Basics of traffic assignment

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Outline

1. Travel demand forecasting
2. Introducing traffic assignment
3. Modeling the transportation network
4. User equilibrium
5. Mathematical models and solution algorithms
6. Overall takeaways

Outline

Travel Demand Forecasting

We divide the geographical region into Transportation Analysis Zones (TAZs).

1. **Trip Generation** : Whether/when to travel? Estimates the number of trips from/to each zone.
2. **Trip Distribution** : Where to travel (which destination)? Estimates the other end of trips (OD trip matrix).
3. **Mode Choice** : How to travel (which mode)? Estimates the share of each mode from OD trips.
4. **Traffic Assignment** : How to travel (which route). Estimates traffic flow in transportation network.

Task

Match the following:

1. Traffic assignment	A. Predicts no. of originating from and destined to each zone
2. Trip Distribution	B. Predicts the traffic on each link of the transportation network
3. Mode choice	C. Predicts the number of travelers using each mode
4. Trip generation	D. Estimates the origin-destination flow matrix

Answer

Match the following:

1. Traffic assignment	A. Predicts no. of originating from and destined to each zone
2. Trip Distribution	B. Predicts the traffic on each link of the transportation network
3. Mode choice	C. Predicts the number of travelers using each mode
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Outline

Traffic Assignment Problem

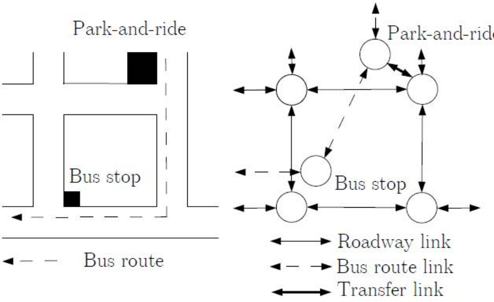
To model how people choose their route in transportation networks and the outcome of their travel in terms of traffic measures (traffic volume, congestion, travel time, etc).

Outline

Modeling the network

Transportation networks are modeled as a directed graph consisting of:

- *Nodes*: intersections or locations where road characteristics change (intersections, freeway merges/diverges)
- *Links*: roadways connecting nodes (highways, streets, bus routes)
- *Commodities*: travelers (cars, motorcycles, transit passengers)

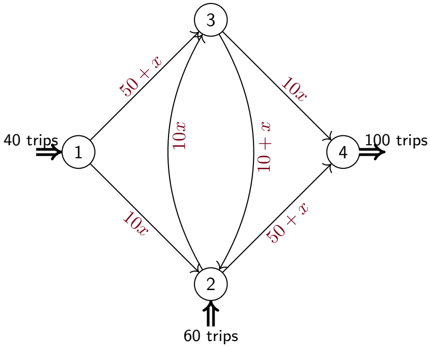


Notations

- $G(N, A)$: graph representing a directed network, where N represents the set of nodes and A represents the set of arcs
- $Z \subseteq N$: set of zones from where the demand is originated and destined to
- d^{rs}: demand between origin-destination pair $(r, s) \in Z^2$
- x_{ij} : flow or volume on link $(i, j) \in A$ during the analysis time period
- t_{ij}: travel time on link $(i, j) \in A$, usually function of x_{ij}
- Π^{rs}: set of paths between origin-destination pair $(r, s) \in Z^2$
- h^{π}: flow or volume on path $\pi \in \Pi$
- c^{π}: travel time on path $\pi \in \Pi$
- $\delta_{ij}^{\pi} = 1,$ if $(i, j) \in \pi, 0,$ otherwise (Link-path incidence matrix)

Example

Write down N, A, d, Π. Assume that demand is equally distributed among paths, compute the value of $\mathbf{x}, \mathbf{c}$.



Bureau of Public Roads (BPR) Function

A common link performance (or volume-delay) function

$$t_{ij}(x_{ij}) = t_{ij}^{0} \left\{ 1 + \alpha \left(\frac{x_{ij}}{u_{ij}} \right)^{\beta} \right\}$$

where, $t_{ij}^{0}, u_{ij}, \alpha, \beta$ represent the free flow travel time on the link, capacity of the link and parameters respectively.

Outline

Which route would you take from CIVIL to India Gate?

