Stochastic User Equilibrium

Pramesh Kumar

IIT Delhi

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Motivation

- In UE, we assume that all travelers choose the minimum travel time path between their O-D pair.
- However, it implicitly assumes each traveler has perfect knowledge of travel times on all possible paths between their O-D pair. Does this seem reasonable?
- For example, if there are two paths between an O-D pair. Path 1 has travel time of 10 minutes and Path 2 has travel time of 9 minutes 59 seconds. UE will assign all the travelers to Path 1. But can all travelers accurately distinguish between these two paths? There could be perception error between belief and actual shortest path.
- Further, travelers may care about factors other than just travel time, e.g., weather, comfort, toll, etc.
- Stochastic User Equilibrium (SUE) tries to relax these assumptions.

A brief introduction to Discrete Choice Modeling (DCM)

Discrete choice modeling

A decision maker often faces choice among various alternatives. For example,

Figure: Which cereal would you buy?

Figure: Which phone would you buy?

Travelers are also often faced with choice among various alternatives including but not limited to

- alternative paths between an O-D pairs
- alternative modes of transportation
- alternative destinations to be visited

Discrete choice modeling help predict the probability of an alternative from a choice set of alternatives. This is a significant area and we restrict our attention to limited aspects that we need for this lecture.

Image sources: https://www.techradar.com/news/best-phone, https://www.cheatsheet.com/money-career/youll-never-guess-americas-favorite-breakfast-cereal.html/

Discrete choice modeling

- Consider an individual who is faced with the decision of making a choice from a set of alternative options. Let us call this finite set of alternatives as choice set denoted by the set C.
- The hypothesis underlying DCM is that individual's preference towards an alternative can be captured using utility or attractiveness measure associated to each alternative.
- The utility of an alternative i, denoted by U_i, is a function of attributes associated to that alternative as well as characteristics of the decision maker.

Discrete choice modeling

- The utility U_i is composed of two components,

$$U_i = V_i + \epsilon_i$$

  1. V_i is observable or systematic utility which is deterministic[1]. This consists of all the objective factors that are known to the modeler. For example, for a buyer purchasing a phone, we know factors such as price, warranty, features, etc.
  2. ϵ_i is unobservable utility which is random. This consists of all the factors that a modeler cannot observe (or choose not to include). For example, buyer may have certain preference how a phone feels in hand.
- Since ϵ is a random variable, we can express the choices in terms of probabilities.
- Each user selects the alternative which has the highest utility among all alternatives in choice set C:

$$P(i) = P(U_i \geq U_k, \forall k \in C) \tag{1}$$

[1]Sometimes U_i is referred to as perceived utility by the decision maker and V_i is referred to as the measured utility

Multinomial choice model

- (McFadden (1974)) The probability that decision maker n chooses alternative i out of choice set C is

$$\begin{aligned} P_{ni} &= \text{Prob}(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj}, \forall j \neq i) \\ &= \text{Prob}(\epsilon_{nj} < \epsilon_{ni} + V_{ni} - V_{nj}, \forall j \neq i) \end{aligned} \tag{2}$$

- Assuming that $\{\epsilon_{nj}\}$ follow Gumbel distribution parameterized by scale parameter θ and zero mean and are independent across alternatives, we get the famous multinomial logit (MNL) choice model.

$$P_{ni} = \int \left(\Pi_{j \neq i} e^{-e^{-\theta(\epsilon_{ni} + V_{ni} - V_{nj})}} \right) \theta e^{-\theta \epsilon_{ni}} e^{-e^{-\theta \epsilon_{ni}}} d\epsilon_{ni}$$

$$\boxed{P_{ni} = \frac{e^{\theta V_{ni}}}{\sum_{j \in C} e^{\theta V_{nj}}}}$$

Remark. If $\theta = 0$, the probability of choosing any alternative is equal (truly random selection). If $\theta \mapsto \infty$, the probability of choosing alternative i with highest V_i approaches 1.

