Inventory management

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Suggested reading

- Goetschalckx Chapter 10
- Chopra et al. Chapters 11-13

Outline

Introduction

EOQ

Newsvendor model

Need for inventory

- To reduce the transportation, setup, and ordering costs.
- To cope up with the demand uncertainty.
- To cope up with the supply uncertainties (shortages, uncertain lead times, price variability, etc.)
- To reduce negative effects of bureaucracy
- To help induce demand
- Sometimes enjoy the quantity discounts

Costs of inventory

- Warehousing cost
- Deterioration of products
- Risk of becoming outdated
- Demand shortfall
- Reduction in cost of production
- Changes in product design specifications
- Tied up capital

Measuring costs

- Production cost c (per unit cost of product)
- Holding cost h (cost per unit per unit time period)
  - capital cost
  - taxes and insurance
  - breakage, spoilage, deterioration, and obsolescence
  - if i is the interest rate every year, then $\boxed{h = ic}$
  - e.g., 18% cost of capital, 2% taxes and insurance, 6% storage cost, and 1% breakage/spoilage cost, then $i = 27\%$ and if $c =$ ₹200 $h = 0.27 \times 200 =$ ₹54.
- Setup (ordering) cost A
- Shortage cost
  - backordering cost b (per unit cost of product)
  - lost sales p (per unit cost of product)

Outline

Assumptions

- Production is instantaneous (no capacity constraints, entire lot is produced simultaneously)
- Delivery is immediate (no time lag between production and availability to satisfy demand)
- Demand is deterministic
- Demand occurs continuously over time with a constant rate
- A production run incurs a fixed cost (regardless of the size of the lot)
- No backorders are allowed (all demand must be satisfied at the time of arrival)
- Multiple products can be analyzed individually and independently of each other

Notations

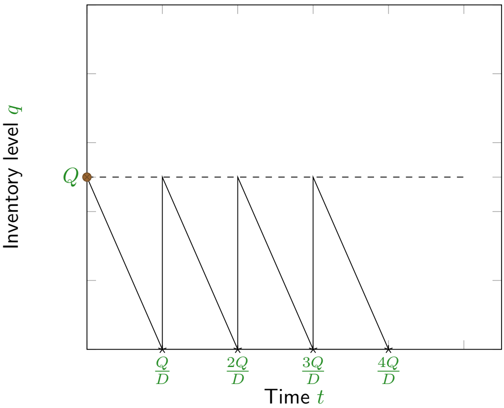
- D: demand rate (in units per time period)
- c: unit production cost (in ₹per unit)
- A: fixed (setup) cost to produce (or purchase) a lot (in ₹)
- h: holding cost (in ₹per unit per unit time period)
  - If the holding cost consists entirely of interest on money tied up in the inventory, then $h = ic$, where i is the annual interest
- Q: lot size (in units); which is a decision variable

Problem

How to choose the order quantity Q so that the overall ordering costs (fixed $+$ variable) and holding costs are minimized?

Remark. The trade off is between fixed ordering cost versus inventory holding.

Inventory level versus time



Remark.

- Length of the cycle (time between consecutive orders) is $\frac{Q}{D}$
- $q(t) = Q - Dt$, $0 \leq 0, \leq \frac{Q}{D}$

Total costs

For every cycle, total costs can be calculated as follows:

- Fixed cost per cycle$= A$
- Production (purchasing) cost per cycle $= cQ$
- Holding cost per cycle $=$

$$h \int_0^{\frac{Q}{D}} (Q - Dt)dt = h\frac{Q^2}{2D}$$

- Total cost per cycle $= A + cQ + h\frac{Q^2}{2D}$

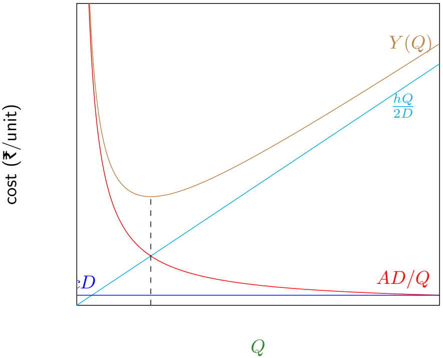
Average costs per unit time

- Fixed cost per unit time $= \frac{A}{\frac{Q}{D}} = \frac{AD}{Q}$
- Production (purchasing) cost per unit time $= \frac{cQ}{\frac{Q}{D}} = cD$
- Holding cost per unit time $= h\frac{\frac{Q^2}{2D}}{\frac{Q}{D}} = h\frac{Q}{2}$
- Total average cost per unit time $= Y(Q) = \frac{AD}{Q} + cD + h\frac{Q}{2}$

