Fundamental theorem of Arithmetic

In sorted order
Only one
is possible

$N = P_1 \cdot P_2 \cdot P_3 \cdots\cdots P_K$

$\left[\text{Ex: } 12 = 2 \cdot 2 \cdot 3 \right]$

$= 3 \cdot 2 \cdot 2$ ✗

all are prime

to avoid this
consider sorted order

$P_1 \le P_2 \le P_3$

Proof by Contradiction

Let,

$N = P_1 \cdot P_2 \cdot P_3 \cdots\cdots P_K$

$N = q_1 q_2 q_3 \cdots\cdots q_K$

need to proove $P_1 \neq q_1$

$\downarrow \quad \downarrow$

$ax + by = c$ এর
solution থাকার শর্ত:-

$\gcd(a, b) \mid c$

from extended
euclidean algorithm

Now,

$ap_1 + bq_1 = 1$ → have a solution

$ap_1 [q_2 \cdots q_K] + bq_1 [q_2 \cdots q_K] = [q_2 \cdots q_K]$

$ap_1 [q_2 \cdots q_K] + bN = [q_2 \cdots q_K]$

$\downarrow$

$P_1 \mid q_2 \cdots q_K$ Or in other case $q_1 \mid P_2 \cdots\cdots P_K$

As it not possible, So

[$p_1 < q_1$ হলে, p_1 $(q_2 \dots q_k)$ এর মধ্যে থাকতে পারবে না]

Similar for $q_1 < p_1$

Proof prime# not finite

Let,

$\Rightarrow$ $P = \{P_1, P_2, P_3, \dots\dots, P_N\}$ $\rightarrow$ set of all primes

$$N = P_1 \cdot P_2 \cdot P_3 \dots\dots P_k + 1$$

So, we get $N > P_k$

thus N is a prime

but not present in that

Set,

Contradiction!!!

$E_p(N)$ = the highest power of p in the number N

$12 = 2^2 \cdot 3$

$E_2(12) = 2$

$E_3(27) = 3$

Now, $E_2(N!) = \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{4} \right\rfloor + \left\lfloor \frac{N}{8} \right\rfloor + \left\lfloor \frac{N}{16} \right\rfloor + \ldots\ldots$

$N! = 1.2.3.4\ldots\ldots N$

$\quad\quad\;\; \downarrow \quad \downarrow$

$\quad\quad\;\; 0 \quad\; 0 \;\;\ldots\ldots$

In General,

$$E_p(N!) = \left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{N}{p^2} \right\rfloor + \left\lfloor \frac{N}{p^3} \right\rfloor + \ldots\ldots$$

$$E_2(N!) = N - \left(\text{num of set bits in } N\right) = N - V_2(N)$$

close form →
sin $E_2(N!)$ দেখাব
$E_p(N!)$ নিজের জন্য

$E_2(10!) = 5+2+1+0$

$\downarrow$

1010

$\downarrow /2$

101

$\downarrow /2$

10

$\downarrow /2$

1

$101 = 5$, $\;1 \to 2^0$

$10 = 2$

$1 = 1$

$\downarrow$

$2^2 + 2^1 + 2^0$

$$\underbrace{2^2+2^1+2^0}_{2^3-1} + \underbrace{2^0}_{2^1-1} = 4+4 = 8$$

$E_2(101010!)$

$\downarrow \quad\quad \downarrow \quad\quad \downarrow$

$2^5-1 \quad 2^3-1 \quad 2^1-1$

$= 2^5+2^3+2^1-3$

$= N-3$

#

$$\epsilon_p(N!) = \left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{N}{p^2} \right\rfloor + \left\lfloor \frac{N}{p^3} \right\rfloor + \cdots\cdots$$

(as যেহেতু floor নিচ্ছি)

$$\leq \left(\frac{N}{p}\right) + \left(\frac{N}{p^2}\right) + \left(\frac{N}{p^3}\right)$$

$$= N\left(\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right)$$

$$= \frac{N}{p-1}$$

1 .... N > কয়টা Prime আছে

$\Rightarrow \dfrac{N}{\ln N}$ → exact answer না, estimate

→ N^{th} prime কত বড় (Size) হবে?

$\Rightarrow$ Let N^{th} prime $= X$

$1 \ldots\ldots X = N$ টা Prime

So, $\dfrac{X}{\ln X} = N$

around $N . \ln N$

→ একটা Number N হলে কয়টা divison?

