Chapter 4
Informed/Heuristic Search

Outline

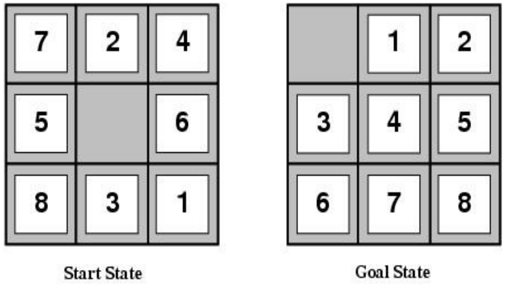
- Limitations of uninformed search methods

- Informed (or heuristic) search uses problem-specific heuristics to improve efficiency
  - Best-first
  - A*
  - RBFS
  - SMA*
  - Techniques for generating heuristics

- Can provide significant speed-ups in practice
  - e.g., on 8-puzzle
  - But can still have worst-case exponential time complexity

- Reading:
  - Chapter 4, Sections 4.1 and 4.2

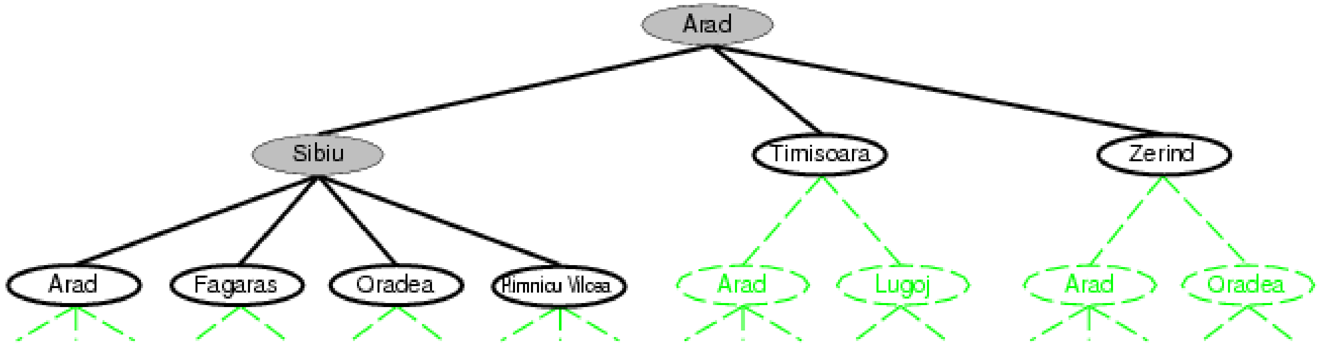
Limitations of uninformed search



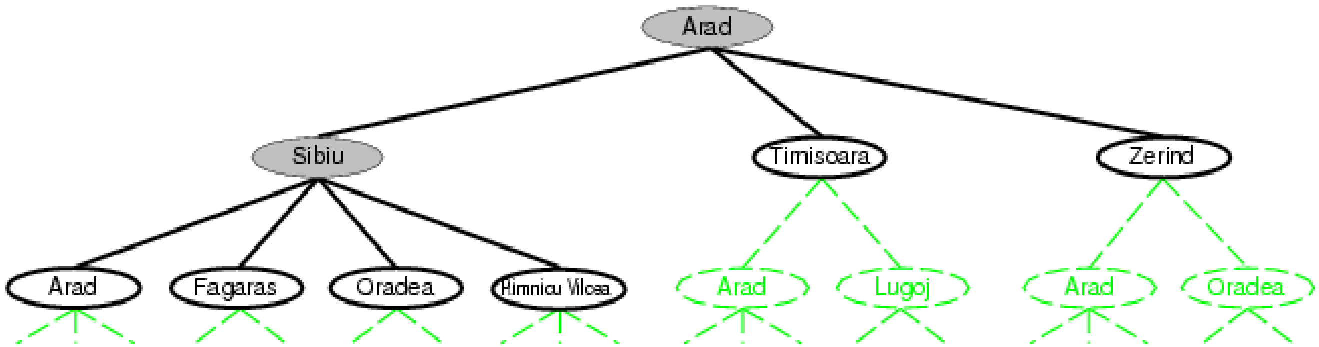
- 8-puzzle
  - Avg. solution cost is about 22 steps
  - branching factor ~ 3
  - Exhaustive search to depth 22:
    - 3.1×10^{10} states
  - E.g., d=12, IDS expands 3.6 million states on average

[24 puzzle has 10^{24} states (much worse)]

Recall tree search...



Recall tree search...



function TREE-SEARCH(*problem*, *strategy*) **returns** a solut

  initialize the search tree using the initial state of *problem*

  loop do

    if there are no candidates for expansion **then return** failu

    choose a leaf node for expansion according to *strategy*

    if the node contains a goal state **then return** the corresponding solution

    else expand the node and add the resulting nodes to the search tree

This "strategy" is what differentiates different search algorithms

Best-first search

- Idea: use an evaluation function f(n) for each node
  - estimate of "desirability“
  - → Expand most desirable unexpanded node
- Implementation:
  - Order the nodes in fringe by f(n) (by desirability, lowest f(n) first)
- Special cases:
  - uniform cost search (from last lecture): f(n) = g(n) = path to n
  - greedy best-first search
  - A^* search
- Note: evaluation function is an estimate of node quality
  - => More accurate name for “best first” search would be “seemingly best-first search”

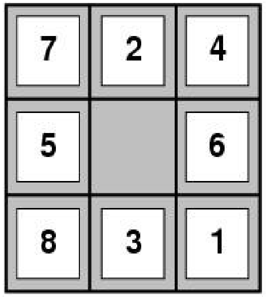
Heuristic function

- Heuristic:
  - Definition: “using rules of thumb to find answers”

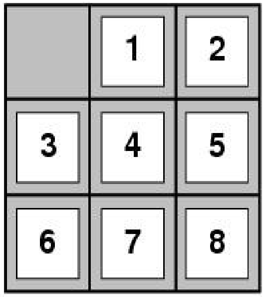
- Heuristic function h(n)
  - Estimate of (optimal) cost from n to goal
  - h(n) = 0 if n is a goal node
  - Example: straight line distance from n to Bucharest
    - Note that this is not the true state-space distance
    - It is an estimate – actual state-space distance can be higher

