Minimum cost flow problem

Pramesh Kumar

IIT Delhi

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Introduction

- Shortest path problems model link cost but not link capacity. On the other hand, the max flow problem models link capacity but not link cost.
- Mincost flow problem (MCFP) models both link costs as well as link capacity.
- It is fundamental problem with numerous applications such as production planning, scheduling, transportation of goods, etc.

Definition (Minimum cost flow problem). Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, lower and upper bounds (capacity) on the flow on links $l : A \mapsto \mathbb{R}$ and $u : A \mapsto \mathbb{R}$ resp., and supply/demand at each node $b : N \mapsto \mathbb{R}$, find the least cost shipment of a commodity.

Assumptions

- All data (costs, capacities, supply/demand) are integral.
- The network is directed.
- Supply/demand balance, i.e., $\sum_{i \in N} b(i) = 0$ and MFCP has a feasible solution.[1]
- All costs are non-negative.

[1] One can find a feasible solution to MCFP by solving a max flow problem on a modified network with a "super source" connecting each supply node using a link with capacity $b(i)$ and a "super sink" connecting each demand node using a link with capacity $b(i)$. If the max flow saturates all the source links, then that flow is a feasible solution to MCFP.

LP formulation

Primal

$$\min_{\mathbf{x}} \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j\in FS(i)} x_{ij} - \sum_{j\in BS(i)} x_{ji} = b(i), \forall i \in N$$

$$0 \le x_{ij} \le u_{ij}, \forall (i,j) \in A$$

where, if $b(i) > 0, b(i) < 0$, and $b(i) = 0$, then i is called supply node, demand node, and transshipment node respectively.

Dual

$$\max_{\mathbf{d},\alpha} \sum_{i\in N} b(i)\pi_i - \sum_{(i,j)\in A} \alpha_{ij} u_{ij}$$

$$\text{s.t. } \pi_i - \pi_j - \alpha_{ij} \le c_{ij}, \forall (i,j) \in A$$

$$\alpha_{ij} \ge 0$$

$$\pi_i \text{ free }, \forall i \in N$$

$$= \max_{\pi} \sum_{i\in N} b(i)\pi_i - \sum_{(i,j)\in A} \max\{-c^{\pi}_{ij}, 0\} u_{ij}$$

where, $\boxed{c^{\pi}_{ij} = c_{ij} - \pi_i + \pi_j}$ is called reduced cost of link $(i,j) \in A$

Residual network

Residual network corresponding to flow $\mathbf{x}$ is created as below:

- Replace each link (i,j) by two links (i,j) and (j,i).
- Put c_{ij} cost on link (i,j) and $-c_{ij}$ cost on link (j,i).
- Put $r_{ij} = u_{ij} - x_{ij}$ as the residual capacity on link (i,j) and $r_{ji} = x_{ij}$ as the residual capacity on link (j,i).
- Remove links with zero residual capacity.

Negative cycle optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ is an optimal solution if and only if the residual network $G(\mathbf{x}^*)$ contains no negative cost (directed) cycles.

Proof.

$\Longrightarrow$ Suppose $\mathbf{x}^*$ is an optimal flow and there still exists a negative cost (directed) cycle. Then, one can improve the minimum cost by augmenting a positive flow along that cycle, which contradicts that $\mathbf{x}^*$ is optimal.

$\Longleftarrow$ Suppose that $\mathbf{x}^*$ is a feasible flow and $G(\mathbf{x}^*)$ contains no negative cost (directed) cycles. Let $\mathbf{x}^{'}$ be an optimal flow and $\mathbf{x}^* \neq \mathbf{x}^{'}$. Then, we can decompose the flow $\mathbf{x}^{'} - \mathbf{x}^*$ into m augmenting cycles wrt the flow $\mathbf{x}^*$ and sum of costs of these augmenting cycles will be equal to $\mathbf{c}^T(\mathbf{x}^{'} - \mathbf{x}^*)$. Since the cost of every augmenting cycle is non-negative, we have $\mathbf{c}^T(\mathbf{x}^{'} - \mathbf{x}^*) \geq 0 \implies \mathbf{c}^T\mathbf{x}^{'} \geq \mathbf{c}^T\mathbf{x}^*$. Also, $\mathbf{x}^{'}$ is optimal, we have $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}^{'}$. Therefore, $\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T\mathbf{x}^{'}$ and $\mathbf{x}^*$ is also optimal flow. $\square$

Reduced costs

Given node potentials (or dual variables corresponding to conservation constraints) $\pi(i)$, $\boxed{c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)}$ is called the reduced cost of link $(i, j) \in A$.