$\Rightarrow \sqrt[3]{N}$

$1 - 10^9$ এর মধ্যে

1330 টি divison হচ্ছে

STERN-BROCOT TREE

দুইটা fraction আনো

$\frac{0}{1}$ $\qquad$ $\frac{0+1}{1+0} = \frac{1}{1}$ $\qquad$ $\frac{1}{0}$

In general:

$$\boxed{\frac{m}{n} \quad \frac{m'}{n'} \quad \rightarrow \quad \frac{m+m'}{n+n'}}$$

$\frac{1}{2}$ $\qquad$ $\frac{2}{1}$

$\frac{1}{3}$ $\quad$ $\frac{2}{3}$ $\quad$ $\frac{3}{2}$ $\quad$ $\frac{3}{1}$

$\frac{1}{4}$ $\quad$ $\frac{2}{5}$ $\quad$ $\frac{3}{5}$ $\quad$ $\frac{5}{3}$ $\quad$ $\frac{5}{2}$ $\quad$ $\frac{4}{1}$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

always sorted

Observation

1) কোনো fraction কাটাকাটি যাচ্ছে না
2) কোনো fraction 2 বার আসবে না
3) সব rational # একবার আসবেই (কোনো fraction missing নাই)

next class

2nd Class

1) কোনো fraction কাটাকাটি যাচ্ছে না

↳ already in reduced format

need to prove : $\gcd(m,n)=1$

Now: already $\gcd = 1$

$$\frac{m}{n} \qquad \frac{m'}{n'}$$

$$\frac{m+m'}{n+n'}$$

always follow a property:

$$\boxed{m'n - mn' = 1}$$

$m = 0$, $n = 1$ $\qquad$ $m' = 1$, $n' = 0$

$$\frac{0}{1} \qquad \frac{1}{0}$$

$$\frac{1}{1}$$

$$\frac{1}{2}$$

$$2\cdot 1 - 1\cdot 1 = 1$$

So following the property:

$$n(m+m') - m(n+n') = 1$$

$$n\cdot a - m\cdot b = 1$$ → Soln থাকবে যখন $\gcd(a,b)=1$

So, $\gcd(m+m', n+n') = 1$

2) কোনো fraction 2 বার আসবে না

$\Rightarrow$ Let,

$$\frac{m}{n} < \frac{m'}{n'}$$

$$\frac{m}{n} < \frac{m'}{n'}$$

$mn' < m'n$

$\left[\frac{m}{n} < \frac{m'}{n'}\right]$

So, we need to prove

$$\frac{m}{n} < \frac{m+m'}{n+n'} < \frac{m'}{n'}$$

$m(n+n') < n(m+m')$

$\cancel{mn} + mn' < \cancel{mn} + m'n$

$mn' < m'n$

need to write in reverse to prove

as strictly increasing, So কোনো fraction 2 বার exist করা possible না [Proved]

3) সব rational # একবার আসবেই

(কোনো fraction missing নাই)

$\Rightarrow$ $\frac{a}{b}$ মাঝে কোথাও আছে,

$$\frac{m}{n} < \frac{a}{b} < \frac{m'}{n'}$$

Let, $\frac{m}{n}$ $\quad \frac{m+m'}{n+n'} \quad$ $\frac{m'}{n'}$

$\frac{a}{b}$ continous recursively খুঁজে বের করব।

need to prove recursive step finite হবে।

$\Rightarrow \frac{m}{n} < \frac{a}{b}$

$mb < an$

$mb - an < 0$

$an - mb > 0$

$an - mb \geq 1$ [as all are integers]

$* (m'+n')$

Similarly

$\frac{a}{b} < \frac{m'}{n'}$

$an' < m'b$

$an' - m'b < 0$

$m'b - an' > 0$

$m'b - an' \geq 1$

$* (m+n)$

$(an - mb) * (m'+n') \geq (m'+n') \quad$ — (i)

$(m'b - an') * (m+n) \geq (m+n) \quad$ — (ii)

(i) + (ii) $\Rightarrow$

$(an - mb) * (m'+n') + (bm' - an') * (m+n) \geq m' + n' + m + n$

$$\Rightarrow an(m'+n') - an'(m+n) + bm(-m'-n') + bm'(m+n) \geq m'+n'+m+n$$

$$\Rightarrow a\{m'n + \cancel{nn'} - mn' - \cancel{nn'}\} + b\{-\cancel{mm'} - mn' + \cancel{mm'} + m'n\} \geq \ldots$$

$$\Rightarrow a\{\underbrace{m'n - mn'}_{1}\} + b\{\underbrace{m'n - mn'}_{1}\} \geq m'+n'+m+n$$

$$\Rightarrow a+b \geq \underbrace{m'+n'+m+n}_{\text{Initially } = 2}$$

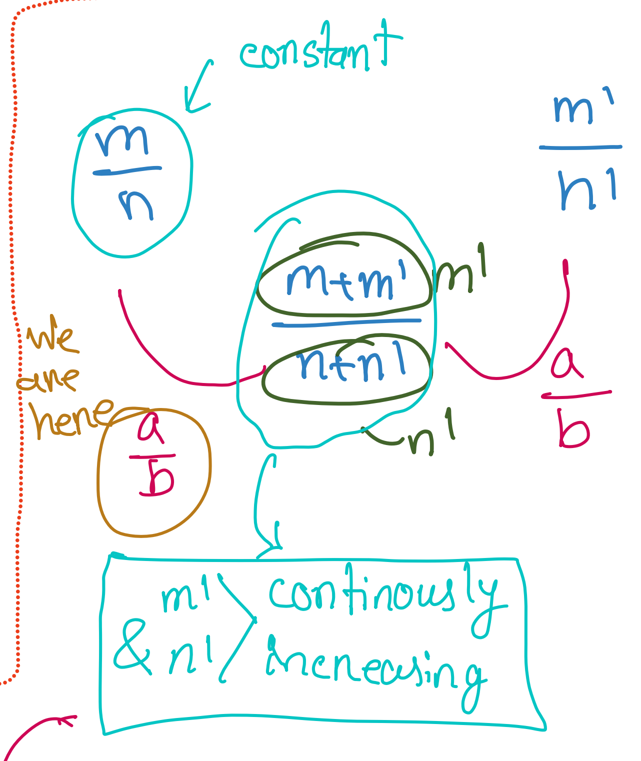
as $\left(\frac{0}{1} \ \& \ \frac{1}{0}\right)$

