User Equilibrium with Link Interaction

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Motivation

Up to this point, we have assumed that link performance functions are independent of each of other, i.e., travel time of a link depends only its own flow but not on the flow of other links. However, this assumption may not hold in many situations. For example,

- Two-way streets: The delay in one direction will increase if the traffic builds up in the opposite direction.
- Multi-class traffic flow: Travel time of one class of vehicle (e.g., motorcycles) will depend on travel time of other vehicle classes in the network (e.g., bus, cars, etc.). Such situations can be modeled using multi-layer network with each layer representing each class of vehicle.
- Junction interaction: The delay on one approach depends on flows from competing approaches. Right turn traffic has to yield to the through movements. Other example is ramp metering.
- Bus stop: Waiting time at a bus stop depends on passengers waiting as wells passengers on-board.

Link interactions

In this case travel time of a link is no longer separable as it depends on not just on its on flow but also on flow of other links as well. We can express the travel times of different links can be written as $\mathbf{t}(\mathbf{x})$ (vector). Note that its Jacobian is given as:

$$J\mathbf{t}(\mathbf{x}) = \begin{bmatrix} \frac{\partial t_1(\mathbf{x})}{x_1} & \frac{\partial t_1(\mathbf{x})}{x_2} & \dots & \frac{\partial t_1(\mathbf{x})}{x_n} \\ \frac{\partial t_2(\mathbf{x})}{x_1} & \frac{\partial t_2(\mathbf{x})}{x_2} & \dots & \frac{\partial t_2(\mathbf{x})}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_n(\mathbf{x})}{x_1} & \frac{\partial t_n(\mathbf{x})}{x_2} & \dots & \frac{\partial t_n(\mathbf{x})}{x_n} \end{bmatrix}$$

Types of link interaction

Jacobian	**Type**	**Convex Optimization formulation known?**
Diagonal	Basic TAP	Yes
Symmetric	Symmetric TAP	Yes
Asymmetric	Asymmetric TAP	No

VI formulation for UE with link interaction

Let $t_{ij} = t_{ij}(x_1, x_2, \cdots, x_{ij}, \cdots, x_{|A|})$. We can formulate UE in terms of path flows, i.e.,

$$\boxed{\mathbf{c}(\mathbf{h}^*)^T(\mathbf{h} - \mathbf{h}^*) \geq 0, \forall \mathbf{h} \in H} \tag{1}$$

In terms of link flows, i.e.,

$$\boxed{\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X} \tag{2}$$

Symmetric Traffic Assignment

Example: Two-way traffic interactions, let (i, j) be the link and $(i^{'}, j^{'})$ be the link in opposite direction. The link performance functions can be given as:

$$t_{ij} = t_{ij}(x_{ij}, x_{i'j'}) \tag{3}$$

$$t_{i'j'} = t_{i'j'}(x_{i'j'}, x_{ij}) \tag{4}$$

The symmetry condition is given by:

$$\frac{\partial t_{ij}(x_{ij}, x_{i'j'})}{\partial x_{i'j'}} = \frac{\partial t_{i'j'}(x_{i'j'}, x_{ij})}{\partial x_{ij}}, \forall (i, j) \in A \tag{5}$$

The marginal effect of link flow x_{ij} on $t_{i'j'}$ is equal to the marginal effect of $x_{i'j'}$ on t_{ij}. In case of symmetric traffic assignment can be formulated as an optimization problem:

Symmetric Traffic Assignment

$$\min_{\mathbf{x}} \quad \frac{1}{2}\sum_{(i,j)\in A}\left(\int_0^{x_{ij}} t_{ij}(\omega, x_{i'j'})d\omega + \int_0^{x_{ij}} t_{ij}(\omega, 0)d\omega\right) \tag{6a}$$

$$\text{s.t.} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s)\in Z^2 \tag{6b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{6c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j)\in A \tag{6d}$$

- The first integral is wrt its own flow while keeping the flow in opposite direction constant. The second integral is wrt to own flow while the flow in opposite direction is kept zero. The objective is halved to scale.
- The objective function is carefully crafted to evaluate UE. It does not have any economic or behavioral interpretation.