Economic interpretation: $-\pi(i)$: cost of obtaining a unit of this commodity at node i. $c_{ij} - \pi(i)$: the cost of obtaining unit commodity to node j from node i.

Lemma

1. *For a directed path P from node k to node l,* $\sum_{(i,j) \in P} c_{ij}^{\pi} = \sum_{(i,j) \in P} c_{ij} - \pi(k) + \pi(l)$.
2. *For a directed cycle W,* $\sum_{(i,j) \in W} c_{ij}^{\pi} = \sum_{(i,j) \in W} c_{ij}$

Proof.

1. *Let $P = \{k = i_1, ..., i_h = l\}$ be a directed path. Then,* $\sum_{(i,j) \in P} c_{ij}^{\pi}$ $= c_{i_1 i_2}^{\pi} + \cdots + c_{i_{h-1} i_h}^{\pi} = (c_{i_1 i_2} - \pi(i_1) + \pi(i_2)) + \cdots + (c_{i_{h-1} i_h} - \pi(i_{h-1}) + \pi(i_h)) = \sum_{(i,j) \in P} c_{ij} - \pi(k) + \pi(l)$.
2. *Trivial.*

□

Remark. 2 implies that if W is a negative cost cycle wrt costs c_{ij}, then it

Reduced costs optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ is an optimal solution if and only if there exists node potentials π which satisfy $c_{ij}^{\pi} \geq 0, \forall (i,j) \in G(\mathbf{x}^*)$.

Proof.

$\Longleftarrow$ Assume that for a feasible solution $\mathbf{x}^*$, $c_{ij}^{\pi} \geq 0, \forall (i,j) \in G(\mathbf{x}^*)$. Then, we know that $\sum_{(i,j) \in W} c_{ij}^{\pi} \geq 0$ for every directed cycle W in $G(\mathbf{x}^*)$ (previous lemma). Then, there does not exists any cycle with negative cost. Using negative cycle optimality conditions, we know that $\mathbf{x}^*$ is optimal.

$\Longrightarrow$ Assume that $\mathbf{x}^*$ is optimal, then using negative cycle optimality conditions, we know that there are no negative cost directed cycles in $G(\mathbf{x}^*)$. Then, find the shortest path from node 1 (w.l.o.g.) to all other nodes. Since, there are no no negative cost directed cycles, we can find shortest path labels $d(i)$ for all nodes satisfying $d(j) \leq d(i) + c_{ij}, \forall (i,j) \in G(\mathbf{x}^*)$. Define $\pi(i) = -d(i), \forall i \in N$. Clearly, $c_{ij} - (-d(i)) + (-d(j)) \geq 0$.

$\square$

Remark. $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j) \geq 0$ means that cost of obtaining the commodity at node j is no more than the cost of the commodity if we obtain at node i and incur the transportation cost in sending it from node i to node j.

Complementary slackness optimality conditions

Theorem

A feasible solution $\mathbf{x}^*$ is an optimal solution if and only if there exists node potentials π that (together with $\mathbf{x}^*$) satisfy the following complementary slackness optimality conditions:

- *If $c_{ij}^{\pi} > 0$, then $x_{ij}^* = 0$.*
- *If $0 < x_{ij}^* < u_{ij}$, then $c_{ij}^{\pi} = 0$.*
- *If $c_{ij}^{\pi} < 0$, then $x_{ij}^* = u_{ij}$.*

Proof.

We'll show that if $\mathbf{x}^*$ and π satisfy the reduced cost optimality conditions, then they also satisfy complementary conditions.

