1] &&= 0 + 1 = 1 \\
&\cdots
\end{aligned}
$$

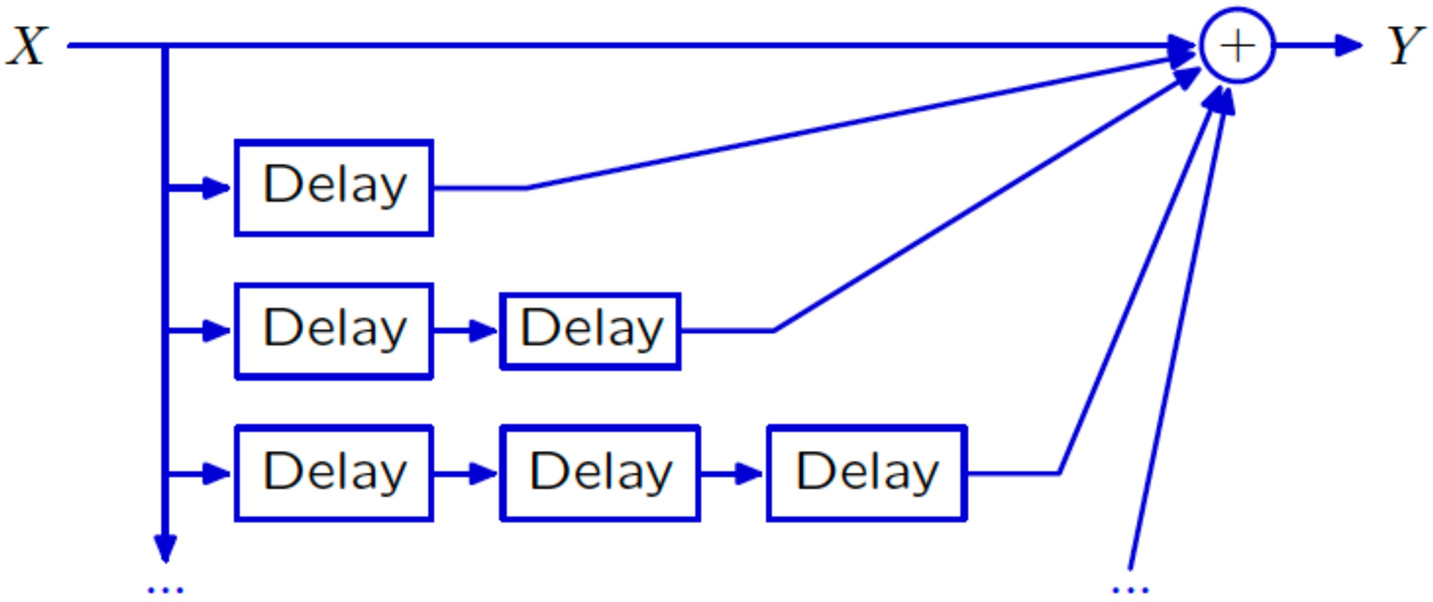



Persistent response to a transient input!

## Example: Accumulator

The response of the accumulator system could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous.

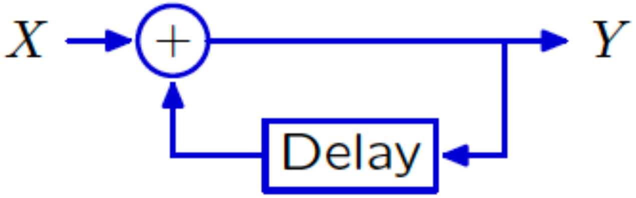



$$Y = (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)\, X$$

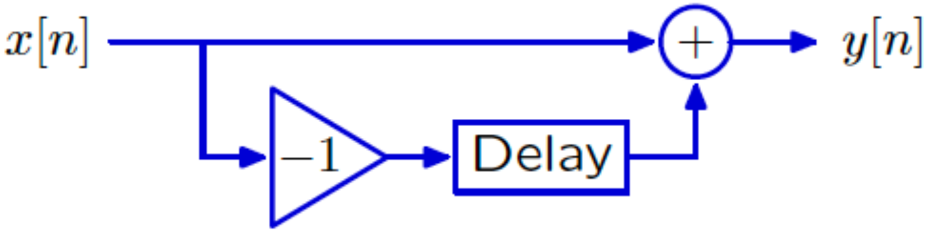
## Feedback

Systems with signals that depend on previous values of the same signal are said to have **feedback**.