$\frac{m}{n}$ adjacent $\frac{m'}{n'}$

$m'n - mn' = 1$

So, we can tell, we can find $\frac{a}{b}$ by $(a+b-2)$ step.

So, finite step $[\text{Proved}]$



So, $\frac{m+m'}{n+n'}$ is increasing

$a+b \not\geq m'+n'+m+n$

Cz তখন

$\frac{m}{n} < \frac{a}{b} < \frac{m'}{n'}$

condition true হবে না

Modular Arithmetic

$a \equiv b \pmod{m} \xrightarrow{\text{means}} a \% m = b$ ······ (1)

$\longrightarrow (a-b) \% M = 0$ ······ (2)

(1) $a + c \equiv b + c \pmod{m}$ ✓

So, using 2, $a + c - b - c = (a-b)$

divisible by M

(2) $ad = bd \pmod{m}$ ✓

as using 2, $(ad - bd) = d(a-b)$ $[d > 1]$

(3) $\frac{a}{d} = \frac{b}{d} \pmod{m}$ ✗

$6 \equiv 2 \pmod{4}$

$3 \not\equiv 1 \pmod{\underset{2}{4}}$

It will be true, if we do it

So, then,

$3 \equiv 1 \pmod{2}$

division এর way:-

$ad \equiv bd \pmod{m}$

$a \equiv b \pmod{m}$

possible when, m & d are coprime

cz: $(ad-bd)$ is divisible by m

$d(a-b)$ is " " m

m divides only $(a-b)$, as $(m$ & $d) \to$ coprime

New Class

Residue Number System

$$M = \{m_1, m_2, m_3, \ldots\ldots, m_k\}$$ all are prime

$$X = \{X \% m_1, X \% m_2, X \% m_3, \ldots\ldots, X \% m_k\}$$

Ex: Let, $M = \{2, 3, 5\}$

$$17 = \{17 \% 2, 17 \% 3, 17 \% 5\} = \{1, 2, 2\}$$

এগুলো দিয়ে represent করতে পারি Given M

Residual number allow to represent a large number by some small numbers

Representation unique হতে হবে, মানে

$\{1, 2, 2\}$ থেকে only 17 ই Produce করা যাবে

$$0 \equiv \{0, 0, 0\}$$
$$30 \equiv \{0, 0, 0\}$$
Problem

consider if we represent 0 ..... 29 uniquely

$\Rightarrow$ $M = \{m_1, m_2, m_3 \ldots\ldots m_k\}$

$P = m_1 . m_2 . m_3 . \ldots\ldots m_k$

need to prove, every representation is distinct

→ Proof by contradiction,

$\Rightarrow$ Let,

$0 \leq Y < Z \leq P-1$ [Y ও Z এর representation same]

$y_i \equiv y \mod m_i$

$z_i \equiv z \mod m_i$

$0 \equiv (y-z) \mod m_i$

So, m_i divides $(z-y)$

$\Rightarrow m_i \mid (z-y)$

As $m_i \mid (z-y)$ (as all are prime)

So, $\underbrace{m_1 m_2}_{1} \mid (z-y)$

#

$P \mid \underbrace{(z-y)}_{0 \ldots\ P-1} \leq P-1$ → $(z-y) > 0$

So, we get $P \mid (z-y)$

but this not possible

as $0 \leq Y < Z \leq P-1$

So, contradiction

if $\gcd(a,b) = k$.

