Linear Programming Duality

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Outline

Motivation

Consider the following example

$$
\begin{aligned}
Z = \underset{x_1, x_2, x_3}{\text{minimize}} \quad & x_1 + 2x_2 + x_3 && \text{(1a)} \\
\text{subject to} \quad & 3x_2 + x_3 \geq 1 && \text{(1b)} \\
& x_1 + 2x_2 - 3x_3 \geq 2 && \text{(1c)} \\
& x_1 - x_2 \geq 1 && \text{(1d)} \\
& x_3 \geq 3 && \text{(1e)}
\end{aligned}
$$

- Multiplying (1b), (1c), (1d), and (1e) by 1, 0, 1, and 0 respectively and adding, we have

$$
\begin{aligned}
& x_1 + 3x_2 - x_2 + x_3 \geq 1 + 1 \\
\implies & Z^* = x_1 + 2x_2 + x_3 \geq 2
\end{aligned}
$$

- Similarly, multiplying (1b), (1c), (1d), and (1e) by 0, 1, 0, and 4 respectively and adding, we have

$$
Z^* = x_1 + 2x_2 + x_3 \geq 14
$$

Motivation

- By multiplying constraints with suitable multipliers and adding, we can obtain a lower bound on the objective value.
- Two important questions arise here:
  - Which multipliers should we multiply the constraints with?
  - How close is the best lower bound to the optimal value?

Procedure

1. Multiply each inequality i by μ_i. Choose the sign of each μ_i so that the inequality sign remains $\geq$.
2. Add all the inequalities. If the resultant matches the objective function, then the r.h.s. of the resultant provides the lower bound on Z^*.

Definition (Dual problem). The dual of a minimization problem is the problem of finding best multipliers for its constraints so that their resultant matches the objective function with maximum r.h.s.

The dual is an optimization problem!!.

For previous problem, we multiply (1b), (1c), (1d), and (1e) by $\mu_1, \mu_2, \mu_3,$ and μ_4 respectively, we obtain

$$(\mu_2+\mu_3)x_1+(3\mu_1+2\mu_2-\mu_3)x_2+(\mu_1-3\mu_2+\mu_4) \geq \mu_1+2\mu_2+\mu_3+3\mu_4$$

- We don't want to change the $\geq$ sign, so we keep $\mu_1, \mu_2, \mu_3, \mu_4 \geq 0$.
- We want l.h.s. to match the objective function, i.e.,

$$\mu_2 + \mu_3 = 1 \tag{2}$$
$$3\mu_1 + 2\mu_2 - \mu_3 = 2 \tag{3}$$
$$\mu_1 - 3\mu_2 + \mu_4 = 1 \tag{4}$$

- Finally, we want to r.h.s to be as maximum as possible. Therefore, the problem becomes

$$\begin{aligned}
Z = \underset{\mu_1,\mu_2,\mu_3,\mu_4}{\text{maximize}} \quad & \mu_1 + 2\mu_2 + \mu_3 + 3\mu_4 && \text{(5a)}\\
\text{subject to} \quad & \mu_2 + \mu_3 = 1 && \text{(5b)}\\
& 3\mu_1 + 2\mu_2 - \mu_3 = 2 && \text{(5c)}\\
& \mu_1 - 3\mu_2 + \mu_4 = 1 && \text{(5d)}\\
& \mu_1, \mu_2, \mu_3, \mu_4 \geq 0 && \text{(5e)}
\end{aligned}$$

Outline

A linear optimization problem (primal)

$$Z = \underset{\mathbf{x}}{\text{minimize}} \quad \sum_{j=1}^{n} c_j x_j \tag{6a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \forall i = 1, \cdots, m \tag{6b}$$

$$x_j \geq 0, \forall j = 1, \cdots, m. \tag{6c}$$

has the associated dual linear program given by

$$Z = \underset{\mu}{\text{maximize}} \quad \sum_{i=1}^{m} b_i \mu_i \tag{7a}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} \mu_i \leq c_j, \forall j = 1, \cdots, n \tag{7b}$$

$$\mu_i \geq 0, \forall i = 1, \cdots, m. \tag{7c}$$

- Each primal constraint correspond to a dual variable.
- Each primal variable correspond to a dual constraint.
- Dual of a minimization problem is the maximization problem and vice-versa.
- Equality constraint correspond to free dual variable variable.
- Dual of the dual is primal.