Example: The accumulator system has feedback.



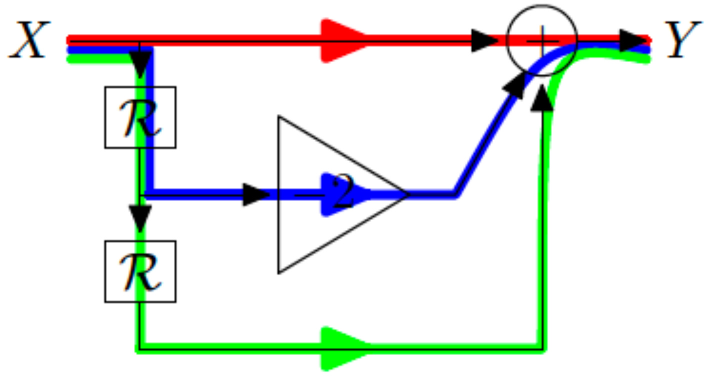

By contrast, the difference machine does not have feedback.



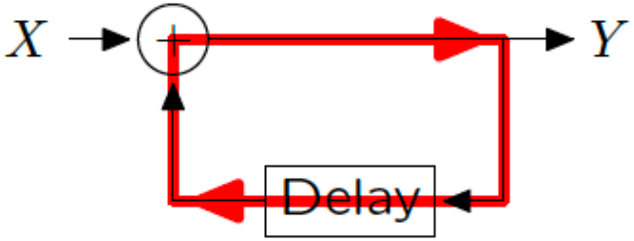
# Cyclic Signal Paths, Feedback, and Modes

Block diagrams help visualize feedback.

Feedback occurs when there is a cyclic signal flow path.