  - Provides problem-specific knowledge to the search algorithm

Heuristic functions for 8-puzzle



Start State



Goal State

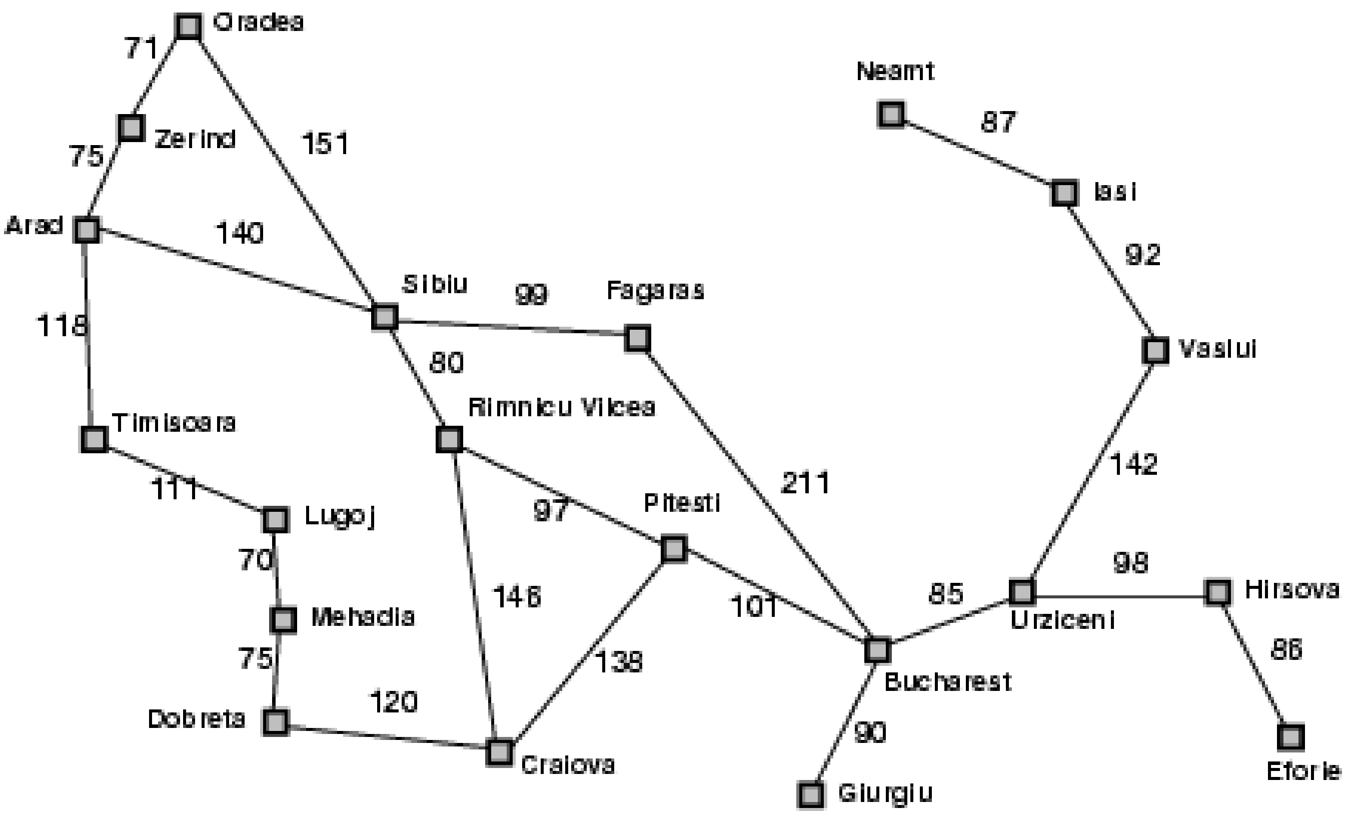
- 8-puzzle
  - Avg. solution cost is about 22 steps
  - branching factor ~ 3
  - Exhaustive search to depth 22:
    - 3.1 x 10^{10} states.
  - A good heuristic function can reduce the search process.

- Two commonly used heuristics
  - h_1 = the number of misplaced tiles
    - $h_1(s)=8$
  - h_2 = the sum of the distances of the tiles from their goal positions (Manhattan distance).
    - $h_2(s)=3+1+2+2+2+3+3+2=18$

Greedy best-first search

- Special case of best-first search
  - Uses h(n) = heuristic function as its evaluation function
  - Expand the node that appears closest to goal

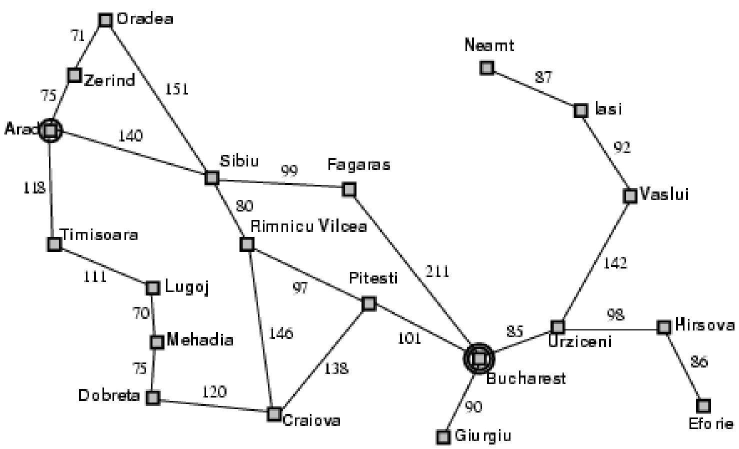
Romania with step costs in km



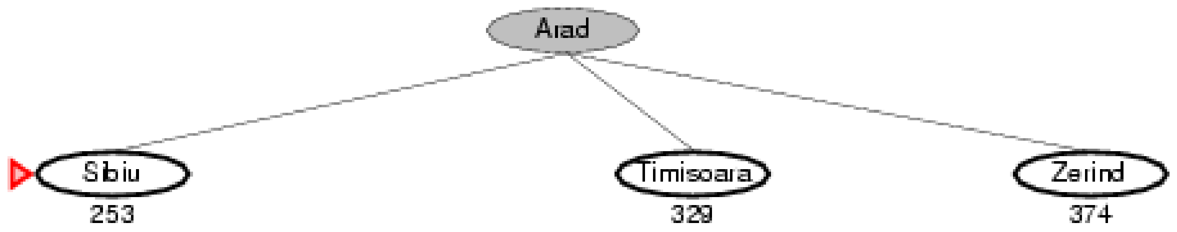
Straight–line distance to Bucharest

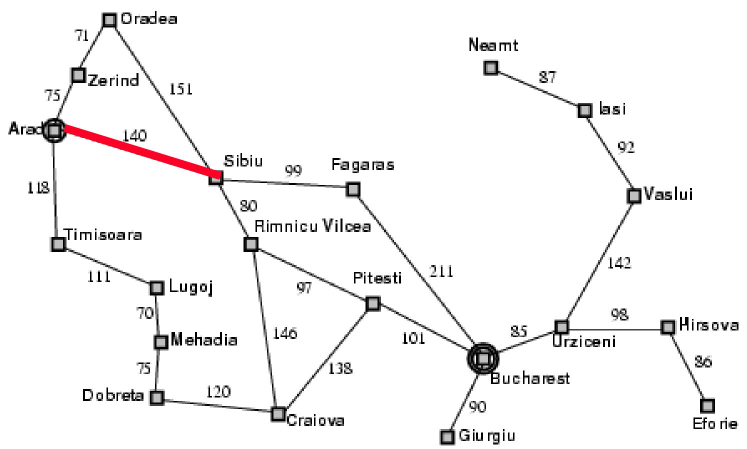
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	10
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy best-first search example