Outline

Weak Duality Theorem

Theorem

If $(x_1, ..., x_n)$ is feasible for the primal and $(\mu_1, ..., \mu_m)$ is feasible for the dual, then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i \mu_i$$

Proof.

If $(\mu_1, ..., \mu_m)$ is feasible for the dual, then $\sum_{i=1}^{m} a_{ij}\mu_j \leq c_j, \forall j = 1, \cdots, n$.
$\therefore \sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\mu_i\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}x_j\right) \mu_i \geq \sum_{i=1}^{m} b_i\mu_i$.
The last inequality follows from the fact that $(x_1, ..., x_n)$ is feasible to the primal. □

Corollary

If primal problem is unbounded then the dual problem is infeasible.

Proof.

Assume that the dual LP is feasible and let $(\hat{\mu}_1, ..., \hat{\mu}_m)$ be a feasible solution to dual problem. Then, by previous theorem, we have $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i \hat{\mu}_i, \forall \mathbf{x}$ feasible to the primal problem. Then, $\exists \{\mathbf{x}^i\}_{i=1}^{\infty}$ such that $\lim_{i \mapsto \infty} \mathbf{c}^T \mathbf{x}^i = -\infty \geq \mathbf{b}^T \mu$, a contradiction. □

Corollary

If dual problem is unbounded then the primal problem is infeasible.

Remark. It is possible for both primal and dual to be infeasible.

Strong Duality Theorem

Theorem

If the primal problem has an optimal solution $\mathbf{x}^* = (x_1^*, ..., x_n^*)$, then its dual also has an optimal solution $\mu^* = (\mu_1^*, ..., \mu_m^*)$ such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i \mu_i^* \tag{8}$$

Proof.

We prove this using the simplex method. Assume that the primal problem is in the standard form. If the primal problem has an optimal solution $\mathbf{x}^*$, then it is associated with some optimal basis B such that $\mathbf{x}_B = A_B^{-1}\mathbf{b}$ ($\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$). We also know that when the simplex method terminates, the reduced costs are all non-negative

$$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A \geq 0 \tag{9}$$

Define $\mu^T = \mathbf{c}_B^T A_B^{-1}$. Using (9), $A^T \mu \leq \mathbf{c}$, which means that μ is feasible to the dual problem. Moreover, $\mu^T \mathbf{b} = \mathbf{c}_B^T A_B^{-1} \mathbf{b} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}^T \mathbf{x}^*$. Using the weak duality theorem, μ must be optimal to the dual problem. □

Remark. The proof shows that the dual solution comes as a by-product of the primal simplex method.

Summary

Relationships	Dual			
		Optimal	Unbounded	Infeasible
Primal	Optimal	✓		
	Unbounded			✓
	Infeasible		✓	✓

Complementarity conditions

Theorem

Suppose that $x^* = (x_1^*, ..., x_n^*)$ and $y = (y_1^*, ..., y_m^*)$ are primal and dual optimal solutions respectively. Let e_i^* be the excess variables for constraints $\sum_{j=1}^n a_{ij} x_j^* \geq b_i, \forall i = 1, ..., m$ and s_j^* be the slack variables for $\sum_{i=1}^m a_{ij} \mu_i^* \leq c_j, \forall j = 1, ..., n$. Then, $e_i^* \cdot \mu_i^* = 0, \forall i = 1, ..., m$ and $s_j^* \cdot x_j^* = 0, \forall j = 1, ..., n$.

Proof.