acyclic



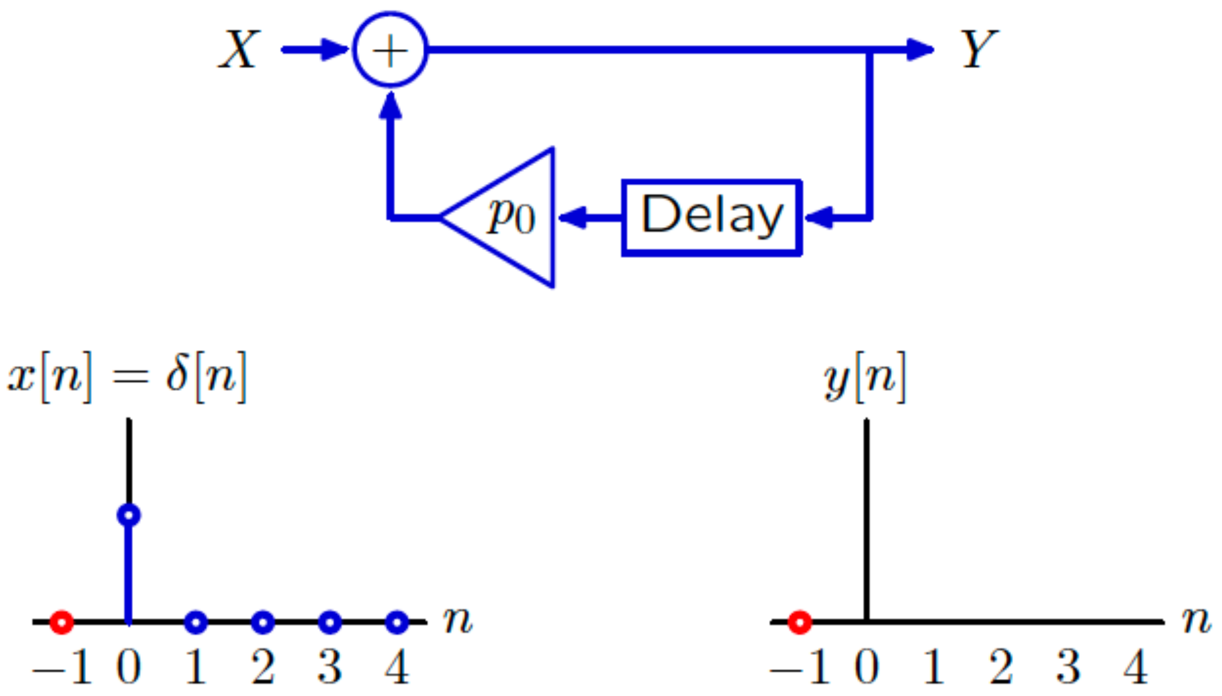

cyclic

**Acyclic:** all paths through system go from input to output with no cycles.

**Cyclic:** at least one cycle.

# Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.




Each cycle creates another sample in the output.

# Feedback, Cyclic Signal Paths, and Modes

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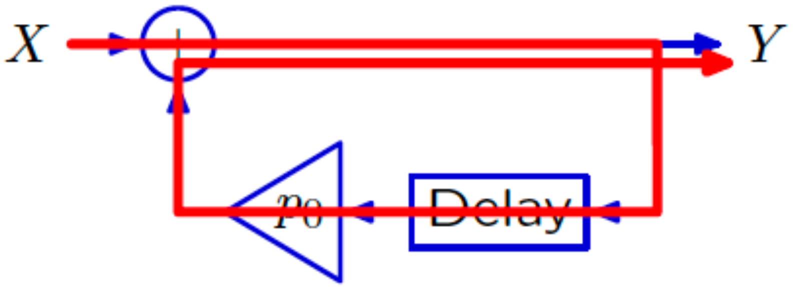


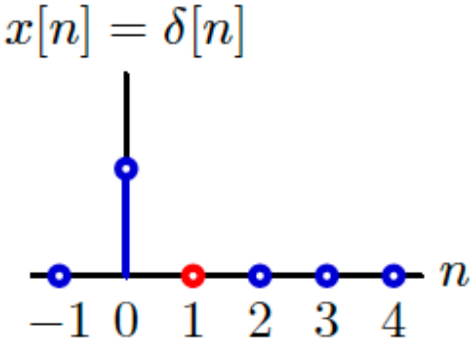


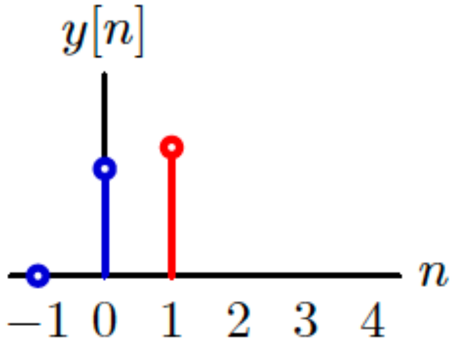


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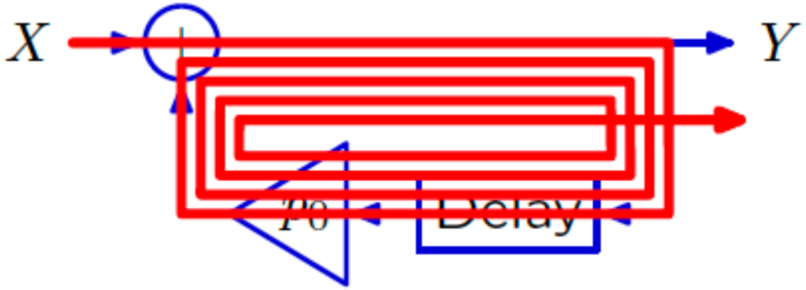
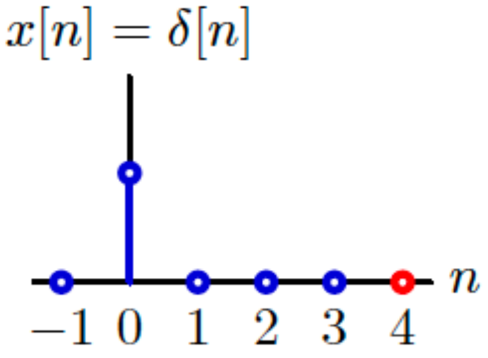
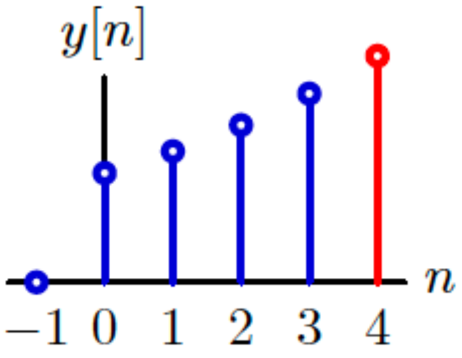


