Sets, Functions, and Relations

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"Indeed there is no royal road through mathematics, but we do not need to break up the asphalt and destroy the signage to make travelling what roads there are a trial of one's skill." — Jordan Bell.

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PREFACE

Mathematics is not a spectator sport, nor is it a collection of arbitrary rules to be memorized and regurgitated. It is a language—the most precise and powerful language we have—for describing structure, pattern, and change.

In the current landscape of competitive examination preparation, particularly for the JEE, mathematics is often reduced to a "bag of tricks"—a disjointed set of shortcuts and formulas designed to hack a test but ill-suited for building a mind. Students are taught *how* to solve a specific problem type, but rarely *why* the solution works or *where* the concepts originate. This book is written as an antidote to that approach.

Who is this book for? This text is aimed at the serious high school student who suspects that there is more to mathematics than rote calculation. It is designed for those preparing for the JEE, ISI, and CMI entrance exams who want to do so not by memorizing distinct cases, but by mastering the fundamental principles that govern them all.

The Philosophy: Visual and Rigorous. Two pillars support this text: *Visual Intuition* and *Rigorous Foundation*. The reader will notice a plethora of high quality diagrams throughout these pages. A graph or a set diagram often communicates a truth instantly that a page of algebra might obscure. However, pictures alone are not proofs. Therefore, this book takes care not to gloss over the "boring" details. Many of the illustrations have been solved in both the analytic (rigorous) way along with the graphical (intuitive) way. We build arguments step-by-step, ensuring that when you reach a complex result, you understand the machinery that makes it work.

Overview of the content.

a) Part I: General Theory. The first three chapters form the grammar of mathematics. Unlike standard high school texts that treat these as dry vocabulary lists, we treat them with the depth usually reserved for undergraduate courses.

   - **Chapter 1 (Sets):** We move beyond roster forms to the logic of mathematics, introducing quantifiers ($\forall, \exists$) and logical connectors. We explore the algebra of sets and the Principle of Inclusion-Exclusion.

- **Chapter 2 (Functions):** This is the heart of the book. We treat functions as mappings, rigorously defining images and preimages. We explore Injectivity, Surjectivity, and Bijectivity, along with their combinatorial applications.
- **Chapter 3 (Relations):** We connect the abstract notion of a Cartesian Product to the intuitive concepts of Equivalence Relations and Partitions.

b) Part II: Real Functions. Here, we apply our general theory to the real number line, building a visual toolkit for analysis.

- **Chapter 4 (Graphs):** This chapter establishes your "visual brain." We learn to manipulate graphs—stretching, shifting, reflecting, and composing them—to understand functions without getting lost in algebra.
- **Chapters 5 & 6 (Linear & Quadratic):** We analyze lines and parabolas not just as equations, but as geometric objects, exploring slopes, roots, Vieta's formulas, and optimization.
- **Chapters 7 & 8 (Modulus & Signum):** We tackle absolute values and signs. Highlights include the Triangle Inequality, the Diamond Relation, and the Wavy Curve Method for solving inequalities.
- **Chapter 9 (Square Roots):** We delve into surds, focusing on the subtleties of irrational equations and determining complex domains and ranges.
- **Chapter 10 (GIF & FPF):** We explore the discrete world of the Floor and Fractional Part functions, covering advanced topics like Hermite's Identity and Legendre's Formula.
- **Chapters 11 & 12 (Exponential & Logarithmic):** We conclude with the study of growth and inverse relationships, covering the laws of indices, base changes, and the solution of transcendental equations.

To the Student. There is no royal road to mathematics, but the road need not be unpaved. This book is a map to a small part of it. It will not just help you pass an exam; it will help you think like a mathematician.

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Part I

General Theory

CHAPTER 1

SETS

Welcome to what is perhaps the most fundamental building block of modern mathematics. To explore the vast and intricate world of mathematical ideas, we first need a language that allows us to communicate with precision and clarity. The language of **sets** provides exactly that foundation. It is the vocabulary we will use to define everything from numbers to functions to complex geometric structures, ensuring that our arguments are rigorous and free from ambiguity.

As we begin, we may encounter notation and a style of writing—a certain *formalese*—that appears daunting. I strongly encourage you not to be deterred by this. This formalism is not designed to be difficult; it is a powerful tool that helps us avoid the misunderstandings common in everyday language. The underlying concepts are often surprisingly simple and intuitive. With a small amount of practice, you will find this language becomes second nature, empowering you to express complex ideas with elegance and simplicity.

In this chapter, we will not attempt to answer the deep logical question of what a set *really* is. That journey leads to a fascinating and highly abstract field known as axiomatic set theory. For our purposes, we must start somewhere, and we will do so by relying on our intuition. Let us agree to understand a set, for now, as simply a collection of distinct objects. This informal starting point is more than sufficient to unlock the powerful machinery of set theory and apply it throughout your study of mathematics.

1.1 BASICS

Definition. A **set** is an unordered collection of well-defined objects.

It is important to recognize that even this seemingly straightforward definition is built upon notions that we have not formally defined, such as *unordered*, *collection*, *well-defined*, and *object*. We are accepting these terms based on their intuitive, everyday meaning. This illustrates a fundamental truth in mathematics: we must always start with some undefined terms. Thankfully, this intuitive foundation is robust enough for us to proceed. Before we move on to examples, let us establish the standard notation.

A set can be described in two primary ways. The first, and most direct, is to list its contents, separated by commas, inside a pair of curly braces: $\{\dots\}$. The objects listed inside the braces are called the **elements** or **members** of the set. Alternatively, a set may be described by writing a sentence that specifies a clear property for membership. It is not always necessary, or even possible, to list every element explicitly using braces. We will later see a more powerful way of describing sets.

Example 1.1.1.

a) $\{1, 2, 3\}$. This is a set whose elements are $1, 2$ and 3.
b) $\{1, 2, 3, a, b\}$. This is a set having five distinct elements in it. By means of this example we emphasize that the elements of a set need not be of the "same type." As in this example, some elements are numbers while others are letters of the English alphabet.
c) Consider the set of all the planets in the solar system, namely

$$\{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}\}$$

This is a slightly informal example, since it is not clear if the set above is the set of certain 8 words or the set of the 8 planets of the solar system. When we get into serious mathematics, we will not see such informal things. But we will indulge such ambiguities for the moment.
d) Now consider the description: *the collection of all beautiful planets in our solar system.* This is **not** a set. The word *beautiful* is subjective and depends on personal opinion. One person might consider the rings of Saturn beautiful, while another may not agree. Since we cannot say with universal certainty whether any given planet belongs to this collection, the objects are not well-defined. Therefore, this collection fails to meet our definition of a set.

1.1.2 Belonging/Membership. Given a set and an object, one can always ask if the object is a member of the set. The answer will either be a 'Yes' or a 'No'. For example, let P be the set of all the planets in the Solar System.

- Is Mercury a member of P? The answer is 'Yes.'
- Is $\sqrt{2}$ a member of P? 'No.'

If x is a member of a set X, then we also write/say:

- x is an element of X.
- x belongs to X.
- x is in X.

We use the notation $x \in X$ to denote this. If x is *not* a member of X, then we write $x \notin X$. ♢

1.1.3 Some standard sets.

- $\mathbb{N} = \{1, 2, 3, \dots\}$, the set of natural numbers.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, the set of whole numbers.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, the set of integers.
- $\mathbb{Q}$ denotes the set of all the rational numbers.
- $\mathbb{R}$ denotes set of real numbers.
- $\mathbb{C}$ the set of complex numbers.[1] $\diamondsuit$

1.1.4 Important remarks.

- A set may contain other sets. E.g. $\{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}$.
- Repetition of an element does NOT alter the set. E.g. $\{1, 1, 2, 3\} = \{1, 2, 3\}$.
- Order does NOT matter. E.g. $\{1, 2, 3\} = \{3, 1, 2\}$.
- Two sets are equal if they have precisely the same elements. E.g. $\{1, 2, 3, 4\} \neq \{1, 2, 3\}$.
- The empty set is a set containing NO elements. It is denoted by $\emptyset$ or $\{\}$. $\diamondsuit$

1.2 ROSTER AND BUILDER FORM

1.2.1 Roster and Set-Builder Form. There are two primary ways to represent a set.

- *Roster Form:* This method involves listing all the elements of the set, separated by commas, within curly braces $\{\}$. This is practical for sets with a small number of elements.
- *Set-Builder Form:* This method describes the elements of the set by stating a property or rule that they must satisfy. The general structure is $\{x : P(x)\}$, which reads "the set of all x such that $P(x)$ is true".

Example 1.2.2 (Examples of Roster and Set-Builder Form)**.**

a) Let S be the set of natural numbers which are less than 100.

- Set-Builder Form: $S = \{n \in \mathbb{N} : n < 100\}$
- Roster Form: $S = \{1, 2, 3, \dots, 99\}$

b) Let S be the set of all the even natural numbers.

- Roster Form: $S = \{2, 4, 6, 8, 10, 12, \dots\}$
- Set-Builder Form: $S = \{n \in \mathbb{N} : 2 \text{ divides } n\}$ or $S = \{2k : k \in \mathbb{N}\}$

c) The set of all the rational numbers, $\mathbb{Q}$.

- Set-Builder Form: $\mathbb{Q} = \{x : \text{there exist } p \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } x = \frac{p}{q}\}$
- This is often written more concisely as: $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$

[1] We are not assuming that the reader is familiar with complex numbers.

Example 1.2.3. Following are few examples where we show how a builder form can be expressed in roster form.

- $\{x : x \text{ is a prime number dividing } 60\} = \{2, 3, 5\}$
- $\{x \in \mathbb{N} : 1 \leq x^2 \leq 100 \text{ and } 3 \text{ divides } x\} = \{3, 6, 9\}$
- $\{x : x \text{ is a month beginning with 'M'}\} = \{\text{March, May}\}$

A few examples which show some sets expressed in roster form and a corresponding builder form.

- $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\} = \{\frac{n}{n+1} : n \in \mathbb{N}, n \leq 6\}$
- $\{4, 5, 6, 7, \dots\} = \{n \in \mathbb{N} : n \geq 4\}$
- $\{1, 4, 9, 16, 25, \dots\} = \{n^2 : n \in \mathbb{N}\}$

Exercise 1.2.1. Write the following sets in Roster form.

- $\{x : x \in \mathbb{R}, \quad x^2 = 1\}$.
- $\{x : x \in \mathbb{N}, x < 12\}$.
- $\{x : x \text{ is the square of an integer, } x < 100\}$.
- $\{x : x \text{ is an integer such that } x^2 = 2\}$.

Exercise 1.2.2. For each of the following sets determine whether 2 is an element of them:

- $\{x : x \in \mathbb{Z}, x > 1\}$.
- $\{x : x \text{ is the square of an integer }\}$.
- $\{2, \{2\}\}$.
- $\{\{2\}, \{\{2\}\}\}$.
- $\{\{2\}, \{2, \{2\}\}\}$.
- $\{\{\{2\}\}\}$

1.3 QUANTIFIERS

1.3.1 The universal quantifier. In mathematics, we often need to make statements that are true for all elements in a particular set. The universal quantifier is the symbol used to express such statements.

- *Symbol:* $\forall$.
- *Meaning:* "For all", "for every", "for each".

The expression "for all $x \in S, P(x)$" means that for every element x in the set S, the property $P(x)$ is true.