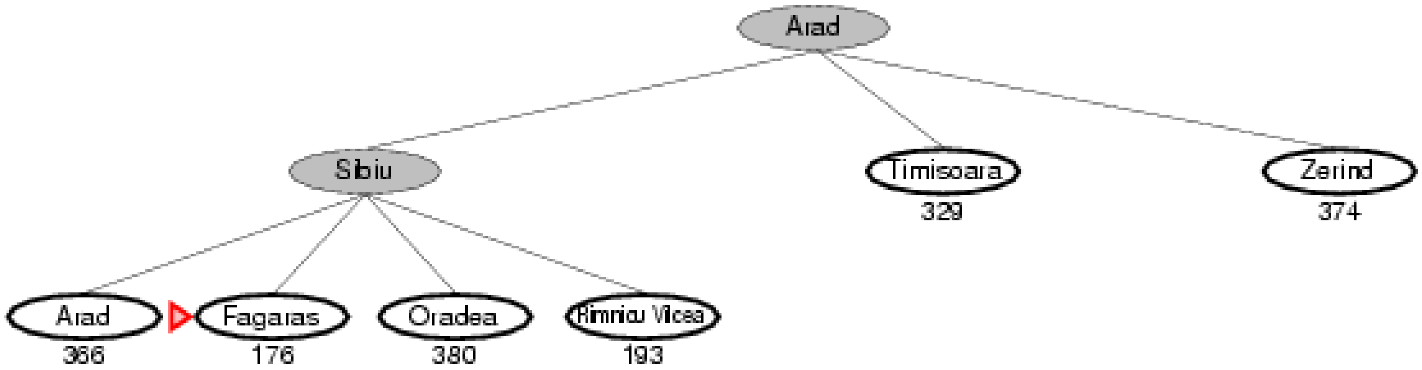


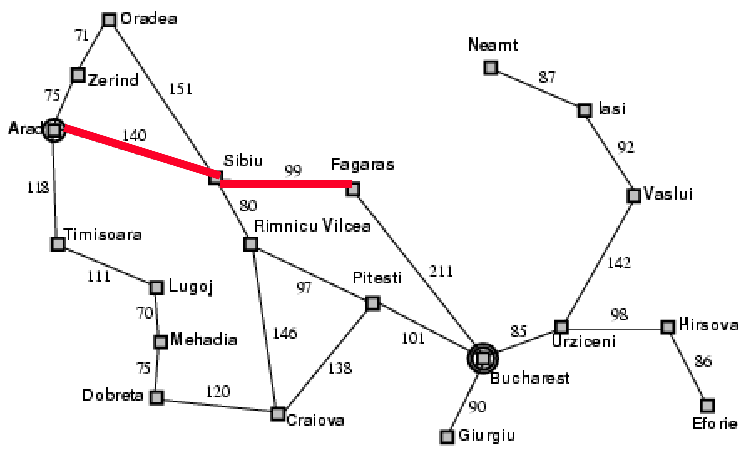
Greedy best-first search example



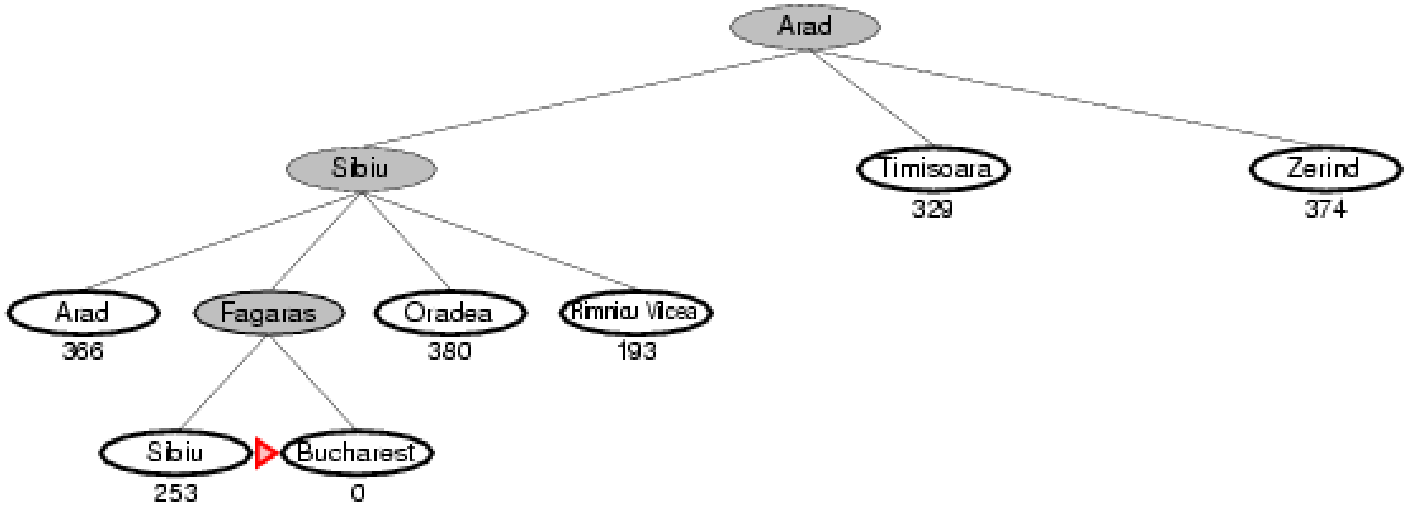


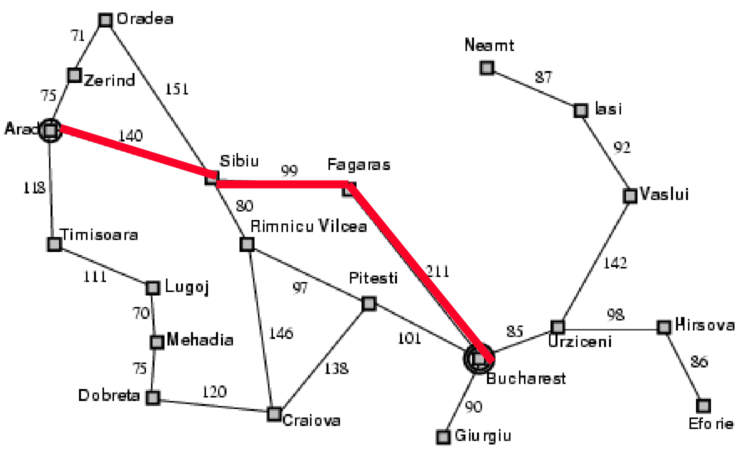
Greedy best-first search example



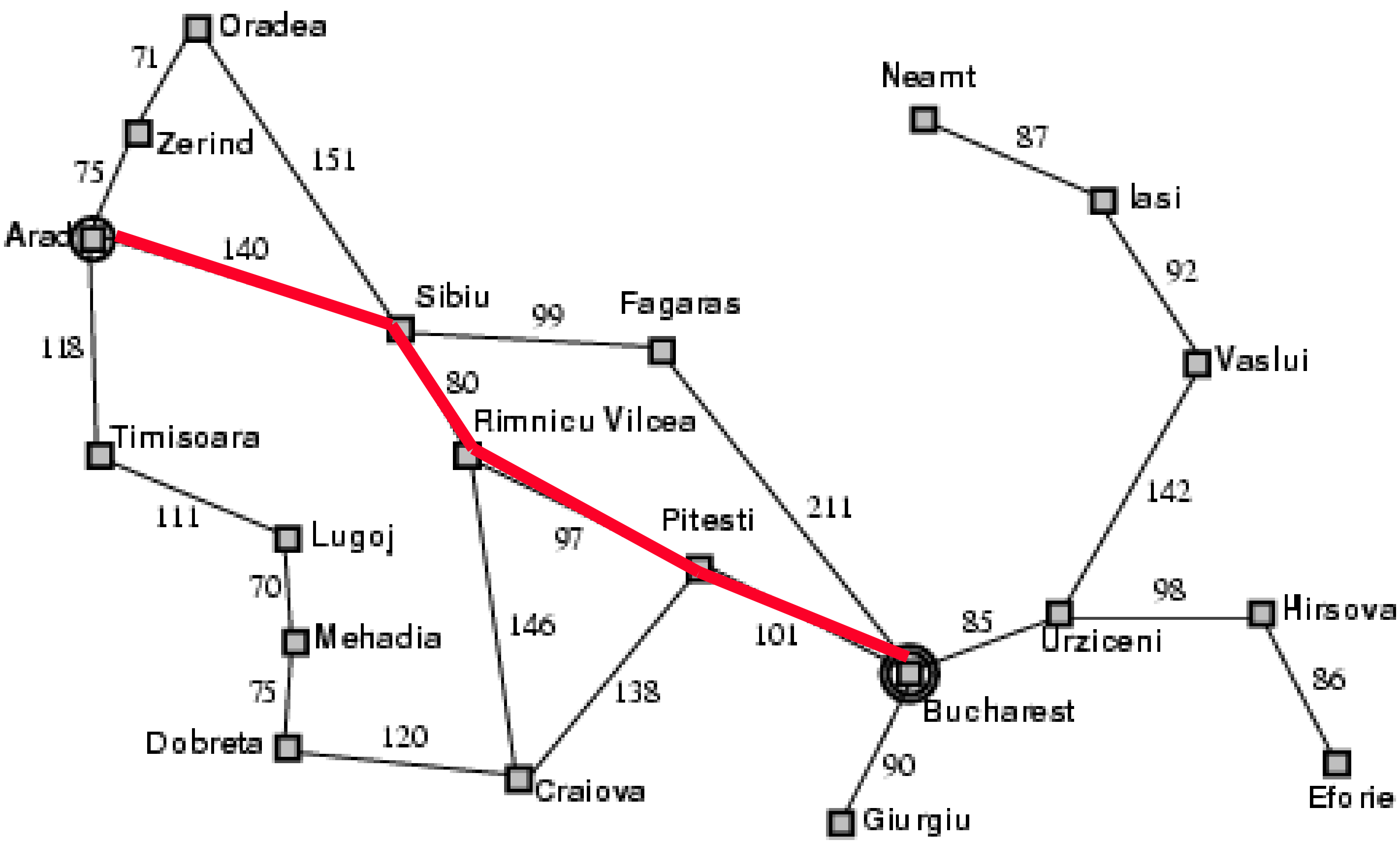


Greedy best-first search example





Optimal Path



Properties of greedy best-first search

- Complete?
  - Not unless it keeps track of all states visited
    - Otherwise can get stuck in loops (just like DFS)
- Optimal?
  - No – we just saw a counter-example
- Time?
  - $O(b^m)$, can generate all nodes at depth m before finding solution