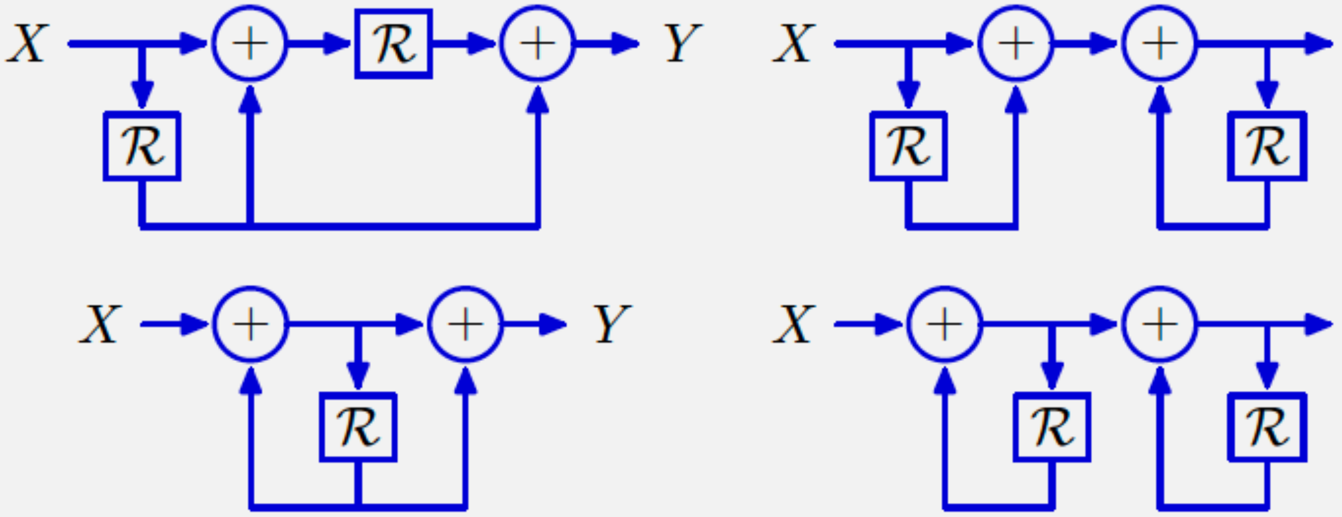




Each cycle creates another sample in the output.

The response will persist even though the input is transient.

