L-3/T-2/CSE Date : 29/10/2023

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations 2021-2022

Sub : **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks : 210 Time : 3 Hours

The figures in the margin indicate full marks.

USE SEPARATE SCRIPTS FOR EACH SECTION

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) A certain letter is equally likely to be in any one of three different files. Let, α_i be the probability that you will find the letter upon making a quick examination of file i, if the letter is, in fact, in file i, $i = 1,2,3$. Given $\alpha_1 = 1/4$, $\alpha_2 = 1/3$, $\alpha_3 = 1/2$. Suppose you look into file 1 and do not find the letter after quick examination. Calculate the probability that the letter is in file 2 or file 3? **(10)**

   (b) Each element in a sequence of binary data is either 1 with probability p or 0 with probability $1\text{-}p$. A maximal subsequence of consecutive values having identical outcomes is called a run. For instance, if the outcome sequence is 1, 1, 0, 1, 1, 1, 0, the first run is of length 2, the second is of length 1, and the third is of length 3. **(10)**

   (i) Compute the expected length of the first run.

   (ii) Compute the expected length of the second run.

   (c) Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5. If the sum of chips drawn is even, the random variable X equals 5; if the sum of chips drawn is odd, $X = -3$. **(15)**

   (i) Formulate the moment-generation function (mgf) for X.

   (ii) Use the mgf to calculate the first and second moments.

   (iii) Compute the expected value and variance of X.

2. (a) At a party n men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. Calculate the probability of no matches. Calculate the probability of exactly k matches. **(8+5=13)**

   (b) A computer receives requests for elements stored in its memory. Consider n elements $e_1, e_2, \ldots, e_n$ are initially arranged in some ordered list. At each unit of time a request is made for one of these elements – e_i being requested, independently of the past, with probability P_i. After being requested, the element is then moved to the front of the list. That is, for instance, if the present ordering is e_1, e_2, e_3, e_4 and e_3 is requested, then the next ordering is e_3, e_1, e_2, e_4. Determine the expected position of the element requested after this process has been in operation for a long time. **(10)**

   (c) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom. **(6+6=12)**

Contd P/2

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Contd ... Q. No. 2(c)

(i) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.3, 0.3, 0.4, compute the expected number of days until he reaches freedom.

(ii) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, compute the expected number of days until he reaches freedom. (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only form doors 2 and 3.)

3. (a) A basketball player makes a shot with the following probabilities: 1/2 if he misses the last two times, 2/3 if he has hit one of his last two shots, 3/4 if he has hit both his last two shots. Formulate a Markov Chain to model his shooting, and compute the limiting fraction of the time he hits a shot. **(15)**

(b) A particle moves among $n + 1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Determine the probability that all states have been visited by time T. **(12)**

(c) Consider a Poisson process with rate λ, and let us denote the time of the first event by T_1. Further, for $n > 1$, let T_n denote the elapsed time between the $(n - 1)$st and the nth event. The sequence $\{T_n, n = 1, 2, \ldots\}$ is called the sequence of inter-arrival times. Show that $T_n, n = 1, 2, \ldots$, are independent identically distributed exponential random variables having mean $1/\lambda$. **(8)**

4. (a) Customers arrive at a single-server station in accordance with a Poisson process with rate λ. All arrivals that find the server free immediately enter service. All service times are exponentially distributed with rate μ. An arrival that finds the server busy will leave the system and roam around "in orbit" for an exponential time with rate θ at which time it will then return. If the server is busy when an orbiting customer returns, then that customer returns to orbit for another exponential time with rate θ before returning again. An arrival that finds the server busy and N other customers in orbit will depart and not return. That is, N is the maximum number of customers in orbit. **(20)**

(i) Show the states.

(ii) Construct the balance equations.

(iii) Determine the proportion of all customers that are eventually served.

(iv) Determine the average time that a served customer spends waiting in orbit.

Contd P/3

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Contd ... Q. No. 4

(b) A supermarket has two exponential checkout counters, each operating at rate μ. Arrivals are Poisson process with rate λ. The counters operate in the following way: (i) One queue feeds both counters; (ii) One counter is operated by a permanent checker and the other by a stock clerk who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever he completes a service, and there are fewer than two customers in the system. **(15)**

(i) Determine P_n, the proportion of time during which there are n customers in the system.

(ii) Calculate the rate at which the number of customers in the system goes from 0 to 1 and from 2 to 1.

(iii) Compute the proportion of time the stock clerk is checking.

SECTION – B

There are **TWELVE** questions in this section. Answer any **THREE** from 5-8, any **THREE** form 9-12, and any **THREE** from 13-16. You may use any formula from the supplementary materials without proving them.

5. In this problem, you will have to show the recurrence relation for the minimum number of moves for the 'Tower of Hanoi' problem with an added constraint: direct moves between the leftmost peg (i.e., the source) and the rightmost peg (i.e., the destination peg) are disallowed. You must explain how you have formulated the recursion. Remember to write the base case! **(10)**

   N.B.: You only need to show the recurrence formulation; there is no need to elaborate the necessity and sufficiency conditions separately.

6. Construct the recurrence relation for the running time of the 'quicksort' algorithm. Remember to write the base case too. **(10)**

7. Show that there are infinitely many primes. **(10)**

8. State and prove the Chinese Remainder Theorem. **(10)**

9. You are presented with a new variation of the Josephus problem: as usual, we begin with a group of n individuals numbered from 1 to n arranged in a circular formation. However, in this variant, we eliminate the first person of a consecutive pair until only one individual remains. For instance, when $n = 10$, the sequence of eliminations is 1, 3, 5, 7, 9, 2, 6, 10, 8, resulting in the survival of individual 4. Formulate the recurrence relations for determining the survivor's number in this new variation of the Josephus problem. Remember to specify the base case too. **(10)**

   N.B.: You are not required to find the closed form.

10. Infer the following sum **(10)**

$$\frac{1}{2}+\frac{3}{2\times4}+\frac{5}{2\times4\times6}+\frac{7}{2\times4\times6\times8}+\ldots\ldots$$

Up to the first n terms.

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11. If $\frac{m}{m}$ and $\frac{m'}{n'}$ are consecutive fractions at any stage of the construction of the *Stern-Brocot tree*, then show that the fraction $\frac{m'n+1}{mn'+1}$ is in the lowest terms. **(10)**

12. You already know how to derive the number of tilings for a $2 \times n$ rectangle using vertical and horizontal dominos. Now, an eccentric collector of $2 \times n$ domino tilings pays 4 for each vertical domino and 1 for each horizontal domino. Calculate the number of tilings that are worth exactly m under this criterion. **(10)**

13. Calculate the maximum number of regions defined by n circles in a 2-dimensional space. You must systematically compute the closed form as a function of n and provide necessary justifications. **(15)**
    Hints:
    - How many regions are there when there are no circles drawn?
    - How many regions are defined by one circle?
    - Can you categorize them by bounded and unbounded regions?
    - What happens to those regions when two circles intersect?
    - Can the n-th circle intersect all previous $(n-1)$ circles?
    - If so, how would it be possible?