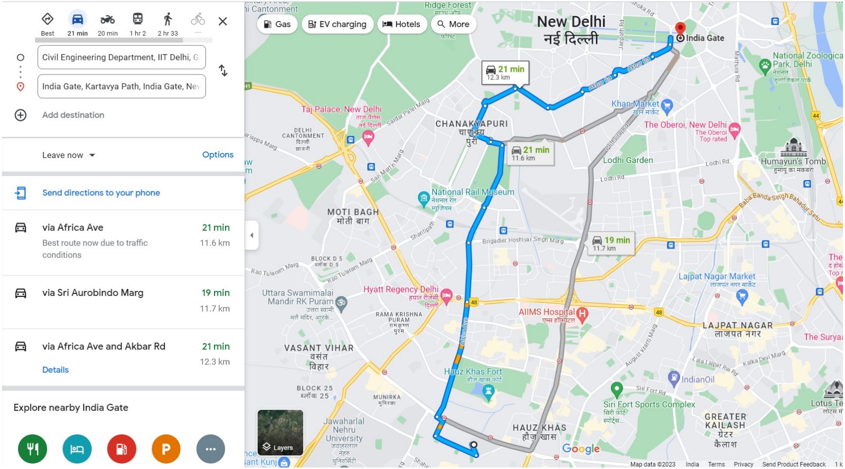


Figure: Which route would you take?

Feasible assignment

An assignment $\{h^\pi\}_{\pi \in \Pi}$ is defined as the feasible assignment if it satisfies the following constraints:

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{1}$$

$$\sum_{\pi \in \Pi^{rs}} h^\pi = d^{rs}, \forall rs \in Z^2 \tag{2}$$

Modeling Route Choice:

Assumptions:

- Users choose the path with minimum travel time
- Users have complete information about transportation system

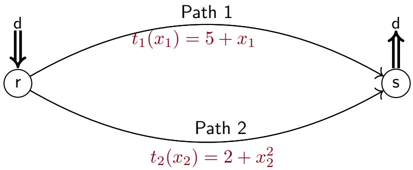
Outcome:

User Equilibrium (*Wardrop equilibrium*)s: No user can decrease his/her own travel time by unilaterally changing the path. In traffic networks, this leads to every used path connecting an origin and destination having equal and minimal travel time.

Example

Three possible cases:

1. All users take path 1
   $t_1(d) < t_2(d)$
2. All users take path 2
   $t_1(d) > t_2(d)$
3. Some users take path 1 and some take path 2
   $t_1(x_1) = t_2(x_2)$



- Let $d = 1 \implies t_1(1) = 6, t_2(1) = 3 \implies x_1^* = 0, x_2^* = 1$
  All users take path 2 ($t_1^* = 5, t_2^*(1) = 3$)
- Let $d = 2 \implies t_1(2) = 7, t_2(2) = 6$, but then $t_1 = 5$.
  Some users take path 1 and some take path 2
  $t_1(x_1^*) = t_2(x_2^*); x_1^* + x_2^* = 2$
  $\implies x_1^* = 1.791, x_2^* = 0.209$ and $t_1^* = t_2^* = 5.21$

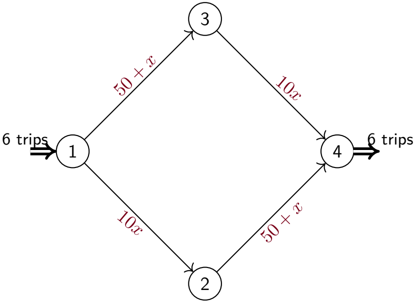
Steps of finding UE solution using its definition:

1. Select a set of paths that are likely to be used according to their free flow travel time. A safe assumption is to select one path for each OD pair.
2. Write the equations for UE conditions, i.e. equal path travel times for the selected paths in each OD pair.
3. Write the equations for conservation of demand for each OD pair.
4. Solve the system of equations to calculate link flows $\mathbf{x}$ and then calculate link travel times $\mathbf{t}$.
5. If the solution satisfies the UE conditions, stop.
6. Otherwise, modify the selected set of paths by adding other likely used paths or removing unused paths and go to step 2.

Example

Task

Find UE. What is the total system travel time (veh-hrs) spent?



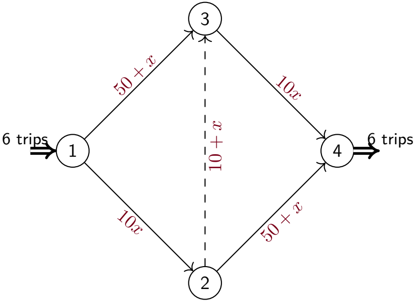
$h^{\pi^1} = h^{\pi^2} = 3; c^{\pi^1} = c^{\pi^2} = 83$

$TSTT = 83.3 + 83.3 = 496$ veh-min.