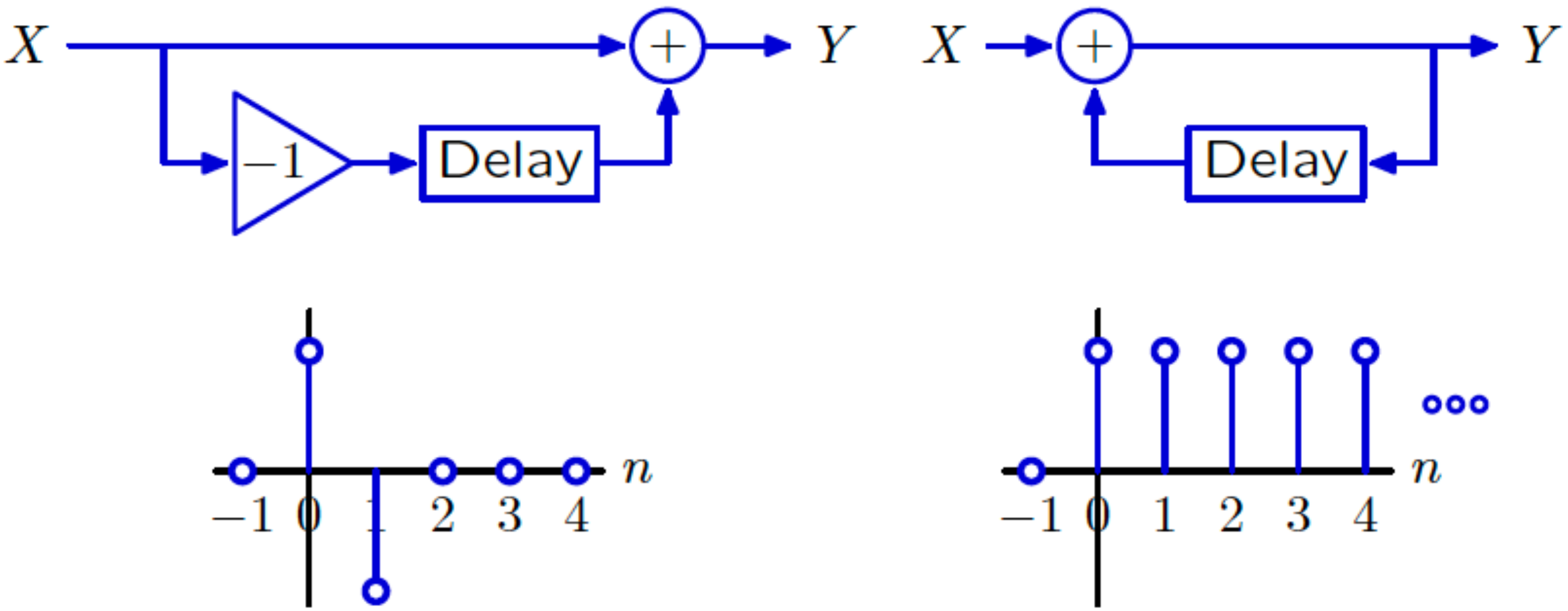
## Check Yourself

How many of the following systems have cyclic signal paths?



# Finite and Infinite Impulse Responses

The impulse response of an acyclic system has finite duration, while that of a cyclic system can have infinite duration.



## Analysis of Cyclic Systems: Geometric Growth

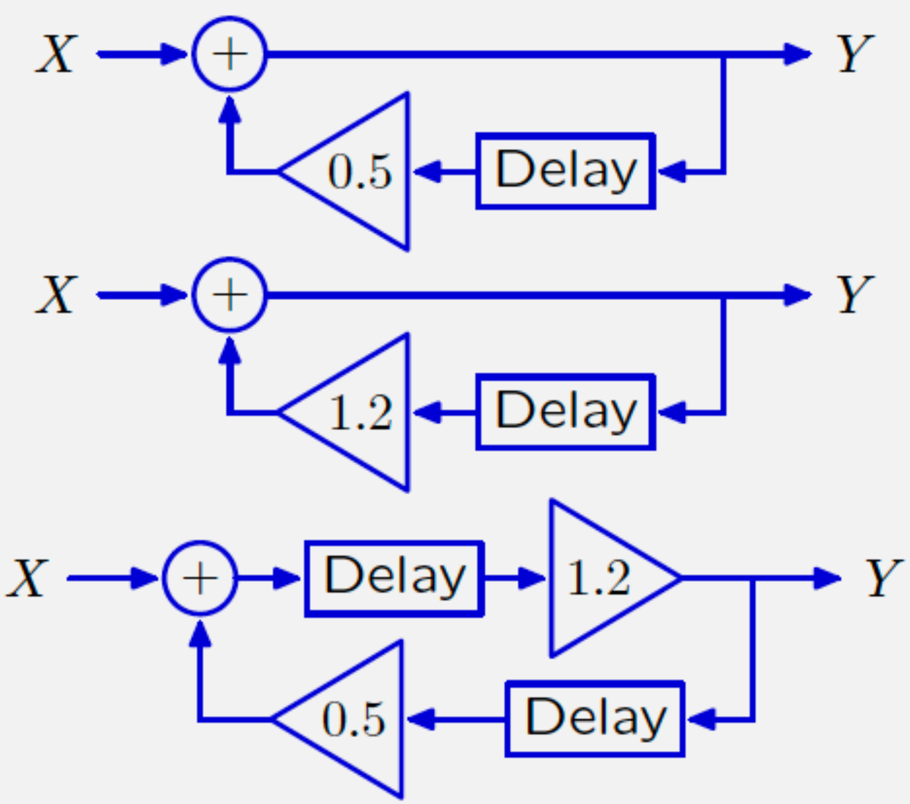
If traversing the cycle decreases or increases the magnitude of the signal, then the fundamental mode will decay or grow, respectively.