Symmetric Traffic Assignment

Proposition

The KKT conditions of (6) *represent the UE conditions.*

Proof.

Let's do it together. □

Remark. If travel time function is strictly increasing i.e., $\frac{\partial t_{ij}(x_{ij}, x_{i'j'})}{\partial x_{ij}} > 0$ and the effect of additional flow on travel time in its own direction is more than the effect on travel time in the other direction, i.e., $\frac{\partial t_{ij}(x_{ij}, x_{i'j'})}{\partial x_{ij}} >> \frac{\partial t_{i'j'}(x_{ij}, x_{i'j'})}{\partial x_{ij}}$, then objective function (6a) is strictly convex and therefor UE solution is unique in terms of link flows.

Remark. We can develop solve (6) using Frank-Wolfe method.

Symmetric Traffic Assignment

Generally, for travel time functions $t_{ij} = t_{ij}(x_1, x_2, \cdots, x_{ij}, \cdots, x_{|A|})$, the symmetry condition can be stated as:

$$\frac{\partial t_{ij}(\mathbf{x})}{\partial x_{i'j'}} = \frac{\partial t_{i'j'}(\mathbf{x})}{\partial x_{ij}}, \forall (i,j) \neq (i^{'}, j^{'}) \tag{7}$$

In other words, the symmetry condition can be stated as a requirement that Jacobian of vector of travel time functions is a symmetric matrix.

$$J\mathbf{t}(\mathbf{x}) = \begin{bmatrix} \frac{\partial t_1(\mathbf{x})}{x_1} & \frac{\partial t_1(\mathbf{x})}{x_2} & \dots & \frac{\partial t_1(\mathbf{x})}{x_n} \\ \frac{\partial t_2(\mathbf{x})}{x_1} & \frac{\partial t_2(\mathbf{x})}{x_2} & \dots & \frac{\partial t_2(\mathbf{x})}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_n(\mathbf{x})}{x_1} & \frac{\partial t_n(\mathbf{x})}{x_2} & \dots & \frac{\partial t_n(\mathbf{x})}{x_n} \end{bmatrix} \in \mathbb{S}^n$$

Symmetric Traffic Assignment

Symmetry condition allows us to formulate an optimization problem to solve for UE. In this case, the vector of ravel times $\mathbf{t}(\mathbf{x})$ is a gradient (or conservative filed) which means $\mathbf{t}(\mathbf{x})$ can be expressed as a gradient of scalar potential function $\Phi(\mathbf{x})$ [1], i.e.,

$$\mathbf{t}(\mathbf{x}) = \nabla\Phi(\mathbf{x})$$

where,

$$t_{ij}(\mathbf{x}) = \frac{\partial\Phi(\mathbf{x})}{\partial x_{ij}}, \forall(i,j) \in A$$

Since $\mathbf{t}(\mathbf{x})$ is a gradient field, then the line integral

$$\sum_{(i,j)\in A} \int_{\mathbf{0}}^{\mathbf{x}} t_{ij}(\mathbf{x})d\mathbf{x} \tag{8}$$

is path-independent (only depends on the end point $\mathbf{x}$ and $\mathbf{0}$). One can minimize (8) to obtain UE. If the Jacobian is positive definite then the minimum will be unique.

[1] since $\frac{\partial^2\Phi(\mathbf{x})}{\partial x_{ij} x_{i^{\prime} j^{\prime}}} = \frac{\partial^2\Phi(\mathbf{x})}{\partial x_{i^{\prime} j^{\prime}} x_{ij}}$ due to symmetry (Schwarz's theorem)

Asymmetric Traffic Assignment

The symmetry condition is difficult to defend in many practical situations. For the asymmetric Jacobian of travel time functions, the integral (8) is dependent on path between $\mathbf{0}$ to $\mathbf{x}$. Hence, it is not possible to formulate an easy convex optimization problem to solve for UE. But we can still pose UE as VI as before. Although uniqueness of UE can be shown if Jacobian is positive definite (even though it is not symmetric).