Let us see how this works through a familiar example. Consider the everyday statement: *every student in this class has submitted the assignment.* Let S denote the set of all students in the class, and let $P(x)$ denote the statement "x has submitted the assignment." The mathematical rendering of the above statement is:

$$\forall x \in S, \ P(x)$$

Read aloud, this says: "for all x in S, $P(x)$ holds." It sounds stiff and unnatural compared to the plain English version—and that is perfectly fine. We are not adopting this language for its charm. In everyday speech, the meaning of "every" is usually clear from context. But in mathematics, where a single misplaced word can change the truth of a statement, we need a notation that leaves no room for ambiguity. The universal quantifier $\forall$ gives us exactly that. Once you grow accustomed to it, you will find it not only tolerable but indispensable. ♢

Example 1.3.2.

a) Consider the set of positive rational numbers, denoted by $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$. We can state that every positive rational number is greater than its negative. Using the universal quantifier, we write:

$$\forall x \in \mathbb{Q}^+, \text{ we have } x > -x$$

b) Let R be the set of all rectangles and P be the set of all parallelograms. The geometric fact that every rectangle is a parallelogram can be written as:

$$\forall x \in R, \text{ we have } x \in P$$

This statement asserts that if we take any object x from the set of rectangles, it will also be an object in the set of parallelograms.

c) The statement "every bird can fly" is a proposition that uses a universal quantifier implicitly. If B is the set of all birds, and $F(x)$ is the statement "x can fly", the assertion is:

$$\forall x \in B, \ F(x)$$

(Note: This is a logical statement, not necessarily a biologically true one, as some birds like penguins and ostriches cannot fly. This highlights the importance of the domain over which we quantify.)

d) For any rational number x, its square is also a rational number. Let $\mathbb{Q}$ be the set of rational numbers.

$$\forall x \in \mathbb{Q}, \text{ we have } x^2 \in \mathbb{Q}$$

1.3.3 The Existential Quantifier. In contrast to making a claim about every element, sometimes we only want to assert that there is at least one element in a set that satisfies a certain property. The existential quantifier is used for this purpose.

- *Symbol:* $\exists$.
- *Meaning:* "There exists", "there is at least one", "for some".

The expression $\exists x \in S, P(x)$ means that there is at least one element x in the set S for which the property $P(x)$ is true. We can also express the negation, that no such element exists.

- *Symbol:* $\nexists$
- *Meaning:* "There does not exist".

To ground this, consider the statement: *there is a student in this class who scored full marks*. Let S be the set of students in the class, and let $P(x)$ denote "x scored full marks." The formal rendering is:

$$\exists x \in S,\ P(x)$$

Note that this does *not* claim every student scored full marks—only that at least one did. If, on the other hand, no student scored full marks, we would write $\nexists x \in S,\ P(x)$.

Example 1.3.4.

a) Is there a rational number x that solves the equation $2x = 1$? Yes, $x = 1/2$. Since $1/2$ is a rational number, we can write:

$$\exists x \in \mathbb{Q} \text{ such that } 2x = 1$$

b) Let us consider the same equation over the set of integers, $\mathbb{Z}$. Is there an integer solution to $2x = 1$? No, because the solution $x = 1/2$ is not an integer. So, we can state:

$$\nexists x \in \mathbb{Z} \text{ such that } 2x = 1$$

c) Does there exist a number whose square is 2? If we are working within the rational numbers $\mathbb{Q}$, the answer is no.

$$\nexists x \in \mathbb{Q} \text{ such that } x^2 = 2$$

However, if we expand our domain to the real numbers $\mathbb{R}$, the answer is yes ($\sqrt{2}$ and $-\sqrt{2}$).

$$\exists x \in \mathbb{R} \text{ such that } x^2 = 2$$

This illustrates that the truth of a quantified statement depends critically on the domain (the set) being considered.

d) The statement "There are parallelograms which are not rectangles" can be formalized. Let P be the set of parallelograms and R be the set of rectangles. We are asserting that there is some element in P that is not in R.

$$\exists x \in P \text{ such that } x \notin R$$

Example 1.3.5 (Nested quantifiers)**.** We can combine universal and existential quantifiers to form more complex statements. The order in which the quantifiers appear is very important and can drastically change the meaning of the statement.

a) Consider the following two English sentences.

(i) *Every person in this room speaks some language.*
(ii) *There is some language that every person in this room speaks.*

Let P be the set of people in the room and L the set of all languages. Let $S(x,l)$ denote "person x speaks language l." Statement (i) says:

$$\forall x \in P,\ \exists l \in L,\ S(x,l)$$

This is a very mild claim: each person speaks *some* language, but different people may speak different languages. Statement (ii) says:

$$\exists l \in L,\ \forall x \in P,\ S(x,l)$$

This is a much stronger claim: there is a *single* language that everyone in the room speaks. Notice that the words "every" and "some" appear in both sentences, but their order is swapped—and the meaning changes dramatically. Statement (ii) implies statement (i), but not vice versa. This is a recurring theme: when a $\forall$ comes before an $\exists$, the choice made by $\exists$ is allowed to depend on the $\forall$ variable; when the order is reversed, a single choice must work universally.

b) Consider the statement: for every integer, we can find a rational number that is half of it. This means for any integer x we pick, there exists a rational number y such that $y = x/2$, or $x = 2y$. We write this as:

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Q} \text{ such that } x = 2y$$

This is a true statement. For any integer x, the number $y = x/2$ is a rational number.

c) Let us analyze the statement "The sum of two natural numbers is always a natural number". Let $\mathbb{N}$ be the set of natural numbers. This statement says that if we take *any* natural number x and *any* natural number y, their sum is also a natural number. This requires two universal quantifiers.

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \text{ we have } x + y \in \mathbb{N}$$

This is often abbreviated as:

$$\forall x, y \in \mathbb{N}, \text{ we have } x + y \in \mathbb{N}$$

d) Let P be the set of parallelograms and R be the set of rectangles. Consider the statement: "For every parallelogram, there exists a rectangle with the same area."

$$\forall x \in P, \exists y \in R \text{ such that } \text{area}(y) = \text{area}(x)$$

Exercise 1.3.1. Let $\mathbb{Q}^+$ denote the set of all the positive rational numbers. Answer true or false.

- $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $y < x$.
- $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Q}^+$such that $y < x$.
- $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ such that $y < x$.
- $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Z}$ such that $y < x$.
- $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{N}$ such that $y < x$.
- $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Q}$ such that $\forall z \in \mathbb{Q}^+$we have $yz < x$.

1.4 LOGICAL CONNECTORS

1.4.1 Implication (If... Then...) The most common way to connect two statements is through implication. An implication is a statement of the form "if A, then B". It asserts that whenever statement A is true, statement B must also be true.

- *Symbol:* $\Rightarrow$
- *Meaning:* $A \Rightarrow B$ is read as "A implies B". This means that the truth of A guarantees the truth of B. In this relationship, A is called the *hypothesis* and B is the *conclusion.*

Here is a familiar real-world instance. Consider the rule: *if it is raining, then the ground is wet.* Let A be the statement "it is raining" and B the statement "the ground is wet." The rule asserts $A \Rightarrow B$. Notice what this does *not* say: it does not say that rain is the *only* cause of wet ground—a sprinkler could do the job just as well. All it promises is that rain is *sufficient* to guarantee wet ground. $\diamondsuit$

Example 1.4.2.

a) Let x be a variable representing a number. If x is greater than 3, then it must also be greater than 2. We can write this as:

$$(x > 3) \Rightarrow (x > 2)$$

b) Let us consider multiple conditions. Let x be an integer. If x is greater than 1 AND x is less than 3, then x must be 2.

$$(x \in \mathbb{Z} \text{ and } x > 1 \text{ and } x < 3) \Rightarrow (x = 2)$$

c) Let R be the set of all rectangles and P be the set of all parallelograms. The geometric fact that every rectangle is a parallelogram can be expressed as an implication:

$$x \in R \Rightarrow x \in P$$

This means that if a shape x is a rectangle, then it is guaranteed to be a parallelogram.

1.4.3 A very important caution. The implication $A \Rightarrow B$ does **not** mean that $B \Rightarrow A$. For example, while it is true that $(x > 3) \Rightarrow (x > 2)$, the reverse is false. A number that is greater than 2 (like 2.5) is not necessarily greater than 3. So, $(x > 2) \nRightarrow (x > 3)$.

A real-world example makes the danger vivid. The statement *if an animal is a dog, then it has four legs* is true. But the reverse—*if an animal has four legs, then it is a dog*—is plainly false; cats, horses, and elephants all have four legs. Confusing an implication with its converse is one of the most common logical errors, and it is worth training yourself to catch it every time. ♢

1.4.4 The Contrapositive. Every implication $A \Rightarrow B$ has a logically equivalent statement called the contrapositive, which is "if B is false, then A is false".

- *Symbolic form:* $(\neg B) \Rightarrow (\neg A)$ (where $\neg$ means "not").

The contrapositive is an extremely useful tool for proofs. Sometimes, proving the contrapositive is more straightforward than proving the original implication. Consider again the implication: *if it is raining, then the ground is wet.* Its contrapositive is: *if the ground is not wet, then it is not raining.* Take a moment to convince yourself that this is just as obviously true as the original. If you step outside and the ground is completely dry, you can confidently conclude that it is not raining—no need to look up at the sky. This is exactly how a contrapositive proof works: instead of directly showing $A \Rightarrow B$, we show $(\neg B) \Rightarrow (\neg A)$, which is logically the same thing. ♢

Example 1.4.5. Suppose we want to prove the statement

"If n^2 is even, then n is even"

for an integer n. The original implication is $A \Rightarrow B$, where A is "n^2 is even" and B is "n is even". The contrapositive is $(\neg B) \Rightarrow (\neg A)$, which is "If n is not even, then n^2 is not even". "Not even" means "odd". So we need to prove "If n is odd, then n^2 is odd". Let us prove this. If n is odd, it can be written as $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2(2k^2 + 2k)$ is an even number, adding 1 makes n^2 an odd number. We have successfully proven the contrapositive. Since the contrapositive is true, the original statement must also be true.

1.4.6 Double Implication (if and only if.) A stronger connection between two statements is the double implication, which states that the two statements are logically equivalent; they are either both true or both false.

- *Symbol:* $\Leftrightarrow$
- *Meaning:* $A \Leftrightarrow B$ is read as "A if and only if B". It is a shorthand for writing that both $A \Rightarrow B$ and $B \Rightarrow A$ are true. The abbreviation "iff" is commonly used for "if and only if".

Consider the statement: *a person is eligible to vote if and only if the person is aged 18 or above.* Let A be "the person is eligible to vote" and B be "the person is aged 18 or above." The double implication $A \Leftrightarrow B$ captures both directions: being an eligible voter *guarantees* that you satisfy the age criteria ($A \Rightarrow B$), and satisfying the age criterion *guarantees* that you are eligible to vote ($B \Rightarrow A$). Neither condition is stronger than the other—they are exactly equivalent. $\diamondsuit$

Example 1.4.7.

a) For a number x, the statement "x is a rational number" is true if and only if "$3x$ is a rational number".

$$x \in \mathbb{Q} \Leftrightarrow 3x \in \mathbb{Q}$$

This is because if x is rational, then $3x$ is rational (the $\Rightarrow$ direction). And if $3x$ is rational, then $x = (3x)/3$ is also rational (the $\Leftarrow$ direction).

b) Let S be the set of squares, R be the set of rectangles, and Rh be the set of rhombii. A shape x is a square if and only if it is both a rectangle and a rhombus.

$$x \in S \Leftrightarrow (x \in R \text{ and } x \in Rh)$$

1.4.8 Negation (Not.) Negation is the logical operator that reverses the truth value of a statement.

- *Symbol:* $\neg$.
- *Meaning:* $\neg A$ means "not A". If A is true, $\neg A$ is false. If A is false, $\neg A$ is true.

A fundamental principle in logic is the **law of the excluded middle**, which states that for any mathematical statement A, either A is true or its negation $\neg A$ is true. There is no third option. For instance, let A be the statement "it is raining right now." Then $\neg A$ is the statement "it is *not* raining right now." Exactly one of these two statements is true at any given moment—there is no middle ground. $\diamondsuit$

Example 1.4.9. Negation has a very important interaction with quantifiers. To negate a quantified statement, we switch the quantifier and negate the property.

a) *Negating a Universal Statement:* The negation of "for all x, $P(x)$ is true" is "there exists an x for which $P(x)$ is false".

$$\neg(\forall x \in A, P(x)) \quad \text{is equivalent to} \quad \exists x \in A, \neg P(x)$$

Let us negate the (false) statement "All integers are natural numbers".