Given primal and dual solutions $\mathbf{x}^*$ and $\mathbf{y}^*$, we have,

$$\sum_{j=1}^n c_j x_j^* = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \mu_i + s_j \right) x_j^* \tag{10}$$

$$= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j^* \right) \mu_i^* + \sum_{j=1}^n s_j x_j^* \tag{11}$$

$$= \sum_{i=1}^m \left(b_i^* + e_i^* \right) \mu_i^* + \sum_{j=1}^n s_j x_j^* \tag{12}$$

$$= \sum_{i=1}^m b_i^* \mu_i^* + \sum_{i=1}^m e_i^* \mu_i^* + \sum_{j=1}^n s_j x_j^* \tag{13}$$

From strong duality theorem, we know that $\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i^* \mu_i^*$. Therefore, $\sum_{i=1}^m e_i^* \mu_i^* + \sum_{j=1}^n s_j x_j^* = 0$. The theorem follows. □

Example

Example(s). Verify the Complementarity conditions:
$\mathcal{P}$ Optimal solution $(1, 0, 1)$

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & 13x_1 + 10x_2 + 6x_3 && (14)\\
\text{subject to} \quad & 5x_1 + x_2 + 3x_3 = 8 && (15)\\
& 3x_1 + x_2 = 3 && (16)\\
& x_1, x_2, x_3 \geq 0 && (17)
\end{aligned}
$$

$\mathcal{D}$ Optimal solution $(2, 1)$

$$
\begin{aligned}
Z = \underset{\mathbf{y}}{\text{maximize}} \quad & 8y_1 + 3y_2 && (18)\\
\text{subject to} \quad & 5y_1 + 3y_2 \leq 13 && (19)\\
& y_1 + y_2 \leq 10 && (20)\\
& 3y_1 \leq 6 && (21)
\end{aligned}
$$

Outline

Alternative systems

Given the system $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$

- Can we say that the system has a solution? One can find a solution (also known as certificate).
- To disprove the existence of a solution, do we have a certificate? Yes!
  - We can find a vector μ such that $\mu^T A \geq 0$ and $\mu^T \mathbf{b} < 0$.
  - Such a system is called alternative system.

Farkas Lemma

Theorem (Farkas Lemma)

Given $A \in \mathbb{R}^{m \times n}$. Then, exactly one of the following two systems has a solution:

1. $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ *for* $\mathbf{x} \in \mathbb{R}^n$
2. $\mu^T A \geq 0$ *and* $\mu^T \mathbf{b} < 0$ *for* $\mu \in \mathbb{R}^m$

Proof.

Using F-M elimination or separation theorem $\square$

There are many alternative theorems. One example is below:

- If the system $A^T \mu \leq \mathbf{c}$ has no solution, then we can find a vector $\mathbf{x}$ such that $A\mathbf{x} = 0, \mathbf{x} \geq 0, \mathbf{c}^T \mathbf{x} < 0$.

LP duality general form

$\mathcal{P}$

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}_1^T\mathbf{x}_1 + \mathbf{c}_2^T\mathbf{x}_2 + \mathbf{c}_3^T\mathbf{x}_3 \\
\text{subject to} \quad & A_{11}\mathbf{x}_1 + A_{12}\mathbf{x}_2 + A_{13}\mathbf{x}_3 \leq \mathbf{b}_1 \\
& A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 + A_{23}\mathbf{x}_3 \geq \mathbf{b}_2 \\
& A_{31}\mathbf{x}_1 + A_{32}\mathbf{x}_2 + A_{33}\mathbf{x}_3 = \mathbf{b}_3 \\
& \mathbf{x}_1 \geq 0, \mathbf{x}_2 \leq 0, \mathbf{x}_3 \text{free}
\end{aligned}
$$