Diagonalization

Diagonalization can be used for solving UE in case of asymmetric traffic assignment. At any iteration k, we know the current vector of link flows $\mathbf{x}^k = (x_1^k, x_2^k, \cdots, x_{ij}^k, \cdots, x_n^k)$. We solve the following optimization problem:

$$Z^{UE} = \underset{\mathbf{x},\mathbf{h}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} \hat{t}_{ij}(x_{ij}) = t_{ij}^k(x_1^k, x_2^k, \cdots, \omega, \cdots, x_n^k) d\omega \tag{9a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s) \in Z^2 \tag{9b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{9c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j) \in A \tag{9d}$$

where, $\hat{t}_{ij}^k(x_{ij})$ is obtained by fixing the flows on links other than (i,j). We solve this diagonalized subproblem.

Diagonalization

1. Set the iteration $k = 0$ and find a feasible link flow vector $\mathbf{x}$ (e.g., by performing all or nothing assignment).
2. Solve (9) by using any of the UE methods and obtain $\mathbf{x}^{k+1}$.
3. Check for convergence by evaluating $\text{gap} = \sum_{(i,j)\in A} \frac{|x_{ij}^k - x_{ij}^{k-1}|}{x_{ij}^k} \le \epsilon$.

Remark. Diagonalization can be proven to converge when Jacobian of travel time functions is positive definite. Otherwise, it may not converge

Uniqueness

Definition (Strict monotonicity). Let $X \subseteq \mathbb{R}^n$. A function $f : X \mapsto \mathbb{R}$ is a strictly monotone on X if

$$[f(\mathbf{x}_1) - f(\mathbf{x}_2)]^T (\mathbf{x}_1 - \mathbf{x}_2) > 0, \forall \mathbf{x}_1, \mathbf{x}_2 \in X, \mathbf{x}_1 \neq \mathbf{x}_2 \tag{10}$$

Theorem

If $\mathbf{f}$ us continuously differentiable function whose domain is convex, then $\mathbf{f}$ is strictly monotone if and only if its Jacobian is positive definite.

Theorem

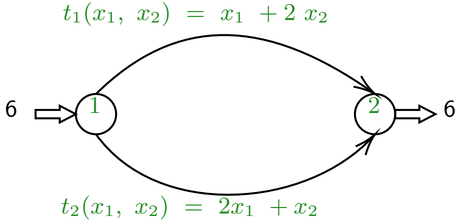
For a traffic assignment problem with continuous and strictly monotone, then there is exactly one User Equilibrium solution.

Proof.

See lecture on "Tools for modeling equlibria" lecture. □

Stability

When the travel time functions are are not strictly monotone, then equilibrium may not be unique. There can be multiple equilibria. For example, consider the following example from BLU book:



1. First UE solution: $x_1^* = x_2^* = 3, t_1^* = t_2^* = 9$
2. Second UE solution: $x_1^* = 6, x_2^* = 0, t_1^* = 6, t_2^* = 12$
3. Third UE solution: $x_1^* = 0, x_2^* = 6, t_1^* = 12, t_2^* = 6$

Remark. An equilibrium is stable is small perturbations to the solution will force travelers to move back to their initial equilibrium, i.e., the travelers shift to another path, they would want to move back to their original path. For first UE solution, if some travelers shift from first path to second, then $x_1^* = 2, x_2^* = 4, t_1^* = 10, t_2^* = 8$. More travelers would want to move to second path. This is not a stable equilibirum.

Optimization formulation for general asymmetric assignment

We can define a class of gap function that can be minimized to obtain equilibrium. Although the gap functions may not be convex and multiple local equilibirum may exist. For more details, refer to the following resources:

- Patriksson Chapter 3 Section 3.1.5
- Fukushima, Masao. "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems." Mathematical programming 53.1 (1992): 99-110.

Thank you!