CVL746 **Fall 2024**

Lecture 9: Frequency-based transit assignment

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9.1 Introduction

Before the development of transit assignment models, it was assumed that the passenger assignment on a transit network can be performed similarly as an auto network using *all or nothing* assignment of the demand to the shortest paths. However, this is not a proper solution to the transit assignment problem due to the phenomenon of waiting for transit service. Spiess and Florian [1989] said *One is tempted to formulate the above problem as: "How does one find the path between two nodes in the transit network that minimizes the expected total travel time?" and doing so one would have already changed the problem. This is because we are accustomed to assuming that the passenger selects a single path between any pair of nodes in a network. This may be true in the case of a car driver, but not in the case of a transit rider.* Consider a passenger wishing to travel between stop A and B in Figure 9.1 served by 5 different transit routes. Which transit route should she choose? Chriqui and Robillard [1975] calls this problem as *common bus lines problem.* They proposed that a passenger will select a subset of transit routes to minimize her expected waiting and travel time and board the first arriving vehicle in that subset.

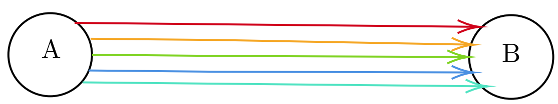

Figure 9.1: Common lines

In this tutorial, we summarize various aspects of the frequency-based (FB) transit assignment. A FB transit system is a system where passengers randomly arrive at bus stops and do not coordinate their arrival time according to any schedule. Before describing the formal concepts, we should understand the phenomenon of waiting for a transit route, which is explained by Larson and Odoni [1981] as a *random incidence process.* They propose the following formula for the wait time distribution based on the headway (time between successive arrival of a bus at a stop) of the bus route:

$$\text{Prob. of bus arrival after } w \text{ min} = \frac{1 - (\text{Prob of headway} \leq w)}{\text{Expected headway}} \tag{9.1}$$

If the headway follows an exponential distribution with the rate (frequency of the bus route) f, then we can compute the distribution of the wait time (using (9.1)) as $fe^{-fw}, w \geq 0$, i.e., an exponential distribution with rate f.

9.2 Strategy/Hyperpath

Spiess and Florian [1989] describes a *strategy* as a set of rules when applied helps a passenger to move from an origin to a destination in a transit network. Chriqui and Robillard [1975] suggests one such strategy as: *Before starting a trip, the passenger has chosen an attractive set of transit lines (boards whichever arrives* *first) for every boarding stop she may encounter on her trip and for every transit line the alighting stop.* Spiess and Florian [1989] showed that this is an optimal strategy minimizing the expected wait and travel time of passengers arriving at a stop randomly. Any strategy induces a sub-network between two nodes in the transit network, which is formalized as a *hyperpath* by Nguyen and Pallottino [1988]. A hyperpath p between origin r and destination k in network $G(N, A)$ is an acyclic sub-network $H_p = (N_p, A_p, \pi_p)$, where $N_p \subset N, A_p \subset A$, and $\pi_p(\pi_{ijp}) \in [0,1]^{|A_p|}$, such that r has no predecessor and k has no successor, there exists a path from every node in N_p to k and, π_p satisfies $\sum_{j \in FS_p(i)} \pi_{ijp} = 1$. Here, π_{ijp} is the probability of selecting transit route link (i, j) out of attractive routes $FS_{p^*}(i)$. If the headway of all transit routes associated to links $a \in FS_{p^*}(i)$ are independent and follows an exponential distribution with frequency f_a, then the probability of taking transit route $a = (i, j) : j \in FS_{p^*}(i) =$ Prob (waiting for a is less than all other transit route) $= \frac{f_a}{\sum_{a'} f_{a'}}, \forall a = (i, j) : j \in FS_{p^*}(i)$.

9.2.1 Computing shortest hyperpath

Nguyen and Pallottino [1988] proved that one can compute the optimal expected cost-to-go from node i to destination node k, u_i^*, using the set of Bellman equations (9.2). As you can see that if i is part of the boarding stops S, then the passenger selects an attractive set of outgoing routes $FS_{p^*}(i)$, for which the expected waiting time, travel time, and rest of the optimal cost-to-go is added. However, finding $FS_{p^*}(i)$ is stated as a combinatorial problem that can be difficult to solve. It turns out that a greedy approach can find $FS_{p^*}(i)$. At every node i, initialize $FS_{p^*}(i) = \phi$ and sort $\{u_j^*\}_{j \in FS(i)}$ in ascending order, then keep adding $j \in FS(i)$ in this sequence as long as the expected cost of travel keeps decreasing. Finally, one can construct an efficient label setting/ label correcting algorithm to compute the solution of (9.2).