14. Compute the following sum: **(15)**

$$\sum_{k=0}^{n}(-1)^{n-k}k(k-1)$$

15. Alice and Bob are sending cryptic messages to each other. Let p and q be distinct primes and $n = pq$ and $t = (p-1)(q-1)$. Let e, d be positive integers such that $ed \equiv 1 \pmod{t}$. Alice takes a message, M (an integer relatively prime to n), and sends $C = M^e \bmod n$ to Bob. Bob receives C and computes $M' = C^d \bmod n$. Can you determine that whether the following statement is true or not: $M \equiv M' \pmod{n}$? **(15)**

16. An inversion of a permutation P of $\{1, 2, 3, .., n\}$ is a pair of numbers (i, j) such tha t $i < j$ and $P(i) > P(j)$. For example, the permutation $\{5, 3, 4, 1, 2\}$ of $\{1, 2, 3, 4, 5\}$ has 8 inversions, namely the pairs (5,3), (5,4), (5,1), (5,2), (3,1), (3,2), (4,1), (4,2). Let $b(n, k)$ be the number of permutations of n letters that have exactly k inversions. Now answer the following: **(15)**
    - Show that $b(n,k) = b(n-1,k) + b(n-1, k-1) + \ldots\ldots + b(n-1,0)$. Calculate the appropriate base cases too.
    - Using the recurrence relation above, show that the generating function $B_n(x) = \sum_k b(n,k)x^k$ is $(1+x)\,(1+x+x^2)\,(1+x+x^2+x^3)\ldots.(1+x+\ldots+x^{n-1})$.

L-3/T-2/CSE **Date: 30/04/2023**

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Sub: **CSE 301** (Mathematical Analysis for Computer Science)

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USE SEPARATE SCRIPTS FOR EACH SECTION

SECTION – A

There are **NINE** questions in this section. Answer any **SEVEN** questions.

1. A radar system is designed such that the probability of detecting the presence of an aircraft in its range is 0.98. However, if no aircraft is present in its range it still reports (falsely) that an aircraft is present with a probability of 0.05. At any time, the probability that an aircraft is present within the range of the radar is 0.07. **(10+5=15)**
   (a) What is the probability that no aircraft is present in the range of the radar given that an aircraft is detected?
   (b) If two such radar systems are set up that operate independently, and if an aircraft enters the area, what is the probability that neither of the radar systems detects it?

2. Prove that when the number of trials n in a binomial distribution is large and the probability of success p is very small, the binomial distribution can be approximated by a Poisson distribution with mean $\lambda = np$. **(15)**

3. (a) Derive the moment generating function of the normal distribution with parameter μ and σ^2, and calculate its first two moments. **(10+5=15)**
   (b) State and explain the central limit theorem.

4. (a) Prove the law of total expectation for discrete random variables. **(5+10=15)**
   (b) A bus arrives at the 1st station with zero passengers on board. At the 1st station, 0, 1, or 2 passengers could get on the bus with probabilities 0.3, 0.5, 0.2, respectively. At every station, each passenger could get off the bus with a probability of 0.1. Find the expected value of the number of passengers that get off the bus at the 2nd station.

5. A professor frequently gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. **(8+7=15)**
   (a) What proportion of exams are of type i, $i = 1, 2, 3$ assuming the process has run for a long time?
   (b) If the first exam is of type 3, then calculate the probability that there is no type 1 exam in the first 10 exams.

Contd P/2

6. Suppose, a data centre has 100 storage devices. The designer of the data center assumed that the time to failure of the storage devices is exponentially distributed with mean two years, that is, $\lambda = 1/2$. **(15)**
   (a) What is the probability that the first failure occurs after 1 year?
   (b) What is the probability that there are exactly 10 failures in the 13^{th} month?
   (c) Do you think modeling failure of storage devices using an exponential distribution is reasonable?

7. (a) Provide the instantaneous transition rates of a linear growth process (birth-death process) with immigration. **(5+10=15)**
   (b) Derive Kolmogorov's forward and backward equations for a general continuous-time Markov chain using Chapman-Kolmogorov equations i.e. $P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s)$. Now show that $\boldsymbol{P}(t) = \boldsymbol{P}(0)e^{\boldsymbol{R}t}$ where $\boldsymbol{P}(t)$ is the transition probability matrix and $\boldsymbol{R}$ is the instantaneous transition rate matrix.

8. Derive the balance equations for a single-server exponential queuing system having finite capacity. Solve the equations to obtain the expression for the limiting probability that there are n customers in the system P_n and the average number of customers in the system L. **(5+10=15)**

9. Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates of 1 and 2 are 8 and 10, respectively. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system; whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise. Determine the limiting probabilities (n customers at server 1 and m customers at server 2), the average number of customers L, and the average amount of time spent by customers in the system W. It is known that the limiting probabilities of an M/M/1 queue is given by $\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$. **(15)**

Contd P/3

SECTION – B

There are **FOUR** questions in this section. Answer any **THREE** questions.

You may use any formula from the supplementary materials without proving them.

10. (a) Find all positive integers n such that $J(n)=\frac{1}{2}n$, where $J(n)$ denotes the survivor's number in the *Josephus problem* with n persons. **(10)**

(b) What is the maximum number of regions defined by n planes in a 3-dimensional space? You must systematically derive the closed form as a function of n and provide necessary justifications. **(10)**

(c) Solve the following recurrence relation defined on the set of non-negative integers: **(15)**

$$a_n = \begin{cases} 1, \text{if } n=0 \\ \dfrac{4}{11}, \text{if } n=1 \\ \dfrac{a_{n-1}a_{n-2}}{2a_{n-2}-a_{n-1}}, \text{otherwise} \end{cases}$$

11. (a) Solve the following recurrence relation defined on the set of non-negative integers: **(10)**

$$T_0 = 5$$

$$2T_n = nT_{n-1} + 3\cdot n! \quad \text{for } n>0$$

(b) Evaluate the following sum: **(10)**

$$\sum_{n=1}^{\infty}\frac{1}{n(n+k)}$$

where k is a positive integer. You are allowed to include *harmonic numbers* in the closed form.

(c) Evaluate the following sum: **(15)**

$$\sum_{k=1}^{n}(-1)^{k+1}(2k-1)$$

12. (a) If $\frac{m}{n}$ and $\frac{m'}{n'}$ are consecutive fractions at any stage of the construction of the *Stern-Brocot tree*, prove that $m'n-mn'=1$. **(10)**

(b) Find all primes p, q such that $p^3+1=q^2$. **(10)**

(c) Let m be an even positive integer. Assume that $\{a_1,a_2,\cdots,a_m\}$ and $\{b_1,b_2,\cdots,b_m\}$ are two complete sets of residue classes modulo m. Prove that the set $\{a_1+b_1,a_2+b_2,\cdots,a_m+b_m\}$ is not a complete residue class. **(15)**

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13. (a) Show that $\sum_{d \backslash m} \varphi(d) = m$, where φ is the *Euler's totient function.* **(10)**

(b) Let p, q be two distinct prime numbers. Show that every integer n satisfies the congruence $n^{pq-p-q+2} \equiv n \pmod{pq}$. **(10)**

(c) A partition of a positive integer n is a way of writing n as a sum of positive integers (each integer in the sum is called a part). For example, 4 can be partitioned in five distinct ways. **(15)**

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Show that for each positive integer n, the number of partitions of n into unequal parts is equal to the number of partitions of n into odd parts. For instance, if $n = 6$, there are four partitions into unequal parts: 6, 5 + 1, 4 + 2, 3 + 2 + 1. And there are also four partitions into odd parts: 5 + 1, 3 + 3, 3 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1.

