COMS 3261 Homework 0

Summer B 2021

This homework set is OPTIONAL.

Problem 1 (Sets; Sipser 0.1, 0.2, 0.4 and 0.5). 1.1. Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

1. $\{1, 3, 5, 7...\}$
2. $\{n \mid n = 2m \text{ for some } m \in \mathcal{N}\}$
3. $\{w \mid w \text{ is a string of 0s and 1s and } w \text{ equals the reverse of } w\}$

1.2. Write formal descriptions of the following sets.

1. The set containing the numbers 1, 10, and 100
2. The set containing all natural numbers less than 5
3. The set containing nothing at all. (There are two ways to write this: the natural way and using a certain special symbol.)

1.3. If the set A has a elements and B has b elements, how many elements are in $A \times B$?

1.4. If C is a set with c elements, how many elements are in the power set of C? (Recall that C's power set is the set containing all subsets of C.)

Problem 2 (Graphs; Sipser 0.8). 2.1. Consider the undirected graph $G = (V, E)$, where V, the set of nodes, is $\{1, 2, 3, 4\}$ and E, the set of edges, is $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$. Draw G. What are the degrees of each node? Indicate a path from node 3 to node 4 on the graph.

Problem 3 (Proofs; Sipser 0.10, 0.12). 3.1. Prove that every graph with two or more nodes contains two nodes with the same degree.

3.2. Given $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ where f is defined as:

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$$

Prove or provide a counterexample that f is (a) injective, (b) surjective, and (3) bijective.

3.3. Let $S(n) = 1+2+\cdots+n$ be the sum of the first n natural numbers and let $C(n) = 1^3+2^3+\cdots+n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n to arrive at the curious conclusion that $C(n) = S^2(n)$ for every n.

1. $S(n) = \frac{1}{2}n(n+1)$
2. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Problem 4 (Relations). 4.1. Let R and S be relations from A to B. Prove that: $(R \subseteq S) \to (R^{-1} \subseteq S^{-1})$.

Problem 5: (Sets and Their Complements). Prove that

$$A \cap ((B \cup A^c) \cap B^c) = \emptyset$$

(A^c denotes the complement of A and B^c denotes the complement of B. $\emptyset$ is the empty set.)

Problem 6 (Proof by Contradiction). Prove using **contradiction** that for all integers n, if 5 divides n^2 then 5 divides n (Hint: what does it mean to be not divisible by 5?).

Problem 7 (Proof by Induction).

1. Assume n is a positive integer. Use induction to prove the following:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdot \frac{1}{n\cdot(n+1)} = 1 - \frac{1}{n+1}$$

2. Prove that if $m, n \in \mathbb{Z}$ such that m is even and n is odd, then $m + n - 2$ is odd.

Problem 8 (Composition of Functions).

1. Prove that if $g \circ f$ is one-to-one (injective), then f is one-to-one (injective).
2. Prove that if $g \circ f$ is onto (surjective), then g is onto (surjective).

Problem 9 (Proof by Construction). Prove that for any positive integer n, there exists a sequence of n consecutive positive composite integers. [Hint: try to construct such a sequence!]