Remark. Probability of selecting any alternative is always positive.

Remark. If we assume $\{\epsilon_{nj}\} \sim$ **MVN**$(\mu, \boldsymbol{\Sigma})$, the resulting choice model is known as probit model which does not have any closed form formula and probabilities are calculated using simulation.

Satisfaction function

Satisfaction is the expected utility that a decision maker gains from a set of alternatives C in which each alternative $i \in C$ has utility U_i.

$$\boxed{S(\mathbf{V}) = \mathbb{E}\left[\max_{i \in C}\{V_i + \epsilon_i\}\right]} \tag{3}$$

- $S(\mathbf{V})$ is convex wrt $\mathbf{V}$
- $\frac{\partial S(\mathbf{V})}{\partial V_i} = P_i = \frac{\exp^{\theta V_i}}{\sum_{j \in C} \exp^{\theta V_j}}$ (for MNL)
- $S(\mathbf{V})$ is monotonic wrt to the size of the choice set, i.e., $S(V_1, V_2, \cdots, V_k, V_{k+1}) \geq S(V_1, V_2, \cdots, V_k)$
- For MNL, $S(\mathbf{V}) = \frac{1}{\theta} \log(\sum_{i \in C} \exp^{\theta V_i})$

Stochastic Network Loading (SNL)

Logit route choice

- Consider a traveler going from origin r to destination s whose is faced with the choice among various paths Π^{rs} available between r and s.
- Let U^π be the perceived utility of path π which is composed of negative of travel time c^π and a random error.

$$U^\pi = -c^\pi + \epsilon^\pi, \forall \pi \in \Pi^{rs}, \forall (r,s) \in Z^2$$

- We assume $\mathbb{E}[\epsilon^\pi] = 0$, implying $\mathbb{E}[U^\pi] = -c^\pi$
- Probability of choosing path $\pi \in \Pi^{rs}$ is given by $P^\pi = \text{Prob}(U^\pi > U^{\pi'}, \forall \pi' \in \Pi^{rs} \backslash \{\pi\}) = \text{Prob}(-c^\pi + \epsilon^\pi > -c^{\pi'} + \epsilon^{\pi'}, \forall \pi' \in \Pi^{rs} \backslash \{\pi\})$
- Assuming ϵ^π are i.i.d. Gumbel Distribution random variables, we get

$$\boxed{P^\pi = \frac{e^{-\theta c^\pi}}{\sum_{\pi' \in \Pi^{rs}} e^{-\theta c^{\pi'}}}}, \forall \pi \in \Pi^{rs}, \forall (r,s) \in Z^2$$

- If d^{rs} travelers are going from r to s, then the total number of travelers traveling on path π is given by $h^\pi = d^{rs} \frac{e^{-\theta c^\pi}}{\sum_{\pi' \in \Pi^{rs}} e^{-\theta c^{\pi'}}}$.
- Expected utility is given by $S(\mathbf{c}) = -\frac{1}{\theta} \log(\sum_{\pi \in \Pi} e^{-\theta c^\pi})$

Logit-based Stochastic Network Loading

$$P^{\pi} = \frac{e^{-\theta c^{\pi}}}{\sum_{\pi' \in \Pi^{rs}} e^{-\theta c^{\pi'}}}, \forall \pi \in \Pi^{rs}, \forall (r,s) \in Z^2 \tag{4}$$

- As per (4), loading the demand on to a path $\pi \in \Pi^{rs}$ requires enumeration of all the paths between r and s.
- This seems like a bad idea from computational point of view. However, by carefully selecting the paths (also known as reasonable paths), we can perform the loading.
- The reasonable path includes links from the set of reasonable links. Let $d(i)$ be the cost of shortest path from origin r and $D(i)$ be the cost of shortest path from node i to destination s. Then, the set of reasonable links are defined as $\hat{A} = \{(i,j) \in A \mid d(i) < d(j) \text{ and } D(i) > D(j)\}$.
- Using these links, Dial (1971) proposed an algorithm to perform the logit-based stochastic network loading efficiently.

