Sensitivity analysis and its applications in Highway Network Design and OD estimation

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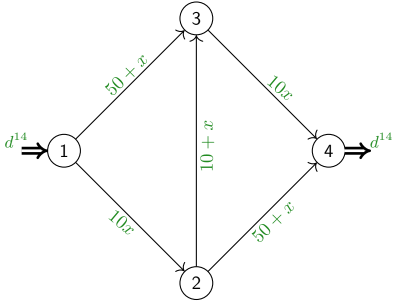
Motivation

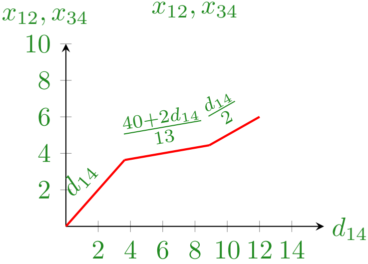
- Sensitivity analysis help study the changes in the model output when model inputs (parameters) are changed.
- In our case, we would like to know how UE flow changes as parameters such as demand or free flow link travel time are changed.
- This will help us understand how "sensitive" UE flows are to the changes in the input value.
- This is useful in OD estimation and network design problems that we will study in the next lecture.

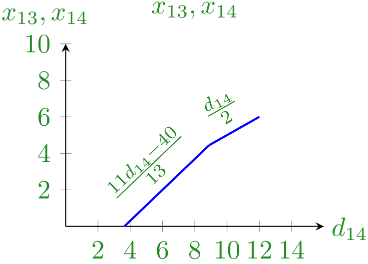
Example

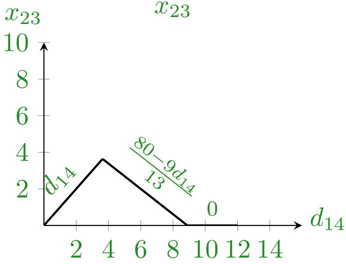
Consider the Braess' network. When $d^{14} = 6$, we evaluated the UE flows as: $x_{13} = x_{23} = x_{24} = 2$ and $x_{12} = x_{34} = 4$ and equilibrium travel times on all used paths was 92 minutes. What if we change the demand values, how would these flow values and equilibirum travel times look different?

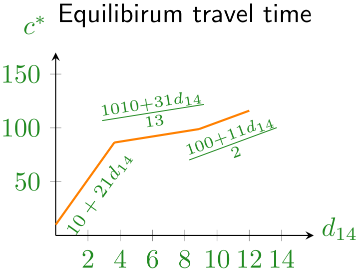


Equilibrium flows and travel times as functions of d_{14}









Observations from previous plots

- For $0 \leq d_{14} < \frac{40}{11}$ only middle path $\pi_3 = \{(1,2),(2,3),(3,4)\}$ is used in equilibrium.
- For $\frac{40}{11} \leq d_{14} < \frac{80}{9}$ All three paths $\pi_1 = \{(1,3),(3,4)\}$, $\pi_2 = \{(1,2),(2,4)\}$, and $\pi_3 = \{(1,2),(2,3),(3,4)\}$ are used in equilibrium.
- For $d_{14} \geq \frac{80}{9}$ Only paths $\pi_1 = \{(1,3),(3,4)\}$ and $\pi_2 = \{(1,2),(2,4)\}$ are used in equilibrium.
- Both equilibrium flows and travel times are piece wise linear functions of demand. These are differentiable within each of these pieces. However, they are not differentiable at degenerate points such as $d^{14} = \frac{40}{11}$ or $d^{14} = \frac{80}{9}$. At these points the travel time on used or unused paths are the same.
- Our goal in the sensitivity analysis is to find the derivatives of equilibrium flows and travel times at a given point (assuming this point is not degenerate).

Sensitivity to demand

- We can use matrix-based methods (Bell and Iida (1997)), Variational inequalities (Patriksson 2004), or bush-based method (BLU book).
- We will study bush-based method to find derivatives. Although same results can be obtained using other approaches.