  - m = maximum depth of search space
- Space?
  - $O(b^m)$ – again, worst case, can generate all nodes at depth m before finding solution

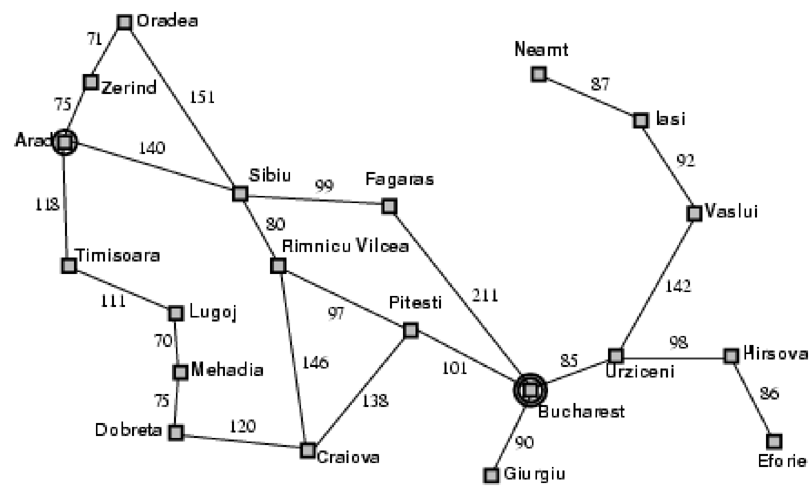
A* Search

- Expand node based on estimate of total path cost through node

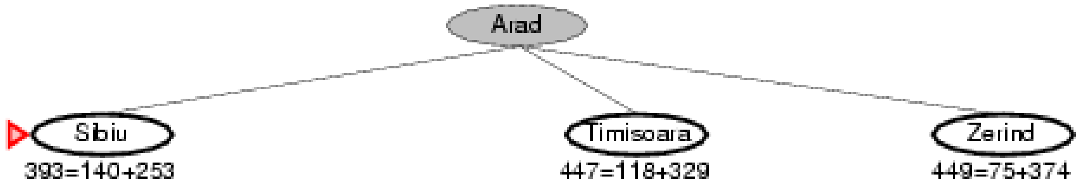
- Evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
  - $h(n)$ = estimated cost from n to goal
  - $f(n)$ = estimated total cost of path through n to goal

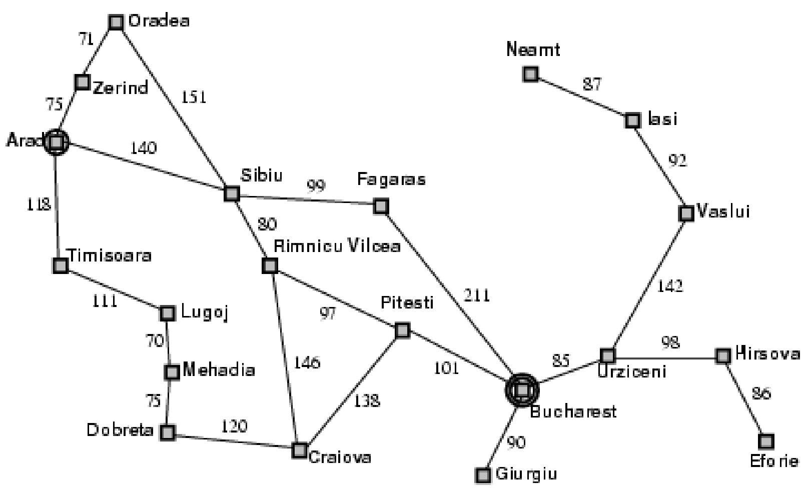
- Efficiency of search will depend on quality of heuristic h(n)

