Stat-581: Probability and Statistical Inference for Data Science
Assignment - 3
Method of Moments

Name: Prakruti Joshi
NetID: phj15

Method of Moments:

Method of moments for parameter estimation is based on the *law of large numbers*. If X_1, X_2,... X_n are independent random variables, the sample mean converges to the distributed mean as the number of observations/samples increases ($n \to \infty$).

If the model has *m* parameters, we compute the *first m moments,* obtaining m equations in m variables.

Definitions.

(1) $E(X^k)$ is the k^{th} **(theoretical) moment** of the distribution (**about the origin**), for $k = 1, 2, ...$

(2) $E\left[(X-\mu)^k\right]$ is the k^{th} **(theoretical) moment** of the distribution (**about the mean**), for $k = 1, 2, ...$

(3) $M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$ is the k^{th} **sample moment**, for $k = 1, 2, ...$

(4) $M_k^* = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^k$ is the k^{th} **sample moment about the mean**, for $k = 1, 2, ...$

Thus, from the above definition, we can infer:

1. The 1st theoretical moment (k=1) of the distribution about the origin converges to the mean of the distribution: E(X) as the number of observations increases.

   Sample mean = E(X) -- (1)

2. The 2nd theoretical moment (k=2) of the distribution about the mean converges to the variance of the distribution as the number of observations increases.

   $E[(X-\mu)^2]$ = Var(X) = Sample Variance(n observations) -- (2)

For a 2- parameter distribution, we can form two equations by equating the theoretical mean with the sample mean and equating the theoretical variance with the population/sample variance (2nd moment about the mean). By solving the equations, we can get the estimated value of the parameters of the distribution based on the data.

For notation purposes:

1. Sample Mean ($1/n \sum_{1}^{n} X_i$) = $\overline{Xn}$ = μ1
2. Sample Variance ($1/n \sum_{i=1}^{n} (X_i - \overline{Xn})^2$) = μ2

The estimated parameters using method of moments estimator for the distributions is shown in the table. The equations are solved for each distribution and the estimation of the parameters in terms of sample mean and 2nd moment about the sample mean is determined.

Parameter estimation using method of moments:

Index	Distribution	Mean	Variance	Parameter 1	Parameter 2
1.	Point Mass(a)	a	0	$\hat{a}$ = μ1	-
2.	Bernoulli (p)	p	p(1-p)	$\hat{p}$ = μ1	-
3.	Binomial (n,p)	np	np(1-p)	$\hat{p}$ = μ2/μ1	$\hat{n}$ = $\mu1^2$/(μ1-μ2)
4.	Geometric (p)	1/p	$(1-p)/p^2$	$\hat{p}$ = 1/μ1	-
5.	Poisson(λ)	λ	λ	$\hat{\lambda}$ = μ1	-
6.	Uniform(a,b)	(a+b)/2	$(b-a)^2/12$	$\hat{a}$ = μ1- $\sqrt{3\mu2}$	$\hat{b}$ = μ1 + $\sqrt{3\mu2}$
7.	Exponential(β)	β	β^2	$\hat{\beta}$ = 1/ μ1	-
8.	Normal(μ,σ^2)	μ	σ^2	$\hat{\mu}$ = μ1	$\hat{}^{2}$ = μ2
9.	Gamma(α, β)	αβ	$\alpha\beta^2$	$\hat{\alpha}$ = $\mu1^2$/μ2	$\hat{\beta}$ = μ2/μ1
10.	Beta(α, β)	α/(α + β)	αβ /(α + β $)^2$(α + β +1)	$\hat{\alpha}$ = $\mu1^2$/μ2(1-μ1) -μ1	$\hat{\beta}$ =(($\mu1^2$/μ2(1-μ1)) -μ1)(1-μ1)/μ1)
11.	Chi-Square (p)	p	2p	$\hat{p}$ = μ1	-

Based on the method of moments estimator in the above table, I have tested the distribution by generating *100 observations/samples* from the distribution. To infer about the accuracy of the estimated parameters, the parameters has been estimated 5 times for the same distribution (randomly generated 100 samples each time) and the average value of the estimated parameter is taken. The estimated parameters and the actual parameters are shown in the following table:

Results:

Index	Distribution	Actual Parameter 1	Actual Parameter 2	Estimated Parameter 1	Estimated Parameter 2
1.	Point Mass(a)	10	-	9.88	-
2.	Bernoulli (p)	p = 0.6	-	$\widehat{p} = 0.61$	-
3.	Binomial (n,p)	n = 10	p = 0.6	$\widehat{n} = 10.573$	$\widehat{p} = 0.585$
4.	Geometric (p)	p = 0.6	-	$\widehat{p} = 0.613$	-
5.	Poisson(λ)	$\lambda = 0.02$	-	$\widehat{\lambda} = 0.02$	-
6.	Uniform(a,b)	a = 0	b = 5	$\widehat{a} = 0.08$	$\widehat{b} = 5.17$
7.	Exponential(β)	β = 6	–	$\widehat{\beta} = 6.59$	-
8.	Normal(μ,σ^2)	$\mu = 0$	$\sigma^2 = 1$	$\widehat{\mu} = -0.12$	$^2\widehat{\ } = 0.93$
9.	Gamma(α, β)	α = 2	β = 0.4	$\widehat{\alpha} = 2.69$	$\widehat{\beta} = 0.33$
10.	Beta(α, β)	α = 3	β = 5	$\widehat{\alpha} = 3.34$	$\widehat{\beta} = 5.78$
11.	Chi-Square (p)	p = 3	-	$\widehat{p} = 2.93$	-

Conclusion:

It can be seen from the estimated values that the method of moments estimator is consistent and accurately estimates value upto approximately ± 0.5 deviation. The parameter estimation values for each distribution (from solving the two equations) is much easier to calculate as compared to the maximum likelihood estimate method.

However, there is still a considerable deviation of the parameters from the actual values and thus better methods for parametric estimation such as maximum likelihood can be used.

References:

1. https://newonlinecourses.science.psu.edu/stat414/node/193/
2. https://www.math.arizona.edu/~jwatkins/M_moments.pdf