Supplementary Materials

Recurrent Problems:

1. Recurrence relation for the *tower of Hanoi* problem: $T_n = 2T_{n-1} + 1$, where $T_0 = 0$. Closed form: $T_n = 2^n - 1$.

2. Recurrence relation for lines in a plane: $L_n = L_{n-1} + n$, where $L_0 = 1$. Closed form: $L_n = \frac{n(n+1)}{2} + 1$.

3. Recurrence relation for the *Josephus problem*:

   $J(2n) = 2J(n) - 1$ and $J(2n+1) = 2J(n) + 1$, where $J(1) = 1$.

   Closed form: $J(2^m + l) = 2l + 1$ for $m \ge 0$ and $0 \le l < 2^m$.

Sums:

1. Any recurrence of the form $a_n T_n = b_n T_{n-1} + c_n$ has the solution $T_n = \dfrac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$, where $s_n = \dfrac{a_{n-1}a_{n-2}\cdots a_1}{b_n b_{n-1}\cdots b_2}$.

2. Laws for manipulation of sums:

   $\sum_{k\in\mathbb{K}} ca_k = c\sum_{k\in\mathbb{K}} a_k$ (distributive law)

   $\sum_{k\in\mathbb{K}} a_k + b_k = \sum_{k\in\mathbb{K}} a_k + \sum_{k\in\mathbb{K}} b_k$ (associative law)

   $\sum_{k\in\mathbb{K}} a_k = \sum_{p(k)\in\mathbb{K}} a_k$ (commutative law). Here p is a permutation of $\mathbb{K}$.

3. *Perturbation method:* If $S_n = \sum_{0\le k\le n} a_k$, then

   $S_n + a_{n+1} = a_0 + \sum_{0\le k\le n} a_{k+1}$.

4. Multiple sums: $\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]$.

5. *Chebyshev's inequalities*:

   $(\sum_{k=1}^{n} a_k)(\sum_{k=1}^{n} b_k) \le n(\sum_{k=1}^{n} a_k b_k)$, if $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_n$.

   $(\sum_{k=1}^{n} a_k)(\sum_{k=1}^{n} b_k) \ge n(\sum_{k=1}^{n} a_k b_k)$, if $a_1 \le \cdots \le a_n$ and $b_1 \ge \cdots \ge b_n$.

6. General sums: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Number Theory:

1. Divisibility: $m \setminus n \iff m > 0$ and $n = mk$ for some integer k.

2. *Greatest common divisor (gcd) and least common multiple*:

   $\gcd(m,n) = \max\{k | k \setminus m \text{ and } k \setminus n\}$.

   $\mathrm{lcm}(m,n) = \min\{k | k > 0, m \setminus k \text{ and } n \setminus k\}$.

3. *Euclid's algorithm*: for $0 \leq m < n$, $\gcd(0, n) = n$;

   $\gcd(m, n) = \gcd(n \bmod m, m)$, for $m > 0$.

4. *Diophantine equation*: $ax + by = n$, has solutions (x, y) iff $\gcd(x, y) \setminus n$.

5. $k \setminus m$ and $k \setminus n \iff k \setminus \gcd(m, n)$.

6. Fundamental theorem of arithmetic: Every positive integer n has a unique prime power factorization $n = \prod_p p^{n_p}$.

7. *Legendre's function*: $\epsilon_p(n!) = \sum_{k \geq 1} \lfloor \frac{n}{p^k} \rfloor$.

8. Relative primality: $m \perp n \iff m, n$ are integers and $\gcd(m, n) = 1$.

   $\frac{m}{\gcd(m,n)} \perp \frac{m}{\gcd(m,n)}$.

   $k \perp m$ and $k \perp n \iff k \perp mn$.

9. Congruence (modular arithmetic):

   $a \equiv b \pmod{m} \iff a \bmod m = b \bmod m$

   $a \equiv b \pmod{m} \iff m \setminus a - b$

   $a \equiv b$ and $c \equiv d \implies a + c \equiv b - d \pmod{m}$

   $a \equiv b$ and $c \equiv d \implies a - c \equiv b - d \pmod{m}$

   $a \equiv b$ and $c \equiv d \implies ac \equiv bd \pmod{m}$

   $ad \equiv bd \iff a \equiv b \pmod{m}$ for $d \perp m$

   $ad \equiv bd \pmod{md} \iff a \equiv b \pmod{m}$ for $d \neq 0$

   $ad \equiv bd \pmod{m} \iff a \equiv b \pmod{\frac{m}{\gcd(d,m)}}$

   $a \equiv b \pmod{md} \implies a \equiv b \pmod{m}$

   $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n} \iff a \equiv b \pmod{\operatorname{lcm}(m, n)}$

   $a \equiv b \pmod{mn} \iff a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$, if $m \perp n$

   $a \equiv b \pmod{m} \iff a \equiv b \pmod{p^{m_p}} \forall$ prime p

10. Residue classes: The m numbers

    $0 \bmod m, n \bmod m, 2n \bmod m, \cdots, (m-1)n \bmod m$

    consist of precisely d copies of the $\frac{m}{d}$ numbers $0, d, 2d, \cdots, m - d$.

11. *Fermat's little theorem*: For a prime p, $n^{p-1} \equiv 1 \pmod{p}$, if $n \perp p$.

12. *Wilson's theorem*: $(n-1)! \equiv -1 \pmod{n} \iff n$ is prime, where $n > 1$.

13. *Euler's totient function*: The numbers of integers from $\{0, 1, \cdots, m-1\}$ relatively prime to m, $\varphi(m)$, is given by $\varphi(m) = m \prod_{p \setminus m} (1 - \frac{1}{p})$.

14. *Euler's theorem*: $n^{\varphi(m)} \equiv 1 \pmod{m}$, if $m \perp n$.

Generating Functions:

1. The generating function T for tiling a $2 \times n$ rectangle with horizontal and vertical dominoes is $T = \frac{1}{1-z-z^2}$, where the co-efficient of z^n in T is the number of possible tiling for the integer n.
2. The generating function C for paying n cents with the denominations of $1, 5, 10, 25, 50$ cents is $C = \frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}}\frac{1}{1-z^{25}}\frac{1}{1-z^{50}}$, where co-efficient of z^n in C is the number of possible ways for paying n cents.
3. Generating function manipulations: $G(z) = \sum_n g_n z^n$, $F(z) = \sum_n f_n z^n$

$$\alpha F(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n$$

$$z^m G(z) = \sum_n g_{n-m} z^n, \text{ integer } m \geq 0$$

$$\frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n \geq 0} g_{n+m} z^n, \text{ integer } m \geq 0$$

$$G(cz) = \sum_n c^n g_n z^n$$

$$G'(z) = \sum_n (n+1) g_{n+1} z^n$$

$$zG'(z) = \sum_n n g_n z^n$$

$$\int_0^z G(t)dt = \sum_{n \geq 1} \frac{1}{n} g_{n-1} z^n$$

$$F(z)G(z) = \sum_n \left(\sum_k f_k g_{n-k}\right) z^n$$

$$\frac{1}{1-z} G(z) = \sum_n \left(\sum_{k \leq n} g_k\right) z^n$$

$$\frac{1}{2}(G(z) + G(-z)) = \sum_n g_{2n} z^{2n}$$

$$\frac{1}{2}(G(z) - G(-z)) = \sum_n g_{2n+1} z^{2n+1}$$

L-3/T-2/CSE **Date: 22/03/2022**

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USE SEPARATE SCRIPTS FOR EACH SECTION

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SECTION – A

There are **FOUR** questions in this section. Answer any **THREE** questions.