A* search example

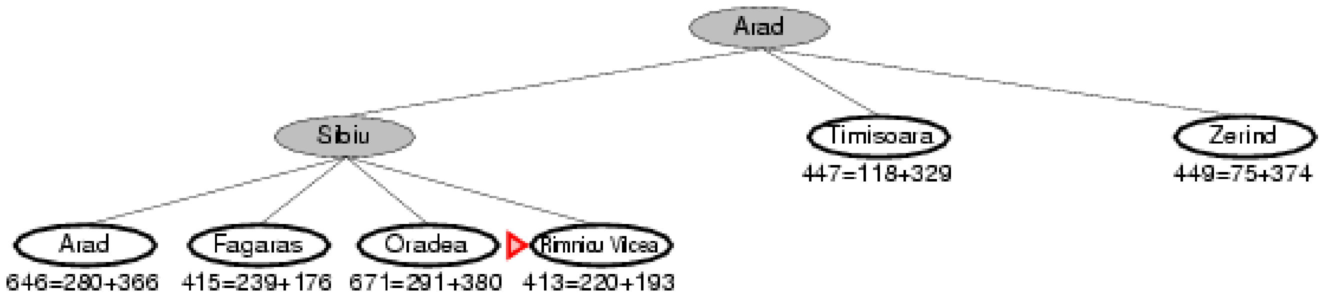


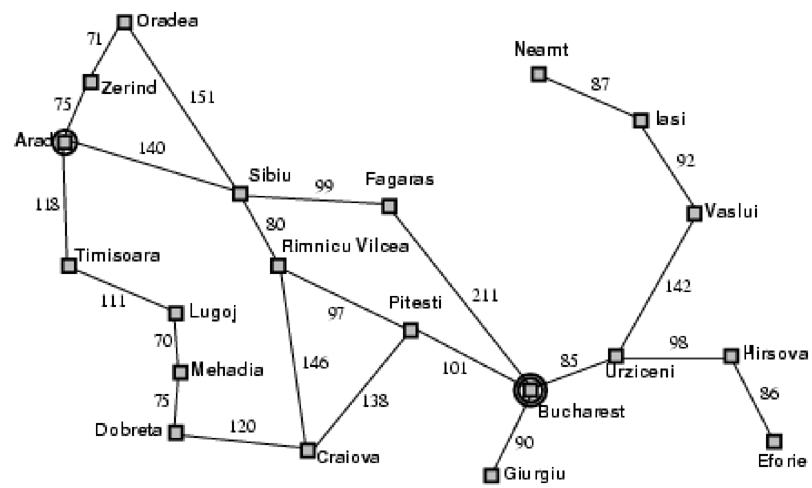
A* search example



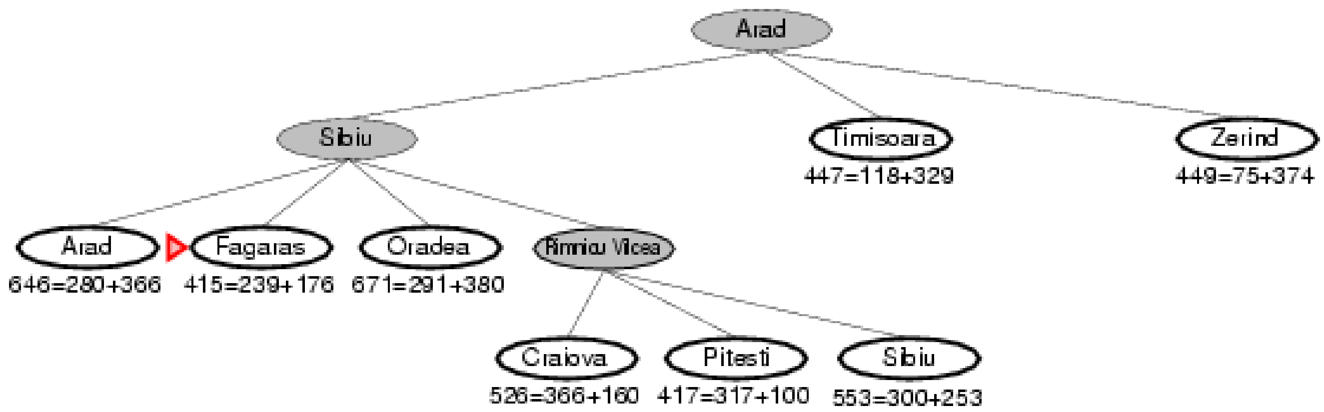


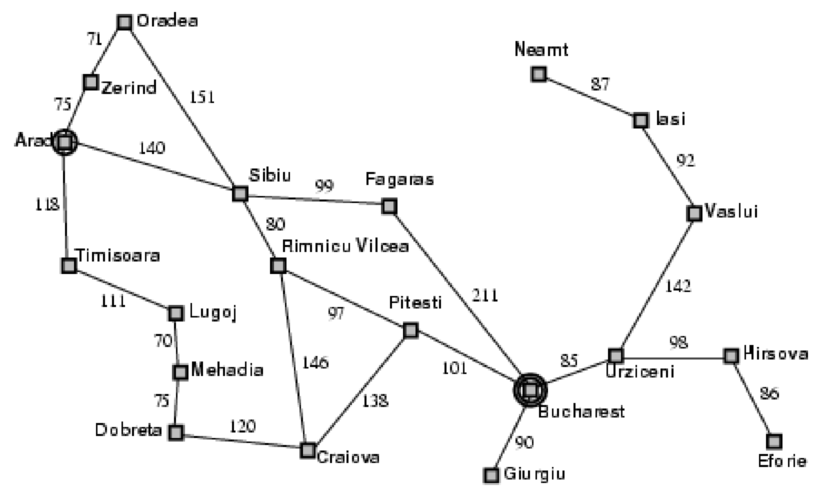
A* search example



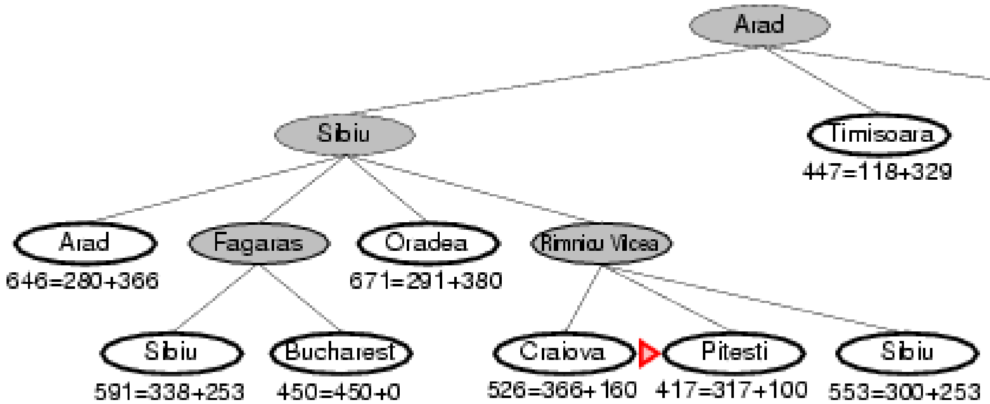


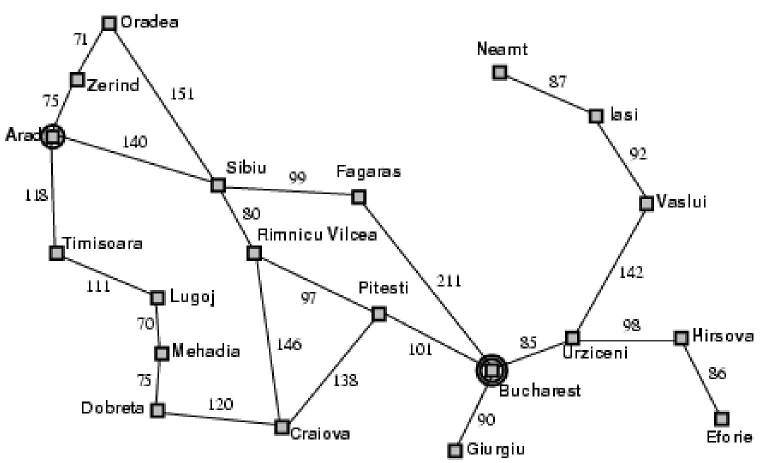
A* search example



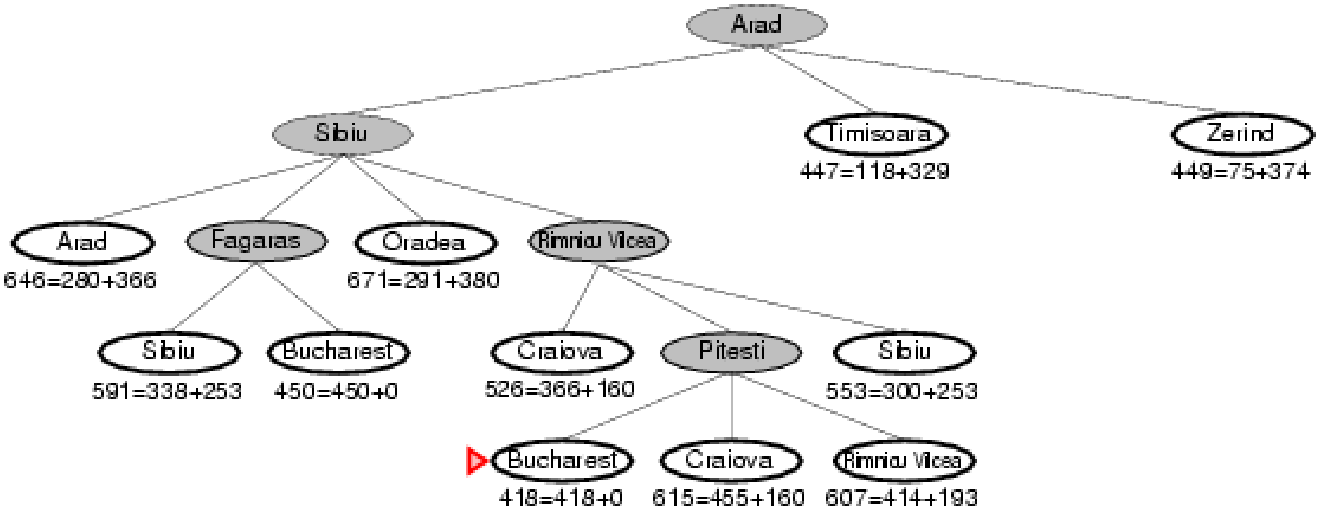


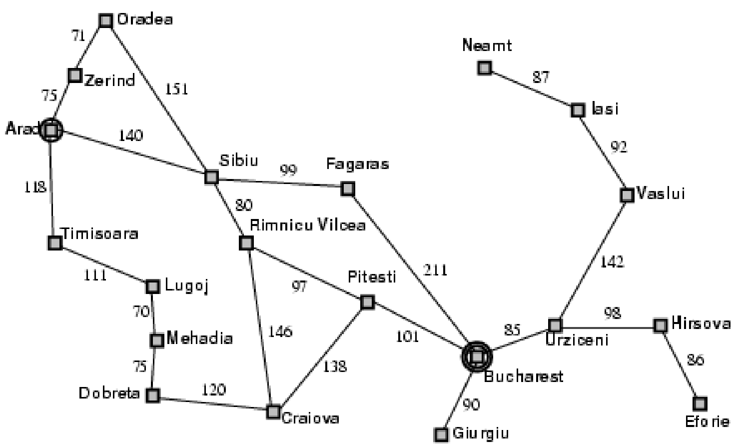
A* search example





A* search example





Admissible heuristics

- A heuristic $h(n)$ is admissible if for every node n,
  $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach the goal state from n.

- An admissible heuristic never overestimates the cost to reach the goal, i.e., it is optimistic

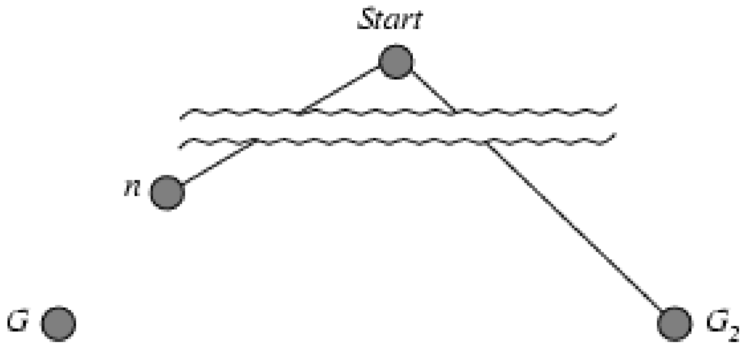
- Example: $h_{SLD}(n)$ *is admissible*
  - never overestimates the actual road distance

- Theorem:

  If $h(n)$ is admissible, A* using TREE-SEARCH is optimal

Optimality of A* (proof)

- Suppose some suboptimal goal G_2 has been generated and is in the fringe. Let n be an unexpanded node in the fringe such that n is on a shortest path to an optimal goal G.



- $f(G_2) = g(G_2)$ since $h(G_2) = 0$