- Original statement: $\forall x \in \mathbb{Z}, x \in \mathbb{N}$.
- Negation: $\exists x \in \mathbb{Z}$ such that $x \notin \mathbb{N}$.

The negation is true, because we can find a counterexample, like -1, which is an integer but not a natural number.

b) *Negating an Existential Statement:* The negation of "there exists an x for which $P(x)$ is true" is "for all x, $P(x)$ is false".

$$\neg(\exists x \in A, P(x)) \quad \text{is equivalent to} \quad \forall x \in A, \neg P(x)$$

Let us negate the true statement "There exists an integer whose square is 4".

- Original statement: $\exists x \in \mathbb{Z}$ such that $x^2 = 4$.
- Negation: $\forall x \in \mathbb{Z},$ we have $x^2 \neq 4$.

The negation is clearly false, because the original statement was true.

Exercise 1.4.1. State true or false.

a) If $x \in \mathbb{R}$, $x \notin \mathbb{Q}$, and $\lambda \in \mathbb{Q}$ with $\lambda \neq 0$, then $\lambda x \notin \mathbb{Q}$.
b) If $x, y \in \mathbb{R}$ and $x, y \notin \mathbb{Q}$, then $xy \notin \mathbb{Q}$.
c) If $x, y \in \mathbb{R}$ with $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x + y \notin \mathbb{Q}$.

1.5 SUBSETS

We now introduce a way to relate two sets to each other. This is the idea of one set being "contained" within another. Think of the set of all students in a school, and within it, the set of students who play cricket. Every cricket-playing student is, of course, a student of the school—but not every student plays cricket. We would say that the cricket players form a *subset* of the school's student body, and in fact a *proper* subset, since there are students who do not play cricket. If, on the other hand, every single student happened to play cricket, the two sets would be equal, and the cricket players would still be a subset—just not a proper one.

Definition. Let A and B be sets. We say that A is a **subset** B, written $A \subseteq B$, if every element of A is also an element of B. We can express this formally using the universal quantifier:

$$A \subseteq B \quad \text{means} \quad \forall x, (\text{ if } x \in A \text{ then } x \in B)$$

The notation $A \subset B$ is also used to denote that A is a subset of B.

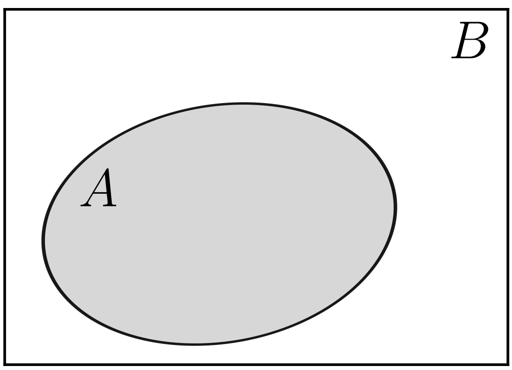


Figure 1.1: The set A is a subset of a set X.

Definition. We say that A is a **proper subset** of B if A is a subset of B, but A is not equal to B. This means that every element of A is in B, and there is at least one element in B that is not in A. The notation for a proper subset is $A \subsetneq B$.

$$A \subsetneq B \quad \text{means} \quad (A \subseteq B \text{ and } A \neq B)$$

Example 1.5.1.

a) *Geometric Shapes:* Let S be the set of all squares, R be the set of all rhombii, and P be the set of all parallelograms. Every square is a rhombus (a rhombus with right angles), and every rhombus is a parallelogram (a parallelogram with all four sides being equal). So we have a chain of subsets:

$$S \subseteq R \subseteq P$$

b) *Number Systems:* Our standard number systems form a chain of proper subsets. Let $\mathbb{N}$ be the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. Then:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

1.5.2 Fundamental Properties of Subsets. There are three foundational rules that subsets always obey.

a) (The Empty Set.) For any set A, the empty set $\emptyset$ is a subset of A.

$$\emptyset \subseteq A$$

This is true because we cannot find any element in $\emptyset$ that is not in A (since $\emptyset$ has no elements).

b) (Reflexivity.) Any set A is a subset of itself.

$$A \subseteq A$$

This is true because every element in A is, trivially, an element in A.

c) (Transitivity.) If A is a subset of B, and B is a subset of C, then A is a subset of C.

$$(A \subseteq B \text{ and } B \subseteq C) \Rightarrow (A \subseteq C)$$

Proof. We only prove (c). To establish the inclusion $A \subseteq C$, we must demonstrate that every element residing in A is necessarily a member of C. Let x be an arbitrary element of A. By the hypothesis that $A \subseteq B$, the membership of x in A guarantees that $x \in B$. But since $B \subseteq C$, the presence of x in B implies that x must also belong to C. As x was chosen arbitrarily, this logic holds for all elements of A, allowing us to conclude that $A \subseteq C$. ♢

Since subsets are themselves mathematical objects, we can create a set that contains all of the subsets of a given set.

Definition. Let X be a set. The **power set** of X, denoted by $\mathcal{P}(X)$, is the set of all subsets of X.

Example 1.5.3.

a) Let $X = \{a, b\}$. The subsets of X are the empty set $\emptyset$, the sets containing one element $\{a\}$ and $\{b\}$, and the set itself $\{a, b\}$. The power set is therefore:

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Note that if X has 2 elements, $\mathcal{P}(X)$ has $2^2 = 4$ elements.

b) Let $X = \{a, b, c\}$. The subsets are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. The power set is:

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Here, X has 3 elements, and $\mathcal{P}(X)$ has $2^3 = 8$ elements.

c) Let $X = \emptyset$. The only subset of the empty set is the empty set itself. Therefore, the power set of $\emptyset$ is the set containing the empty set.

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

Note that the power set of the empty set is not empty; it contains one element.

Exercise 1.5.1. For each of the following pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither.

a) The set of people who speak English; the set of people who speak Hindi.
b) The set of natural numbers; the set of integers.
c) The set of all fruits; the set of all citrus fruits.

Exercise 1.5.2. Let $A = \{2, 4, 6\}$, $B = \{2, 6\}$, and $C = \{4, 6\}$. Determine the subset relationships between these three sets.

Exercise 1.5.3. Determine whether the following statements are true or false.

a) $\emptyset \in \emptyset$
b) $\emptyset \subseteq \emptyset$
c) $\{\emptyset\} \subseteq \emptyset$
d) $\emptyset = \{0\}$
e) $\{0\} \in \{0\}$
f) $\{\emptyset\} = \{\emptyset\}$
g) $\emptyset \subseteq \{\emptyset\}$

1.6 UNION AND INTERSECTION

Suppose a teacher asks: "Who in this class plays cricket *or* football (or both)?" The students who raise their hands form, informally, the *union* of the cricket players and the football players—everyone who belongs to at least one of the two groups.

Definition. Let A and B be two sets. The **union** of A and B is the set containing all elements that are in set A, or in set B, or in both. The keyword for union is **OR**. We write $A \cup B$ to denote the union of A and B. In symbols

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

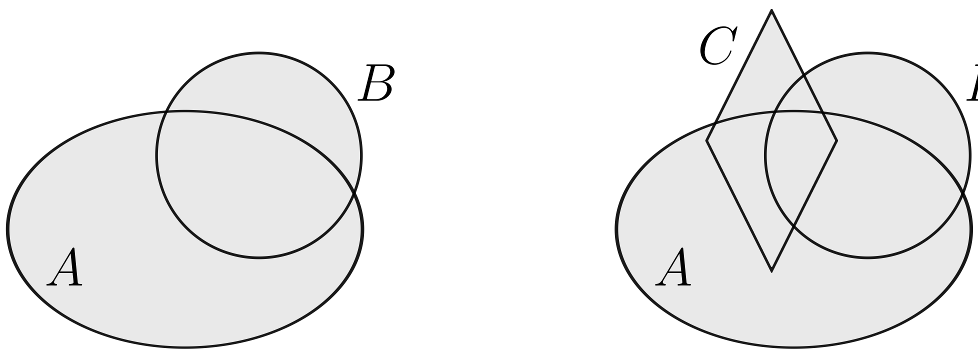


Figure 1.2: The shaded region on the left depicts the union of two sets A and B. The one on the right depicts the union of three sets A, B and C.

Example 1.6.1. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Their union is:

$$A \cup B = \{1, 2, 3, 4, 5\}$$

Note that the element 3, which is in both sets, is listed only once.

1.6.2 Properties of union. The following are some properties enjoyed by the operation of union and the proofs of these are left to the reader.

a) (Commutativity.) The order in which we unite sets does not matter.

$$A \cup B = B \cup A$$

b) (Associativity.) When uniting three or more sets, the grouping does not matter.

$$(A \cup B) \cup C = A \cup (B \cup C)$$

c) (Idempotence.) The union of a set with itself is the set itself.

$$A \cup A = A$$

d) (Identity Element.) The empty set $\emptyset$ is the identity element for union.

$$A \cup \emptyset = A$$

$\diamondsuit$

Continuing the classroom analogy, suppose the teacher now asks: "Who plays *both* cricket *and* football?" The students who raise their hands this time form the *intersection* of the two groups—only those who belong to both.

Definition. Let A and B be two sets. The **intersection** of A and B is the set containing all elements that are in *both* set A and set B. The keyword for intersection is *and*. We denote the intersection of A and B as $A \cap B$. In symbols

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Two sets A and B are said to be **disjoint** if they have no elements in common, i.e., their intersection is the empty set.

$$A \text{ and } B \text{ are disjoint if } A \cap B = \emptyset$$

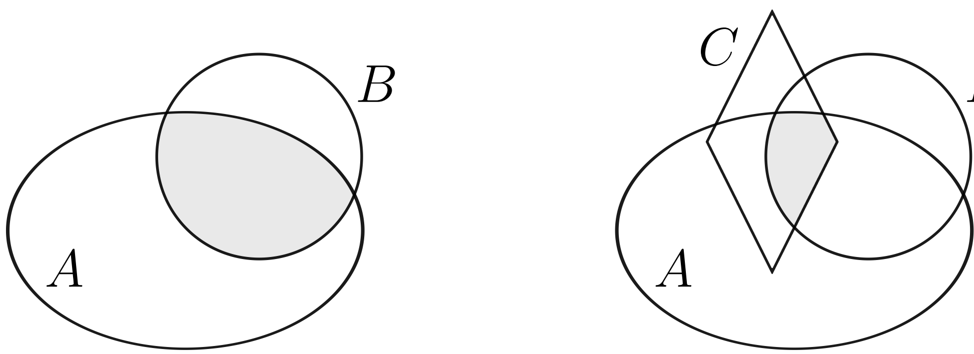


Figure 1.3: The shaded region on the left depicts the intersection of two sets A and B.a The one on the right shows the intersection of three sets A, B and C.

Example 1.6.3.

a) Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Their intersection is:

$$A \cap B = \{3\}$$

b) Let R be the set of all rectangles and $\tilde{R}$ be the set of all rhombii. Their intersection is the set of all shapes that are both a rectangle and a rhombus.

$$R \cap \tilde{R} = S \quad \text{(where } S \text{ is the set of all squares)}$$

1.6.4 Properties of intersection. Following are some properties of the operation of intersection whose proofs we skip since they are straightforward.

a) (Commutativity.) $A \cap B = B \cap A$

b) (Associativity.) $(A \cap B) \cap C = A \cap (B \cap C)$

c) (Idempotence.) $A \cap A = A$

d) (Annihilation Element.) The empty set $\emptyset$ is the annihilation element for intersection.

$$A \cap \emptyset = \emptyset$$

1.6.5 Distributive Laws Union and intersection interact with each other in a way that is similar to how addition and multiplication interact in arithmetic. These are known as the distributive laws.

Lemma. Let A, B and C be sets. Then

a) (Intersection distributes over union.)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

b) (Union distributes over intersection.)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof. We give a formal argument below but we encourage the reader to first see the truth of these statements by means of diagrams.