Example

Task

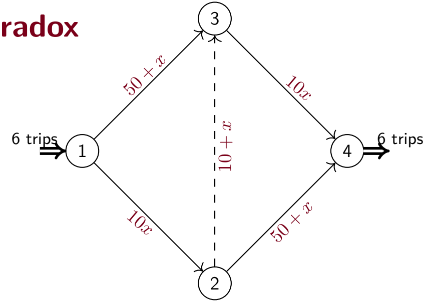
We try to improve congestion by building a bridge. Compute the total system travel time now.



Braess' paradox

Task

We try to improve congestion by building a bridge. Compute the total system travel time now.



$h^{\pi^1} = h^{\pi^2} = h^{\pi^3} = 2; c^{\pi^1} = c^{\pi^2} = c^{\pi^3} = 92$

$TSTT = 92.2 + 92.2 + 92.2 = 552$ veh-min.

The situation actually got worse.

This is known as Braess' paradox! Adding a new link may increase the travel time of commuters.

This happens because of the selfish behavior of commuters.

Braess' paradox can be observed in springs!

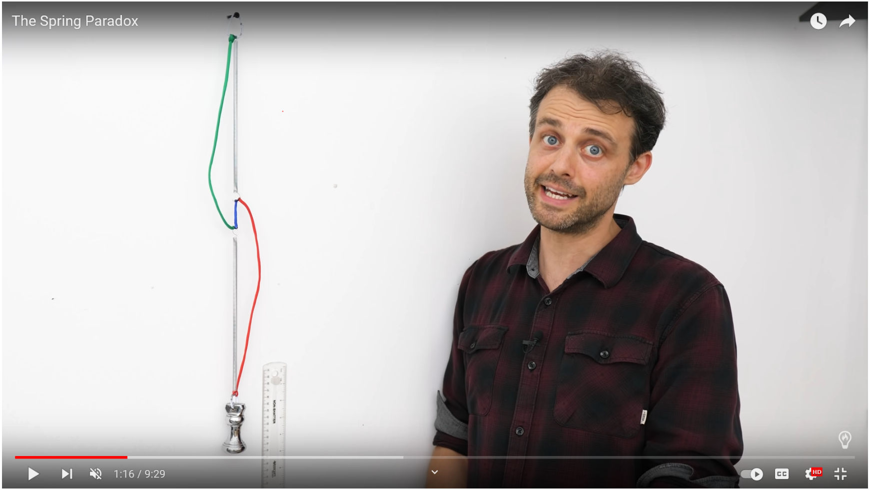


Figure: https://www.youtube.com/watch?v=Cg73j3QYRJc

System optimal (SO) traffic assignment:
In this assignment, the total system travel time is minimized. Although some commuters may face higher travel time than others.

Price of Anarchy (ρ):
The ratio of TSTT from the UE solution to the SO solution is called the *Price of Anarchy*, which is always greater than or equal to 1.

$$\rho = \frac{TSTT_{UE}}{TSTT_{SO}} \tag{3}$$

This measure tells us how bad is selfish routing for the network.

Proposition

(Roughgarden (2004)) For linear, non-negative, and non-decreasing travel time functions, ρ is at most $4/3$.

Outline

A few definitions

Definition (Fixed point problem) Let $X \subseteq \mathbb{R}$ and $f : X \mapsto X$. A fixed point is a point $x \in X$ such that $f(x) = x$.

Definition (Variational inequality problem) Let $X \subseteq \mathbb{R}^n$ be a non-empty, closed and convex set and $f : X \mapsto \mathbb{R}^n$ be a continuous mapping on X. The variational inequality problem is to find a vector $\mathbf{x}^* \in X$ such that

$$\boxed{f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X} \tag{4}$$

First order optimality condition

Let $X \subseteq \mathbb{R}^n$ be a closed and convex set and $f : X \mapsto \mathbb{R}$ be a differentiable and convex function to be minimized. Further, let $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$. Then, $\mathbf{x} \in X$ is optimal if and only if

$$\nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) \geq 0, \forall \mathbf{y} \in X$$

UE as a fixed point problem

Let $X \subseteq \mathbb{R}^n$ be the set of feasible flows (is it closed and convex set? Try proving it!). Then, UE can be modeled as the following fixed point problem: x^* is equilibrium flow vector.

$$\boxed{\mathbf{x}^* = \mathsf{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}^*))} \tag{5}$$

UE as a variational inequality problem

Let $X \subseteq \mathbb{R}^n$ be the set of feasible flows. Then, $\mathbf{x}^* \in X$, which is the equilibrium flow vector, is a solution to the following variational inequality problem.

$$\boxed{\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in X} \tag{6}$$

UE as a convex optimization problem

$$Z^{UE} = \underset{\mathbf{h},\mathbf{x}}{\text{minimize}} \quad \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx \tag{7a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^\pi = d^{rs}, \forall (r,s) \in Z^2 \tag{7b}$$

$$h^\pi \geq 0, \forall \pi \in \Pi \tag{7c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^\pi h^\pi, \forall (i,j) \in A \tag{7d}$$