- If $c_{ij}^{\pi} > 0$, then residual network cannot contain link (j, i) because $c_{ji}^{\pi} = -c_{ij}^{\pi} < 0$ contradicting the reduced cost optimality conditions. Therefore, $x_{ij}^* = 0$.
- If $0 < x_{ij}^* < u_{ij}$, then the residual network contains both (i, j) and (j, i). Further, reduced cost optimality conditions state that $c_{ij}^{\pi} \geq 0$ as well as $c_{ji}^{\pi} \geq 0$. But we know that $c_{ij}^{\pi} = -c_{ji}^{\pi}$. Therefore, $c_{ij}^{\pi} = 0$.
- If $c_{ij}^{\pi} < 0$, then residual network cannot contain link (i, j) since its reduced cost violates the reduced cost optimality conditions. Therefore, $x_{ij}^* = u_{ij}$.

□

Evaluating optimal node potentials given optimal flows

- Construct $G(\mathbf{x}^*)$
- Solve the shortest path from node 1 (pick arbitrarily) to all other nodes and compute distance labels $d(i)$ (which are well defined since no negative cycle exists at optimality).
- Assign $\pi(i) = -d(i)$

Remark. Above node potentials are optimal because $c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j) = c_{ij} - (-d(i)) + (-d(j)) \geq 0 \implies d(j) \leq d(i) + c_{ij}$ which are shortest path optimality conditions.

Evaluating optimal flows given optimal node potentials

- Compute reduced cost c_{ij}^{π} of each link $(i,j) \in A$.
- If $c_{ij}^{\pi} > 0$, then assign $x_{ij}^{*} = 0$. Remove (i,j) from the network.
- If $c_{ij}^{\pi} < 0$, then assign $x_{ij}^{*} = u_{ij}$. Remove (i,j) from the network. Reduce $b(i)$ by u_{ij} and increase $b(j)$ by u_{ij}
- The network $G^{'}(N, A^{'})$ with modified supply/demand $b^{'}(i)$ at nodes.
- Add new links from "super source" to supply nodes (with capacity $b^{'}(i)$) and demand nodes to "super sink" (with capacity $-b^{'}(i)$).
- Solve the max flow problem from super source to super sink. Assign x_{ij}^{*} equal to the optimal solution of max flow problem.

Remark. Above node potentials are optimal because $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j) = c_{ij} - (-d(i)) + (-d(j)) \geq 0 \implies d(j) \leq d(i) + c_{ij}$ which are shortest path optimality conditions.

Cycle-canceling algorithm

```
procedure CycleCanceling(G, c, u, b)
    find a feasible flow x[2] in the network.
    while G(x) contains a negative cycle do
        find a negative cycle W[3].
        δ = min{r_ij : (i, j) ∈ W}
        augment δ units of flow along W
        update G(x)
    end while
end procedure
```

- The upper bound on the initial cost of flow is mCU.
- The lower bound on the optimal cost of flow is $-mCU$.
- Each iteration of above algorithm changes the objective value by $\left(\sum_{(i,j)\in A} c_{ij}\right)\delta < 0$.
- Finding the cycle takes $O(mn)$ time using label correcting algorithm.
- Since data is integral, total time in running the algorithm is $O(mn \times 2mCU) = O(m^2nCU)$.

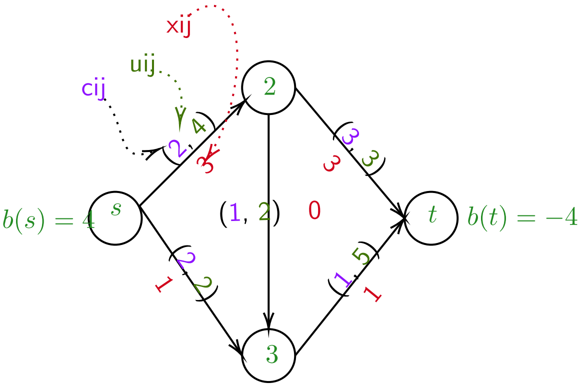
[2] possibly by solving max flow problem on a modified network

[3] possibly using label correcting algorithm

Theorem

If all link capacities and supplies/demands of nodes are integer, the minimum cost flow problem has always integer minimum cost flow.