Bush-based approach for sensitivity to demand

- In bush-based approach, an equilibrated bush flow for origin $r \in Z$ is given by $\mathbf{x}^r$, where for each $(i,j) \in A$ if $x_{ij} > 0 \implies d^r(j) = d^r(i) + t_{ij}(x_{ij})$., where $d^r(i)$ represent the equilibrium travel time to node i from origin r. Let $A^r = \{(i,j) \in A : x^r_{ij} > 0\}$
- We assume non-degeneracy. Due this assumptions, if the changes to the demand is small, then the original equilibrium is also non-degenerate, i.e., all the equilibrium bushes remain unchanged.
- For bush-based method, we can state the UE conditions as follows:

$$d^r(j) - d^r(i) - t_{ij}(x_{ij}) = 0, \forall (i,j) \in A^r, \forall r \in Z \tag{1}$$

$$d^r(r) = 0 \tag{2}$$

$$\sum_{j \in FS(i)} x^r_{ij} - \sum_{j \in BS(i)} x^r_{ji} = \begin{cases} -d^{ri}, & \text{if } i \in Z \\ \sum_{s \in Z} d^{rs}, & \text{if } i = r \\ 0, & \text{otherwise} \end{cases} , \forall i \in N, \forall r \in Z \tag{3}$$

$$x^r_{ij} = 0, \forall (i,j) \notin A^r, \forall r \in Z \tag{4}$$

Bush-based approach for sensitivity to demand

Let $\xi_{ij}^r = \frac{dx_{ij}^r}{dd^{rs}}$ (derivative of link flows wrt demand) and $\Lambda_i^r = \frac{dd^r(i)}{dd^{rs}}$ (derivative of travel time wrt
demand). Differentiate each equation (2)-(4) wrt d^{rs}, we get the following:

$$\Lambda_j^r - \Lambda_i^r - t_{ij}^{'} \sum_{r' \in Z} \xi_{ij}^r = 0, \forall (i,j) \in A^r, \forall r \in Z \tag{5}$$

$$\Lambda_r^r = 0 \tag{6}$$

$$\sum_{j \in FS(i)} \xi_{ij}^r - \sum_{j \in BS(i)} \xi_{ji}^r = \begin{cases} -1, & \text{if } i = s \\ 1, & \text{if } i = r \\ 0, & \text{otherwise} \end{cases}, \forall i \in N, \forall r \in Z \tag{7}$$

$$\xi_{ij}^r = 0, \forall (i,j) \notin A^r, \forall r \in Z \tag{8}$$

- (5)-(6) enforces that bushes remain same.
- (8) signify flow conservation at each node. It tells that a unit increase in demand will create additional vehicle being sent from origin to destination.

Bush-based approach for sensitivity to demand

- (5)-(8) represent a system of linear equations which can be solved to evaluate the derivatives ξ_{ij}^r and Λ_i^r.
- It turns out that (5)-(8) represent the optimality conditions of the following optimization problem:

$$\min_{\xi} \quad \sum_{(i,j)\in A} \int_0^{\xi_{ij}} t_{ij}^{'} \xi \, d\xi \tag{9a}$$

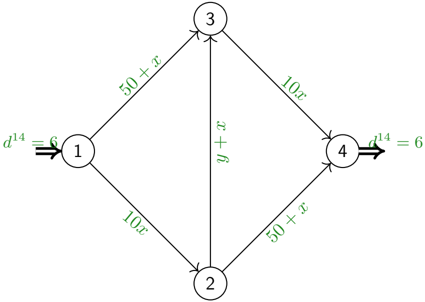
$$\text{s.t.} \quad \sum_{j\in FS(i)} \xi_{ij}^r - \sum_{j\in BS(i)} \xi_{ji}^r \qquad = \begin{cases} 1, & \text{if } i = s \\ -1, & \text{if } i = r \\ 0, & \text{otherwise} \end{cases} , \forall i \in N, \forall r \in Z \tag{9b}$$