then, $\frac{ab}{k} \not\mid$ → always true

cz, common factor গুলো একবার ই নিচ্ছি

Solve: $x^2 \equiv 1 \pmod{p^k}$ $[x \to \text{odd}]$

$$x^2 - 1 \equiv 0 \pmod{p^k}$$

$$(x+1)(x-1) \equiv 0 \pmod{p^k}$$

So, $p \neq 2$ হলে,

$p^k \mid (x-1)$ OR, $p^k \mid (x+1)$

$x - 1 \equiv 0 \pmod{p^k}$ $\quad$ $x \equiv -1 \pmod{p^k}$

$x \equiv 1 \pmod{p^k}$

Now, $p = 2$ হলে,

$$\underbrace{(x+1)}_{x \text{ div by } 4}(x-1) \equiv 0 \pmod{2^k}$$

$p=2$ হলে both $(x+1)$ & $(x-1)$ p দিয়ে divisible, এছাড়া possible না

$k = 1 \Rightarrow x \equiv 1 \pmod 2$

$k = 2 \Rightarrow x \equiv 1, 3 \pmod 4$

$k \geq 3$

$\boxed{\begin{matrix} 1, 5 \\ 3, 7 \end{matrix}} \bmod 8$

if $(x+1)$ is not div. by 4,

$(x+1) = 2^1 \cdot \boxed{}$

$2^{k-1} \mid (x-1)$

এগুলোর square করলেও
1 mod 8 আসবে

9 skip করা
cz, $9 \equiv 1$

$$x - 1 \equiv 0 \pmod{2^{k-1}}$$

$$x \equiv 1 \bmod 2^{k-1}$$

if $(x-1)$ is not div by 4,

(Similar)

at last > $x \equiv -1 \pmod{2^{k-1}}$

↳ মানে 3 পাওয়া

FERMAT'S LITTLE Theorem

fermat এর আরেকটা theorem আছে,

$$a^n + b^n \neq c^n \quad [n > 2]$$

$$a^{p-1} \equiv 1 \pmod{p}$$

$a, p \rightarrow$ coprime

$p \rightarrow$ prime

Let,

$$K_1 \cdot n \equiv x \pmod{p}$$
$$K_2 \cdot n \equiv x \pmod{p}$$

K_1, K_2, $\{1 \ldots (p-1)\}$ এর মধ্যে যেকোনো number

$$(k_1 - k_2) \cdot n \equiv 0 \pmod{p}$$

$$p \nmid (k_1 - k_2) \cdot n$$

$n = a$ দেওয়া উচিত ছিলো → Sir bolche

$p, n \rightarrow$ coprime

p doesn't dived either $(k_1 - k_2)$ on n

$(k_1 - k_2)$ p থেকে ছোট

→ So $k_1 \cdot n \equiv x \pmod{p}$ হলে
$k_2 \cdot n \not\equiv x \pmod{p}$ হবে

$n \cdot 2n \cdot 3n \cdot 4n \ldots\ldots (p-1)n \equiv (p-1)! \pmod{p}$

p দিয়ে mod করলে

$1 \ldots (p-1)$ এর permutation হবে, So,

$$n^{p-1} \cdot (p-1)! \equiv (p-1)! \mod p$$

$$\Rightarrow n^{p-1} \equiv 1 \mod p$$

$ad \equiv bd \mod p$

$\rightarrow a \equiv b \pmod{p}$ হবে যদি

$b, d \rightarrow$ coprime হয়

New class

$\phi(n)$

= How many numbers from 1 to n are coprime with n

$\phi(6) = 1 + 1 = 2$

$(1,6)$ $(5,6)$

$\phi(p) = p - 1$

# of p এর multiple $= \frac{p^k}{p} = p^{k-1}$

$\phi(p^k) = 1, 2, 3, 4, \ldots, p, \ldots, p^2, \ldots, p^3, \ldots, p^k$

$2p, 3p \ldots$ $(p+1)p$

$= p^k - p^{k-1}$

x

$1 \ldots\ldots N$

$\left\lfloor \frac{N}{x} \right\rfloor$

$\gcd(p^k, x)$ ← যখন x, p^k coprime না

$= p^y$

$\phi(n)$ is multiplicative function

$$f(ab) = f(a) \cdot f(ab) \; ; \; a \perp b$$

coprime

Now,

$$\boxed{\phi(m.n) = \phi(m) . \phi(n)} \quad [m \perp n]$$

$$\phi(17) = 16$$

$$\phi(15) = \phi(3) . \phi(5)$$
$$= 2 . 4 = 8$$

Prove: $\phi(m.n) = \phi(m) \cdot \phi(n) \; ; \; m \perp n$

$\Rightarrow \quad 1, 2, 3, 4, \ldots\ldots\ldots, mn$

$\gcd(x, mn) = 1$

$\gcd(x, m) = 1$

&

$\gcd(x, n) = 1$

Row-1 $\quad 0.m+1 \quad 1.m+1 \quad 2.m+1 \quad \cdots\cdots \quad (n-1)m+1$

Row-2 $\quad 0.m+2 \quad 1.m+2 \quad 2.m+2 \quad \cdots\cdots \quad (n-1)m+2$

$\vdots$

Row-m $\quad 0.m+m \quad 1.m+m \quad 2.m+m \quad \cdots\cdots \quad (n-1)m+m$

Now, for Row-n,

$$\underline{0.m+n \quad 1.m+n \quad 2.m+n \quad \cdots\cdots \quad (n-1)m+n}$$

$$k.m+n \; ; \; 0 \le k \le n-1$$

$$\gcd(km+n, m)$$
$$= \gcd(n, m)$$

$$\boxed{\gcd(a,b) = \gcd(a\%b, b)}$$

So, $\gcd(n,m) > 1$ হলে

$\gcd(km+n,m) > 1$ হবে,

পুরো row Count থেকে বাদ যাবে

So, $\gcd(R,m)=1$ এরকম $\boxed{n \times \phi(m)}$ সংখ্যক row আছে

m এর সাথে coprime

এখন n এর সাথেও co-prime কিনা বের করব

কতগুলো n এর সাথে coprime বের করব

Row-R:

$0.m+R \quad 1.m+R \quad 2.m+R \quad \ldots\ldots\ldots \quad (n-1)m+R$

$\gcd(km+R, n)$

$\%n \quad \%n \quad \%n \quad \%n$

- permutation of $[0 \ldots (n-1)]$
- no duplicate

$\#\gcd(km+R, n)$

$= \#\gcd\{(km+R)\%n, n\}$

$= \#\gcd(R, n)$

$= \phi(n)$

$k_1 m + \cancel{R} \equiv k_2 m + \cancel{R} \pmod{n}$

$(k_1 - k_2) m \equiv 0 \pmod{n}$

তখন $k_1 = k_2$ হলো,

তাই, $k_1 m + R \not\equiv k_2 m + R$

এজন্য প্রত্যেক $\%n$ different হবে

- no duplicate

Finally $\phi(n)$ সংখ্যক row

প্রতি row তে $\phi(n)$ সংখ্যক element

তাই total $\phi(mn) = \phi(m)\cdot\phi(n)$ [Proved]

Prove:

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

$$\phi(30) = 30.\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$\phi(12) = 12.\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$$

$$\phi(n) = \phi(p_1^{\alpha_1})\,\phi(p_2^{\alpha_2})\ldots\phi(p_k^{\alpha_k})$$

$$= \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right)\ldots$$

$$= p^{\alpha_1}\left(1 - \frac{1}{p}\right) p_2^{\alpha_2}\left(1 - \frac{1}{p_2}\right)\ldots$$

$$= p_1^{\alpha_1}.p_2^{\alpha_2}\ldots p_k^{\alpha_k}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right)$$

CT upto this

New Class

Sample ct question:

- prove: $\phi(n)$ is always even ; $n>2$

$\Rightarrow$ $n = 2^k$; $k > 1$

$\phi(2^k) = 2^k - 2^{k-1} = 2(2^{k-1} - 2^{k-2})$; even

$n = p^k \cdot n'$ $\rightarrow$ coprime ; p = odd prime

$\phi(n) = \phi(p^k) \cdot \phi(n')$

$= (p^k - p^{k-1})\, \phi(n')$

$= (\text{even})$

$\text{odd}^{\text{odd}} = \text{odd}$

So, odd $-$ odd $=$ even

if $a \mid b$, prove $\phi(a) \mid \phi(b)$

$a \mid b$ $\downarrow$ a divides b

$\Rightarrow$ $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots\cdots p_k^{\alpha_k}$

$b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \cdots\cdots p_k^{\beta_k} \cdot q_1^{\gamma_1} \cdot q_2^{\gamma_2}$

$\beta_i \geq \alpha_i$

$4 \mid 12$

$4 = 2^2$

$12 = 2^2 \cdot 3$

$$\phi(a) = \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right) \cdots\cdots$$

$$\phi(b) = \left(p_1^{\beta_1} - p_1^{\beta_1 - 1}\right)\left(p_2^{\beta_2} - p_2^{\beta_2 - 1}\right) \cdots\cdots \phi\left(q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdots\cdots\right)$$

$\boxed{\text{need to prove } \phi(a) \mid \phi(b)}$

$$\phi(b) = p_1^{\beta_1}\left(1 - \frac{1}{p_1}\right) p_2^{\beta_2}\left(1 - \frac{1}{p_2}\right) \cdots\cdots$$

$$\phi(a) = p_1^{\alpha_1}\left(1 - \frac{1}{p_1}\right) p_2^{\alpha_2}\left(1 - \frac{1}{p_2}\right) \cdots\cdots$$

$$\frac{\phi(b)}{\phi(a)} = p_1^{\beta_1 - \alpha_1} \cdot p_2^{\beta_2 - \alpha_2} \cdots\cdots \phi\left(q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdots\cdots\right)$$

ct

STARS & BARS

★ ★ ★ ★ ★ ★ → 6

n objects (identical)
k bins (not identical)