Example



Definition (pseudoflow). A pseudoflow is a function $x : A \mapsto \mathbb{R}_+$ satisfying only capacity and non-negativity constraints; conservation constraints may or may not be satisfied.

Definition (Imbalance). For any pseudoflow, we define the imbalance of node i as $e(i) = b(i) + \sum_{j \in BS(i)} x_{ji} - \sum_{j \in FS(i)} x_{ij}$.

- If $e(i) > 0$, we call i as excess node. Let E be set of excess nodes.
- If $e(i) < 0$, we call i as deficit node. Let D be set of deficit nodes.
- If $e(i) = 0$, we call i as balanced node.

Clearly, $\sum_{i \in N} e(i) = \sum_{i \in N} b(i)$ and $\sum_{i \in D} e(i) = -\sum_{i \in D} e(i)$

Lemma

Suppose pseudoflow $\mathbf{x}$ satisfies the reduced cost optimality conditions wrt some node potentials π. Let $\mathbf{d}$ represent the SP distance label from some node s to all other nodes in the residual network $G(\mathbf{x})$ with c_{ij}^{π} as the link costs. Then,

1. *The pseudoflow also satisfies the reduced cost optimality conditions wrt the node potentials $\pi^{'} = \pi - \mathbf{d}$.*
2. *The reduced costs $c_{ij}^{\pi^{'}}$ are zero for all links (i, j) in a SP from node s to every other node.*

Proof.

1. Since $\mathbf{x}$ satisfies the reduced cost optimality conditions wrt some node potentials π, $c_{ij}^{\pi} \geq 0, \forall (i,j) \in G(\mathbf{x})$. Since $\mathbf{d}$ represent the shortest path labels, we have $d(j) \leq d(i) + c_{ij}^{\pi}$. Plugging the value of c_{ij}^{π}, we get $d(j) \leq d(i) + c_{ij} - \pi(i) + \pi(j)$
$\implies c_{ij}^{\pi} - (\pi(i) - d(i)) + (\pi(j) - d(j)) \geq 0$.
2. Consider a SP P from k to l. We know that
$d(j) = d(i) + c_{ij}^{\pi}, \forall (i,j) \in P$. Plugging c_{ij}^{π}, we get
$d(j) = d(i) + c_{ij} - \pi(i) + \pi(j) \implies c_{ij}^{\pi} = 0$.

□

Lemma

Let pseudoflow $\mathbf{x}$ satisfy the reduced cost optimality conditions and we obtain flows $\mathbf{x}^{'}$ from $\mathbf{x}$ by sending flow along a SP P from s to some other node k; then $\mathbf{x}^{'}$ also satisfies the reduced cost optimality conditions.

Proof.

Define π and $\pi^{'}$ as in the previous lemma. We know that
$c_{ij}^{\pi^{'}} = 0, \forall (i,j) \in P$. Augmenting flow on any link (i,j) in P might add
(j,i) to the residual network but since $c_{ij}^{\pi^{'}} = 0 \implies c_{ji}^{\pi^{'}} = 0$. □

Successive shortest path algorithm

procedure SUCCESSIVESHORTESTPATH($G, \mathbf{c}, \mathbf{u}, \mathbf{b}$)
    $\mathbf{x} = 0$; $\pi = 0$
    $e(i) = b(i), \forall i \in N$
    while $E \neq 0$ **do**
        select a node $k \in E$ and $l \in D$
        determine SP distance labels $\mathbf{d}$ from k to all nodes in $G(\mathbf{x})$ wrt costs c^{π}_{ij}.
        let P be the shortest path from k to l
        Update $\pi = \pi - \mathbf{d}$
        $\delta = \min\{e(k), -e(l), \min\{r_{ij} \ : \ (i,j) \in P\}\}$
        augment δ units of flow along the path P
        update $\mathbf{x}$, $G(\mathbf{x}), E, D$ and the reduced costs
    end while
end procedure