Average costs per unit

- Fixed cost per unit $= \frac{A}{Q}$
- Production (purchasing) cost per unit $= \frac{cQ}{Q} = c$
- Holding cost per unit $= h\frac{\frac{Q^2}{2D}}{Q} = h\frac{Q}{2D}$
- Total average cost per unit $= \frac{A}{Q} + c + h\frac{Q}{2D}$

Economic Order Quantity (EOQ)



- Holding-cost term $\frac{hQ}{2D}$ increases linearly in Q and eventually becomes the dominant component.
- Setup-cost term AD/Q diminishes quickly in Q, indicating initial savings in cost if we increase the lot size and but returns reduces later
- Unit-cost term cD does not affect the relative cost for different lot sizes, since it does not depend on Q
- $Y(Q)$ is minimum at a point where holding cost and setup cost are exactly balanced.

Economic Order Quantity (EOQ)

Let's find the value of Q for which $Y(Q)$ is minimum.

$$\frac{dY(Q)}{dQ} = 0$$

$$\implies \frac{h}{2} - \frac{AD}{Q^2} = 0$$

$$\implies \boxed{Q^* = \sqrt{\frac{2AD}{h}}}$$

Check if the second order condition is satisfied, i.e.,

$$\frac{d^2Y(Q)}{dQ} = 2\frac{AD}{Q^3} > 0$$

Remark. The optimal average cost per unit is given by

$$\begin{aligned} Y(Q^*) =& \frac{AD}{Q^*} + cD + h\frac{Q^*}{2} \\ =& \sqrt{2ADh} + cD \end{aligned}$$

Sum of holding and setup costs is fairly insensitive to lot size

Let's assume we use $Q^{'}$ (smaller or larger than) Q^*, then sum of holding and setup costs is given by $\frac{hQ^{'}}{2} + \frac{AD}{Q^{'}}$. Then the ratio

$$\begin{aligned}
\frac{cost(Q^{'})}{cost(Q^*)} =& \frac{\frac{hQ^{'}}{2} + \frac{AD}{Q^{'}}}{\sqrt{2ADh}} && (1)\\
=& \frac{Q^{'}}{2}\sqrt{\frac{h^2}{2ADh}} + \frac{1}{Q^{'}}\sqrt{\frac{A^2D^2}{2ADh}} && (2)\\
=& \frac{Q^{'}}{2}\sqrt{\frac{h}{2AD}} + \frac{1}{2Q^{'}}\sqrt{\frac{2AD}{h}} && (3)\\
=& \frac{1}{2}\left(\frac{Q^{'}}{Q^*} + \frac{Q^*}{Q^{'}}\right) && (4)
\end{aligned}$$

Remark. If $Q^{'} = 2Q^*$, then $\frac{cost(Q^{'})}{cost(Q^*)} = \frac{1}{2}\left(2 + \frac{1}{2}\right) = 1.25$, i.e. 100% error in lot size will result in only 25% error in cost.

Optimal order interval (cycle length)

$$T^* = \frac{Q^*}{D} = \frac{\sqrt{\frac{2AD}{h}}}{D}$$

$$\boxed{T^* = \sqrt{\frac{2A}{hD}}}$$

Total cost in terms of order interval

$$\begin{aligned} Y(Q) &= \frac{AD}{Q} + cD + h\frac{Q}{2} \\ &= \frac{A}{T} + cD + \frac{hDT}{2} \end{aligned}$$

Remark. $\frac{\text{Annual cost under } T'}{\text{Annual cost under } T^*} = \frac{1}{2}\left(\frac{T'}{T^*} + \frac{T^*}{T'}\right)$

Limitations of EOQ

- Replenishment is instantaneous
- Constant, deterministic, continuous demand
- Ordering (fixed) cost is deterministic and independent of order size
- Multiple products ordering not considered
- Backorders not allowed

Extensions of EOQ

- EOQ model with backorders

$$\boxed{Q^* = \sqrt{\frac{2AD}{h\alpha}}}, \alpha = \frac{h}{b+h}$$

where, b is the penalty cost incurred per unit backordered item per unit time

- Economic production quantity (EPQ) model

$$\boxed{Q^* = \sqrt{\frac{2AD}{h\left(1 - \frac{D}{P}\right)}}}$$

where, P is the production rate (units per unit time)
Remark. When $P = \infty$ (instantaneous replenishment), it reduced to basic EOQ model.