$\boxed{\begin{array}{l} n=6 \\ k=2 \end{array}}$

Box 1	Box 2
0	6
1	5
2	4
⋮	⋮
6	0

7 ভাবে

k=2 হলে ans = n+1

bar → ⦚ |

different

$\boxed{\begin{array}{l} n=6 \\ k=3 \end{array}}$

_ ★ _ ★ _ ★ | ⦚ | ★ _ ★ _ ★ _

left/right

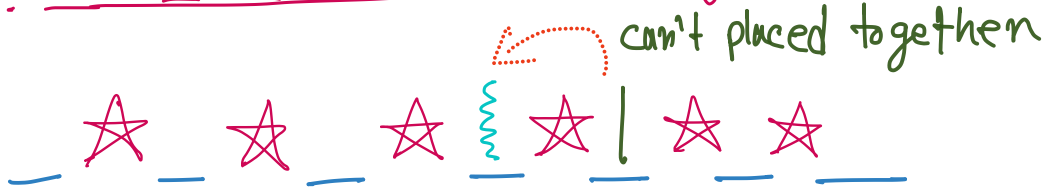
So, way $= \dfrac{7\cdot 8}{2}$

★ | ★★★ | ★★

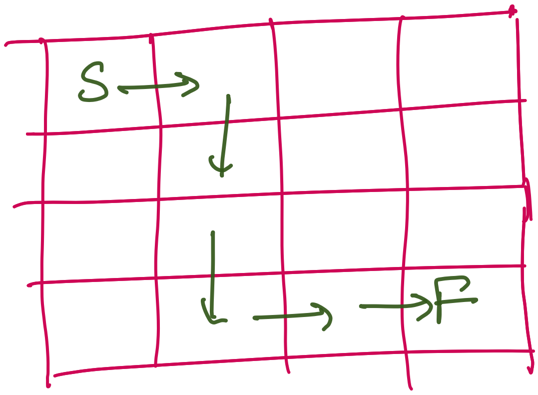
6 star, 2 bar

$$\frac{\cancel{8!}\ 7\cdot 8}{\cancel{6!}\,2!}$$

if we don't allow empty ban:-



So, way = $\frac{7 \cdot 6}{2}$



way to reach F = $\frac{(n+m-2)!}{(n-1)!\,(m-1)!}$

New Class

Catalan Numbers

Balanced Bracket Seq:-

((   )) → এখানে mathmatical term add করে দেখব valid হয় কিনা

(1+(1+1))

Now,

n → opening bracket

n → closing "

$S \to \epsilon \mid (S)S$

for valid bracket sequence

n = 0   1

n = 1   ( )

n = 2   (( )) ( )( )

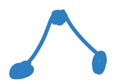
n = 3

Some Problem can be solved by catalan #:

(i) How many binary tree can be constructed with n internal nodes?

⇒ $n=0$

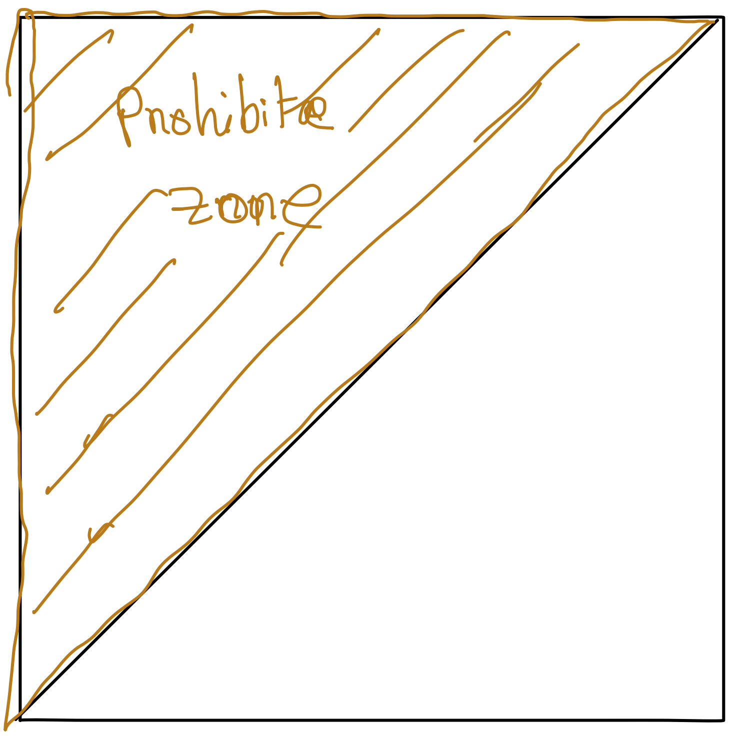
$n=1$



$n=2$

$n=3$

(ii) # of ways to reach from $(0,0)$ to (n,n)