$$u_i^* = \begin{cases} 0, & \text{if } i = k, \\ \min\limits_{j \in FS(i)} \{c_{ij} + u_j^*\} & \text{if } i \notin S \\ \min\limits_{j \in FS_{p^*}(i) \subseteq FS(i)} \left\{ \frac{1}{\sum_{\in FS_{p^*}(i)} f_{ij}} + \sum_{j \in FS_{p^*}(i)} \left(\frac{f_{ij}}{\sum_{j \in FS_{p^*}(i)} f_{ij}} \left(c_{ij} + u_j^* \right) \right) \right\} & \text{if } i \in S \end{cases} \tag{9.2}$$

9.2.2 Other remarks on hyperpaths

First, finding the optimal strategy/hyperpath is a stochastic shortest path problem (undiscounted Markov Decision Process with a termination state). Here, states are the nodes in the network, actions are $FS_{p^*}(i)$ at each node i, and costs are the waiting and travel time. In this case, a policy/strategy/hyperpath maps each state i to a subset $FS'(i) \subseteq FS(i)$. Corresponding to each policy, the transition matrix is given by $\{\pi_{ijp}\}_{j \in FS'(i)}$. The Bellman equation (9.2) can be solved using the value iteration, policy iteration, and Linear Programming.

Second, finding the optimal strategy/hyperpath can also be viewed as a zero-sum game, in which a passenger at every node i is playing against an adversary/nature/demon (Schmöcker et al. [2009]). The passenger plays each strategy (selects a transit route) with certain probability π_{ijp} to minimize the worst-case expected cost of travel. The payoff matrix has u_j in every column of row $j \in FS(i)$, except the diagonal entries where the payoff is equal to $u_j + \frac{1}{f_{ij}}$. Since it is a zero-sum game, it can be formulated as a Linear Program.

Third, the above model assumes that each passenger has a correct perception about the travel time and wait time. This assumption can be relaxed using discrete choice models. One such example is logit-based hyperpaths proposed by Nguyen et al. [1998].

Finally, the assumption of wait time following exponential distribution can be too restrictive. For example, if a bus route has a fixed headway of 10 minutes, then the probability of the bus arriving after 10 minutes should be high. However, the exponential distribution has memoryless property and the elapsed time has no effect on the probability. Billi et al. proposed to use Erlang distribution for the wait time and emphasized the importance of the *dynamic strategy*, i.e., passengers keeps updating the strategy until boarding. Hickman and Bernstein [1997] proposes a methodology to compute such a dynamic strategy.

9.3 Uncongested FB transit assignment

Spiess and Florian [1989] formulated the FB transit assignment as a Linear Program (9.3) for the case when travel times are constant, there is no denied boarding due to capacity limits, and headways of various transit routes follow an exponential distribution. Let O and D be the set of origins and destinations resp., and v_{ak} and W_{ik} be the flow on link $(i,j) \in A$ and wait time of the passengers boarding at node i destined to $k \in D$ respectively. Then, the assignment can be formulated as (9.3) and solved using Algorithm 1.

$$\underset{\mathbf{v},\mathbf{W}}{\text{minimize}} \quad \sum_{k\in D}\left(\sum_{a\in A} c_a v_{ak} + \sum_{i\in S} W_{ik}\right) \tag{9.3a}$$

$$\text{subject to} \quad \sum_{a\in FS(i)} v_{ak} = \sum_{a\in BS(i)} v_{ak} + g_{ik}, \forall i \in N, \forall k \in D \tag{9.3b}$$

$$v_{ak} \leq f_a W_{ik}, \forall a = (i,j) : j \in FS(i), \forall i \in S, \forall k \in D \tag{9.3c}$$

$$v_{ak} \geq 0, \forall a \in A, \forall k \in D \tag{9.3d}$$

$$g_{ik} = \begin{cases} d_{ik}, & \text{if } i \neq k, (i,k) \in O \times D \\ -\sum_{o\in O} d_{ok}, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$$

Algorithm 1 Algorithm for optimal strategy and link flows (Spiess and Florian)