(a): We first show that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

Let $x \in A \cap (B \cup C)$. We want to show that x is in $(A \cap B)$ or in $(A \cap C)$. We know that x is in A and $x \in B \cup C$. There are two cases. Either $x \in B$ or $x \notin B$. First suppose that $x \in B$. Then since x is already in A, we have $x \in A \cap B$ and we are done. So we may assume that $x \notin B$. Then since $x \in B \cup C$, we deduce that x must be in C. But then $x \in A \cap C$ and we are done.

Now we show the reverse containment. Let $x \in (A \cap B) \cup (A \cap C)$. We want to show that $x \in A \cap (B \cup C)$. We know that $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$. Then $x \in A$ and $x \in B \cup C$, giving $x \in A \cap (B \cup C)$ and we are done. Similarly, if $x \in A \cap C$ then also we get $x \in A \cap (B \cup C)$. This shows the reverse containment.

We have thus shown that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad \text{and} \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

finishing the proof. The proof of (b) is an exercise. ■ ♢

Exercise 1.6.1. Let $A_1, A_2, \ldots, A_n$ and B be sets. The following problems concern the generalization of the distributive laws. You are asked to show the following identities are true.

a) $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$

This can be written more compactly as:

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap B = \bigcup_{i=1}^{n} (A_i \cap B)$$

b) $(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B)$
This can be written more compactly as:

$$\left(\bigcap_{i=1}^{n} A_i\right) \cup B = \bigcap_{i=1}^{n} (A_i \cup B)$$

1.7 SET DIFFERENCE

Returning to our classroom, suppose we want the list of students who play cricket but *not* football. We start with the cricket players and remove anyone who also plays football. What remains is the *difference* of the cricket players and the football players.

Definition. The **difference** of two sets A and B is the set of elements that are in A but not in B. We denote this by $A \setminus B$ or $A - B$. In symbols

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

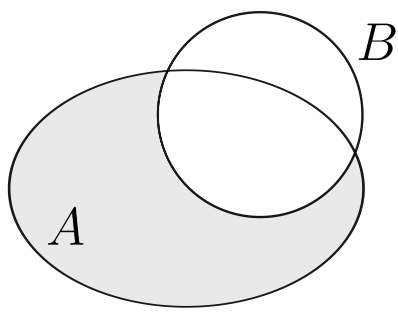


Figure 1.4: The set $A \setminus B$ shown as shaded.

Example 1.7.1. Let $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. The set of elements in A but not in B is:

$$A \setminus B = \{5\}$$

Conversely, the set of elements in B but not in A is

$$B \setminus A = \{2\}$$

1.7.2 Properties of set difference. The following properties are easy to verify. The reader is encouraged to draw pictures in order to foster understanding.

a) (Non-Commutativity.) In general, set difference is not commutative. As seen in the example, $A \setminus B \neq B \setminus A$.
b) The difference of a set with itself is the empty set: $A \setminus A = \emptyset$.
c) The difference of a set with the empty set is the set itself: $A \setminus \emptyset = A$. ♢

Now suppose we ask: "Who plays *exactly one* of cricket or football—but not both?" This picks out the cricket-only players together with the football-only players, while leaving out anyone who plays both sports. This operation is called the *symmetric difference.*

Definition. The **symmetric difference** of two sets A and B is the set of elements that are in either of the sets, but not in their intersection. It represents the elements that are unique to each set. We denote the symmetric difference of A and B as $A\Delta B$. Note that

$$A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

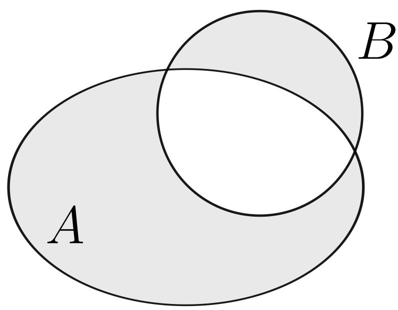


Figure 1.5: The symmetric difference of two sets A and B depicted by the shaded region.

Example 1.7.3. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then

$$A\Delta B = (\{1, 2\}) \cup (\{4, 5\}) = \{1, 2, 4, 5\}$$

1.7.4 Properties of symmetric difference. The following are some basic properties of symmetric difference.

a) (Commutativity.) $A\Delta B = B\Delta A$.
b) (Associativity.) $(A\Delta B)\Delta C = A\Delta(B\Delta C)$.
c) The symmetric difference of a set with itself is the empty set: $A\Delta A = \emptyset$.
d) The empty set is the identity element: $A\Delta\emptyset = A$.

All the of the above except the associativity are clear. The proof of associativity can be given directly from first principles, but is cumbersome and we skip. We encourage the reader, however, to draw diagrams and convince himself/herself of the truth of this statement. Later, when we develop a few basic things about functions, we will be able to give a very simple proof of this fact. ♢

Finally, consider the set of *all* students in the class (the "universal" set for this discussion) and, within it, the set of cricket players. The students who do *not* play cricket form what we call the *complement* of the cricket players—everything in the universal set that is left over once we remove the specified subset.

Definition. Let A be a subset of a set X. The **complement** of A in X is the set of all elements in X that are not in A. When the set X is clear from context, we denote the complement of A in X as A^c or A'. In symbols

$$A^c = X \setminus A = \{x \in X \mid x \notin A\}$$

Note. Let A be a subset of a set X. Then

$$A \cup A^c = X \quad \text{and} \quad A \cap A^c = \emptyset$$

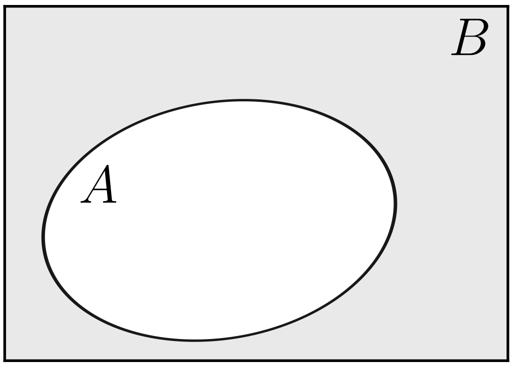


Figure 1.6: The complement of A in a set X shown as the shaded region.

Example 1.7.5. Let X be the set of all the natural numbers. Let $A = \{n \in \mathbb{N} : \ n \geq 11\}$. The complement of A in X is:

$$A^c = \{n \in \mathbb{N} \mid n < 11\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Before we state De Morgan's laws formally, let us see them in action with an everyday example. Consider a class of students, and let C be the set of cricket players and F the set of football players. The union $C \cup F$ is the set of students who play *at least one* sport. Its complement $(C \cup F)^c$ is the set of students who play *neither* cricket *nor* football—equivalently, students who are not cricket players *and* not football players. That is precisely $C^c \cap F^c$. Similarly, the intersection $C \cap F$ is the set of students who play *both* sports. Its complement $(C \cap F)^c$ is the set of students who do *not* play both—meaning they are missing at least one sport, i.e., they are not a cricket player *or* not a football player. That is $C^c \cup F^c$. These two identities are De Morgan's laws.

1.7.6 De Morgan's Laws. We now discuss De Morgan's laws, which are a pair of fundamental rules that relate the complement operation to union and intersection. They provide a way to simplify the negation of complex set expressions.

Lemma. Let A and B be subsets of a universal set X.

a) (Complement of a Union.) The complement of the union of two sets is the intersection of their complements.

$$(A \cup B)^c = A^c \cap B^c$$

b) (Complement of an Intersection.) The complement of the intersection of two sets is the union of their complements.

$$(A \cap B)^c = A^c \cup B^c$$

Proof. (a)

$$\begin{aligned} x \in (A \cup B)^c \quad &\Leftrightarrow \quad x \in X, x \notin A \cup B \\ &\Leftrightarrow \quad x \in X, x \notin A \text{ and } x \notin B \\ &\Leftrightarrow \quad x \in X, x \notin A \text{ and } x \in X, x \notin B \\ &\Leftrightarrow \quad x \in A^c \text{ and } x \in B^c \end{aligned}$$

The proof of (b) is similar and is left as an exercise. ■

Figure 1.7 illustrates both laws. In each row, the left and right panels shade the same region, confirming the identity.

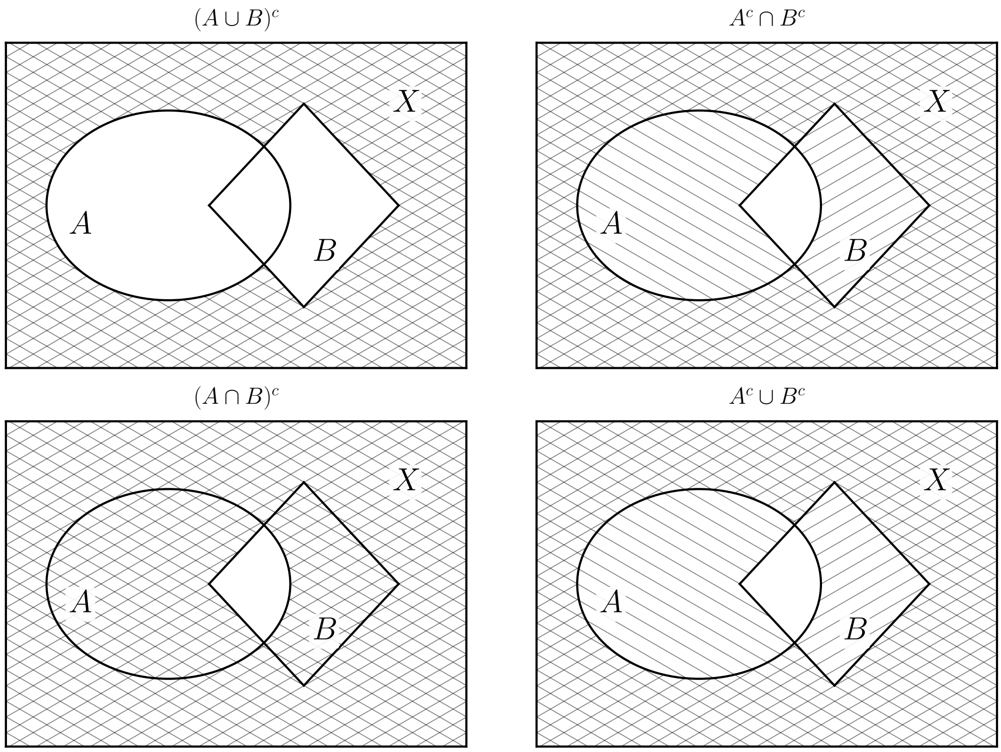


Figure 1.7: De Morgan's Laws. Top row: $(A \cup B)^c = A^c \cap B^c$. Bottom row: $(A \cap B)^c = A^c \cup B^c$.

Exercise 1.7.1. This problem asks to prove the generalized version of De Morgan's Law. Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set X. Show that the complement of the union of these sets is equal to the intersection of their complements.

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

This can be written more compactly as:

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$$

1.8 SIZE OF A SET

We often want to know how many elements are in a set. This leads to the distinction between finite and infinite sets.

Definition. A set A is called **finite** if it contains a specific, countable number of elements. We can, in principle, list all its elements and finish. Examples include the set of letters in the Greek alphabet, $\{\alpha, \beta, \gamma\}$, or the set of all integers up to a very large number, like $\{1, 2, \ldots, 10^{10^{10}}\}$.

Definition. A set is called **infinite** if it is not finite. The elements of an infinite set cannot be counted to completion. The standard number systems are classic examples of infinite sets: the natural numbers ($\mathbb{N}$), the integers ($\mathbb{Z}$), the rational numbers ($\mathbb{Q}$), and the real numbers ($\mathbb{R}$).

Definition. For a finite set A, its **size** or **cardinality** is simply the number of elements it contains. The size of a set A is denoted by $|A|$ or $\#A$.