Dial's STOCH algorithm

procedure $\textsc{ForwardPass}(G, \mathbf{t}, r)$
  $\mathbf{d} \leftarrow \textsc{Dijkstra}(G, \mathbf{t}, r)$
  $\hat{A} \leftarrow \{(i,j) \in A \mid d(i) < d(j)\}$
  $L_{ij} \leftarrow e^{\theta(d(j)-d(i)-t_{ij})}, \forall (i,j) \in \hat{A}$
  $order \leftarrow \textsc{TopologicalOrdering}(G(\hat{N}, \hat{A}))$
  if $i == r$ **then**
    $w_r \leftarrow 1$
  else
    $w_i \leftarrow \sum_{j \in BS(i):(j,i) \in \hat{A}} W_{ji}$
  end if
  for each $i \in order$ **do**
    for $j \in FS(i) : (i,j) \in \hat{A}$ **do**
      $W_{ij} = w_i \times L_{ij}$
    end for
  end for
  return $\mathbf{w}, \mathbf{W}$
end procedure

Dial's STOCH algorithm

```
procedure BackwardPass($G, \mathbf{w}, \mathbf{W}, r$)
    for $s \in Z$ do
        // Process nodes in reverse order starting from $s$
        for each $i \in reverse\ order$ do
            $\mathbf{Z}_i \leftarrow d^{ri} + \sum_{j \in FS(i):(i,j) \in \hat{A}} z_{ij}$
            for $j \in BS(i) : (j,i) \in \hat{A}$ do
                $y_{ji} \leftarrow Z_i \frac{W_{ji}}{w_i}$
                if $i == r$ then
                    break
                end if
            end for
        end for
    end for
    return $\mathbf{Z}, \mathbf{z}$
end procedure
```

Dial's STOCH algorithm

procedure $\mathrm{STOCH}(G, \mathbf{t})$
    $y_{ij} \leftarrow 0, \forall (i,j) \in A$ $\triangleright$ Initialization
    for $r \in Z$ **do**
        $\mathbf{w}, \mathbf{W} \leftarrow \textsc{ForwardPass}(G, \mathbf{t}, r)$
        $\mathbf{Z}, \mathbf{z} \leftarrow \textsc{BackwardPass}(G, \mathbf{w}, \mathbf{W}, r)$
        for $(i,j) \in \hat{A}$ **do**
            $y_{ij} \leftarrow y_{ij} + z_{ij}$
        end for
    end for
    return $\mathbf{y}$
end procedure

Logit-based SUE using MSA

procedure $\mathrm{FW}(G, \mathbf{t}, \mathbf{d}, \text{tol})$

  $k = 1$, $x_{ij}^k = 0$, $t_{ij}^k = t_{ij}(0), \forall (i,j)$, and $\text{gap} = \infty$ ▷ Initialization

  while $\text{gap} > \text{tol}$ **do**

    $y_{ij}^k \leftarrow \mathrm{STOCH}(G, \mathbf{t}), \forall (i,j) \in A$ ▷ Auxiliary flows

    $\lambda = \frac{1}{k}$

    for $(i,j) \in A$ **do**

      $x_{ij}^{k+1} \leftarrow (1-\lambda) x_{ij}^k + \lambda y_{ij}^k$ ▷ Update link flows

      $t_{ij}^{k+1} \leftarrow t_{ij}(x_{ij}^{k+1})$ ▷ Update link travel times

    end for

    Evaluate $\text{gap} = \sum_{(i,j) \in A} \frac{|x_{ij}^k - x_{ij}^{k-1}|}{x_{ij}^k}$

    $k \leftarrow k + 1$

  end while

end procedure

SUE as a Fixed Point Problem

Let $\mathbf{h}^*$ be the equilibrium flow vector, the SUE can be formulated as a FPP

$$\mathbf{h}^* = \mathbf{d}P(\mathbf{c}(\mathbf{h}^*)) \tag{5}$$

In case of logit-based route choice, we have

$$h^{\pi *} = d^{rs} \frac{e^{-\theta c^{\pi}(\mathbf{h}^*)}}{\sum_{\pi' \in \Pi^{rs}} e^{-\theta c^{\pi'}(\mathbf{h}^*)}}, \forall \pi \in \Pi^{rs}, \forall (r, s) \in Z^2 \tag{6}$$

Remark. Since link performance functions are continuous and feasible path set is also compact and convex, we have at least one solution to above FPP using Brouwer's fixed point theorem.