- only possible move → up, right
- do not cross diagonal

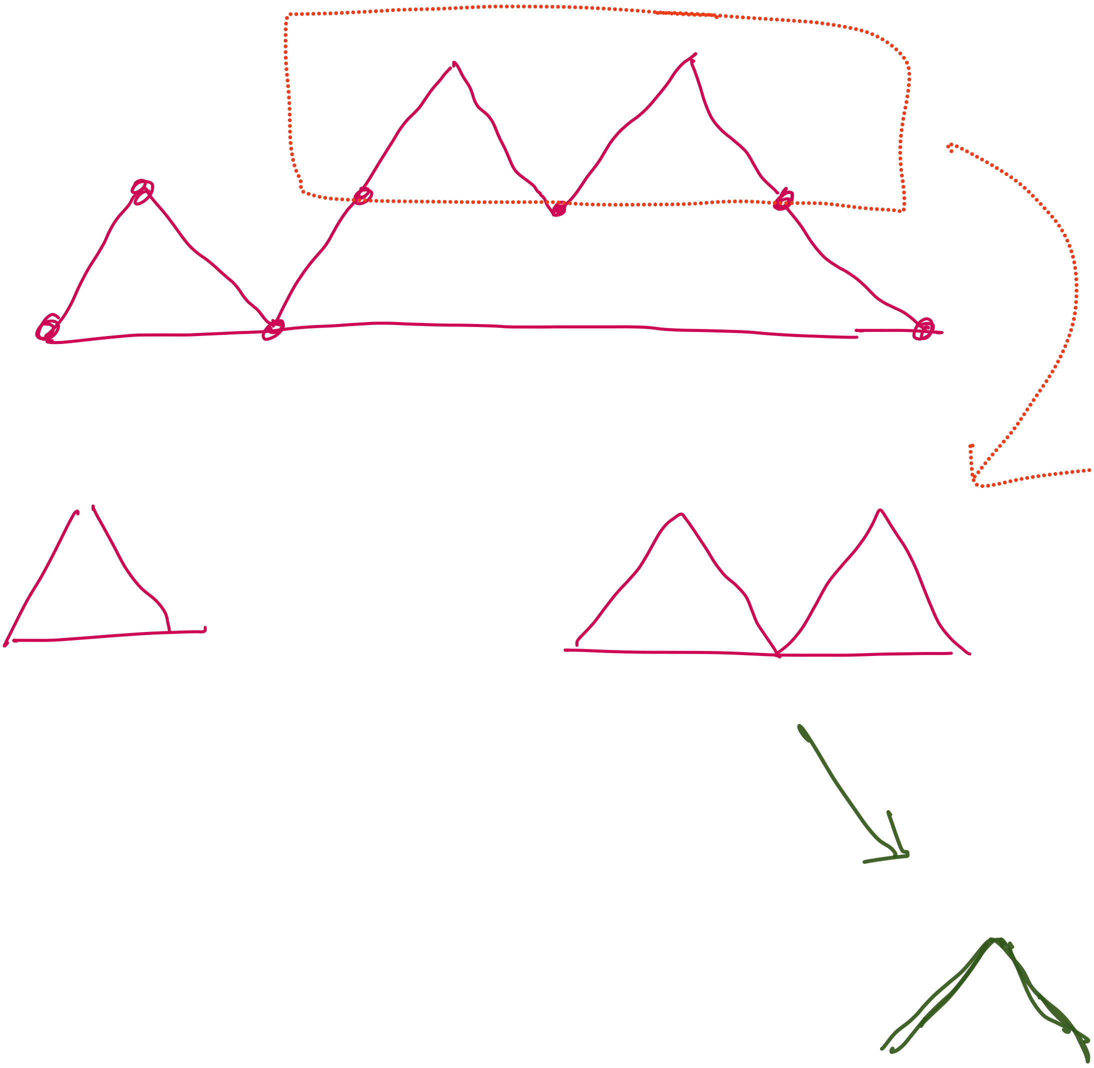
$$\underbrace{\frac{(2n)!}{n!\,n!}}_{\text{total way}} - \underbrace{\frac{(2n)!}{(n+1)!\,(n-1)!}}_{\text{invalid way}}$$

Visit :- https://cses.fi/problemset/task/2064

Soln :- $\frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$

→ Youtube এ video আছে ☺

(ii) mountain range problem



initial Problem :-

$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \dots\dots + C_{n-1}C_0$$

diff. with stars & Bars
as n objects
are non-identical

Stirling Number

New Class

Second kind:

$$\left\{ {n \atop k} \right\} = \text{\# of ways to divide } n \text{ items into } k \text{ partitions}$$

$\{1, 2, 3, 4\}$

$\{1,2\}, \{3,4\} = \{3,4\}, \{1,2\}$

same partition

Ex: $\{1,2,3,4\}$

$n=2$, $k=2$

$\left.\{1\}\ \{2\}\right\}$ 1 way

$n=3$, $k=2$

$$\left.\begin{array}{l} \{1\}\ \{2,3\} \\ \{1,2\}\ \{3\} \\ \{1,3\}\ \{2\} \end{array}\right\} \text{3 way}$$

$n=4$, $k=2$

$$\left.\begin{array}{l} \{1\}\ \{2,3,4\} \\ \{1,2\}\ \{3,4\} \\ \{1,3\}\ \{2,4\} \\ \{1,4\}\ \{2,3\} \\ \{1,2,3\}\ \{4\} \\ \{1,3,4\}\ \{2\} \\ \{1,2,4\}\ \{3\} \end{array}\right\} \text{7 way}$$