1.8.1 Sum rule for disjoint sets If A and B are finite, disjoint sets, then the size of their union is the sum of their sizes.

$$|A \cup B| = |A| + |B|$$

A direct consequence of this is the following. For any finite set A within a universal set X, A and its complement $A^c = X \setminus A$ are disjoint, and their union is X. Therefore:

$$|X| = |A \cup A^c| = |A| + |A^c|$$

Rearranging this gives a formula for the size of the complement:

$$|A^c| = |X| - |A|$$

1.8.2 Size of the Power Set. A fascinating question is how the size of a set relates to the size of its power set. Recall that the power set, $\mathcal{P}(X)$, is the set of all subsets of X.

> **Lemma.** Let X be a finite set with size $|X| = n$. Then the size of its power set is $|\mathcal{P}(X)| = 2^n$.

Proof. Let us first give an informal argument. To form a subset of X, we must decide for each of the n elements whether to include it in the subset or not. For the first element, we have 2 choices (in or out). For the second element, we also have 2 choices, and so on for all n elements. Since each choice is independent, the total number of possible subsets is $2 \times 2 \times \cdots \times 2$ (n times), which is 2^n. As we make the choices of including an element or not, we can think of traversing down a binary tree as shown in Figure 1.8. Let us now

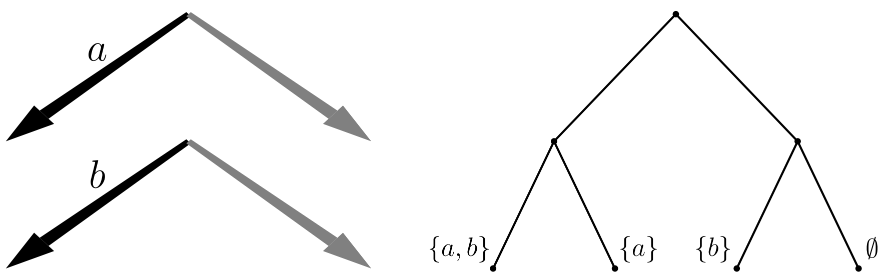


Figure 1.8

give a formal proof. If $|X| = 1$ then the proposition is clearly true. Let $n \geq 1$ Suppose the proposition is true for sets X whose size is n. Let X be a set with size $n+1$. Let x be an element of X. Note that

$$\mathcal{P}(X) = \{A \subseteq X : \ x \in A\} \cup \{A \subseteq X : x \notin A\}$$

The second of the two sets on the right is exactly $\mathcal{P}(X \setminus \{x\})$. On the other hand, the first is obtained by inserting x to each element of $\mathcal{P}(X \setminus \{x\})$. Thus both of them have their sizes same as that of $\mathcal{P}(X \setminus \{x\})$, which is nothing but 2^n since $X \setminus \{x\}$ has size n. Also, these two sets are disjoint and hence

$$|\mathcal{P}(X)| = |\{A \subseteq X : \ x \in A\}| + |\{A \subseteq X : x \notin A\}| = 2 \times 2^n = 2^{n+1}$$

and we are done. ■

Figure 1.9 shows the tree corresponding to the power set of a set of size 3. ♢

1.9 INCLUSION AND EXCLUSION: MOST BASIC VERSIONS

What happens when we want to find the size of the union of two sets that are *not* disjoint? If we simply add their sizes, we will have counted the elements in their intersection twice. The Principle of Inclusion-Exclusion provides a way to correct for this overcounting.

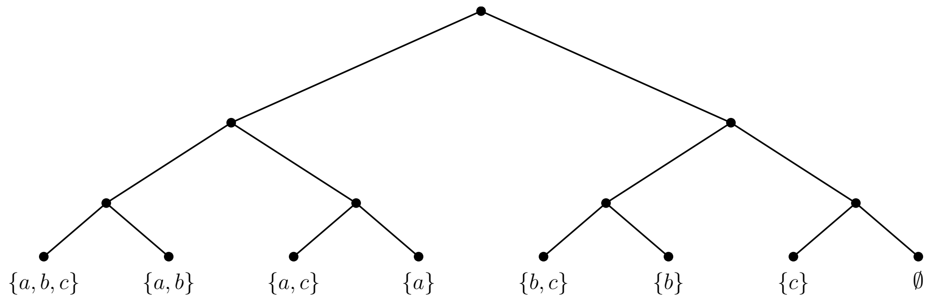


Figure 1.9: The power set of $\{a, b, c\}$ as a binary tree. At each level, we decide whether to include the next element. The 8 leaves correspond to all $2^3 = 8$ subsets.

Lemma 1.9.1. Let A and B be finite sets. The size of their union is given by:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Intuitively, this formula works by first *including* the sizes of both sets, and then *excluding* the size of their intersection to correct the double count. Here is a way to prove this formally using Article 1.8.1. We have

$$A \cup B = (A \setminus B) \cup B = (A \setminus (A \cap B)) \cup B$$

and the sets $A \setminus (A \cap B)$ and B are disjoint. Thus

$$|A \cup B| = |A \setminus (A \cap B)| + |B| = |A| - |A \cap B| + |B|$$

and we are done. ∎

Illustration 1.9.2 Among 50 patients admitted to a hospital, 25 have pneumonia, 30 have bronchitis and 10 have both. Find

a) The number of patients which have at least one of the two diseases.

b) The number of patients which have neither of the two diseases.

Solution. Let X be the set of all the patients. Let P be the set of all the patients with pneumonia and B be the set of all the patients with bronchitis. We are given that

$$|P| = 25, \quad |B| = 30, \quad |X| = 50$$

By the inclusion-exclusion principle we have

$$\begin{aligned} |P \cup B| &= |P| + |B| - |P \cap B| \\ &= 25 + 30 - 10 \\ &= 55 - 10 = 45 \end{aligned}$$

Thus the number of patients with at least one disease is 45. Also

$$\begin{aligned} |X \setminus (P \cup B)| &= |X| - |P \cup B| \\ &= 50 - 45 = 5 \end{aligned}$$

So the number of patients that don't have any of the two diseases is 5. ∎

Lemma 1.9.3. Let A, B and C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof. Say $A \cup B = E$. Then

$$\begin{aligned}
|A \cup B \cup C| = |E \cup C| &= |E| + |C| - |E \cap C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= (|A| + |B| - |A \cap B|) + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - [|A \cap C| + |B \cap C| - |A \cap B \cap C|] \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\end{aligned}$$

which finishes the proof. ■

The principle extends to more than three sets but we will see this general statement when we discuss functions.

Illustration 1.9.4 Out of a 100 programmers

- 45 know Java.
- 30 know Python.
- 10 know both Java and Python.
- 5 know Java and C.
- 7 know Python and C.
- 2 know Java, Python and C.

How many know at least one of Java or Python, but not C.

Solution. Let X be the set of all the 100 programmers. Let P, J and C be the set of all the programmers who know Python, Java and C respectively. We are interested in the size of $(J \cup P) \cap C'$, where C' denotes the complement of C in X. See Figure 1.10. We have

$$\begin{aligned}
(J \cup P) \cap C' &= (J \cap C') \cup (P \cap C') \\
&= (J \backslash C) \cup (P \backslash C) \\
&= (J \backslash J \cap C) \cup (P \backslash P \cap C)
\end{aligned}$$

Now applying inclusion-exclusion we have

$$\begin{aligned}
\left|(J \cup P) \cap C'\right| &= |(J \backslash J \cap C) \cup (P \backslash P \cap C)| \\
&= |J \backslash J \cap C| + |P \backslash P \cap C| - |(J \backslash J \cap C) \cap (P \backslash P \cap C)| \\
&= |J \backslash J \cap C| + |P \backslash P \cap C| - |(J \cap P) \backslash (J \cap P \cap C)| \\
&= (|J| - |J \cap C|) + (|P| - |P \cap C|) - (|J \cap P| - |J \cap P \cap C|) \\
&= (45 - 5) + (30 - 7) - (10 - 2) = 55
\end{aligned}$$

■

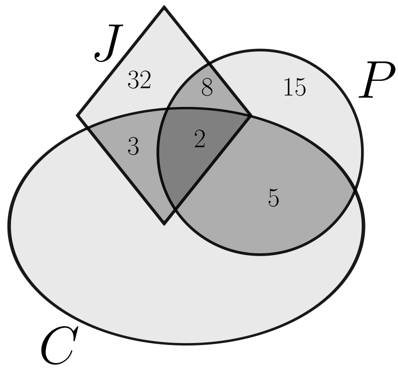


Figure 1.10

Exercise 1.9.1. **[UGA 2022]** In a class of 45 students, three students can write well using either hand. The number of students who can write well only with the right hand is 24 more than the number of those who write well only with the left hand. Then, the number of students who can write well with the right hand is:

(a) 33 (b) 36 (c) 39 (d) 41

Exercise 1.9.2. **[CMI 2020 Part A]** Each student in a small school has to be a member of at least one of THREE school clubs. It is known that each club has 35 members. It is not known how many students are members of two of the three clubs, but it is known that exactly 10 students are members of all three clubs. What is the largest possible total number of students in the school? What is the smallest possible total number of students in the school?

CHAPTER 2

FUNCTIONS

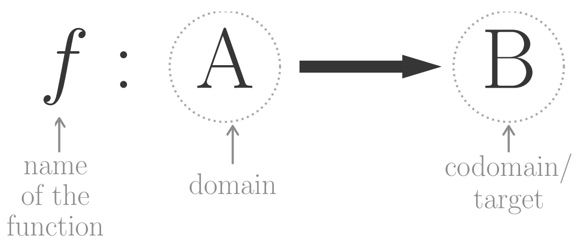
The language of functions is a tool which facilitates clear thinking and allows us to communicate mathematical ideas efficiently. Thus it is of fundamental importance in mathematics to be conversant in this language. Before we begin, let me mention that the importance of the notion of a function will not be clear immediately. However, as we go along, we will inevitably be able to come to appreciate it.

2.1 BASICS

Consider a class of students and a teacher who has just graded their final exam. Each student receives exactly one grade—A, B, C, D, or F. The act of assigning grades is, in essence, a *function*: it takes each student (the input) and produces a grade (the output). Two different students may receive the same grade, but no student can receive two different grades at once. This simple example already captures the heart of the matter: a function is a rule that assigns to each input exactly one output.

We give a somewhat informal definition of a function below. A rigorous definition of a function would take us into the realm of foundations of mathematics, a subject, in my opinion, that should be undertaken for study only after one has learnt a fair amount of mathematics.

Definition. Let A and B be non-empty sets. A **function** (or **map**) *from A to B* is a rule that assigns exactly one element of B to each element of A.



2.1.1 Terminology. This definition has two key parts:

i) We use a special notation for functions: If f is a function from a set A to a set B, then we write

$$f : A \to B$$

ii) The set A is called the **domain** of f. It is the set of all possible inputs.

iii) The set B is called the **codomain** or **target** of f. It is the set where all the outputs must live.

iv) The function can "consume" *any* element of the domain and output *exactly one* element of the codomain. The output of f, when it is "fed" an element $a \in A$, is denoted by $f(a)$. This is also called the **value** of f at a. Thus $f(a)$ is the unique element of B assigned to the element a of A under the function f.

v) Functions are also sometimes called **maps**.

2.1.2 An unfortunate abuse of notation. You'll often read the phrase "Let $f(x)$ be a function..." This is an abuse of notation, and a source of endless confusion. The function is f, while $f(x)$ is the value (output) of the function f at the input x. The two should not be confused.

Why does this matter? Because the phrase "let $f(x)$ be a function" conflates two entirely different things: the function itself (the rule, the machine, the assignment) and a particular output (the value it produces at one specific input). Saying "let $f(x)$ be a function" is like saying "let 3 be a vending machine." The number 3 is what comes out of the machine when you press a particular button; it is not the machine.

The damage goes beyond sloppy language. Once you think of $f(x)$ as "the function," you lose the ability to talk about f as an object in its own right—something you can compose with other functions, invert, restrict to a smaller domain, or compare with another function. You also lose clarity about what the domain and codomain are, because $f(x)$ draws your attention to a single input x rather than to the full sets A and B.