Logit-based SUE optimization formulation by Fisk(1980)

$$Z^{SUE} = \underset{\mathbf{x},\mathbf{h}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx + \frac{1}{\theta} \sum_{\pi\in\hat{\Pi}} h^\pi \log h^\pi \tag{7a}$$

$$\text{subject to} \quad \sum_{\pi\in\hat{\Pi}^{rs}} h^\pi = d^{rs}, \forall (r,s) \in Z^2 \tag{7b}$$

$$h^\pi \geq 0, \forall \pi \in \hat{\Pi} \tag{7c}$$

$$x_{ij} = \sum_{\pi\in\hat{\Pi}} \delta_{ij}^\pi h^\pi, \forall (i,j) \in A \tag{7d}$$

Remark. We have added another entropy term for reasonable paths.
Remark. The objective function is strictly convex in terms of path flows. Therefore, the SUE solution is unique in terms of path flows!

Proposition

The KKT conditions of (7) *satisfies logit-based SUE.*

SUE link-based optimization formulation

$$\min_{\mathbf{x}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx + \frac{1}{\theta}\sum_{s\in Z}\left(\sum_{(i,j)\in A} x_{ij}^s \log x_{ij}^s - \sum_{i\in N: i\neq s} x_i^s \log x_i^s\right) \tag{8a}$$

$$\text{s.t.} \quad x_{ij}^s \geq 0, \forall (i,j)\in A, s\in Z \tag{8b}$$

$$\sum_{j\in FS(i)} x_{ij}^s - \sum_{j\in BS(i)} x_{ji}^s = \begin{cases} d^{rs}, & \text{if } i = r \\ -\sum_{r\in Z} d^{rs}, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \forall i\in N, \forall s\in Z \tag{8c}$$

$$x_{ij} = \sum_{s\in Z} x_{ij}^s, \forall (i,j)\in A \tag{8d}$$

Remark. The above formulation can be solved using any link-based methods we studied for UE.

General optimization formulation of SUE

The objective function for error terms with different distribution can be written as:

$$Z(\mathbf{x}) = -\sum_{(r,s)\in Z^2} d^{rs}\mathbb{E}\left[\min_{\pi\in\Pi^{rs}}\{c^\pi+\epsilon\}\;\middle|\;\mathbf{c}^{rs}(\mathbf{x})\right] + \sum_{(i,j)\in A} t_{ij}(x_{ij}) - \sum_{(i,j)\in A}\int_0^{x_{ij}} t_{ij}(x)dx \tag{9}$$

where, $S^{rs}(\mathbf{c}^{rs}(\mathbf{x})) = \mathbb{E}\left[\min_{\pi\in\Pi^{rs}}\{c^\pi+\epsilon\}\;\middle|\;\mathbf{c}^{rs}(\mathbf{x})\right]$

We know that $\frac{\partial S^{rs}(\mathbf{c}^{rs}(\mathbf{x}))}{\partial c^\pi} = P^\pi, \forall\pi\in\Pi^{rs}$. We we take the derivative wrt x and equate it to zero, we get

$$\boxed{x_{ij} = \sum_{(r,s)\in Z^2}\sum_{\pi\in\Pi^{rs}} d^{rs}P^\pi\delta_{ij}^\pi} \tag{10}$$

Final remarks

- There also exists a Braess-like paradox in case of SUE.
- Logit-based SUE may not represent reality since the paths have overlaps and the error term defining utility may not be independent.
- One can solve probit-based SUE. However, it does not have closed-form formulation, so we may require simulation or other tools to evaluate SUE.
- Finally, is Stochastic User Equilibrium really stochastic?