- $g(G_2) > g(G)$ since G_2 is suboptimal
- $f(G) = g(G)$ since $h(G) = 0$
- $f(G_2) > f(G)$ from above

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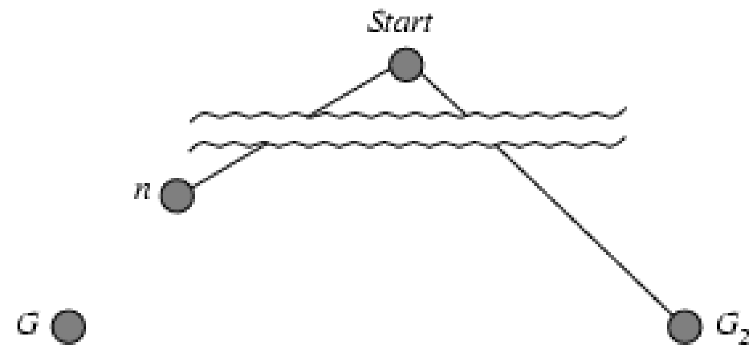


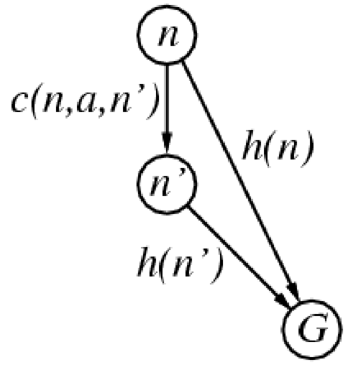
- $f(G_2)$ $> f(G)$ from above
- $h(n)$ $\leq h^*(n)$ since h is admissible
- $g(n) + h(n)$ $\leq g(n) + h^*(n)$
- $f(n)$ $\leq f(G)$

Hence $f(G_2) > f(n)$, and A* will never select G_2 for expansion

Optimality for graphs?

- Admissibility is not sufficient for graph search
  - In graph search, the optimal path to a repeated state could be discarded if it is not the first one generated
  - Can fix problem by requiring consistency property for h(n)

- A heuristic is consistent if for every successor *n'* of a node *n* generated by any action *a*,

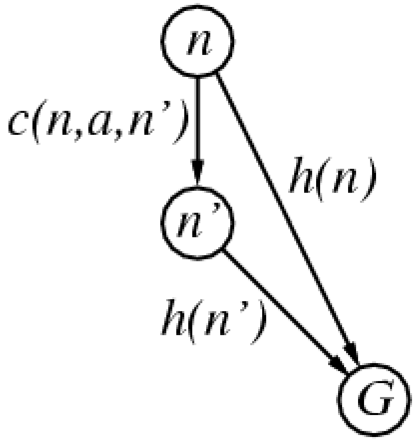


$$h(n) \leq c(n,a,n') + h(n')$$

(aka "monotonic")

- admissible heuristics are generally consistent

A* is optimal with consistent heuristics



- If *h* is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n,a,n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

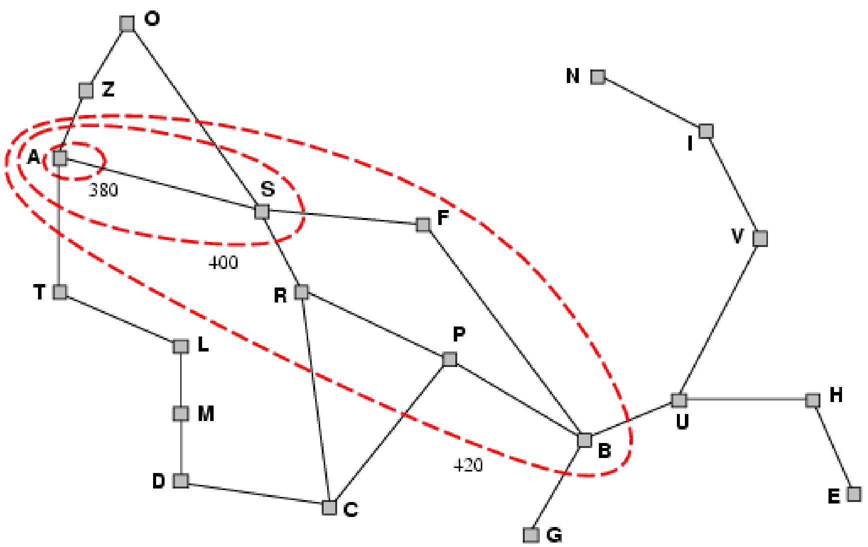
i.e., *f(n)* is non-decreasing along any path.

