Modeling Optimization Problems

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Outline

What is optimization?

(Merriam-Webster Dictionary) An act, process, or methodology of making something (such as a design, system, or decision) as fully perfect, functional, or effective as possible.

Example 1

(Maximum Area Problem)

You have 80 meters of wire and want to enclose a rectangle as large as possible (in area). How should you do it?

Example 2

(Production Problem) A factory can produce two products, A and B. The production of each item of A takes 2 hours, and that of item B takes 7 hours. Further, each item of products A and B takes 22 and 41 ft^3 storage capacity, respectively. The manager gets a profit of \$30 and \$50 by producing each item of A and B resp. Assuming that there is an 88-hour limit on the number of hours of operating the factory and the maximum storage capacity of the factory is 9,000ft^3, how many items of A and B should the manager decide to produce to maximize the profit?

Outline

Common Framework

Components of an optimization problem

- Decisions
- Constraints
- Objective

Optimization seeks to choose some decisions to optimize (maximize or minimize) an objective subject to certain constraints.

Common Framework

Given $f, g_i, h_i : \mathbb{R}^n \mapsto \mathbb{R}$

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad & f(\mathbf{x}) && \text{(1a)}\\
\text{subject to} \quad & g_i(\mathbf{x}) \leq 0, \forall i = 1, 2, ..., p && \text{(1b)}\\
& g_j(\mathbf{x}) \geq 0, \forall j = 1, 2, ..., q && \text{(1c)}\\
& h_k(\mathbf{x}) = 0, \forall k = 1, 2, ..., r && \text{(1d)}
\end{aligned}
$$

- Decisions: $\mathbf{x}$, Objective: $f(\mathbf{x})$, and Constraints: (1b)-(1d)
- (1b), (1c), and (1d): set of "$\leq$", "$\geq$", and equality constraints
- $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : (1\text{b}) - (1\text{d})\}$ define the feasible region.
- Any $\hat{\mathbf{x}}$ satisfying all the constraints is a feasible solution.
- Any $\mathbf{x}^* \in \mathcal{X}$ satisfying $f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathcal{X}$ is an optimal solution.
- $f(\mathbf{x}^*)$ is known as optimal objective value.

A few classes of optimization problems

- Linear optimization: f, g_i, h_i are all linear functions of continuous variables x.
- Non-linear optimization: At least one of f, g_i, h_i is non-linear function of continuous variables x.
  - Convex optimization: All functions are convex and feasible region is a convex set
- (Mixed) Integer optimization: Some of the variables x are restricted to be integers.
- (Mixed) Integer Non-linear optimization: Some of the variables x are restricted to be integers and at least one of f, g_i, h_i is non-linear.

Difficulty of solving above classes rises significantly as we go from above to below.

A few definitions

Definition (Maximum) Let $S \subseteq \mathbb{R}$. We say that x is a maximum of S iff $x \in S$ and $x \geq y, \forall y \in S$.

Definition (Minimum) Let $S \subseteq \mathbb{R}$. We say that x is a minimum of S iff $x \in S$ and $x \leq y, \forall y \in S$.

Definition (Bounds) Let $S \subseteq \mathbb{R}$. We say that u is an upper bound of S iff $u \geq x, \forall x \in S$. Similarly, l is a lower bound of S iff $l \leq x, \forall x \in S$.

Definition (Supremum) Let $S \subseteq \mathbb{R}$. We define the supremum of S denoted by $\textbf{sup}(S)$ to be the smallest upper bound of S. If no such upper bound exists, then we set $\textbf{sup}(S) = +\infty$.

Definition (Infimum) Let $S \subseteq \mathbb{R}$. We define the infimum of S denoted by $\textbf{inf}(S)$ to be the largest lower bound of S. If no such lower bound exists, then we set $\textbf{inf}(S) = -\infty$

Definition If $x \in S$ such that $x = \textbf{sup}(S)$, we say that supremum of S is achieved (which means that there is a maximum to the problem). Similar definition for whether infimum is achieved.

General Formulation of LP

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize/maximize}} \quad & \mathbf{c}^T\mathbf{x} && \text{(2a)}\\
\text{subject to} \quad & \mathbf{a}_i^T\mathbf{x} \leq b_i, \forall i \in C_1 && \text{(2b)}\\
& \mathbf{a}_j^T\mathbf{x} \geq b_j, \forall j \in C_2 && \text{(2c)}\\
& \mathbf{a}_k^T\mathbf{x} = b_k, \forall k \in C_3 && \text{(2d)}\\
& x_u \geq 0, \forall u \in N_1 && \text{(2e)}\\
& x_v \leq 0, \forall v \in N_2 && \text{(2f)}\\
& x_w \text{ free }, \forall w \in N_3 && \text{(2g)}
\end{aligned}
$$

where, $C_1, C_2, C_3 \subseteq \{1, ..., m\}$, $N_1, N_2, N_3 \subseteq \{1, ..., n\}$

More definitions

Definition (Hyperplane) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$

Definition (Halfspace) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \geq b\}$

Definition (Polyhedron) A set $P \subseteq \mathbb{R}^n$ is called a polyhedron if P is the intersection of a finite number of halfspaces. $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$

Definition (Polytope) A bounded polyhedron is called a polytope.
Question Is $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ a polyhedron?