$$\xi_{ij}^r = 0, \forall (i,j) \notin A^r, \forall r \in Z \tag{9c}$$

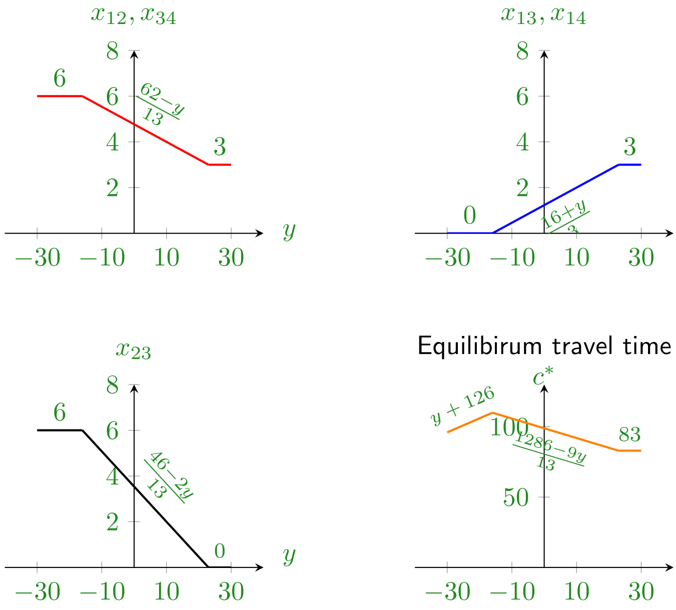
- $\{\Lambda_i^r\}_{i\in N, r\in Z}$ are the dual variables corresponding to constraints (9b)-(9c).
- The formulation (9) is essentially the bush-based UE-based TAP formulation with linear travel time functions whose slope represent the derivative of original link travel time functions ar the original equilibrium solution and unit demand between each O-D pair.
- Note that $\{\xi_{ij}^r\}_{(i,j)\in A^r, r\in Z}$ and $\{\Lambda_i^r\}_{i\in N, r\in Z}$ can take the negative values as the derivative can be negative as well.

Sensitivity to link travel time function parameters

- We can also change the parameter values in the link travel time functions and we would like to know the change in the equilibrium flow and travel time values.
- Similar to demand, we expect to see the equilibrium flow and travel time values vary piecewise linear wrt link travel time function parameter. For sensitivity, we again assume non-degeneracy
- Example: Consider the Braess' network. Assume that demand is fixed at $d^{14} = 6$. Let $t_{23}(x_{23}, y) = y + x_{23}$, where y is the free flow travel time for link $(2,3)$. One can change y to reflect the improvement or deterioration of link $(2,3)$.



Equilibrium flows and travel times as functions of y



Bush-based approach for sensitivity to y

Let $\xi^r_{ij} = \frac{dx^r_{ij}}{dy}$ (derivative of link flows wrt y) and $\Lambda^r_i = \frac{dd^r(i)}{dy}$ (derivative of travel time wrt y). Further, let $t^{'}_{ij,x}$ be the derivative of link travel time function wrt link flows and $t^{'}_{ij,y}$ be the derivative of link travel time function wrt free flow travel time y. Differentiate each equation (2)-(4) wrt y, we get the following:

$$\Lambda^r_j - \Lambda^r_i - t^{'}_{ij,x} \sum_{r' \in Z} \xi^r_{ij} - t^{'}_{ij,y} = 0, \forall (i,j) \in A^r, \forall r \in Z \quad (10)$$

$$\Lambda^r_r = 0 \quad (11)$$

$$\sum_{j \in FS(i)} \xi^r_{ij} - \sum_{j \in BS(i)} \xi^r_{ji} = 0, \forall i \in N, \forall r \in Z \quad (12)$$

$$\xi^r_{ij} = 0, \forall (i,j) \notin A^r, \forall r \in Z \quad (13)$$

- (10)-(11) enforces that bushes remain same.
- (12) signify that flow conservation at each node still holds since there is no change in the demand.ϵ form a circulation.