The correct phrasing is: "let $f : A \to B$ be a function." Here f is the function, A is its domain, B is its codomain, and $f(x)$ is the value of f at a particular element $x \in A$. Train yourself to keep these apart from the very beginning, and a great deal of later confusion will simply never arise. $\diamondsuit$

2.1.3 Visualizing functions. We should try to think in terms of pictures as much as possible. There are various ways in which one can visualize a function, and one is more useful than the other depending on the context. Following are three common ways of visualization.

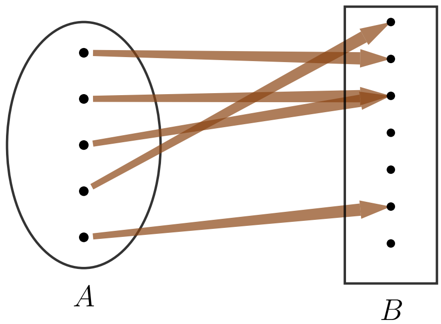


Figure 2.1: We imagine the elements of A and B as dots and draw arrows to indicate which points in A map to which points in B.

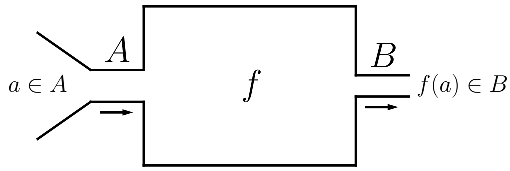


Figure 2.2: We visualize the function $f : A \to B$ which takes a point $a \in A$ as input, processes it somehow, and outputs an element in B, which we write as $f(a)$.

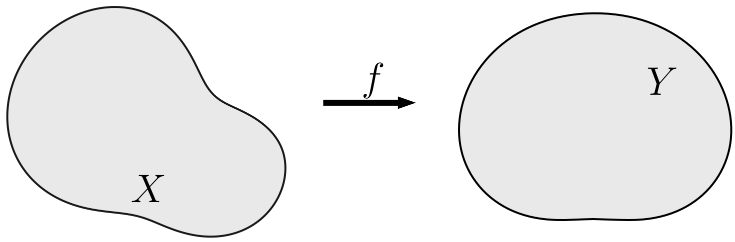


Figure 2.3: A very useful and abstract way to visualize a function is by drawing two "blobs" side by side, denoting the domain and the target of the function respectively, and indicating the function by an arrow as shown above.

Example 2.1.4.

a) *Gravitational force.* Let us describe the gravitational force exerted by a large mass M on a smaller mass m. The force depends on the distance r between them. We can define a function g for this:

$$g : \mathbb{R}_{>0} \to \mathbb{R}$$

The rule is given by Newton's Law of Universal Gravitation:

$$g(r) = \frac{GmM}{r^2}$$

Here, the domain $\mathbb{R}_{>0}$ represents all possible positive distances r. The output $g(r)$ is the (magnitude of) force, which is a real number. G, m, and M are constants in this context.

b) *Assigning Grades.* Let S be the set of all students in a class. Let $G = \{A, B, C, D, F\}$ be the set of possible grades. We can define a grading function, let's call it γ:

$$\gamma : S \to G$$

where for any student $s \in S$, $\gamma(s)$ is the grade that student received. This is a function because every student ($s \in S$) gets exactly one grade ($\gamma(s) \in G$).

c) *Prime sensitive.* Let us define a function that checks if a number is prime. Let the domain be the set of natural numbers up to 100, so $A = \{n \in \mathbb{N} : n \leq 100\}$. Let the codomain be the set {Yes, No}. We define the function f as:

$$f : A \to \{\text{Yes, No}\}$$

with the rule:

$$f(n) = \begin{cases} \text{Yes} & \text{if } n \text{ is a prime number} \\ \text{No} & \text{if } n \text{ is not a prime number} \end{cases}$$

For example:

- $f(1) = \text{No}$ (by definition, 1 is not prime)
- $f(2) = \text{Yes}$
- $f(3) = \text{Yes}$
- $f(4) = \text{No}$

Example 2.1.5 (Some numerical examples)**.**

i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the rule $f(x) = x^2$ for all $x \in \mathbb{R}$. Clearly, every real number input has exactly one output.

ii) Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as $f(n) = n + 1$ for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Every natural number input has a unique integer output.

iii) Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. Let $f : \mathbb{Z} \to \mathbb{N}_0$ be defined as $f(n) = n^2$. For any integer n (positive, negative, or zero), its square n^2 is a unique non-negative integer.

iv) Let $f : \mathbb{Z} \to \mathbb{Q}$ be defined as $f(x) = x/2$. For every integer x, $x/2$ is a unique rational number. So this is a valid function.

Example 2.1.6 (A piecewise Function)**.** A function's rule can be split into two (or more) different cases. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$$

This is still a valid function. Any given x falls into exactly one of the two cases. For instance, to find $f(5)$, we use the first rule since $5 \geq 0$. So, $f(5) = 5 + 1 = 6$. To find $f(-3)$, we use the second rule since $-3 < 0$. So, $f(-3) = (-3)^2 = 9$.

2.1.7 Some non-examples. Understanding what is NOT a function is just as important as knowing what is. These examples violate our definition in some way.

a) Let us try to define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 1/x$. This is not a function. Why? Because the rule must work for *every* element in the domain. Here, the domain is $\mathbb{R}$, which includes the number 0. But $f(0) = 1/0$ is undefined. So, we have an element in the domain with no assigned output, which violates the definition. (To make it a function, we would have to change the domain to $\mathbb{R} \setminus \{0\}$.)

b) Let us try to define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = \sqrt{x}$. This is not a function. The reason is similar. The domain $\mathbb{R}$ includes negative numbers. If we take $x = -1$ from the domain, $f(-1) = \sqrt{-1}$ is not a real number. So, not every element of the domain can be mapped to an element of the codomain $\mathbb{R}$.

c) Let us try to define $f : \mathbb{Z} \to \mathbb{R}$ by the rule $f(n) = \pm n$. This is not a function. Why? Because it violates the "exactly one" output rule. For instance, what is $f(3)$? According to the rule, $f(3)$ could be $+3$ or -3. Since one input leads to multiple possible outputs, this is not a function. A function must be deterministic.

d) Let us try to define $f : \mathbb{Z} \to \mathbb{R}$ by the rule $f(n) = \frac{1}{n^2-4}$. This is not a function. The domain is $\mathbb{Z}$, the set of all integers. This set includes $n = 2$ and $n = -2$. For these inputs, the denominator becomes $2^2 - 4 = 0$ and $(-2)^2 - 4 = 0$. Division by zero is undefined, so the rule fails for these elements of the domain.

2.1.8 The set of all functions between two sets. Let X and Y be non-empty sets. We define $\text{Maps}(X, Y)$ as the set of all the functions whose domain is X and codomain is Y. A natural question arises: if X and Y are finite sets, how many different functions can we create between them?

Lemma. Let $m, n \geq 1$. Let X and Y be sets of sizes m and n respectively. Then

$$|\text{Maps}(X, Y)| = n^m$$

Proof. We argue slightly informally. Let us think about how to build a function $f : X \to Y$. We need to decide, for each element in X, where it should go in Y. Since X has size m, we may write $X = \{x_1, x_2, \ldots, x_m\}$.

- For the first element, x_1, we have to choose an output $f(x_1)$ from the set Y. Since $|Y| = n$, we have n choices.
- For the second element, x_2, we have to choose an output $f(x_2)$ from Y. Again, we have n choices. The choice for x_2 is completely independent of the choice for x_1.
- We continue this for all m elements in X. For each of the m elements, we have n independent choices for its image.

To get the total number of ways to define the function, we multiply the number of choices for each element:

$$\underbrace{n \times n \times \cdots \times n}_{m \text{ times}} = n^m$$

Thus, there are n^m possible functions from X to Y. ■ ◇

2.2 IMAGE AND PREIMAGE

Let us return to the grading function from the beginning of this chapter. The teacher assigns each student a grade from $\{A, B, C, D, F\}$. A natural question is: *which grades were actually given out?* Perhaps no student received an F, so F is in the codomain but not in the image. The set of grades that were actually assigned is exactly the *image* of the grading function.

We can also ask a more targeted question: *what grades did the boys in the class receive?* Here we are looking at the image of a particular *subset* of the domain (the boys), not the image of the entire domain.

Definition. Let $f : X \to Y$ be a function. The **image** of f, often denoted as $\text{Image}(f)$ or sometimes $\text{Range}(f)$, is the set of all elements in the codomain Y that are mapped to by some element in the domain X. Formally, we define it as:

$$\text{Image}(f) = \{f(x) : x \in X\}$$

Notice that the image is a subset of the codomain, i.e., $\text{Image}(f) \subseteq Y$. See Figure 2.4. We can also talk about the image of a specific subset of the domain.

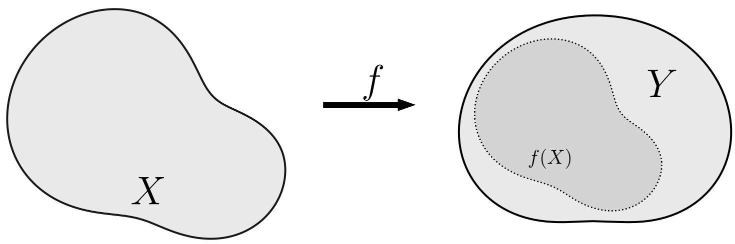


Figure 2.4: Diagram showing a function $f : X \to Y$ and its image as a subset of its codomain.

Definition. Let $f : X \to Y$ be a function and let A be a subset of X. The image of the set A under the function f is the set of all outputs we get when we apply the function to only the elements in A (See Figure 2.5). Formally, we define it as:

$$f(A) = \{f(x) : x \in A\}$$

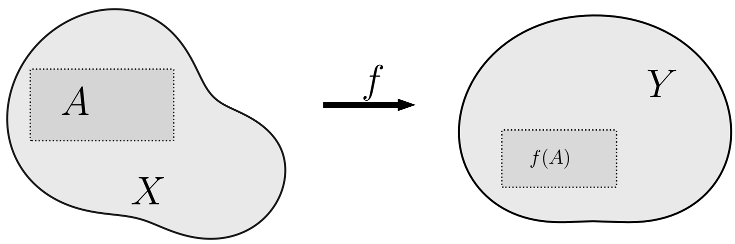
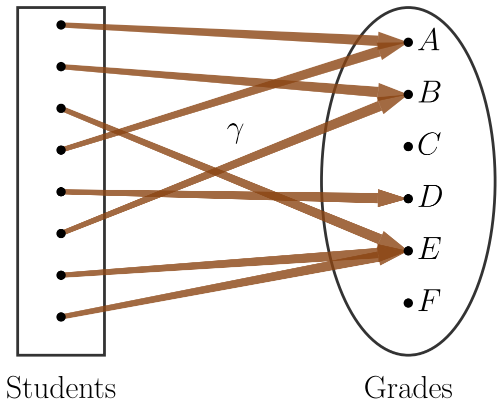


Figure 2.5: Diagram showing a function $f : X \to Y$ and the image $f(A)$ of a subset A of X. Note that the $f(X)$ is nothing but $\text{Image}(f)$.

Example 2.2.1. Imagine a function γ that maps a set of students to the grades they received. Let us say the possible grades are $\{A, B, C, D, E, F\}$. If no student fails, the grade F is not in the image of the function. If the grades awarded were A, B, D, and E, then the image would be:

$$\text{Image}(\gamma) = \{A, B, D, E\}$$



Example 2.2.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. The domain and codomain are both the set of all real numbers. However, what is the image? Since the square of any real number (positive or negative) is non-negative, the function can only output values greater than or equal to zero. Therefore, the image is the set of all non-negative real numbers.

$$\text{Image}(f) = \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$$

2.2.3 A note on terminology. When we have an element $x \in X$, the corresponding output $f(x) \in Y$ is called the **image of the element x under f**. This is different from the image of the entire function, which is a set of all such outputs. ♢

Now, let's see how the concept of an image interacts with basic set operations like union and intersection. We take two subsets, A and B, of our domain X.