Definition (Convex Sets) A set $S \subseteq \mathbb{R}^n$ is a convex set if for any $\mathbf{x}, \mathbf{y} \in S$, and $\lambda \in [0, 1]$, we have $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$.
Question Is polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ a convex set?

Definition (Convex combination) $\mathbf{x} \in \mathbb{R}^n$ is said to be convex combination of $\mathbf{x}^1, ..., \mathbf{x}^p \in \mathbb{R}^n$ if for $\lambda_1, ..., \lambda_p \geq 0$ s.t. $\sum_i^n \lambda_i = 1$, $\mathbf{x}$ can be expressed as $\mathbf{x} = \sum_i^n \lambda_i \mathbf{x}^i$.

Definition (Extreme point) Let P be a polyhedron. Then, $\mathbf{x} \in P$ is an extreme point of P if we cannot express $\mathbf{x}$ as a convex combination of other points in P.

Theorem

Let P be a non-empty polyhedron. Consider LP $\max\{\mathbf{c}^T\mathbf{x} \textit{ s.t. } \mathbf{x} \in P\}$. Suppose the LP has at least one optimal solution and P has at least one extreme point. Then, above LP has at least one extreme point of P that is an optimal solution.

Possible states of optimization problems

An optimization problem may have the following states:

- Infeasible (max problems, $Z = -\infty$ and min problems, $Z = +\infty$)
- Feasible, optimal value finite but not attainable
- Feasible, optimal value finite and attainable
- Feasible, but optimal value is unbounded (max problems, $Z = +\infty$ and min problems, $Z = -\infty$)

Outline

Standard Form of LP[1]

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && \text{(3a)}\\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(3b)}\\
& \mathbf{x} \geq 0 && \text{(3c)}
\end{aligned}
$$

where, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ ($m < n$ fat matrix), $\mathbf{b} \in \mathbb{R}^m$

[1] Following the convention by Bertsimas and Tsitsikilis

Transformation to Standard Form

- To convert maximization of $\mathbf{c}^T\mathbf{x}$ to minimization , write $\min \; -\mathbf{c}^T\mathbf{x}$
- $A\mathbf{x} \le \mathbf{b} \implies A\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{s} \ge 0$, $\mathbf{s}$ are called slack variables
- $A\mathbf{x} \ge \mathbf{b} \implies A\mathbf{x} - \mathbf{s} = \mathbf{b}, \mathbf{s} \ge 0$
- $x_i \le 0$. Define $y_i = -x_i$, write $y_i \ge 0$
- Eliminating free variables. Define $x_i = x_i^+ - x_i^-$, write $x_i^+, x_i^- \ge 0$

Pointwise maximum/minimum, no problem!

How to linearize functions such as $\max_i \{\mathbf{a}_i^T x + b_i\}$ & $\min_i \{\mathbf{a}_i^T x + b_i\}$?

- Define $y = \max_i \{\mathbf{a}_i^T \mathbf{x} + b_i\} \implies y \geq \mathbf{a}_i^T \mathbf{x} + b_i, \forall i$
- Define $y = \min_i \{\mathbf{a}_i^T \mathbf{x} + b_i\} \implies y \leq \mathbf{a}_i^T \mathbf{x} + b_i, \forall i$

How about the following problem?

$$
\begin{aligned}
Z = \underset{\mathbf{x} \geq 0}{\text{minimize}} \quad & \|\mathbf{x}\|_1 = \sum_i |x_i| && \text{(4a)} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(4b)}
\end{aligned}
$$

- Note $|x_i| = \max\{x_i, -x_i\}$. Define $y_i = |x_i|$

$$
\begin{aligned}
Z = \underset{\mathbf{x} \geq 0, \mathbf{y}}{\text{minimize}} \quad & \sum_i y_i && \text{(5a)} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(5b)} \\
& y_i \geq x_i, \forall i && \text{(5c)} \\
& y_i \geq -x_i, \forall i && \text{(5d)}
\end{aligned}
$$

Pointwise maximum/minimum, no problem!