If the response decays toward zero, then we say that it **converges**.

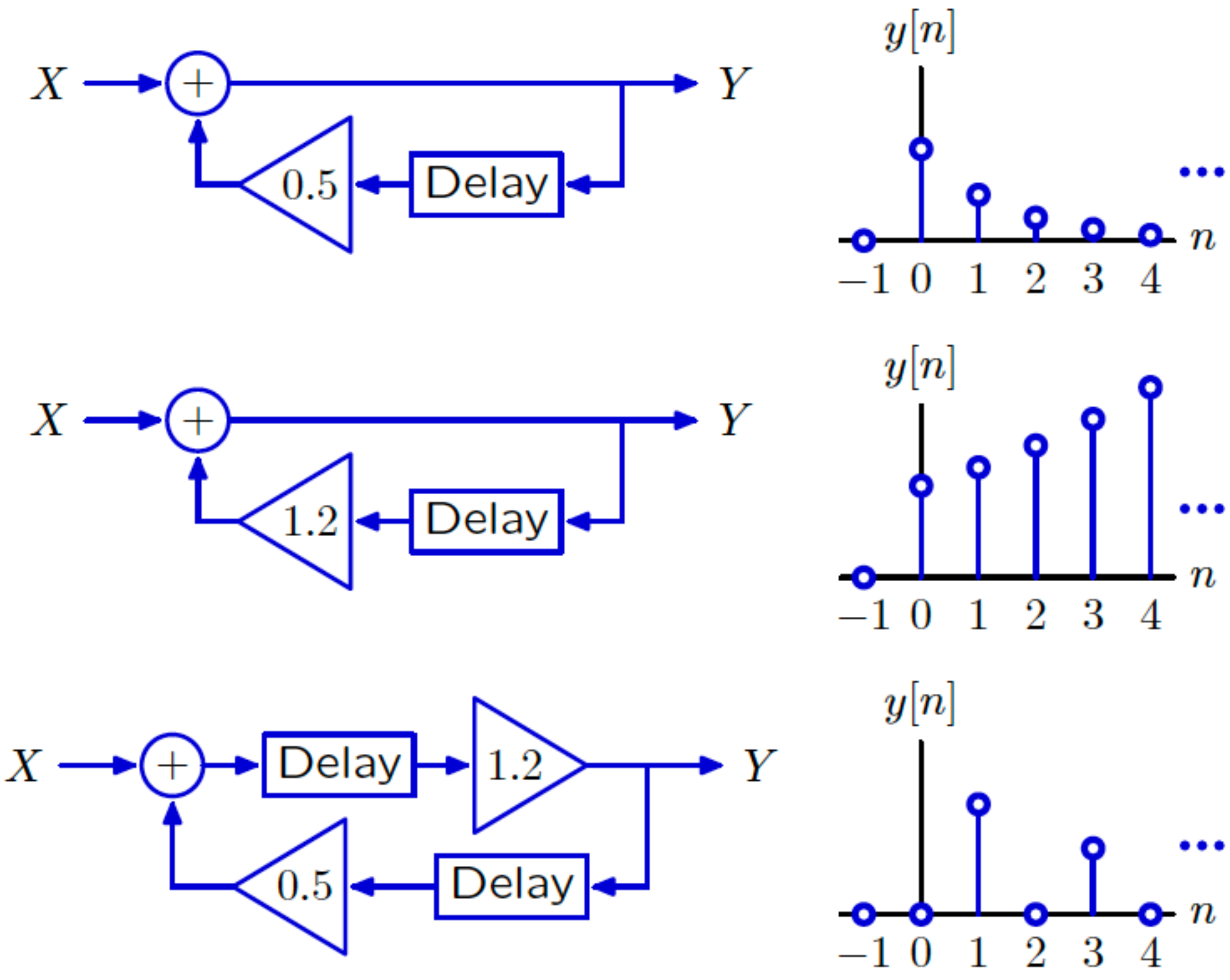
Otherwise, we it **diverges**.