Thus, first goal-state selected for expansion must be optimal

- Theorem:
  - If *h(n)* is consistent, A* using GRAPH-SEARCH is optimal
  -

Contours of A* Search

- A* expands nodes in order of increasing *f* value
- Gradually adds "*f*-contours" of nodes
- Contour *i* has all nodes with $f = f_i$, where $f_i < f_{i+1}$



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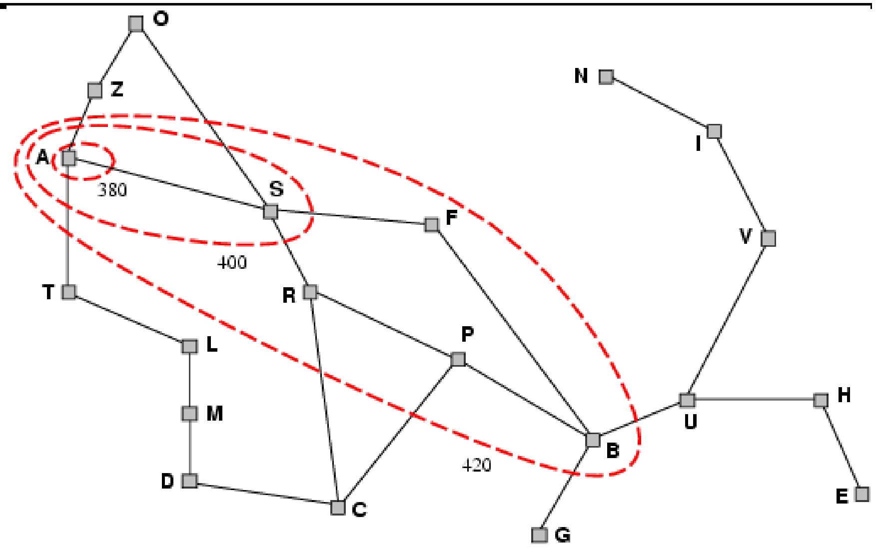


- With uniform-cost (h(n) = 0, contours will be circular
- With good heuristics, contours will be focused around optimal path
- A* will expand all nodes with cost f(n) < C*

Properties of A*

- Complete?
  - Yes (unless there are infinitely many nodes with f ≤ *f(G)*)
- Optimal?
  - Yes
  - Also optimally efficient:
    - No other optimal algorithm will expand fewer nodes, for a given heuristic
- Time?
  - Exponential in worst case
- Space?
  - Exponential in worst case

Comments on A*

- A* expands all nodes with f(n) < C*
  - This can still be exponentially large

- Exponential growth will occur unless error in h(n) grows no faster than log(true path cost)
  - In practice, error is usually proportional to true path cost (not log)
  - So exponential growth is common

Memory-bounded heuristic search

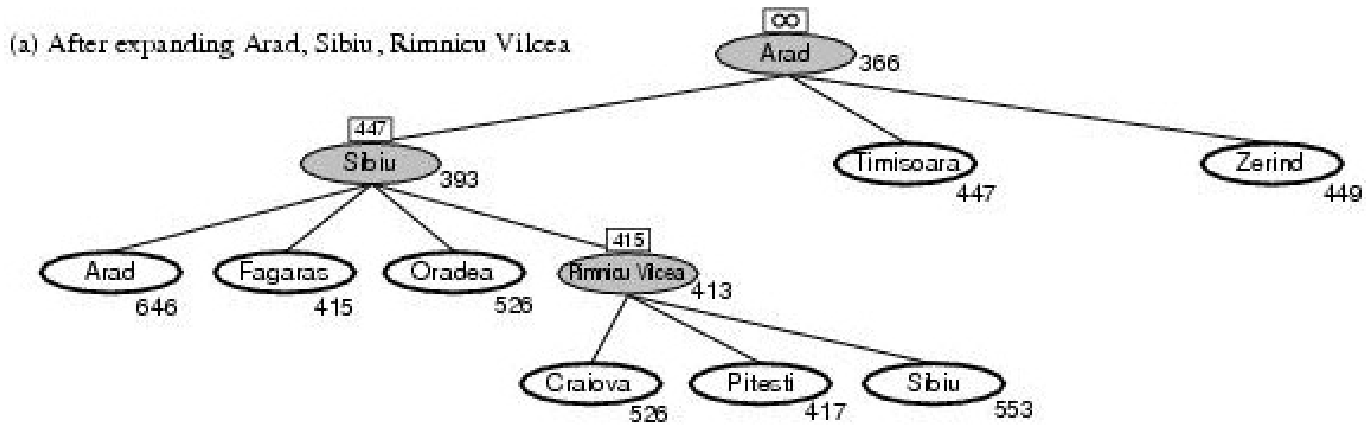
- In practice A* runs out of memory before it runs out of time
  - How can we solve the memory problem for A* search?

- Idea: Try something like depth first search, but let’s not forget everything about the branches we have partially explored.

Recursive Best-First Search (RBFS)

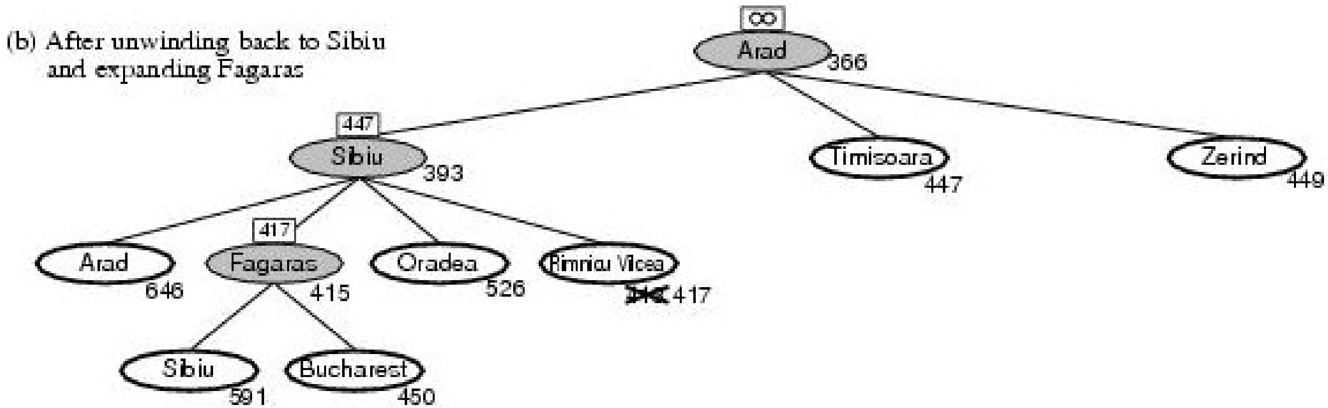
- Similar to DFS, but keeps track of the f-value of the best alternative path available from any ancestor of the current node
- If current node exceeds f-limit -> backtrack to alternative path
- As it backtracks, replace f-value of each node along the path with the best f(n) value of its children
  - This allows it to return to this subtree, if it turns out to look better than alternatives

Recursive Best First Search: Example



- Path until Rumnicu Vilcea is already expanded
- Above node; *f*-limit for every recursive call is shown on top.
- Below node: *f(n)*
- The path is followed until Pitesti which has a *f*-value worse than the *f-limit*.