How about the following problem?

$$\begin{aligned} Z = \underset{\mathbf{x}\geq 0}{\text{minimize}} \quad & \|\mathbf{x}\|_\infty = \max_i \{x_i\} && \text{(6a)} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(6b)} \end{aligned}$$

- Define $y = \max_i\{x_i\}$

$$\begin{aligned} Z = \underset{\mathbf{x}\geq 0, y}{\text{minimize}} \quad & y && \text{(7a)} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(7b)} \\ & y \geq x_i, \forall i && \text{(7c)} \end{aligned}$$

Linear Fractional Program, no problem!

(Assume that $\mathbf{e}^T\mathbf{x} + f > 0$ for any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$)

$$Z = \underset{\mathbf{x}\geq 0}{\text{minimize}} \quad \frac{\mathbf{c}^T\mathbf{x} + d}{\mathbf{e}^T x + f} \tag{8a}$$

$$\text{subject to} \quad A\mathbf{x} = \mathbf{b} \tag{8b}$$

- Define $\mathbf{y} = \frac{\mathbf{x}}{\mathbf{e}^T x + f}, z = \frac{1}{\mathbf{e}^T\mathbf{x} + f}$. We can equivalently write above program as an LP.

$$Z = \underset{\mathbf{y},z}{\text{minimize}} \quad \mathbf{c}^T\mathbf{y} + dz \tag{9a}$$

$$\text{subject to} \quad A\mathbf{y} - \mathbf{b}z = 0 \tag{9b}$$

$$\mathbf{e}^T\mathbf{y} + fz = 1 \tag{9c}$$

$$z \geq 0 \tag{9d}$$

Linear Integer Program

$$
\begin{aligned}
Z = \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && \text{(10a)}\\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b} && \text{(10b)}\\
& x_i \in \mathbb{Z}_+, i = 1, ..., p && \text{(10c)}\\
& x_i \in \mathbb{R}_+, i = p+1, ..., n && \text{(10d)}
\end{aligned}
$$

- Generally, solving IP is more difficult than solving an LP. We use various tools from LP to approach this difficult problem.
- Better formulating the problem makes a lot of difference.

Outline

Example 3

(Fleet sizing Problem) CEGE 5214 A transit agency is going to optimize its fleet size and type to maximize its revenue. Possible vehicle types are:

- Vans, capacity 6, purchase cost \$20, projected revenue \$96
- Regular buses, capacity 28, purchase cost \$120, projected revenue \$400
- Articulated buses, capacity 56, purchase cost \$220, projected revenue \$900

Constraints:

- Available budget is \$2,000
- The agency has 25 drivers who have 20% vacation/sick/no-show rate
- The fleet should provide a minimum capacity of 450
- At least 30% of the fleet should be vans for demand-responsive service
- At least 10 regular buses are needed for the fixed routes
- Exactly 2 articulated buses are needed for an express route

Example 4

(Support Vector Machine Problem) Given two groups of data points in $\mathbb{R}^d$, $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_m\}$, find a plane that separates them.

Network Flow Problems

Example 5

(Minimum cost flow problem) Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, lower and upper bounds (capacity) on the flow on links $l : A \mapsto \mathbb{R}$ and $u : A \mapsto \mathbb{R}$ resp., and supply/demand at each node $b : N \mapsto \mathbb{R}$, find the least cost shipment of a commodity. Note $b(i) > 0$ for a supply nodes, $b(i) < 0$ for demand nodes, and $b(i) = 0$ for transshipment nodes.

Example 6

(Shortest path problem) Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, find the shortest path from $s \in N$ to $t \in N$.

Example 7

(Maximum flow problem) Given a directed graph $G(N, A)$, cost of traversing links $c : A \mapsto \mathbb{R}$, and capacities of links $u : A \mapsto \mathbb{R}$, find the maximum flow possible to send from $s \in N$ to $t \in N$.

Example 8

(Assignment problem) Given a bipartite graph $G(N_1 \cup N_2, A)$ and cost of assignment $c : A \mapsto \mathbb{R}$, find the least cost assignment of items in N_1 to items in N_2.

Example 9

(Transportation Problem) We have n factories each supplying a_i units of construction lumber and m cities each with b_i demand of lumber. If the transportation cost of each unit from factory i to city j is c_{ij}, formulate a program that minimizes the total transportation cost while serving the demand in all the cities.

Integer Problems

Example 10

(Vertex cover problem) Given a graph $G(N, A)$, find the smallest set of vertices that touch every edge of the graph.

Example 11

(Traveling Salesman Problem) A salesman needs to visit a number of places in a day. How should he schedule her trip so that the total distance is shortest (or the total cost is smallest)?

Example 12

(Knapsack Problem) Given a set of items N, each with a weight w_i and a value a_i, determine which items to include in the collection so that the total weight is less than or equal to a given limit W and the total value is as large as possible.

Thank you!