1. (a) Morpheus has one pill in his palm: it is either blue or red with equal probability. He puts another red pill in his palm (Now there are two pills in his palm) and randomly takes one out. If its color is red, what is the probability that the color of the other pill is also red? **(10)**

(b) Let X be a random (uniform) number x; $0 < x < 1$ and let $Y = X^2$. Find the covariance of X and Y. Also, specify if X and Y are positively or negatively correlated, or uncorrelated? **(10)**

(c) At a party n men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of at most k matches? Assume that n is sufficiently large for the ease of calculation. **(15)**

2. (a) The number of traffic accidents on any given day is a Poisson random variable with mean 2, and these random variables for different days are independent. **(8)**

(i) What is the probability that there is a total of six accidents over two days?

(ii) Take some 5 days, for instance, SAT-WED this week. What is the probability that at least three of these five days each have exactly two accidents?

(b) A miner is trapped in a mine (M1) containing three doors. The first door (D1) leads to a tunnel that takes him to safety after 4 hours of travel. The second door (D2) leads to a tunnel that returns him to the mine after 2 hours of travel. However, the third door (D3) leads to another mine (M2) after 3 hours. Now, the second mine (M2) has two other doors. First one (D4) leads to safety after 1 hour of travel. But the second door (D5) leads to a tunnel that returns him to the same mine (M2) after 2 hours of travel. Assume that the miner is at all times equally likely to choose any one of the doors (in M1, the miner can choose between D1, D2, D3, and in M2, the miner can choose between D3, D4, D5). What is the expected duration of time until the miner reaches safety? **(15)**

Contd …… P/2

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Contd... Q. No. 2

(c) Let N be a Poisson random variable with mean λ. Given N, let X be another binomial random variable with parameters N and p (p being the probability).

Let $Y = N - X$ be a random variable and given that Y is non-negative. Prove that Y is a Poisson random variable with mean $\lambda(1 - p)$. **(12)**

3. (a) Three players play a game in which they take turns and draw cards from an ordinary deck of 52 cards, successively, at random, and with replacement. Player-I draws cards until an ace is drawn. Then Player-II draws cards until a diamond is drawn. Next, Player-III draws cards until a face card is drawn. At that point, the deck is returned to Player-I and the game continues. **(15)**

i. Draw a state diagram with transition probability identifying the relevant states.

ii. Determine the long-term proportion of cards drawn by each player out of the total number of cards drawn by the three players.

(There are 4 aces, 13 diamond cards, and 12 face cards in a standard deck).

(b) Consider a population of individuals each of whom possesses two genes that can be either type A or type a. Suppose that in outward appearance type A is dominant and type a is recessive. (That is, an individual will have only the characteristics of the recessive gene if its pair is aa.) Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs AA, aa, and Aa are p, q, and r. Call an individual dominant or recessive depending on the outward characteristics it exhibits. Let S_{11} denote the probability that an offspring of two dominant parents will be recessive; and let S_{10} denote the probability that the offspring of one dominant and one recessive parent will be recessive. Compute S_{11} and S_{10} in terms of p, q, r to show that $S_{11} = S_{10}^2$. **(10)**

(c) Define counting process and illustrate its properties. When do we consider a counting process as a Poisson process? **(6)**

(d) Prove that Exponential distribution is memoryless. **(4)**

4. (a) Customers arrive at a bank at a Poisson rate λ. Suppose three customers arrived during the first hour. What is the probability that exactly two customers arrived during the first 20 minutes? **(8)**

(b) Consider a shoe shop consisting of two chairs. Suppose that an entering customer first will go to chair 1. When his work is completed in chair 1, he will go either to chair 2 if that chair is empty or else wait in chair 1 until chair 2 becomes empty. Suppose that a potential customer will enter this shop as long as chair 1 is empty. (Thus, for instance, a potential customer might enter even if there is a customer in chair 2.)

Contd P/3

CSE 301

Contd… Q. No. 4(b)

Suppose that potential customers arrive in accordance with a Poisson process at rate λ, and that the service times for the two chairs are independent and have respective exponential rates of μ_1 and μ_2. **(15)**

i. Draw a state diagram of the system and write down balance equations for each state.
ii. What proportion of potential customers enters the system?
iii. What is the mean number of customers in the system?
iv. What is the average amount of time that an entering customer spends in the system?
v. What proportion of entering customers are blockers?

(c) Consider a network of three stations with a single server at each station. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 6, 8 and 8. The service times at the three stations are exponential with respective rates 12, 60 and 120. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system. A customer departing service at station 2 always goes to station 3. A departure from service at station 3 is equally likely to either go to station 2 or leave the system. **(12)**

i. What is the probability that at a certain time, station 1 is empty, there are 2 customers in station 2, and 1 customer in station 3.
ii. What is the average number of customers in the system (consisting of all three stations)?
iii. What is the average time a customer spends in the system?

SECTION – B

There are **FOUR** questions in this section. Answer any **THREE**.

5. (a) E(n) denotes the maximum number of regions that can be formed on a plane by drawing n intersecting ellipses. **(10+3=13)**

i. Find the recurrence formula for E(n).
ii. Derive a closed form of E(n).

(b) Given that, **(12)**

$$x^4 = x^{\underline{4}} + 6\,x^{\underline{3}} + 7\,x^{\underline{2}} + + x^{\underline{1}}$$

and,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

Contd …… P/4

CSE 301

Contd… Q. No. 5(b)

Evaluate the following sum,

$$S = \sum_{0 \le k \le n} k^5$$

(c) Let $f(x)$ be any continuous, monotonically strictly increasing function with the property that, **(10)**

If $f(x)$ is an integer, then x is an integer.

Prove that,

$$\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil$$

6. (a) We have an ant and a worm on the tip of two different horizontal rubber strings. Both strings are of length 100 cm. The ant moves 1 cm per second and the worm moves 2 cm per second. The rubber string of the ant is stretched 100cm each second. On the other hand, the rubber string of the worm is stretched 100 cm each 0.5 second. During the stretching operation the ant and the worm maintain their relative positions on their respective strings. **(15)**

Prove or disprove that, after n seconds (where $n \geq 1$ and $n \in \mathbb{N}$),

(The fraction travelled by worm – the fraction travelled by ant) < 2%

(b) Prove that, there are at least $n - 1$ composite integers between $n!$ and $n! + n$, where n is a positive integer. **(10)**

(c) Prove the recurrence of the Eulerian Numbers stated below **(10)**

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$$

7. (a) A Double Tower of Hanoi contains 2n disks of n different sizes, two of each size. We are required to move only one disk at a time, without putting a larger one over a smaller one. **(5+10=15)**

i. How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?

ii. What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement? How many moves does it take?

(b) Prove that, for any $m, k \in \mathbb{N}$ **(10)**

$$\phi(m^k) = \phi(m) \cdot m^{k-1}$$

Where ϕ is the Euler's totient function.

Contd …… P/5

CSE 301

Contd… Q. No. 7

(c) Evaluate B_1 and B_2 **(5+5=10)**

$$B_1 = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

$$B_2 = \sum_{k=0}^{n} 3^{k-n} \binom{n}{k}$$

8. (a) Evaluate the following sum, **(13)**

$$S = \sum_{1 \le j \le k < n} \frac{1}{j(k+1)(k+2)}$$

(b) How many integers n are there such that, **(12)**

$\lfloor \sqrt[3]{n} \rfloor$ divides n and $1 \le n \le 2000$?