2.2.4 Image of union. The image of the union of two sets is the union of their images. This is a very clean and straightforward property. Let $f : X \to Y$ be a function and let $A, B \subseteq X$. Then:

$$f(A \cup B) = f(A) \cup f(B)$$

Proof. To prove that these two sets are equal, we must show that each is a subset of the other. First we show that

$$f(A \cup B) \subseteq f(A) \cup f(B)$$

Let y be an arbitrary element in $f(A \cup B)$. By the definition of the image of a set, this means there exists some element $x \in A \cup B$ such that $y = f(x)$. Since $x \in A \cup B$, we know that either $x \in A$ or $x \in B$. If $x \in A$, then $y = f(x) \in f(A)$. The same argument goes through if $x \in B$. In either case, y must be in $f(A)$ or $f(B)$, which means $y \in f(A) \cup f(B)$. Thus, we have shown that $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now we show that

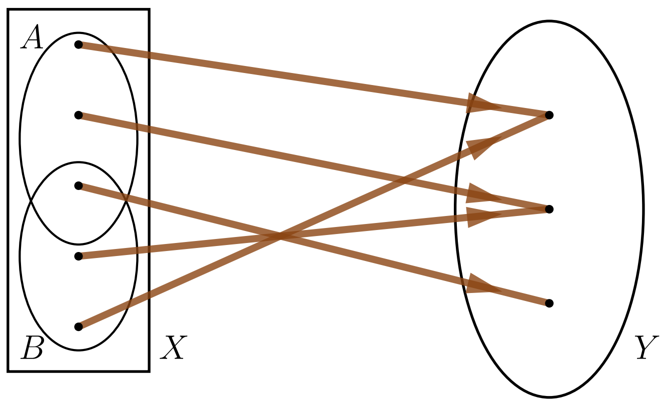
$$f(A) \cup f(B) \subseteq f(A \cup B)$$

Let y be an arbitrary element in $f(A) \cup f(B)$. This means that either $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there exists some $a \in A$ such that $y = f(a)$. But a is also in $A \cup B$. Therefore, $y = f(a) \in f(A \cup B)$. The same conclusion follows if $y \in f(B)$. In both cases, $y \in f(A \cup B)$. Thus, we have shown that $f(A) \cup f(B) \subseteq f(A \cup B)$. Since we have shown containment in both directions, we can conclude that the sets are equal. ♢

2.2.5 Image of intersection. **Caution!** The relationship with intersection is not as simple. It is **not** always true that $f(A \cap B) = f(A) \cap f(B)$. Instead, we only have a subset relationship:

$$f(A \cap B) \subseteq f(A) \cap f(B)$$

The equality might not hold. The following shows a simple example where the equality fails to hold.

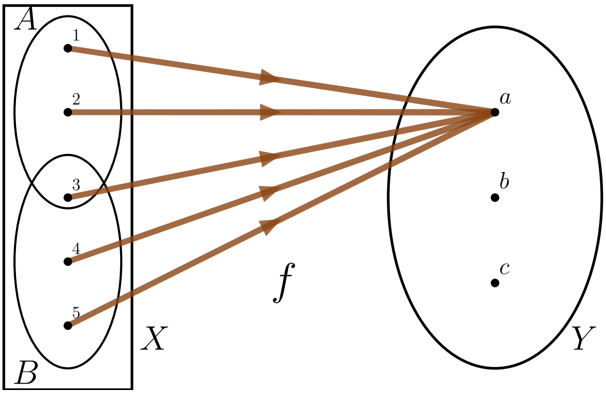


Let us see a numerical example too. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ for all $x \in \mathbb{R}$. Let $A = [-2, 0]$ and $B = [0, 2]$.

- The intersection of these sets is $A \cap B = \{0\}$.
- The image of the intersection is $f(A \cap B) = f(\{0\}) = \{0^2\} = \{0\}$.
- Now let's find the image of each set individually. $f(A) = f([-2, 0])$. The squares of numbers from -2 to 0 range from 0 to 4. So, $f(A) = [0, 4]$. $f(B) = f([0, 2])$. The squares of numbers from 0 to 2 also range from 0 to 4. So, $f(B) = [0, 4]$.
- The intersection of their images is $f(A) \cap f(B) = [0, 4] \cap [0, 4] = [0, 4]$.

Comparing the results, we see that $f(A \cap B) = \{0\}$ while $f(A) \cap f(B) = [0, 4]$. Clearly, they are not equal. This happens because different elements in the domain (e.g., -2 and 2) can map to the same element in the codomain (4). ♢

2.2.6 Image of set difference. Similar to intersection, the image does not distribute nicely over set difference. It is **not** necessarily true that $f(A \setminus B) = f(A) \setminus f(B)$. A simple example is shown below



Explicitly, define a function $f : X \to Y$ with the following sets and mappings. Let the domain be $X = \{1, 2, 3, 4, 5\}$ and the codomain be $Y = \{a, b, c\}$. Define $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Let the function f map every element in X to the single element $a \in Y$. So, $f(1) = a, f(2) = a, f(3) = a, f(4) = a, f(5) = a$. Observe that

$$A \setminus B = \{1, 2\}, \quad f(A \setminus B) = f(\{1, 2\}) = \{a\}$$

and

$$f(A) = f(\{1, 2, 3\}) = \{a\}, \quad f(B) = f(\{3, 4, 5\}) = \{a\}, \quad f(A) \setminus f(B) = \{a\} \setminus \{a\} = \emptyset$$

We can see that $f(A \setminus B) = \{a\}$ and $f(A) \setminus f(B) = \emptyset$. They are not equal. This issue arises because an element in $f(A)$ (namely, a) might also be the image of an element in B, and so it gets removed when we take the set difference on the right-hand side. ♢

Continuing with the grading function, suppose the teacher wants to know: *which students failed?* That is, given the set $T = \{F\}$ inside the codomain, the teacher wants to find all students whose grade lands in T. Or consider a more generous question: *which students scored A or B?* Here $T = \{A, B\}$, and the answer is the set of all students mapped into T. This "reverse lookup"—starting from a set of outputs and finding all the inputs that produce them—is the notion of *preimage*.

Definition. Let $f : X \to Y$ be a function, and let T be a subset of the codomain Y (i.e., $T \subseteq Y$). The **preimage** of T under f, denoted $f^{-1}(T)$, is the set of all elements in the domain X that map into the set T. Formally,

$$f^{-1}(T) = \{x \in X : f(x) \in T\}$$

2.2.7 A possible confusion. The notation f^{-1} for the preimage does **not** mean that the function f has an inverse function. An inverse function f^{-1} only exists if f is bijective (one-to-one and onto). The preimage $f^{-1}(T)$ is defined for *any* function f and is a set of elements from the domain. ♢

2.2.8 Notation for singletons. When we find the preimage of a set containing just one element, say $T = \{y\}$, we often simplify the notation from $f^{-1}(\{y\})$ to just $f^{-1}(y)$. This is an abuse of notation, but is ubiquitous. ♢

Example 2.2.9.

a) *Students and Grades:* Let us use our function γ from students to grades from Example 2.2.1.

   i) What is the preimage of $\{B\}$? This is asking "Which students got the grade B?" If students 1 and 7 are the only students who got a B, then $\gamma^{-1}(B) = \{1, 7\}$.
   ii) What is the preimage of $\{F\}$? This asks "Which students got an F?" If no one failed, then $\gamma^{-1}(F) = \emptyset$. However, usually this set is not empty.
   iii) What is the preimage of $\{A\}$? If only student 2 got an A, then $\gamma^{-1}(A) = \{2\}$.

b) *Squaring Function:* Let us again use $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

   i) What is $f^{-1}(4)$? We are looking for all x such that $x^2 = 4$. The solutions are $x = 2$ and $x = -2$. So, $f^{-1}(4) = \{-2, 2\}$.
   ii) What is $f^{-1}(-1)$? We are looking for all x such that $x^2 = -1$. There are no real numbers that satisfy this. So, $f^{-1}(-1) = \emptyset$.
   iii) What is $f^{-1}([1, 9])$? We are looking for all x such that $f(x) \in [1, 9]$, which means $1 \le x^2 \le 9$. This inequality is satisfied if $1 \le x \le 3$ or if $-3 \le x \le -1$. So, $f^{-1}([1, 9]) = [-3, -1] \cup [1, 3]$.

2.2.10 Properties of preimages. One of the convenient properties of preimage is that it behaves perfectly with respect to all standard set operations. Unlike the image, there are no caveats or special conditions.

Lemma. Let $f : X \to Y$ be a function, and let P and Q be subsets of the codomain Y. The following properties hold:

a) (Union.) $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$
b) (Intersection.) $f^{-1}(P \cap Q) = f^{-1}(P) \cap f^{-1}(Q)$
c) (Set Difference.) $f^{-1}(P \setminus Q) = f^{-1}(P) \setminus f^{-1}(Q)$
d) (Symmetric Difference.) $f^{-1}(P \Delta Q) = f^{-1}(P) \Delta f^{-1}(Q)$

Proof. We prove (a) and (c) and leave (b) and (d) as exercises.

(a): We will prove this by showing that an element x is in the left-hand side set if and only if it is in the right-hand side set. This chain of logical equivalences proves the sets are identical.

$$\begin{aligned}
x \in f^{-1}(P \cup Q) &\iff f(x) \in P \cup Q && \text{(by definition of preimage)} \\
&\iff f(x) \in P \text{ or } f(x) \in Q && \text{(by definition of union)} \\
&\iff x \in f^{-1}(P) \text{ or } x \in f^{-1}(Q) && \text{(by definition of preimage)} \\
&\iff x \in f^{-1}(P) \cup f^{-1}(Q) && \text{(by definition of union)}
\end{aligned}$$

This completes the proof for union.

(c): The logic is exactly the same. We just translate the definitions step-by-step.

$$\begin{aligned}
x \in f^{-1}(P \setminus Q) &\iff f(x) \in P \setminus Q && \text{(by definition of preimage)} \\
&\iff f(x) \in P \text{ and } f(x) \notin Q && \text{(by definition of set difference)} \\
&\iff x \in f^{-1}(P) \text{ and } x \notin f^{-1}(Q) && \text{(by definition of preimage)} \\
&\iff x \in f^{-1}(P) \setminus f^{-1}(Q) && \text{(by definition of set difference)}
\end{aligned}$$

This completes the proof for set difference. The proofs for intersection and symmetric difference follow a very similar structure. ■ ♢

Definition. Let $f : X \to Y$ be a function, and let y be a single element in the codomain Y (i.e., $y \in Y$). The **fiber** of f above y is simply the preimage of the singleton set $\{y\}$. As we saw with our notation simplification, this is written as $f^{-1}(y)$.

$$\text{Fiber above } y = f^{-1}(y) = \{x \in X : f(x) = y\}$$

2.2.11 Visualizing fibers. The name "fiber" comes from a geometric way of thinking about functions.

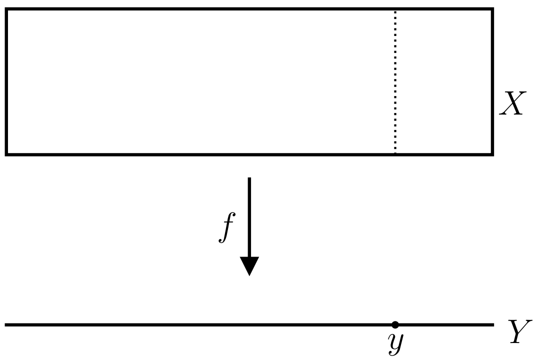


Figure 2.6: A popular way to visualize the fiber above a point.