1: **for** $k \in D$ **do**
2:     (*Finding optimal strategy/hyperpath* (A_p))
3:     (*Initialize*) $u_i = \infty, \forall i \in N \backslash \{r\}; u_k = 0; f_i = 0, \forall i \in N; \mathcal{S} = A; A_p = \phi$
4:     **while** $\mathcal{S} \neq \phi$ **do**
5:         (*Get next link*) Find $a = (i,j) \in \mathcal{S}$ which satisfies $u_j + c_a \leq u_{j'} + c_{a'}, \forall a' = (i', j') \in \mathcal{S}$
6:         $\mathcal{S} = \mathcal{S} \backslash \{a\}$
7:         **if** $u_i \geq u_j + c_a$ **then** (*Update node label*)
8:             $u_i = \frac{f_i u_i + f_a(u_j + c_a)}{f_i + f_a}$
9:             $f_i = f_i + f_a$; $A_p = A_p \cup \{a\}$
10:             Go to step 5
11:     (*Assign demand according to optimal strategy*)
12:     (*Initialize*) $V_i = g_{ik}, \forall i \in N$
13:     **for** every link $a \in A$ in decreasing order of $(u_j + c_a)$ **do** (*Loading*)
14:         **if** $a \in A_p$ **then**
15:             $v_a = v_a + \frac{f_a}{f_i} V_i$
16:             $V_i = V_i + v_a$
17:         **else**
18:             Continue

(9.2) can be used to derive the Spiess-Florian model. To see this, we can cast the dual of the S-F model using the (9.2).

$$\underset{\mathbf{u}}{\text{maximize}} \quad \sum_{k\in D}\sum_{i\in S} u_i^k g_{ik} \tag{9.4a}$$

$$\text{subject to} \quad u_i^k \leq \frac{1}{\sum_{j\in FS(i)}} + \sum_{j\in FS(i)} \frac{f_j\left(c_{ij}+u_j^k\right)}{\sum_{j\in FS(i)} f_j}, \forall i \in S, \forall k \in D \tag{9.4b}$$

(9.4b) can be written as

$$\frac{\sum_{j\in FS(i)} f_j}{\sum_{j\in FS(i)} f_j} u_i^k \leq \frac{1}{\sum_{j\in FS(i)}} + \sum_{j\in FS(i)} \frac{f_j\left(c_{ij}+u_j^k\right)}{\sum_{j\in FS(i)} f_j}, \forall i \in S, \forall k \in D \tag{9.5}$$

Assuming $\nu_{ak} = u_i^k - u_j^k - c_a$, we can write:

$$\underset{\mathbf{u},\nu}{\text{maximize}} \quad \sum_{k\in D}\sum_{i\in S} u_i^k g_{ik} \tag{9.6a}$$

$$\text{subject to} \quad \nu_{ak} \geq u_i^k - u_j^k - c_a, \forall a \in FS(i), \forall k \in D \tag{9.6b}$$

$$\sum_{a\in FS(i)} f_a \nu_{ak} \leq 1, \forall i \in S, \forall k \in D \tag{9.6c}$$

This is the dual of (9.3).

9.4 Congested FB transit assignment

Previous models assume that travel time and wait time functions are constant and are not affected by the flow of passengers. This could result in unrealistic passenger flows on some transit routes, especially when the number of passengers boarding a transit route is more than its capacity. Due to congestion, some of the transit routes which were not attractive before may become attractive. The congestion in FB transit assignment is modeled through the following ways:

9.4.1 Discomfort functions

Spiess and Florian [1989] considered an increasing *discomfort function* (in-vehicle travel time) $c_{ij}(v_{ij})$ to capture growing discomfort with increasing number of passengers on-board. They found the equilibrium by minimizing $\sum_{k\in D}\left(\sum_{ij\in A}\int_0^{v_{ak}} c_a(x_{ak})dx_{ak} + \sum_{i\in S} W_{ik}\right)$ instead of (9.3a). Further, Wu et al. [1994] expressed travel time on in-vehicle links as the function of its flow, whereas they expressed wait time as sum of two components:

1. a BPR-type asymmetric function of on-board flow and waiting flow
2. expected wait time due to multiple transit route in the strategy

They formulated a variational inequality (VI) problem for the equilibrium and solved it using a symmetric linearizion procedure.

9.4.2 Effective frequency

To incorporate congestion, De Cea and Fernández [1993] considered a transit network composed of route sections. A route section is a unique link between two transfer nodes served by various transit routes. The wait time for any route section s is equal to the sum of nominal wait time ($\frac{1}{\sum_{l \in s} f_l}$) and a BPR-type waiting time ($\phi_s(v) = (\frac{v^s + \tilde{v}_s}{K_s})^n$), where v^s is the waiting flow, $\tilde{v}_s$ is the competing flow (passengers already on-board and passengers using route section s for a different route), K_s is the practical capacity and ϕ_s^{-1} is the *effective frequency* of the route section. Since this is an asymmetric assignment problem, they propose to use *diagonalization method* to solve the problem.