RBFS example



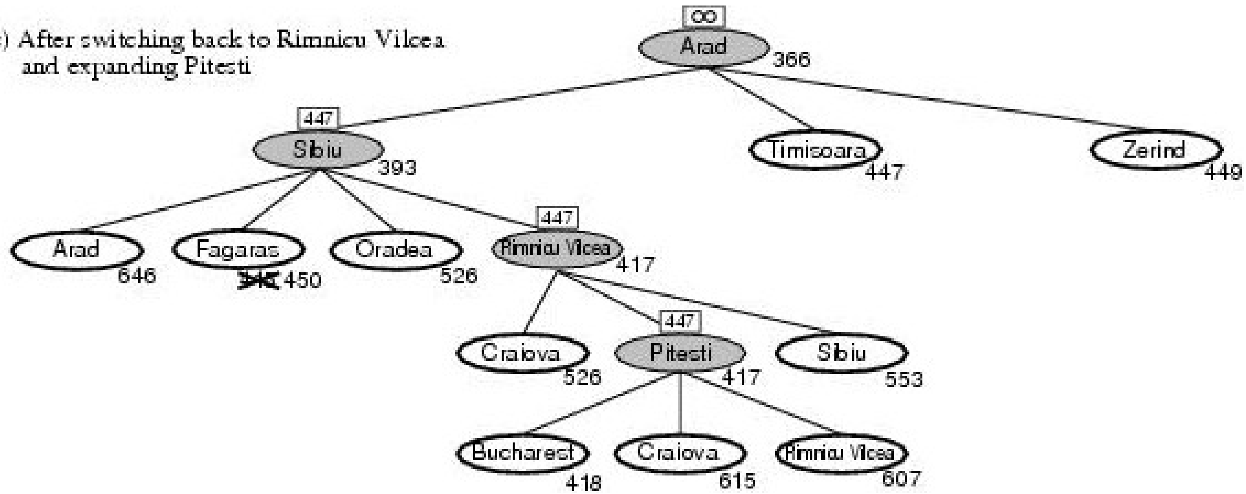
- Unwind recursion and store best *f*-value for current best leaf Pitesti

  result, *f* [*best*] ← RBFS*(problem, best,* min(*f_limit, alternative*))

- *best* is now Fagaras. Call RBFS for new *best*
  - *best* value is now 450

RBFS example

(c) After switching back to Rimnicu Vilcea
and expanding Pitesti



- Unwind recursion and store best *f*-value for current best leaf Fagaras
  result, *f* [*best*] ← RBFS*(problem, best,* min(*f_limit, alternative*))

- *best* is now Rimnicu Viclea (again). Call RBFS for new *best*
  - Subtree is again expanded.
  - Best *alternative* subtree is now through Timisoara.

- Solution is found since because 447 > 418.

RBFS properties

- Like A*, optimal if *h(n)* is admissible

- Time complexity difficult to characterize
  - Depends on accuracy if h(n) and how often best path changes.
  - Can end up “switching” back and fortyh

- Space complexity is *O(bd)*
  - Other extreme to A* - uses ***too little*** memory.

(Simplified) Memory-bounded A* (SMA*)

- This is like A*, but when memory is full we delete the worst node (largest f-value).

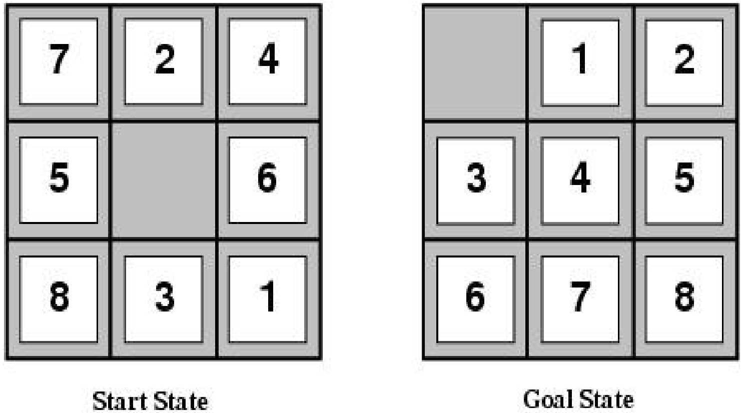
- Like RBFS, we remember the best descendant in the branch we delete.

- If there is a tie (equal f-values) we delete the oldest nodes first.

- simplified-MA* finds the optimal *reachable* solution given the memory constraint.

- Time can still be exponential.

Heuristic functions



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Notion of dominance

- If $h_2(n) \geq h_1(n)$ for all n (both admissible) then h_2 dominates h_1

  h_2 is better for search

- Typical search costs (average number of nodes expanded) for 8-puzzle problem

  d=12 IDS = 3,644,035 nodes
  $A^*(h_1)$ = 227 nodes
  $A^*(h_2)$ = 73 nodes

  d=24 IDS = too many nodes
  $A^*(h_1)$ = 39,135 nodes
  $A^*(h_2)$ = 1,641 nodes

Effective branching factor

- Effective branching factor b*
  - Is the branching factor that a uniform tree of depth *d* would have in order to contain *N+1* nodes.

$$N + 1 = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$$

  - Measure is fairly constant for sufficiently hard problems.
    - Can thus provide a good guide to the heuristic’s overall usefulness.

Effectiveness of different heuristics

	Search Cost			Effective Branching Factor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	–	539	113	–	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

- Results averaged over random instances of the 8-puzzle

Inventing heuristics via "relaxed problems"

- A problem with fewer restrictions on the actions is called a relaxed problem

- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem

- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution

- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution

- Can be a useful way to generate heuristics
  - E.g., ABSOLVER (Prieditis, 1993) discovered the first useful heuristic for the Rubik's cube puzzle

More on heuristics

- h(n) = max{ h1(n), h2(n),……hk(n) }
  - Assume all h functions are admissible
  - Always choose the least optimistic heuristic (most accurate) at each node
  - Could also learn a convex combination of features
    - Weighted sum of h(n)'s, where weights sum to 1
    - Weights learned via repeated puzzle-solving

- Could try to learn a heuristic function based on "features"
  - E.g., x1(n) = number of misplaced tiles
  - E.g., x2(n) = number of goal-adjacent-pairs that are currently adjacent
  - $h(n) = w_1\ x1(n) + w_2\ x2(n)$
    - Weights could be learned again via repeated puzzle-solving
    - Try to identify which features are predictive of path cost

Summary

- Uninformed search methods have their limits

- Informed (or heuristic) search uses problem-specific heuristics to improve efficiency
  - Best-first
  - A*
  - RBFS
  - SMA*
  - Techniques for generating heuristics

- Can provide significant speed-ups in practice
  - e.g., on 8-puzzle
  - But can still have worst-case exponential time complexity

- Next lecture: local search techniques
  - Hill-climbing, genetic algorithms, simulated annealing, etc