(c) Compute the value of 7^{10010} mod 11. **(10)**

L-3/T-2/CSE Date: 10/01/2021

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-1 B. Sc. Engineering Examinations (January 2020 Term)

Sub: **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks: 180 Section Marks: 90 Time: 2 Hours (Sections A + B)

USE SEPARATE SCRIPTS FOR EACH SECTION

The figures in the margin indicate full marks.

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) Find integers a, b, and c such that for all m, the following equation is satisfied. 15

$$m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$$

Also, find out the sum of the series $1^3 + 2^3 + 3^3 + ... + n^3$ by using this equation.

(b) Use combinatorial reasoning to prove the following identity, 10

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

(c) What is the coefficient of $x_1^3 x_2^3 x_3\, x_4^2$ in the expansion of $(x_1 - x_2 + 2x_3 - 2x_4)^9$? 5

2. (a) Prove that the derangement number D_n is an even number if and only if n is an odd number. 10

(b) Suppose a class of n boys takes a walk every day. The boys walk in a line so that every boy except the first is preceded by another. In order that a boy can not see the same person in front of him, on the second day the boys decide to switch positions so that no boy is preceded by the same boy who preceded him on the first day. Let Q_n denote the number of how many ways they can switch positions. Find an expression for Q_n. Also, prove that $Q_n = D_n + D_{n-1}$ ($n \geq 2$), where D_n is the derangement number. 20

3. (a) What is the number of ways to place six non-attacking rooks on the 6 × 6 board with forbidden positions as shown? 10

x	x				
x	x				
		x	x		
		x	x		
				x	x
				x	x

(b) Suppose the n men and the n women at the party check their hats before they dance. At the end of the party, their hats are returned randomly. In how many ways can they be returned if each man gets a male hat and each woman gets a female hat, but no one gets the hat he or she checked? 10

(c) Assume that there are four communication devices A, B, C, and D in the network. The total bandwidth of the communication channel available to these four devices is known and it is a function of the physical channel capacity. A bandwidth manager is used to configure and limit the traffic from these devices which determines the upper bound. There is a need to keep certain service levels for users which determines the lower bound. How many solutions are possible if only integer solutions are accepted as valid for the following five equations which represent the scenario? 10

$$X_A + X_B + X_C + X_D = 20,\ 1 \leq X_A \leq 6,\ 0 \leq X_B \leq 7,\ 4 \leq X_C \leq 8,\ 2 \leq X_D \leq 6$$

4. (a) Express the number h_n of regions that are created by n mutually overlapping circles in general positions in the plane as a recurrence relation and find a formula for h_n in terms of n. 10

(b) Formulate a combinatorial problem that leads to the following generating function: 10

$$(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+...)(x+x^2+x^3+...)$$

(c) Show that the sums of the entries along the diagonals of Pascal's triangle running upward from the left are Fibonacci numbers. 10

The deeper one penetrates into nature's secrets, the greater becomes one's respect for God. – Albert Einstein

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations 2018-2019

Sub: **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks: 180 Section Marks: 90 Time: 2 Hours (Sections A + B)

USE SEPARATE SCRIPTS FOR EACH SECTION

The figures in the margin indicate full marks.

SECTION – B

There are **FOUR** questions in this section. Answer any **THREE**.

5. (a) Twenty tickets are sold in a lottery, numbered 1 to 20, inclusive. Five tickets are drawn for prizes. Find out the probability that two of the five winning tickets are in the range between 1 to 5, two are in the range between 6 to 10, and one is in the range between11 to 20. (15)

(b) Consider the following bus ridership model: (15)

- Suppose the bus is empty when it arrives at its first stop.
- Assume the bus is so large that it never becomes full, so the new passengers can always get on.
- At each stop, each passenger gets down from the bus, independently, with a probability of 0.2.
- Either 0, 1 or 2 new passengers get on the bus, with probabilities 0.5, 0.4 and 0.1, respectively. Passengers at successive stops are independent.
- Suppose the bus company charges Tk.15/- for each passenger who boards (gets on) at the first stop, but charges Tk.10/- for a passenger who joins at the second stop, regardless of the destination.

Find the expected total revenue from the first two stops.

6. (a) Consider the numbers 1, 2, . . . ,N written around a ring. Consider a Markov Chain X_n that at any point jumps with equal probability to one of the two adjacent numbers. (15)

(i) What is the expected number of steps that X_n will take to return to its starting position?

(ii) What is the probability that X_n will visit all the other states before returning to its starting position?

(b) A taxicab driver moves between the airport A and two hotels B and C according to the following rules: if the taxicab is at the airport, go to one of the hotel with equal probability, and if the taxicab is at one hotel, go to the airport with probability 1/4, and to the other hotel with probability 3/4. (15)

(i) Find out the transition matrix.

(ii) Suppose the driver starts (time 1) at the airport. Find the probability for each of the three possible locations at time 2. Find the probability that the taxicab is at the airport at time 3.

7. (a) Each element in a sequence of binary data is either 1 with probability p or 0 with probability $1-p$. A maximal subsequence of consecutive values having identical outcomes is called a run. For instance, if the outcome sequence is 1, 1, 0, 1, 1, 1, 0, the first run is of length 2, the second is of length 1, and the third is of length 3. (15)
   (i) Find the expected length of the first run.
   (ii) Find the expected length of the second run.

   (b) Suppose we are given a set of n elements, numbered 1 through n, which are to be arranged in some ordered list. At each unit of time a request is made to retrieve one of these elements, element i being requested (independently of the past) with probability P_i. After being requested, the element then is moved one closer to the front of the list; for instance, if the present list ordering is 1, 3, 4, 2, 5 and element 2 is requested, then the new ordering becomes 1, 3, 2, 4, 5. Find the long-run average position of the element requested. (15)

8. (a) The financial condition of a store is affected significantly during pandemic situation. During non-pandemic times, customers arrive at a certain single-server queueing system in accordance with a Poisson process with rate λ_1, and during the pandemic times, they arrive in accordance with a Poisson process with rate λ_2. A non-pandemic time period lasts for an exponentially distributed time with rate α_1, and a pandemic time period lasts for an exponential time with rate α_2. An arriving customer will only enter the queueing system if the server is free both in pandemic and non-pandemic cases; an arrival finding the server busy goes away. All service times are exponential with rate μ. (20)
   (i) Define the states so as to be able to analyze this system.
   (ii) Give a set of linear equations whose solution will yield the long-run proportion of time the system is in each state.
   (iii) What proportion of time is the system empty?
   (iv) What is the average rate at which customers enter the system?

   (b) 100 items are simultaneously put on a lifetime duration test. Suppose the lifetimes of the individual items are independent exponential random variables with mean 200 hours. The test will end when there have been a total of 5 failures. Find out the expected time at which the test ends. (10)

You are the best nation produced [as an example] for mankind. You enjoin what is right and forbid what is wrong and believe in Allah. If only the People of the Scripture had believed, it would have been better for them. Among them are believers, but most of them are defiantly disobedient. [Al Quran 110]

L-3/T-2/CSE **Date : 25/03/2019**

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations 2017-2018

Sub : **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks : 210 Time : 3 Hours

The figures in the margin indicate full marks.