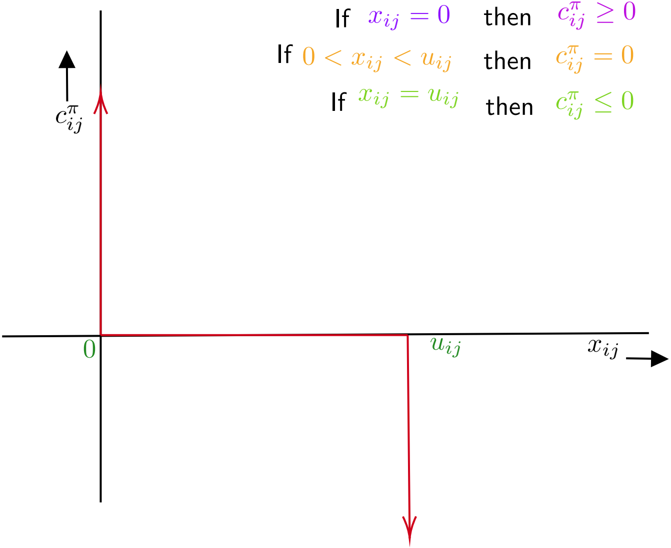
- Let U be the largest supply of any node.
- Each iteration strictly decreases the excess of some node.
- The algorithm will terminate in at most nU iterations.
- Let $O(mn)$ be the complexity of label correcting SP algorithm.
- Therefore, overall complexity is $O(mn^2U)$.
- One can use A^* to find SP from k to l to save some time.

Kilter diagram

If $x_{ij} = 0$ then $c^{\pi}_{ij} \geq 0$

If $0 < x_{ij} < u_{ij}$ then $c^{\pi}_{ij} = 0$

If $x_{ij} = u_{ij}$ then $c^{\pi}_{ij} \leq 0$



If a point (x_{ij}, c^{π}_{ij}) lies in the red lines in the kilter diagram, the link is in-kilter; otherwise, it is out-of-kilter.

Definition (Kilter number). The kilter number of link $(i, j) \in A$ is the magnitude of the change in x_{ij} required to make a link an in-kilter link while keeping c^{π}_{ij} fixed.

- If $c^{\pi}_{ij} > 0$, then $k = |x_{ij}|$
- If $c^{\pi}_{ij} < 0$, then $k = |u_{ij} - x_{ij}|$
- If $c^{\pi}_{ij} < 0$ and $x_{ij} > u_{ij}$, then $k_{ij} = x_{ij} - u_{ij}$
- If $c^{\pi}_{ij} < 0$ and $x_{ij} < 0$, then $k_{ij} = -x_{ij}$

The kilter number of any in-kilter link is 0.

Out-of-kilter algorithm

procedure OutOfKilter($G, \mathbf{c}, \mathbf{u}, \mathbf{b}$)
  $\pi = \mathbf{0}$
  find a feasible flow $\mathbf{x}$ in the network.
  define the feasible network $G(\mathbf{x})$ and compute $k_{ij}, \forall (i,j) \in A$.
  while the network contrains an out-of-kilter link **do**
    select an out-of-kilter link (p,q) in $G(\mathbf{x})$
    assign cost of each link $(i,j) \in A$ in $G(\mathbf{x})$ as $\max\{0, c_{ij}^{\pi}\}$
    find the shortest path from q to all nodes in $G(\mathbf{x}) \setminus \{(p,q)\}$
    determine distance labels $\mathbf{d}$
    let P be the SP from q to p
    update $\pi^{'}(i) = \pi(i) - d(i), \forall i \in N$
    if $c_{pq}^{\pi^{'}} < 0$ **then**
      $W = P \cup \{(p,q)\}$
      $\delta = \min\{r_{ij} \ : \ (i,j) \in W\}$
      augment δ units of along W
      update $\mathbf{x}$, $G(\mathbf{x})$, and the reduced costs
    end if
  end while
end procedure

Runs in $O(m^2 nC)$ time.

Final remarks

- We did not study many other algorithms to solve this problem. I suggest that you study the following from AMO book.
  - Primal-dual algorithm
  - Lagrangian relaxation-based algorithm
  - Network simplex algorithm
- The algorithms we studied had pseudo-polynomial complexity. The scaling algorithms have polynomial complexity.
  - Minimum cost scaling algorithm
  - Cost scaling algorithm
  - Double scaling algorithm

Suggested reading

- AMO Chapter 9 and 10

Thank you!