Bush-based approach for sensitivity to y

- (10)-(13) represent a system of linear equations which can be solved to evaluate the derivatives ξ_{ij}^r and Λ_i^r.
- It turns out that (10)-(13) represent the optimality conditions of the following optimization problem:

$$\min_{\xi} \quad \sum_{(i,j)\in A} \int_0^{\xi_{ij}} (t'_{ij,x}\xi + t'_{ij,y})\, d\xi \tag{14a}$$

$$\text{s.t.} \quad \sum_{j\in FS(i)} \xi_{ij}^r - \sum_{j\in BS(i)} \xi_{ji}^r = 0, \forall i \in N, \forall r \in Z \tag{14b}$$

$$\xi_{ij}^r = 0, \forall (i,j) \notin A^r, \forall r \in Z \tag{14c}$$

- $\{\Lambda_i^r\}_{i\in N}^{r\in Z}$ are the dual variables corresponding to constraints (16c)-(14c).
- The formulation (14) is essentially the bush-based UE-based TAP formulation with affine travel time functions whose slope represent the derivative of original link travel time function wrt x at the original equilibrium solution and intercept represent the derivative of original link travel time functions wrt y at the original equilibrium solution.
- In (14), demand is 0 for each OD pair.
- Note that $\{\xi_{ij}^r\}_{(i,j)\in A^r, r\in Z}$ and $\{\Lambda_i^r\}_{i\in N, r\in Z}$ can take the negative values as the derivative can be negative as well.

Applications

Highway Network Design

- The sensitivity analysis wrt parameters of link travel time functions can be used for determining how much to improve each link so as to optimize the system level objective.
- Assuming that we have limited budget to invest in highway projects, the goal is to find the best investment incorporating the effect on travel behavior of drivers who route themselves according to User Equilibrium (UE).
- Let K_{ij} be the improvement in capacity of link $(i,j) \in A$ per ₹ invested. Our decision variable will be to determine y_{ij} which represent how much investment (in ₹) required on link $(i,j) \in A$ (e.g, through widening of highway link $(i,j) \in A$). To capture this, we can possibly have the following BPR function of link travel times.

$$t_{ij}(x_{ij}, y_{ij}) = t_{ij}^0 \left[1 + \alpha \left(\frac{x_{ij}}{u_{ij} + K_{ij} y_{ij}} \right)^{\beta} \right] \tag{15}$$

- The version of the network design problem we present here is also referred to as Continuous Network Design Problem (CNDP) since improvement in highway are continuous variables.

Continuous Network Design Problem (CNDP)

$$\min_{\mathbf{x},\mathbf{y}} \quad \gamma \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}, y_{ij}) + \sum_{(i,j)\in A} y_{ij} \tag{16a}$$

$$\text{s.t.} \quad \mathbf{x} \in \operatorname{argmin}_{\mathbf{x}\in X} \left\{ \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x, y_{ij})\, dx \right\} \tag{16b}$$

$$\mathbf{y} \in Y \tag{16c}$$

- In CNDP formulation (16), we aim to minimize the total cost which is composed of monetary value of TSTT (obtained by multiplying TSTT and value of time (₹/min) γ) and total investment.
- It captures the effect of improvements $\mathbf{y}$ on the TSTT by evaluating the UE link flows through TAP (16b).
- X is the set of feasible link flows.
- Y in (16c) is the feasible space of possible improvements. For example, Y can contain budget constraints and non-negativity constraints.