USE SEPARATE SCRIPTS FOR EACH SECTION

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) A certain word is equally likely to be in any one of three different files. Let, p_i be the probability that you will find the word upon making a quick examination of file i, if the word is, in fact, in file i, for i = 1,2,3, Given $p_1 = 1/2$, $p_2=1/3$, $p_3 = 1/4$. Suppose you look into file 2 and do not find the word. What is the probability that the word is in file 2? **(10)**

(b) Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5. If the sum of chips drawn is even, the random variable X equals 5; if the sum of chips drawn is odd, then X = -3. **(10)**

(i) Find the moment-generating function (mgf) for X.

(ii) Use the mgf to find the first and second moments.

(iii) Find the expected value and variance of X.

(c) It is known that DVDs produced by a certain company will be defective with probability 0.01, independent of each other. The company sells the DVDs in packages of size 10 and offers a money-back guarantee that if at least 1 of the 10 DVDs in a package is defective, money will be returned. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them? **(15)**

2. (a) Each element in a sequence of binary data is either 1 with probability p or 0 with probability 1 - p. A maximal subsequence of consecutive values having identical outcomes is called a run. For instance, if the outcome sequence is 1, 1, 0, 1, 1, 1, 0, the first run is of length 2, the second is of length 1, and the third is of length 3. **(10)**

(i) Find the expected length of the first run.

(ii) Find the expected length of the second run.

(b) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom. Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.) **(10)**

Contd P/2

CSE 301

Contd ... Q. No. 2

(c) A computer receives requests for elements stored in its memory. Consider n elements $e_1, e_2 \ldots . e_n$ are initially arranged in some ordered list. At each unit of time a request is made for one of these elements, e_i independently of the past, with probability P_i. After being requested, the element is then moved to the front of the list. That is, for instance, if the present ordering is e_1, e_2, e_3, e_4 and e_3 is requested, then the next ordering is e_3, e_1, e_2, e_4. Determine the expected position of the element requested after this process has been in operation for a long time. **(15)**

3. (a) A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done will or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type i, for i = 1, 2, 3? **(10)**

   (b) A particle moves among n + 1 vertices that are situated on a circle in the following manner. At each step it moves to the next vertex either in the clockwise direction with probability p or the counterclockwise direction with probability q = 1 - p. Starting at a specified vertex, call it vertex 0, let T be the time of the first return to vertex 0. Find the probability that all vertices have been visited by time T. **(10)**

   (c) Consider a Poisson process with rate λ, and let us denote the time of the first event by T_1. Further, for n > 1, let T_n denote the elapsed time between the (n-1)st and the nth event. The sequence $\{T_n, n=1,2\ldots\}$ is called the sequence of inter-arrival times. Show that T_n, n=1,2 ..., are independent identically distributed exponential random variables having mean $1/\lambda$. **(7)**

   (d) Suppose that X_1 and X_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$; what is $P\{X_1 < X_2\}$? **(8)**

4. (a) Consider a shoe shine shop consisting of two chairs. Suppose that an entering customer will first go to chair 1. When his work is completed in chair 1, he will go either to chair 2 if that chair is empty or else wait in chair 1 until chair 2 becomes empty. Suppose that a potential customer will enter this ship as long as chair 1 is empty. (Thus, for instance, a potential customer might enter even if there is a customer in chair 2.) Suppose that potential customers arrive in accordance with a Poisson process at rate λ, and that the service times for the two chairs are independent and have respective exponential rates of μ_1 and μ_2. **(10)**

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Contd ... Q. No. 4(a)

(i) Draw a state diagram of the system and write down balance equations for each state.

(ii) What proportion of potential customers enters the system?

(iii) What is the mean number of customers in the system?

(iv) What is the average amount of time that an entering customer spends in the system?

(b) Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ. That is the times between successive arrivals are independent exponential random variables having mean $1/\lambda$. Each customer, upon arrival, goes directly into service if the server is free and, if not, the customer joins the queue. When the server finishes serving a customer, the customer leaves the system, and the next customer in line, if there is any, enters service. The successive service times are assumed to be independent exponential random variable having mean $1/\mu$. This system is called the M/M/1 queue. For the M/M/1 queuing system, compute **(10)**

(i) the average number of customers in the system,

(ii) the average time a customer spends in the system,

(iii) the average number of customers in the queue and

(iv) the average time a customer spends in the queue.

(c) Consider a network of three stations with a single server at each station. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 5, 10, and 15. The service times at the three stations are exponential with respective rates 10, 50, and 100. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the station. A customer departing service at station 2 always goes to station 3. A departure from service at station 3 is equally likely to either go to station 2 or leave the system. **(15)**

(i) What is the average number of customers in the system (consisting of all three station)?

(ii) What is the average time a customer spends in the system?

SECTION – B

There are **FOUR** questions in this section. Answer any **THREE**.

5. (a) Find the solution to the following recurrence relation. **(20)**

$$a_0 = 1$$

$$a_1 = 9$$

$$a_{n+2} - 17a_{n+1} + 70a_n = 6 \quad \text{where } (n \geq 2)$$

(b) What is spectrum (*Spec*( )) of a real number? Let n be an arbitrary positive integer. Let p be the number of elements of *Spec*($\sqrt{2}$) that are less than or equal to n. Let q be the number of elements of *Spec*($2+\sqrt{2}$) that are less than or equal to n. Prove that p + q = n. **(15)**

CSE 301

6. (a) Fine a closed form of the sum $\sum_{1 \le k \le n} k^2 c^{dk}$ where c and d are non-negative integer constants. **(20)**

   (b) Derive, with detailed reasoning, the **L** and **R** matrices for Stern-Brocot tree. Using these, write down the algorithm for representing any positive fraction a/b, with a $\perp$ b, as a string of the letters L and R. **(15)**

7. (a) Find, through detailed steps, the number of ways a rooted ordered binary tree can be constructed from n vertices. **(15)**

   (b) Deduce the recurrence relation for the Stirling numbers of the first and second kind. **(10)**

   (c) State and prove the inversion formula. **(10)**

8. (a) Prove, with detailed reasoning, that the number of partitions of a positive integer into distinct summands is identical to the number of partitions of that integer into odd summands. **(20)**

   (b) A ship carries 48 flags, 12 each of the colors red, white, blue and black. 12 of these flags are placed on a vertical pole in order to communicate a signal to other ships. How many of the signals have at least three white flags or no white flags at all? **(15)**

L-3/T-2/CSE **Date : 18/02/2018**

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations 2016-2017

Sub : **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks: 210 Time : 3 Hours

USE SEPARATE SCRIPTS FOR EACH SECTION

The figures in the margin indicate full marks.

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) A man has 89 quarreling cows that he wants to separate with straight fencing. What is the minimum number of fences he needs. You must deduce any formula you use. **(15)**

(b) Show solution of Multi-Peg Tower of Hanoi problem with (n,p)=(347,7) using a binary tree as used in the class specifying values of K_{max}, $N_a(K_{max})$ and all the inequalities relevant quantities should satisfy. **(20)**

2. (a) Expand the binomial coefficient representing (m+n+1) things taken n at a time in a series with the help of combinatorial argument. **(15)**

(b) Deduce inversion formula. Use this formula for solving the following problem: n men throw their hats and pick them randomly. What is the probability that nobody will end up in his own hat? **(20)**

3. (a) A cricket team of 16 persons is supposed to be accommodated in 5 **distinguishable** rooms with no room remaining empty. In how many ways can we do it? **(15)**

(b) Elaborately explain your deduction of the size of the maximum overhang that n cards, each of length 2, can make when placed on a table. **(20)**

4. (a) Discuss Euler numbers and deduce recurrences they satisfy. **(15)**

(b) Compute in logarithmic time the value of F_n for large n using recurrence relation. **(20)**

SECTION-B

There are **FOUR** questions in this section. Answer any **THREE** questions.