Suggested Readings

- Sheffi Chapters 10 and 11
- BLU book Section 9.3

Dial (1971)

A PROBABILISTIC MULTIPATH TRAFFIC ASSIGNMENT MODEL WHICH OBVIATES PATH ENUMERATION

ROBERT B. DIAL†

Alan M. Voorhees and Associates, Inc., McLean, Virginia

(*Received* 28 *May* 1970)

$$a(e)=\begin{cases}\exp\theta[p(j)-p(i)-t(i,j)] & \text{if } p(i)<p(j),\quad q(j)<q(i)\\ 0 & \text{otherwise}\end{cases}$$

Having thus defined $a(e)$, the algorithm can be described as a two-pass process, which need concern itself only with those links whose $a(e)$ is not zero.

1. (Forward pass.) By examining all nodes i in *ascending* sequence with respect to $p(i)$, their distance from the origin, calculate for each link e in I_i its "link weight",

$$w(e)=\begin{cases}a(e) & \text{if } i=o \text{ (the origin node)}\\ a(e)\sum\limits_{e' \text{ in } F_i} w(e') & \text{otherwise}\end{cases}$$

When the destination node d is reached, go to Step 2.

2. (Backward pass.) Starting with the destination node d, examine all nodes j in *descending* sequence with respect to $p(j)$. Assign a trip volume $x(e)$ to each link e in F_j as follows:

$$x(e)=\begin{cases}y.w(e)/\sum\limits_{e' \text{ in } F_j} w(e'') & \text{if } j=d \text{ (the destination node)}\\ w(e)\sum\limits_{e' \text{ in } I_j} x(e')/\sum\limits_{e' \text{ in } F_j} w(e') & \text{otherwise}\end{cases}$$

Daganzo and Sheffi (1977)

On Stochastic Models of Traffic Assignment

CARLOS F. DAGANZO

University of California, Berkeley, California

and

YOSEF SHEFFI

Massachusetts Institute of Technology, Cambridge, Massachusetts

$$p_k = \frac{\exp\{\theta^*(T_o - T_k)\}}{\sum_{j \in P} \exp\{\theta^*(T_o - T_j)\}} \qquad \text{if } k \in P$$

$$= 0 \qquad \text{if } k \notin P$$

Fisk (1980)

SOME DEVELOPMENTS IN EQUILIBRIUM TRAFFIC ASSIGNMENT†

CAROLINE FISK

Centre de recherche sur les transports, Université de Montréal, Canada

(*Received* 15 *November* 1978; *in revised form* 30 *June* 1979)

$$Z_2 = \frac{1}{\theta} \sum_{pq} \sum_{k} h_{k,pq} \ln h_{k,pq} + \sum_{a} \int_0^{v_a} s_a(v)\, \mathrm{d}v \tag{2}$$

where the expression $h \ln h$ is assigned the value zero at $h = 0$. The problem P_2 consists in minimizing Z_2 subject to the constraints

$$\sum_k h_{k,pq} = g_{pq} \qquad \text{all } p \in \{O\}, q \in \{D\} \tag{3a}$$

$$h_{k,pq} \geq 0 \qquad \text{all } p \in \{O\},\ q \in \{D\},\ k \in K_{pq} \tag{3b}$$

and the link flow relationship

$$v_a = \sum_{pq} \sum_k \delta_{ak,pq} h_{k,pq} \qquad \text{all } a \in \{A\}.$$

Nobel Prize for discrete choice in Economic Sciences

Figure: Daniel Little McFadden

Source: UCB Economics

Daniel Little McFadden, an American econometrician and a doctorate from the University of Minnesota won Nobel Memorial Prize in Economic Sciences for his development of theory and methods for analyzing discrete choice.

Thank you!