Continuous Network Design Problem (CNDP)

- CNDP (16) is a difficult optimization problem to solve exactly.
- Notice that one of its constraints (16b) is itself an optimization problem. Due to this constraints the feasible region of (16) becomes non-convex set. Therefore, there can be multiple local optimal solutions.
- CNDP belongs to the class of optimization problems known as Mathematical Program with Equilibrium Constraints (MPEC) since one (more) constraints of this problem is (are) itself equilibrium constraint(s).
- We use the results from sensitivity analysis to develop a heuristic algorithm to solve this CNDP.
- Note that for a given value of $\mathbf{y}$, the link flow values $\mathbf{x}$ are unique (since they are the solution to the Beckmann's formulation of UE). We can write the objective function (16a) in terms of $\mathbf{y}$ only as $f(\mathbf{x}(\mathbf{y}), \mathbf{y})$. The partial derivative of this function wrt to improvement in any link (i, j) is given as:

$$\frac{\partial f}{\partial y_{ij}} = \gamma \sum_{(i,j)\in A} \frac{\partial f}{d x_{ij}} \frac{\partial x_{ij}}{\partial y_{ij}} + 1 \tag{17}$$

$$= \gamma \left[\sum_{(i,j)\in A} \frac{\partial x_{ij}}{\partial y_{ij}} \left\{ t_{ij}(x_{ij}, y_{ij}) + x_{ij} \frac{\partial t_{ij}(x_{ij}, y_{ij})}{\partial x_{ij}} \right\} \right] + 1 \tag{18}$$

- $\frac{\partial x_{ij}}{\partial y_{ij}}$ can be found using the sensitivity analysis wrt link travel time function parameters discussed in the previous slides. Let $\nabla_{\mathbf{y}} f = \left\{ \frac{\partial f}{\partial y_{ij}} \right\}_{(i,j)\in A}$.

Gradient-based heuristic to solve CNDP

1. Initialize $\mathbf{y} = \mathbf{0}$
2. Calculate the link flows $\mathbf{x}(\mathbf{y})$ by solving the TAP with link travel time functions $\mathbf{t}(\mathbf{x}, \mathbf{y})$.
3. For each link $(i, j) \in A$, evaluate $\frac{\partial f}{\partial y_{ij}}$ by solving the sensitivity analysis problem.
4. Update $\mathbf{y} \leftarrow \max\{0, \mathbf{y} - \mu \nabla_y f\}$, where μ is a step size.
5. Test for convergence or return to step (2) if not converged.

Above algorithm can help evaluate a local optimal solution. There are global solution methods but they tend to be slow for large-scale problems.

Discrete Network Design Problem (DNDP)

In this formulation, the decisions y_{ij} takes the binary variables which represent whether to build a link between $(i,j) \in A$ or not. Let b_{ij} represent the total cost (in ₹) required for building highway of capacity u_{ij} between (i,j) whose travel time function is given as:

$$t_{ij}(x_{ij}) = t_{ij}^0 \left[1 + \alpha \left(\frac{x_{ij}}{u_{ij}} \right)^{\beta} \right] \tag{19}$$

Further, we assume that the total budget available for the high improvement as B. Then DNDP can be formulated as:

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) \tag{20a}$$

$$\text{s.t.} \quad \sum_{(i,j)\in A} b_{ij} y_{ij} \leq B \tag{20b}$$

$$\mathbf{x} \in \operatorname{argmin}_{\mathbf{x}\in X} \left\{ \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x, y_{ij}) dx \right\} \tag{20c}$$

$$x_{ij} \leq u_{ij} y_{ij}, \forall (i,j) \in A \tag{20d}$$

$$\mathbf{y} \in Y \tag{20e}$$

(20b) refers to the budget constraint, (20d) forces the flow values in the traffic assignment problem to take 0 values if link $(i,j) \in A$ is decided not to build, otherwise it forces the flow on (i,j) to be less than its capacity u_{ij}. In general, (20) is a hard problem to solve exactly. With some reformulations and outer approximation approach, it is possible to solve it for mid-sized networks.

Highway Origin-Destination (OD) matrix estimation

OD matrix tells the number of travelers going from one zone to another. Traditionally, OD matrix was estimated by conducting surveys. They used growth factor methods to project demand to future years. One of the common approach was gravity method combined with Iterative Proportional Fitting (IPF) to estimate the OD matrix.