5. (a) It is known that DVDs produced by a certain company will be defective with probability 0.01, independent of each other. The company sells the DVDs in packages of size 10 and offers a money-back guarantee that if at least 1 of the 10 DVDs in a package is defective, money will be returned. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them? **(13)**

(b) Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5. If the sum of chips drawn is even, the random variable X equals 5; if the sum of chips drawn is odd, X = -3. **(10)**

(i) Find the moment-generating function (mgf) for X.

(ii) Use the mgf to find the first and second moments.

(iii) Find the expected value and variable of X.

Contd P/2

CSE 301

Contd... Q. No. 5

(c) Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1-p$. The winner of the series is the first team to win i games. Find the expected number of games that are played when (a) $i = 2$, and (b) $i = 3$. **(6+6)**

6. (a) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom. **(6+6)**

    (i) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?

    (ii) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)

    (b) A computer receives requests for elements stored in its memory. Consider n elements $e_1, e_2, \ldots, e_n$ are initially arranged in some ordered list. At each unit of time a request is made for one of these elements — e_i being requested, independently of the past, with probability, P_i. After being requested, the element is then moved to the front of the list. That is, for instance, if the present ordering is e_1, e_2, e_3, e_4 and e_3 is requested, then the next ordering is e_3, e_1, e_2, e_4. Determine the expected position of the element requested after this process has been in operation for a long time. **(13)**

    (c) The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over (0, 100). Find the mean and variance of the amount of money that the store takes in on a given day. **(10)**

7. (a) A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type i, i = 1, 2, 3? **(10)**

    (b) Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach 0 before reaching N? **(10)**

    (c) Consider a Poisson process with rate λ, and let us denote the time of the first event by T_1. Further, for $n > 1$, let T_n denote the elapsed time between the (n – 1)st and the nth event. The sequence $\{T_n, n = 1,2,\ldots\}$ ia called the sequence of inter-arrival times. Show that T_n, $n = 1,2,\ldots$, are independent identically distributed exponential random variables having mean $1/\lambda$. **(7)**

Contd P/3

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Contd... Q. No. 7

(d) Suppose that X_1 and X_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$; what is $P\{X_1 < X_2\}$? **(8)**

8. (a) Consider a single-server exponential system in which ordinary customers arrive at a rate λ and have service rate μ. In addition, there is a special customer who has a service rate μ_1. Whenever this special customer arrives, she goes directly into service (if anyone else is in service, then this person is bumped back into queue). When the special customer is not being serviced, she spends an exponential amount of time (with mean $1/\theta$) out of the system. **(10)**

   (i) What is the average arrival rate of the special customer?

   (ii) Define an appropriate state space and set up balance equations.

   (iii) Find the probability that an ordinary customer is bumped n times.

   (b) Potential customers arrive to a single-server hair salon according to a Poisson process with rate λ. A potential customer who finds the server free enters the system; a potential customer who finds the server busy goes away. Each potential customer is type i with probability p_i, where $p_1 + p_2 + p_3 = 1$. Type 1 customers have their hair washed by the server; type 2 customers have their hair cut by the server; and type 3 customers have their hair first washed and then cut by the server. The time that it takes the server to wash hair is exponentially distributed with rate μ_1, and the time that it takes the server to cut hair is exponentially distributed with rate μ_2. **(10)**

   (i) Explain how this system can be analyzed with four states.

   (ii) Give the equations whose solution yields the proportion of time the system is in each state.

   (iii) Find the proportion of time the server is cutting hair.

   (iv) Find the average arrival rate of entering customers.

   (c) Customers arrive at a two-server station in accordance with a Poisson process with a rate of two per hour. Arrivals finding server 1 free begin service with that server. Arrivals finding server 1 busy and server 2 free begin service with server 2. Arrivals finding both servers busy are lost. When a customer is served by server 1, she then either enters service with server 2 if 2 is free or departs the system if 2 is busy. A customer completing service at server 2 departs the system. The service times at server 1 and server 2 are exponential random variables with respective rates of four and six per hour. **(15)**

   (i) What fraction of customers do not enter the system?

   (ii) What is the average amount of time that an entering customer spends in the system?

   (iii) What fraction of entering customers receives services from server 1?

L-3/T-2/CSE Date : 05/08/2017

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations 2015-2016

Sub : **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks: 210 Time : 3 Hours

USE SEPARATE SCRIPTS FOR EACH SECTION

The figures in the margin indicate full marks.

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) Establish all recurrence relations satisfied by J(n) of Josephus problem. **(10)**

(b) Compute J_5 (10,000) where every 5th man is deleted. **(10)**

(c) Construct a binary free sharing how multi-peg tower of Hanoi is solved using presumed optimal solution where p =7 and n = 489. **(15)**

2. (a) Deduce average case complexity of Quick sort algorithm using summation factor. **(10)**

(b) Deduce $\sum x^2 H_x \delta_x$ by summation by parts. **(10)**

(c) Use combinational argument to establish the value of $\sum_{k \le n} \binom{k}{m}$. **(15)**

3. (a) Find the multiplicity of 72 in 200!. **(10)**

(b) Deduce recurrence relations satisfied by the Stirling numbers of the first and second kind. **(10)**

(c) A group of n fans of the winning football team throw their hats high into the air. The hats came back randomly, one hat to each of the n fans. How many ways are there for all n fans not to end up in having their hats? Solve it using generating function. **(15)**

4. (a) Construct a generating function for Fibonacci numbers and find their values. **(10)**

(b) Given n cards and a table what is the largest possible overhang by stacking the cards up over the tables edge? **(10)**

(c) prove that $x^{\bar{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k$, integers $n \ge 0$. **(15)**

SECTION-B

There are **FOUR** questions in this section. Answer any **THREE** questions.

5. (a) It is known that DVDs produced by a certain company will be defective with probability 0.01, independent of each other. The company sells the DVDs in packages of size 10 and offers a money-back guarantee that if at least 1 of the 10 DVDs in a package is defective, money will be returned. if someone buys 3 packages, what is the probability that he or she will return exactly 1 of them? **(13)**

Contd P/2

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Contd... Q. No. 5

(b) An airline knows that *5* percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell *52* tickets for a flight that can hold only *50* passengers. What is the probability that there will be a seat available for every passenger who shows up? **(10)**

(c) Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1-p$. The winner of the series is the first team to win i games. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. **(6+6)**

6. (a) At a party n men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of no matches? What is the probability of exactly k matches? **(8+5)**

(b) A computer receives requests for elements stored in its memory. Consider that n elements $e_1, e_2, \ldots, e_n$ are initially arranged in some ordered list. At each unit of time a request is made for one of these elements – e_i, being requested, independently of the past, with probability P_i. After being requested, the element is then moved to the front of the list. That is, for instance, if the present ordering is e_1, e_2, e_3, e_4 and e_3 is requested, then the next ordering is e_3, e_1, e_2, e_4. Determine the expected position of the element requested after this process has been in operation for a long time. **(12)**