# Check Yourself

How many of these systems have divergent unit-sample responses?



# Check Yourself



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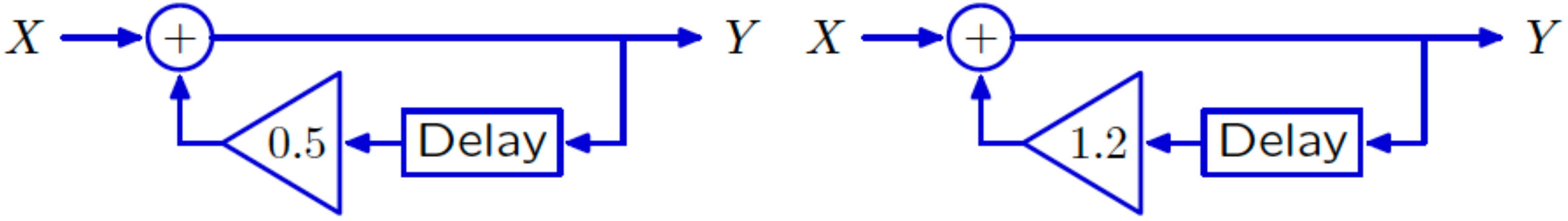
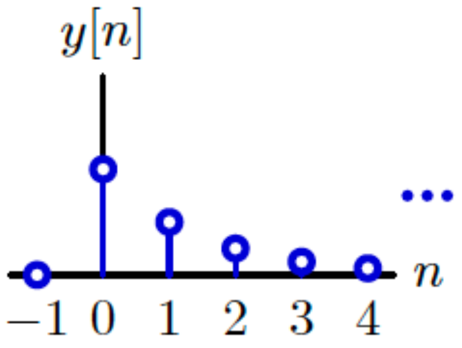



These are geometric sequences: $y[n] = (0.5)^n$ and $(1.2)^n$ for $n \geq 0$.

These geometric sequences are called **fundamental modes**.

# Multiple Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

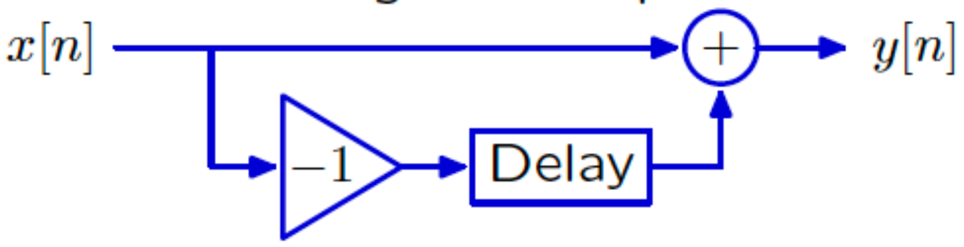
**Verbal descriptions:** preserve the rationale.

> "To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences."

**Difference equations:** mathematically compact.

$$y[n] = x[n] - x[n-1]$$

**Block diagrams:** illustrate signal flow paths.




**Operator representations:** analyze systems as polynomials.

$$Y = (1 - \mathcal{R})\,X$$