$\mathcal{D}$

$$
\begin{aligned}
Z = \underset{\mu}{\text{maximize}} \quad & \mathbf{b}_1^T\mu_1 + \mathbf{b}_2^T\mu_2 + \mathbf{b}_3^T\mu_3 \\
\text{subject to} \quad & A_{11}^T\mu_1 + A_{21}^T\mu_2 + A_{31}^T\mu_3 \leq \mathbf{c}_1 \\
& A_{12}^T\mu_1 + A_{22}^T\mu_2 + A_{32}^T\mu_3 \geq \mathbf{c}_2 \\
& A_{13}^T\mu_1 + A_{23}^T\mu_2 + A_{33}^T\mu_3 = \mathbf{c}_3 \\
& \mu_1 \leq 0, \mu_2 \geq 0, \mu_3 \text{free}
\end{aligned}
$$

Remark. Memorization technique: Normal, Weird, Bizarre

Outline

Solving LP equivalent to finding a feasible solution

Solving

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad & A\mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned} \tag{22}
$$

is equivalent to finding a feasible solution to the the following system ($\star$)

1. (Primal feasibility) $A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0$
2. (Dual feasibility) $A^T\mu \leq \mathbf{c}^T, \mu \geq 0$
3. (Strong duality) $\mathbf{c}^T\mathbf{x} \leq \mu^T\mathbf{b}$

- If $(\hat{\mathbf{x}}, \hat{\mu})$ is feasible to above system ($\star$), then $\mathbf{x}$ is optimal to (22).
- If the system ($\star$) is infeasible, then (22) can be infeasible or unbounded.
  - If $A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0$ is feasible, then (22) is unbounded.

Outline

Connection to finite two person zero-sum game

Matrix game

- 2 players
- Each player selects, independently of other, an action out of finite set of possible actions.
- Both reveal to each other their actions simultaneously
- Let i and j be the actions taken by player 1 and 2 resp.
- Then, player 1 has to pay a_{ij} rupees to player 2.
- The payoff matrix $A = [a_{ij}]_{i=1,\ldots,m \text{ and } j=1,\ldots,n}$ is known beforehand. Of course a_{ij} can be positive or negative.
- A randomized strategy means that players choose their actions at random according to a known probability distribution.
- Let μ_i be the probability with which player 1 plays action $i = 1, \ldots, m$ and let x_j be the probability with which player 2 plays action $j = 1, \ldots, n$.
- $\mu_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^{m} \mu_i = 1$ and $x_j \geq 0, j = 1, \ldots, n, \sum_{j=1}^{n} x_j = 1$
- Expected payoff to player 1 is $\sum_{i,j} \mu_i a_{ij} x_j = \mu^T A \mathbf{x}$
- Assume players are rational and want to maximize their expected payoff.
- Player possible strategies and payoffs are common knowledge.

Example

Rock-Paper-Scissors

- 2 players
- Players simultaneously choose an action $\in \{R, P, S\}$.
- If both players choose the same action, then the game is drawn $a_{ij} = 0, \forall i = j$.
- Rock beats scissors, scissors beat paper, and paper beats rock.
- Payoff matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Optimal strategies

Given $\mathbf{x}$, player 1 solves the following LP:

$$Z(\mathbf{x}) = \underset{\mu}{\text{minimize}} \quad \mu^T A\mathbf{x} \tag{23}$$

$$\text{subject to} \quad \mathbf{e}^T\mu = 1 \tag{24}$$

$$\mu \geq 0 \tag{25}$$

$$\tag{26}$$

Then, player 2 tries to maximizes her payoff

$$\max_{\mathbf{x}} \min_{\mu} \mu^T A\mathbf{x}$$

such that $\mathbf{e}^T\mu = 1, \mu \geq 0$ and $\mathbf{e}^T\mathbf{x} = 1, \mathbf{x} \geq 0$.

Theorem (Minimax theorem)

There exist stochastic vectors μ and $\mathbf{x}$ for which

$$\max_{\mathbf{x}} {\mu^*}^T A\mathbf{x} = \min_{\mu} \mu^T A\mathbf{x}^*$$

Proof.

Use LP strong duality □

Origins of LP Duality

Figure: (From left to right) John von Neumann, George B. Dantzig, David Gale, Harold W. Kuhn and Albert W. Tucker (Pictures source: Wiki)

Thank you!