(c) In an election, candidate A receives n votes, and candidate B receives m votes where $n>m$. Assuming that all orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n-m)/(n+m)$. **(10)**

7. (a) A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportions of days are sunny? What proportions are cloudy? **(8)**

(b) Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1-p$ of losing one unit. Assume that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0? **(12)**

(c) Define Markov chain. Derive the Chapman-Kolmogorov equations for computing n-step transition probabilities in a Markov chain. **(3+4)**

Contd P/3

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Contd... Q. No. 7

(d) Consider a large population of individuals, each of whom possesses a particular pair of genes, of which each individual gene is classified as being of type α or type β. Assume that the proportions of individuals whose gene pairs are αα, ββ, or αβ, respectively are p_o, q_o and $r_o (p_o + q_o + r_o = 1)$ respectively. When two individuals mate, each contributes one of his or her genes, chosen at random, to the resultant offspring. Assuming that the mating occurs at random, in which each individual is equally likely to mate with other individual, determine the proportions p, q, and r of individuals in the next generation whose genes are αα, ββ, or αβ respectively. **(8)**

8. (a) If X and Y are independent Poisson random variables with respective means λ_1 and λ_2, calculate the conditional expected value of X given that $X + Y = n$. **(10)**

(b) Define moments of a Random Variable. How can we obtain them from moment generating function? Derive the moment generating function for the binomial distribution with parameters n and p. If X and Y are independent binomial random variables with parameters (n,p) and (m,p), respectively, then what is the distribution of $X+Y$. **(2+3+4+4=13)**

(c) Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5. If the sum of chips drawn is even, the random variable X equals 5; if the sum of chips drawn is odd, X = -3. **(12)**

(i) Find the moment-generating function (MGF) for X.

(ii) Use the MGF to find the first and second moments.

(iii) Find the expected value and variance of X.

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

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Full Marks: 210 Time : 3 Hours

USE SEPARATE SCRIPTS FOR EACH SECTION

The figures in the margin indicate full marks.

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) Deduce the maximum number of regions n planes can divide the 3-D space. **(15)**

   (b) Let us have Josephus problem where among n persons every 9th person is deleted, and let $J_q(n)$ be the ultimate survivor. Deduce $J_q(n)$ and computed $J_8(1000)$. **(20)**

2. (a) Discuss multiple Tower of Hanoi problem. Deduce the properties of the Presumed Optimal Solution. Solve the problem for n = 378 and p = 8 showing the solution in a binary tree of level 2. **(15)**

   (b) Discuss the properties of finite calculus with examples. Compute the value of $\sum_{0\le k<n} k^2 H_k$. **(20)**

3. (a) Using combinatorial arguments establish the identities **(15)**

$$\sum_{0\le k\le n}\binom{k}{m}=\binom{n+1}{m+1}$$

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$$

   (b) If n men throw their hats and randomly picks up hats. In how many ways can they end up in receiving wrong hats? Use inversion formula to deduce the results. **(20)**

4. (a) Discuss Stirling numbers in detail and deduce recurrence relations they satisfy. **(15)**

   (b)(i) Discuss Harmonic numbers with their properties. Calculate maximum overhang that can be created by placing n cards on n table. **(20)**

   (ii) Discuss Euler number and establish recurrence relations it satisfies.

Contd P/2

SECTION – B

There are **FOUR** questions in this section. Answer any **THREE**.

5. (a) Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1\text{-}p$. The winner of the series is the first team to win i games. Find the expected number of games that are played with (i) $i = 2$, and (ii) i = 3. **(5+5)**

   (b) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads. What is the probability that it is the fair coin? Suppose that he flips the same coin a second time and this time it shows tail. Now what is the probability that it is the fair coin? **(5+5)**

   (c) Define moments of a Random Variable. How can we obtain them from moment generating function? Derive the moment generating function for the Poisson distribution with parameter λ? If X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively, then what is the distribution of $X+Y$? **(2+3+5+5)**

6. (a) Each element in a sequence of binary data is either 1 with probability p or 0 with probability $1\text{-}p$. A maximal subsequence of consecutive values having identical outcomes is called a run. For instance, if the outcome sequence is 1, 1, 0, 1, 1, 1, 0, the first run is of length 2, the second is of length 1, and the third is of length 3. **(6+6)**
   (i) Find the expected length of the first run.
   (ii) Find the expected length of the second run.

   (b) In an election, candidate A receives n votes, and candidate B receives m votes where $n > m$. Assuming that all orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n-m)/(n+m)$. **(11)**

   (c) A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom. **(6+6)**
   (i) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?
   (ii) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)

Contd P/3

7. (a) A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy? **(8)**

(b) Consider a large population of individuals, each of whom possesses a particular pair of genes, of which each individual gene is classified as being of type α or type β. Assume that the proportions of individuals whose gene pairs are αα, ββ or αβ are respectively p_0, q_0, and $r_0 \left(p_0 + q_0 + r_0 = 1\right)$. When two individuals mate, each contributes one of his or her genes, chosen at random, to the resultant offspring. Assuming that the mating occurs at random, in that each individual is equally likely to mate with any other individual, determine the proportions p, q, and r of individuals in the next generation whose genes are αα, ββ, or αβ respectively. **(12)**

(c) Consider a Poisson process with rate λ, and let us denote the time of the first event by $T1$. Further, for $n>1$, let T_n denote the elapsed time between the $(n-1)$st and the nth event. The sequence $\{T_n, n = 1, 2, \ldots,\}$ is called the sequence of inter-arrival times. Show that $T_n, n = 1, 2, \ldots,$ are independent identically distributed exponential random variables having mean 1/λ. **(7)**

(d) Suppose that X_1 and X_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$; what is $P\{X_1 < X_2\}$? **(8)**

8. (a) Customers arrive at a single-server station in accordance with a Poisson process with rate λ. All arrivals that find the server free immediately enter service. All service times are exponential distributed with rate μ. An arrival that finds the server busy will leave the system and roam around "in orbit" for an exponential time with rate θ at which time it will then return. If the server is busy when an orbiting customer returns, then that customer returns to orbit for another exponential time with rate θ before returning again. An arrival that finds the server busy and N other customers in orbit will depart and not return. That is, N is the maximum number of customers in orbit. **(4+3+3+3)**

   (i) Define states.
   (ii) Give the balance equations.

In terms of the solution of the balance equations, find

   (iii) the proportion of all customers that are eventually served;
   (iv) the average time that a served customer spends waiting in orbit.

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Contd... Q. No. 8

(b) Consider a two-server system in which customers arrive at a Poisson rate λ at server 1. After being served by server 1 they then join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server serves one customer at a time with server i taking an exponential time with rate μ_i for a service, $i = 1, 2$. Such a system is called a tandem or sequential system. For the tandem queuing system, compute **(12)**

(i) the average number of customers in the system,

(ii) the average time a customer spends in the system,

(c) Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates of 1 and 2 are, respectively, 8 and 10. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system (i.e., $P_{11} = 0, P_{12} = 1/2$), whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise (i.e., $P_{21} = 1/4, P_{22} = 0$). Determine **(10)**

(i) the probability that there are n customers at server 1 and m customers at server 2.

(ii) the average number of customers in the system L, and

(iii) the average time a customer spends in the system W.