Set→1 : 1 থাকবে

Set→2 : 1 থাকবে না

{1, } { }

1 বাদে বাকি (n-1) item এর জন্য way 2^{n-1}, কারণ হয় set-1 অথবা set-2 তে যাবে।

আর at last empty set allow করছি না

~~{1, 2, 3, 4} { }~~

So, way, $2^{n-1} - 1$

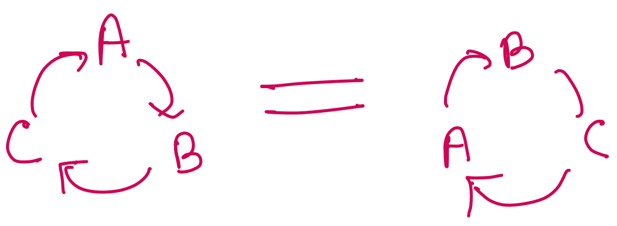
$$\frac{2^n - 2}{2}$$

different চিন্তা ২

-2, কারণ 2 টা empty set থাকে

2 দিয়ে ভাগ কারণ symmetric

[A, B, C, D, E]



$$\begin{bmatrix} n \\ k \end{bmatrix} = n \text{ cycle } k$$ = # of ways to partition n items into k cycles

$$\begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$$

n item থেকে k cycle

$$\begin{bmatrix} n \\ k \end{bmatrix} \geq \left\{ {n \atop k} \right\}$$

$$\begin{bmatrix} n \\ n \end{bmatrix} = \left\{ {n \atop n} \right\} = 1$$

$\{a, b\}$ দিয়ে একটাই cycle বানানো যায়

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = {}^nC_2$$

n item থেকে n-1 টা cycle এ divide করব, 1 টা cycle এ only 2 টা থাকবে, বাকিগুলোতে 1 টা করে থাকবে

cycle এর arrangement matter করছে না, only partition matter করছে

একটা set এ যে 2 জন থাকবে, তাদের choose করার পর arrangement fixed হয়ে যাবে, তাই ${}^nC_2$

2nd kind :-

$$\underbrace{1, 2, 3, 4, \ldots\ldots, n-1, n}_{k \text{ sets}}$$

আগে n-1 কে ভাগ করব

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \cdot k$$

case-1 : n^{th} item নিজেই একটা set তাহলে বাকি থাকবে $(k-1)$ set

case-2 : n^{th} item অন্য কারো সাথে shared হতে পারে। তখন 1st এ $(n-1)$ item কে k সংখ্যক set এ ভাগ করব। তারপর k set এর যেকোনো একটাতে n^{th} item কে insert করব

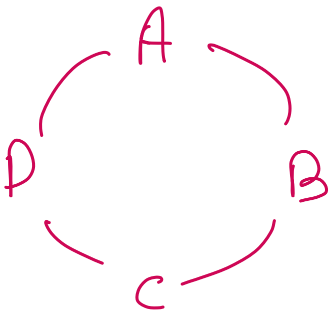
1st kind :-

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot \left(\text{\# of ways to insert } n^{th} \text{ item into any of the } k \text{ cycles} \right)$$

$n-1$

case-1: last item কে separate রাখব, বাকিগুলো আগের মতো

case-2: (2nd kind) এর মতো



নতুন একটা element এখানে 4 ভাবে insert করা যায়

element count $\rangle$ x_1 x_2 x_k

নতুন item insert করব $\rangle$ x_1 way $+$ x_2 way $+ \ldots\ldots\ldots + x_k$ $= n-1$

multiply হবে না কারণ independent

Prove, $\displaystyle\sum_{k=1}^{n} \left[{n \atop k}\right] = n!$

$\Rightarrow$ 1

$n=3, k=1$, 2 way

$k=2$, 3 way

$k=3$, 1 way

---

6 way

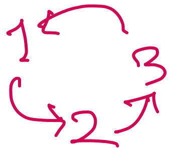
প্রতিটা Permutation একটা Cyclic arrangement কে map করে

1, 2, 3

1, 2, 3

2, 3, 1

1, 2, 3

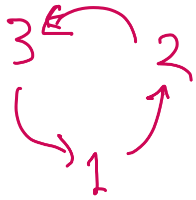


1, 3, 2

1, 2, 3

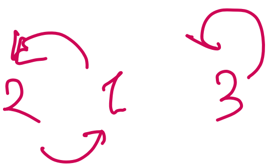
3, 1, 2

1, 2, 3



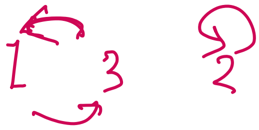
2, 1, 3

1, 2, 3

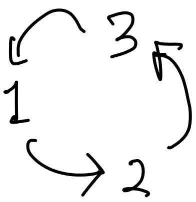


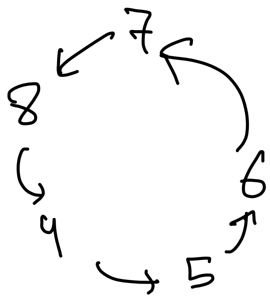
3, 2, 1

1, 2, 3



Now,





2 3 1 5 6 7 8 4 → permutation

1 2 3 4 5 6 7 8

So, cycle $\rightleftharpoons$ permutation

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \begin{Bmatrix} n \\ n-1 \end{Bmatrix} = \binom{n}{2}$$

$$\begin{Bmatrix} n \\ n-2 \end{Bmatrix}$$

1 set : 3 item
rest : 1 item

$$\downarrow$$

$$^{n}C_{3}$$

2 set : 2 item
rest : 1 item

$$\downarrow$$

$$\left(^{n}C_{4}\right) \times 3$$

$$\frac{^{4}C_{2}}{2}$$