One of the criticism of De Cea and Fernández [1993]'s approach is that the equilibrium flows may exceed capacity of transit route due to soft capacity constraints imposed by the effective frequency. Cominetti and Correa [2001] propose to use an effective frequency $f_a(v_a)$ which is a non-increasing function of waiting (v_{ba}) and on-board (v_{xa}) passenger flow. The effective frequency approaches 0 when the total flow on the transit route reaches the capacity limit. Using the equilibrium conditions proposed in Cominetti and Correa [2001], Cepeda et al. [2006] propose the following gap function (9.7) to compute the equilibrium flows. The problem can be solved using a MSA-based heuristic, i.e., fix the flow v^k, compute $f_a(v_k)$ and $c_a(v_k)$, then compute solution $\hat{v}^k$ of (9.7) using Algorithm 1, then update $v^{k+1} = \frac{1}{k} v^k + (1 - \frac{1}{k}) \hat{v}^k$, and repeat until the gap function is close to 0.

$$\min_v \sum_{k \in D} \left[\sum_{a \in A} c_a(v) v_{ak} + \sum_{i \neq k} \max_{a \in FS(i)} \frac{v_{ak}}{f_a(v)} - \sum_{i \neq d} g_{ik} u^*_{ik} \right] \tag{9.7}$$

Consider $X = \left\{ v \in \mathbb{R}_+^{|A||D|} \mid (9.3b), \sum_{k \in D} v_{ak} \leq u_a \right\}$, where u_a is the total capacity of route a and $V = \underset{k \in D}{\times} \left\{ v^k \in \mathbb{R}_+^{|A|} \mid \sum_{a \in FS(i)} v_{ak} = \sum_{a \in BS(i)} v_{ak} + g_{ik}, \forall i \in N \right\}$. Also, $S_i^k = \left\{ \alpha \in \mathbb{R}_+^{|FS(i)|} \mid \sum_{a \in FS(i)} \alpha_a = 1 \right\}$ and $S = \underset{k \in D}{\times} \underset{i \in S}{\times} S_i^k$. Let $F_v(v, \zeta) = \left\{ \begin{matrix} c_a(v) + \zeta_a^k w_a(v); a \in FS(i), i \in S, k \in D \\ c_a(v); a \in FS(i), i \in N \backslash S, k \in D \end{matrix} \right\}$ and $F_\zeta(v) = \left\{ -w_a(v) v_a^k; a \in FS(i), i \in S, k \in D \right\}$. Codina [2013] formulated the capacitated transit assignment problem as a variational inequality (VI) problem given below: Find $(v^*, \zeta^*) \in (V \cap X) \times S$ s.t.,

$$F_v(v^*, \zeta^*)^T (v - v^*) + F_\zeta(v^*)^T (\zeta - \zeta^*) \geq 0, \forall (v, \zeta) \in (V \cap X) \times S \tag{9.8}$$

Under certain assumptions, Codina and Rosell [2017] showed that the variational inequality can be formulated as the following non-convex problem (9.9) that can be solved using MSA-based heuristic.

$$\min_{v,u} \sum_{k \in D} \left[\sum_{a \in A} c_a(v) u_{ak} + \sum_{i \neq k} \max_{a \in FS(i)} \frac{u_{ak}}{f_a(v)} \right] \tag{9.9}$$

9.4.3 Failure to board probabilities

Kurauchi et al. [2003] suggests handling the capacity constraints using failure-to-board probabilities $q_i =$

$\min\left\{1, 1 - \frac{\text{residual capacity}}{\text{boarding passengers}}\right\}$. They split each stop nodes into two nodes, namely, *boarding node* S and *failure-to-board node* E. The failure-to-board node is then connected to the destination node using a *failure arc*. The modified Bellman equations (9.10) are specified below. Here, θ represents the risk of failure to board.

$$u_i^* = \begin{cases} 0, & \text{if } i = s, \\ \min_{j \in FS(i)} \{c_{ij} + u_j^*\} & \text{if } i \notin S \cup E \\ \min_{j \in FS_{p^*}(i) \subseteq FS(i)} \left\{ \frac{1}{\sum_{\in FS_{p^*}(i)} f_{ij}} + \sum_{j \in FS_{p^*}(i)} \left(\frac{f_{ij}}{\sum_{j \in FS_{p^*}(i)} f_{ij}} \left(c_{ij} + u_j^*\right) \right) \right\} & \text{if } i \in S \\ -\theta \log(1 - q_i) + u_j^*, & \text{if } i \in E \end{cases} \tag{9.10}$$

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Acknowledgment

The scriber would like to thank Dr. Niao He for the LaTeXtemplate.