UE as a convex optimization problem

$$
\begin{aligned}
Z^{UE} = \underset{\mathbf{h},\mathbf{x}}{\text{minimize}} \quad & \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx && \text{(8a)}\\
\text{subject to} \quad & \sum_{\pi\in\Pi^{rs}} h^{\pi} = d^{rs}, \forall (r,s) \in Z^2 && \text{(8b)}\\
& h^{\pi} \geq 0, \forall \pi \in \Pi && \text{(8c)}\\
& x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi}, \forall (i,j) \in A && \text{(8d)}
\end{aligned}
$$

SO as a convex optimization problem

$$Z^{SO} = \underset{\mathbf{h},\mathbf{x}}{\text{minimize}} \quad \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) \tag{9a}$$

$$\text{subject to} \quad \sum_{\pi\in\Pi^{rs}} h^{\pi} = d^{rs}, \forall (r,s) \in Z^2 \tag{9b}$$

$$h^{\pi} \geq 0, \forall \pi \in \Pi \tag{9c}$$

$$x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi}, \forall (i,j) \in A \tag{9d}$$

Proposition

(Roughgarden (2004)) For linear, non-negative, and non-decreasing travel time functions, ρ is at most 4/3.

Congestion pricing

1. SO can be solved by using marginal travel time functions $\tilde{t}_{ij}(x) = t_{ij}(x) + x\frac{dt_{ij}(x)}{dx}$ in the UE formulation.
2. This means by imposing the toll equal to $x^{SO}\frac{dt_{ij}(x^{SO})}{dx}$ on each link, one can achieve system optimal traffic state.

Algorithms for solving UE traffic assignment problem

1. Link-based algorithms
   - Requires less memory but higher iterations for convergence
2. Path-based algorithms
   - Requires high memory but faster convergence
3. Bush-based algorithms
   - Fast convergence as well as low memory requirements

Method of successive averages

1. (Initialization) We set iteration $k = 1$, and initialize $x_{ij}^k = 0, t_{ij}^k = t_{ij}(0), \forall (i,j) \in A$
2. Find the shortest path for each origin-destination pair $(r,s) \in Z^2$ using the link travel times $\{t_{ij}^k\}$ and assign the demand $\{d^{rs}\}$ to it. The result will be auxiliary flows $\{y_{ij}^k\}$ on shortest path links.
3. Update the link flows using $x_{ij}^{k+1} = \frac{k-1}{k} x_{ij}^k + \frac{1}{k} y_{ij}^k, \forall (i,j) \in A$.
4. Update the link travel times $t_{ij}^{k+1} = t_{ij}(x_{ij}^{k+1}), \forall (i,j) \in A$.
5. (Termination) Check if the new solution is close enough to UE. If so, terminate, otherwise, go to step 2.

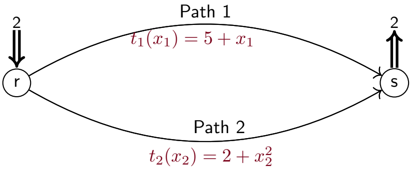
For the termination criterion, we can check if the following gap is small enough (e.g., $1e^{-4}$).

$$\boxed{\text{Relative Gap} = \frac{\text{TSTT -SPTT}}{\text{SPTT}}}$$

Example

Task

Apply MSA algorithm to find UE. First two iterations are shown below:



k	x_1	x_2	t_1	t_2	SP	y_1	y_2	α	Gap
1	0	0	5	2	π_2	0	2	1	
2	0	2	5	6	π_1	2	0	$\frac{1}{2} = 0.5$	$\frac{12-10}{12} = 0.167$

Demonstration of traffic assignment

- Transportation networks repository
- Python program
- Other softwares
  - TransCAD
  - PTV Visum
  - Emme

Conclusions

- Traffic assignment help us understand how commuters travel in the network
- Building more and more roads may not always reduce congestion!
- Tolls can be used to remove the negative externalities in the network

If you'd like to study more on traffic assignment

Here are some books...

1. Boyles, S. D., N. E. Lownes, and A. Unnikrishnan, Transportation Network Analysis, Volume I, Version 0.90, 2022 (https://sboyles.github.io/blubook.html)
2. Sheffi, Y., Urban Transportation Networks: Equilibrium Analysis With Mathematical Programming, Longman Higher Education,1985
3. Patriksson M., The Traffic Assignment Problem: Models and Methods, Dover Publications, 2015

Thank you!