With modern technologies such as loop detectors, Bluetooth, GPS, etc., we started collecting data about travel pattern of drivers. One of the most common approach in the literature is to use link counts obtained from these sensors to estimate an O-D matrix. Note that number of links in the network are quite less as compared to number of OD pairs in the network. Using optimization (with objectives such maximizing entropy, maximizing likelihood, minimizing weighted least squares, etc.), an OD matrix is sought. A seed matrix (possibly obtained using surveys) is used in the regularizer to estimate the actual OD matrix using link counts.

Highway OD matrix estimation

Let us assume a seed matrix $\hat{D} = \{\hat{d}^{rs}\}_{(r,s)\in Z^2}$ is available with us. Further, we have link counts $\tilde{x}_{ij}$ on a subset of links $\tilde{A} \subseteq A$ obtained using sensors. The goal is to estimate an OD matrix $D = \{d^{rs}\}_{(r,s)\in Z^2}$.

$$\min_{\mathbf{x},\mathbf{d}} f(\mathbf{d}, \mathbf{x}) \qquad \gamma \sum_{(r,s)\in Z^2} (d^{rs} - \hat{d}^{rs})^2 + (1-\gamma) \sum_{(i,j)\in \tilde{A}} (x_{ij} - \tilde{x}_{ij})^2 \tag{21a}$$

$$\text{s.t.} \qquad \mathbf{x} \in \operatorname{argmin}_{\mathbf{x}\in X(\mathbf{d})} \left\{ \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx \right\} \tag{21b}$$

$$d^{rs} \geq 0, \forall (r,s) \in Z^2 \tag{21c}$$

- In this formulation, we use least squares to minimize the deviation of the estimated OD $\hat{d}^{rs}$ from seed OD matrix d^{rs} and deviation between actual link flow observed $\tilde{x}_{ij}$ and predicted link flows x_{ij} obtained from UE.
- The parameter γ is used for assigning weight to different terms in the objective function.
- This is also a bi-level program since the constraint (21b) is itself an optimization problem.

Gradient-based heuristic to solve OD estimation problem

We can develop a gradient descent heuristic to solve this problem. Note that for a given value of $\mathbf{d}$, the link flow values $\mathbf{x}$ are unique (since they are the solution to the Beckmann's formulation of UE). We can write the objective function (21a) in terms of $\mathbf{d}$ only as $f(\mathbf{x}(\mathbf{d}), \mathbf{d})$. The partial derivative of this function wrt to demand d^{rs} between any OD pair (r, s):

$$\frac{\partial f}{\partial d^{rs}} = 2\gamma(d^{rs} - \hat{d}^{rs}) + 2(1-\gamma) \sum_{(i,j) \in \tilde{A}} (x_{ij} - \tilde{x}_{ij}) \frac{\partial x_{ij}}{\partial d^{rs}} \tag{22}$$

$$\tag{23}$$

$\frac{\partial x_{ij}}{\partial d^{rs}}$ can be found using the sensitivity analysis wrt demand discussed in the previous slides. Let $\nabla_{\mathbf{d}} f = \left\{ \frac{\partial f}{\partial d^{rs}} \right\}_{(r,s) \in Z^2}$.

Gradient-based heuristic to solve OD estimation problem

1. Initialize $\mathbf{d} \leftarrow \hat{\mathbf{d}}$
2. Calculate the link flows $\mathbf{x}(\mathbf{d})$ by solving the TAP with OD $\mathbf{d}$.
3. For each link $(r,s) \in Z^2$, evaluate $\frac{\partial f}{\partial d^{rs}}$ by solving the sensitivity analysis problem.
4. Update $\mathbf{d} \leftarrow \max\{0, \mathbf{d} - \mu \nabla_{\mathbf{d}} f\}$, where μ is a step size.
5. Test for convergence or return to step (2) if not converged.

Above algorithm can help evaluate a local optimal solution. There are global solution methods but they tend to be slow for large-scale problems.

Suggested reading

The lecture is based on Chapter 8 of BLU book

Thank you!