Extensions of EOQ

- EPQ for multiple products

$$\boxed{Q_i^* = \sqrt{\frac{2A_iD_i}{h\left(1-\frac{D_i}{P_i}\right)}} \text{ s.t. } \sum_{i=1}^{n}\frac{D_i}{P_i} \leq 1}$$

  where, i is the index for product i and $\frac{D_i}{P_i}$ is the fraction of time production is busy with product i.

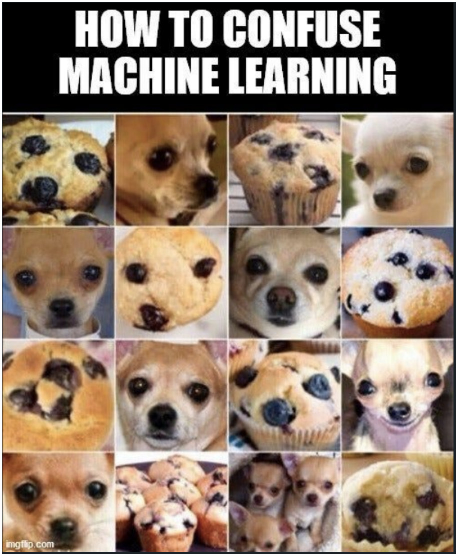
- EOQ with quantity discounts (depends on how discounts are applied)
- Many others including, EOQ with joint replenishment, capacity constraints, multiple stages, and price dependent demand.

Economic Order Quantity (EOQ)

Figure: Ford W. Harris[2], an American production engineer, proposed the EOQ model in 1913

[1] https://controlinventarios.wordpress.com/2020/10/28/ford-whitman-harris-y-r-h-wilson/

Let's relax a bit!



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EOQ

Newsvendor model

Inventory control for stochastic demand

Demand (per unit time) is stochastic, i.e., there is possibility of overstocking (excess) or understocking (shortages). There is cost associated to both overstocking as wells as understocking.

Types of stochastic models

- Single period models
  - perishable goods, seasonal goods
- Multi period models
  - Goods with recurring demand (stochastic)
  - The status of the system is reviewed periodically (at the end/beginning of each period)
  - Problem is to decide how much to order in each time interval
- Continuous time models
  - Goods with recurring and continuously occurring demand with variable inter-arrival times between consecutive orders
  - System status is reviewed continuously (e.g., inventory level is continuously monitored)
  - Continuous decisions on how much to order

The Newsvendor model

Decision How much to order? Q

Parameters

- Single period, random demand denoted by D with known distribution, cost per unit of leftover inventory (overage cost) c_o, and cost per unit of excess demand (shortage cost) c_s

Objective
Minimize the sum of expected shortage and overage costs.

Discrete demand

Let $\mathbb{P}(D = d)$ be the probability that demand is d units.

Objective function

$$
\begin{aligned}
Y(Q) &= \text{expected overage cost} \qquad\qquad\qquad\quad + \text{ expected shortage cost} \\
&= c_o \sum_{d=0}^{\infty} \max\{Q-d, 0\} \mathbb{P}(D=d) + c_s \sum_{d=0}^{\infty} \max\{d-Q, 0\} \mathbb{P}(D=d) \\
&= c_o \sum_{d=0}^{Q} (Q-d) \mathbb{P}(D=d) + c_s \sum_{d=Q}^{\infty} (d-Q, 0) \mathbb{P}(D=d)
\end{aligned}
$$

Optimize $Y(Q)$ to get the optimal order quantity.

Optimality conditions

The optimal value of Q is the smallest integer that satisfies

$$Y(Q+1) - Y(Q) \geq 0$$

$$\begin{aligned} Y(Q+1) - Y(Q) = & c_o \sum_{d=0}^{Q+1} (Q+1-d)\mathbb{P}(D=d) + c_s \sum_{d=Q+1}^{\infty} (d-Q-1)\mathbb{P}(D=d) \\ & - c_o \sum_{d=0}^{Q} (Q-d)\mathbb{P}(D=d) - c_s \sum_{d=Q}^{\infty} (d-Q)\mathbb{P}(D=d) \geq 0 \\ \implies & \boxed{\sum_{d=0}^{Q} \mathbb{P}(D=d) \geq \frac{c_s}{c_s+c_o}} \end{aligned}$$

i.e., order quantity Q should be large enough to guarantee that the probability of not stocking out is at least $\frac{c_s}{c_s+c_o}$.

Continuous demand

$F(d) = \mathbb{P}(D \leq d)$ is cumulative distribution function.
$f(d) = F^{'}(d)$ is probability density function.

Objective function

$$
\begin{aligned}
Y(Q) = &\text{ expected overage cost} \qquad\qquad\qquad + \text{ expected shortage cost} \\
=& c_o \int_0^\infty \max\{Q-d,0\} f(d)\mathsf{d}d + c_s \int_0^\infty \max\{d-Q,0\} f(d)\mathsf{d}d \\
=& c_o \int_0^Q (Q-d) f(d)\mathsf{d}d + c_s \int_Q^\infty (d-Q) f(d)\mathsf{d}d
\end{aligned}
$$

Optimize $Y(Q)$ to get the optimal order quantity. To do so, you might have to use Leibnitz's Rule.

Leibnitz's Rule

$$
\boxed{\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x)) \cdot \frac{d}{dx} b(x) - f(x,a(x)) \cdot \frac{d}{dx} a(x) + \left(\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt\right)} \tag{5}
$$

Optimality conditions

The optimal value of Q is the smallest integer that satisfies

$$\frac{d}{dQ}Y(Q) = 0$$

$$\begin{aligned}\frac{d}{dQ}Y(Q) =& - c_o \int_0^Q -f(d)\mathrm{d}d_{+} c_s \int_Q^\infty -f(d)\mathrm{d}d \\ =& c_o F(Q) - c_s(1 - F(Q)) = (c_o + c_s)F(Q) - c_s = 0 \\ \implies & \boxed{F(Q^*) = \frac{c_s}{c_o + c_s}} \implies \boxed{Prob(D \leq Q^*) = \frac{c_s}{c_o + c_s}}\end{aligned}$$

Remark. If $D \sim \mathcal{N}(\mu, \sigma^2)$. Then, we have

$$Prob(D \leq Q^*) = \frac{c_s}{c_o + c_s} \tag{6}$$

$$Prob(\frac{D - \mu}{\sigma} \leq \frac{Q^* - \mu}{\sigma}) = \frac{c_s}{c_o + c_s} \tag{7}$$

$$\Phi\left[\frac{Q^* - \mu}{\sigma}\right] = \frac{c_s}{c_o + c_s} \implies \boxed{Q^* = \mu + z_\alpha \sigma} \tag{8}$$

where, $\alpha = \frac{c_s}{c_o + c_s}$ also called critical fractile.

Profit maximization model for discrete demand

Let p and c be the price and cost of unit product. The objective function is maximize the expected profit.

$\Pi(Q) = p\sum_{d=0}^{Q} d\mathbb{P}(D=d) + p\sum_{d=Q+1} Q\mathbb{P}(D=d) - cQ$. To maximize this, we need to find the smallest value of Q for which
$\Pi(Q+1) - \Pi(Q) \leq 0$
This is equivalent to choosing the smallest integer Q that satisfies

$$\boxed{\mathbb{P}(D \leq d) = \sum_{d=0}^{Q} \mathbb{P}(D=d) \geq \frac{p-c}{p}} \tag{9}$$

Remark. $p-c$ can be viewed as the cost of being short and c as the cost of being over. If we let $c_s = p-c$ and $c_o = c$, then optimal order quantity is smallest integer quantity that satisfies
$\sum_{d=0}^{Q} \mathbb{P}(D=d) \geq \frac{c_s}{c_s+c_o} = \frac{p-c}{p}$.
Remark. The optimal order quantity is the one for which marginal benefit from increasing the order quantity by one unit beyond the optimal order quantity is negative.

Extensions

- Newsvendor problem can be solved for multiple finite or infinite horizon. Due to lack of background in stochastic programming, we are skipping this here.

Other important models in inventory management

- Inventory pooling
- Inventory systems with renewal demand and continuous review
- Inventory control with time-varying demand
- Control of production-inventory systems with multiple echelons
- and many more

Credits

- Chapter 2, Factory physics by Hopp, Wallace J., and Mark L. Spearman.
- IE5551 lectures taught by Prof. Saif Benjaafar

Thank you!