If we imagine the domain X as a space "above" the codomain Y, and the function f as a projection downwards, then the fiber above a point $y \in Y$ is the set of all points in X that are "hanging" directly above y. ♢

2.2.12 Fibers partition the domain. The fibers of a function partition the domain X. This means that every element of X belongs to exactly one fiber, and the fibers for different elements of the image are disjoint. ♢

Exercise 2.2.1. Let X and Y be finite sets and let $f : X \to Y$ be an arbitrary function. State true or false.

a) Let $A \subseteq X$. Then $|f(A)| \leq |A|$.
b) Let $A \subseteq X$. Then $|f(A)| < |A|$.
c) Let $A \subseteq X$. Then $|f(A)| \geq |A|$.
d) Let $A \subseteq X$. Then $|f(A)| > |A|$.
e) Let $B \subseteq Y$. Then $|f^{-1}(B)| \leq |B|$.
f) Let $B \subseteq Y$. Then $|f^{-1}(B)| \geq |B|$.
g) Let $B \subseteq Y$. Then $|f^{-1}(B)| < |B|$.
h) Let $B \subseteq Y$. Then $|f^{-1}(B)| > |B|$.

Exercise 2.2.2. Let X and Y be finite sets and let $f : X \to Y$ be a function. Assume that Image$(f) = Y$. State true or false:

a) Let $A \subseteq X$. Then $|f(A)| \leq |A|$.
b) Let $A \subseteq X$. Then $|f(A)| < |A|$.
c) Let $A \subseteq X$. Then $|f(A)| \geq |A|$.
d) Let $A \subseteq X$. Then $|f(A)| > |A|$.
e) Let $B \subseteq Y$. Then $|f^{-1}(B)| \leq |B|$.
f) Let $B \subseteq Y$. Then $|f^{-1}(B)| \geq |B|$.
g) Let $B \subseteq Y$. Then $|f^{-1}(B)| < |B|$.
h) Let $B \subseteq Y$. Then $|f^{-1}(B)| > |B|$.

2.3 COMPOSITION

Let us extend the grading example one step further. The grading function g assigns each student a grade from $\{A, B, C, D, F\}$. Now suppose the university has a separate rule c that classifies each grade into a category: A and B are "good," C and D are "acceptable," and F is "fail." To find out a student's category, we first apply g to get their grade, then apply c to that grade. The result is a new function that goes directly from students to categories—this is the *composition* of c and g.

2.3.1 Composition of Two Functions. Often in mathematics and programming, we perform a sequence of operations. Function composition is the formal way to describe this process of chaining functions together. You can think of it like an assembly line: the output of one function becomes the input for the next. ♢

Definition. Let us say we have two functions, $f : A \to B$ and $g : B \to C$. The **composition** of g and f, denoted as $g \circ f$ (read as "g circle f" or "g composed with f"), is a new function that maps elements directly from the starting set A to the final set C defined as

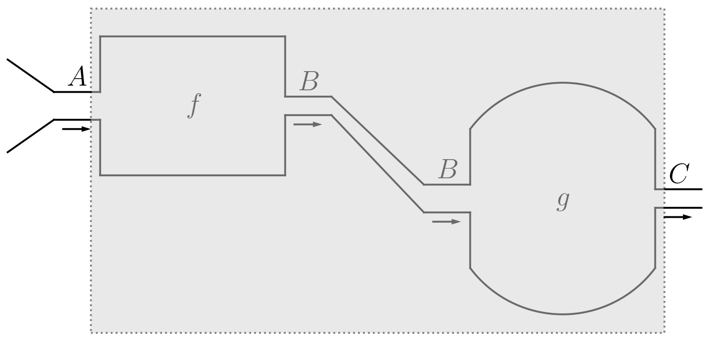
$$(g \circ f)(a) = g(f(a)), \quad \text{for all } a \in A$$

This means to evaluate $g \circ f$ at a, we first apply f to a to get an element $f(a)$ in B. Then, we take that result and apply g to it. It is crucial in the above that the target of f is same as the domain of g, so that $g(f(a))$ is always meaningful.[a]

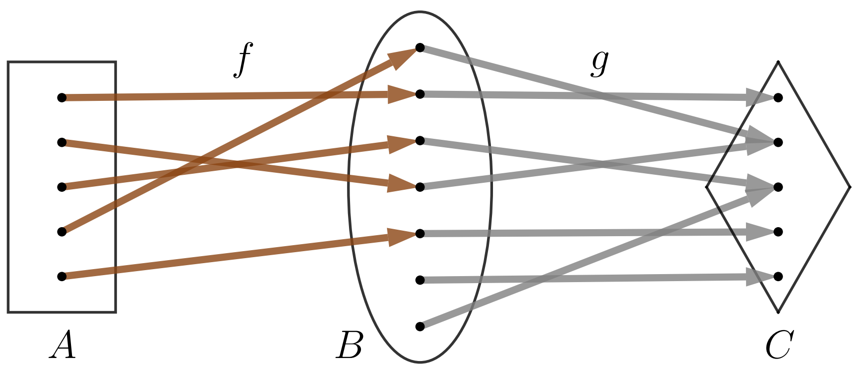
[a]We can relax this requirement a little bit and only demand that the domain of g *contains* the target of f.

2.3.2 Visualizing composition. Let us look at how we can depict the operation of composition diagrammatically. We saw various ways to think of a function, one of which was a to think of a function as a "machine." Suppose we have two machines, $f : A \to B$ and $g : B \to C$, shown as below By connecting the output nozzle of the first with the

input nozzle of the second, we can create a compound machine $g \circ f : A \to C$, shown below



Another way we would visualize a function is by means of drawing arrows between the domain and the target. The picture above would take the following form



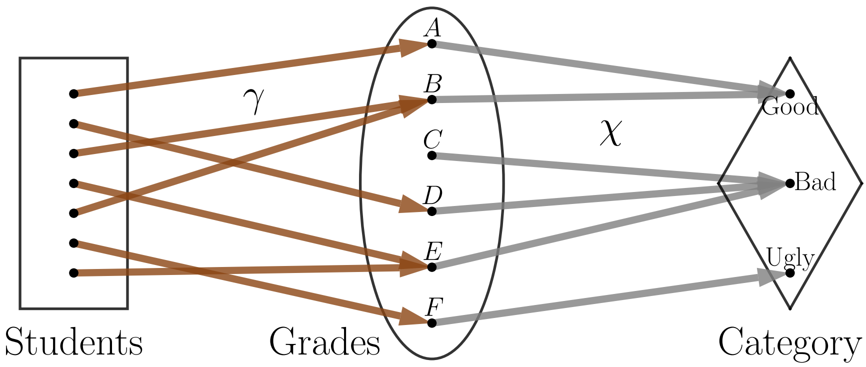
However, a most robust and simultaneously formal way of visualizing a function is by making a simple commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

◇

Example 2.3.3. Let us revisit our example of students and grades (Example 2.2.1) and add another layer.

- Let γ be a function from a set of students to a set of grades, $\{A, B, C, D, E, F\}$.
- Let χ be a function from the set of grades to a set of categories, like {Good, Bad, Ugly}. For instance, $\chi(A) =$ Good, $\chi(B) =$ Good, $\chi(C) =$ Bad, etc.



The composition $\chi \circ \gamma$ is a function that maps a student directly to a category. If a student gets a grade B, and the grade B is categorized as 'Good', then the composite function maps that student to 'Good.'

$$(\chi \circ \gamma)(\text{Student}) = \chi(\gamma(\text{Student})) = \chi(\text{Grade}) = \text{Category}$$

Example 2.3.4. Unit conversion is a perfect real-world example of function composition. Suppose we want to convert a duration from seconds into hours. We can do this in two steps: (1) Convert seconds to minutes, (2) Convert minutes to hours. Let us define these as functions. Let our domain be the set of non-negative real numbers, $\mathbb{R}_{\geq 0}$.

- Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function that converts seconds to minutes. The formula is $f(s) = s/60$.
- Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function that converts minutes to hours. The formula is $g(m) = m/60$.

The composition $g \circ f$ will be a function that converts seconds directly to hours. Let us find its formula:

$$(g \circ f)(s) = g(f(s)) = g(s/60) = \frac{(s/60)}{60} = \frac{s}{3600}$$

This is exactly the formula for converting seconds to hours, created by composing two simpler conversions.

2.3.5 Associativity of composition. When we compose three or more functions, does the order of grouping matter? For example, is $\gamma \circ (\beta \circ \alpha)$ the same as $(\gamma \circ \beta) \circ \alpha$? The answer is yes, function composition is *associative*. Let us have three functions chained together:

$$\alpha : P \to Q, \quad \beta : Q \to R \quad \text{and} \quad \gamma : R \to S$$

The associativity property states that:

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$

Proof. To prove that these two composite functions are equal, we must show that they produce the same output for every input $p \in P$. First we evaluate the left-hand side

$$\begin{aligned} [\gamma \circ (\beta \circ \alpha)](p) &= \gamma((\beta \circ \alpha)(p)) && \text{(by definition of the outer composition)} \\ &= \gamma(\beta(\alpha(p))) && \text{(by definition of the inner composition)} \end{aligned}$$

Now we evaluate the right-hand side

$$\begin{aligned} [(\gamma \circ \beta) \circ \alpha](p) &= (\gamma \circ \beta)(\alpha(p)) && \text{(by definition of the outer composition)} \\ &= \gamma(\beta(\alpha(p))) && \text{(by definition of the inner composition)} \end{aligned}$$

Since both sides simplify to the exact same expression, $\gamma(\beta(\alpha(p)))$, the two functions are indeed equal. This means we can just write $\gamma \circ \beta \circ \alpha$ without any parentheses. ♢

2.3.6 Composition and preimage. How does composition interact with the concept of a preimage? There is a very neat formula for it. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Let C be a subset of the final set Z (i.e., $C \subseteq Z$). Then the preimage of C under the composition $g \circ f$ is given by:

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$$

Notice the order: the preimage under the composition is found by taking the preimage under g first, and then taking the preimage of the resulting set under f. The order of functions is reversed!

Proof. We will use a chain of logical equivalences ('iff' or $\iff$) to prove this equality. Let x be an element in the initial domain X.

$$\begin{aligned} x \in (g \circ f)^{-1}(C) &\iff (g \circ f)(x) \in C && \text{(by def. of preimage)} \\ &\iff g(f(x)) \in C && \text{(by def. of composition)} \\ &\iff f(x) \in g^{-1}(C) && \text{(by def. of preimage for } g) \\ &\iff x \in f^{-1}(g^{-1}(C)) && \text{(by def. of preimage for } f) \end{aligned}$$

Since an element x is in the left-hand set if and only if it is in the right-hand set, the two sets must be equal. ♢

2.3.7 A "functorial" view of preimage (Optional.) We discuss a more abstract, but powerful, way to think about preimages. For any given function $f : X \to Y$, we can define a corresponding "lifted" function that works on the **power sets** of X and Y. Recall that the power set of a set S, denoted $P(S)$, is the set of all possible subsets of S.

We define a new function, let's call it f^*, which maps the power set of the codomain to the power set of the domain.

$$f^* : P(Y) \to P(X)$$

The rule for this function is defined using the preimage:

$$f^*(B) = f^{-1}(B) \quad \text{for any subset } B \subseteq Y$$

So, f^* is just a formalization of the preimage operation as a function between power sets. Now let us look at the composition of $f : X \to Y$ and $g : Y \to Z$ through this new lens. We can define three lifted functions:

- $f^* : P(Y) \to P(X)$ defined by $f^*(B) = f^{-1}(B)$
- $g^* : P(Z) \to P(Y)$ defined by $g^*(C) = g^{-1}(C)$
- $(g \circ f)^* : P(Z) \to P(X)$ defined by $(g \circ f)^*(C) = (g \circ f)^{-1}(C)$

What is the relationship between these three? Let us consider the composition of the lifted functions, $f^* \circ g^*$. Note the order matches the direction of the arrows between the power sets:

$$P(Z) \xrightarrow{g^*} P(Y) \xrightarrow{f^*} P(X)$$

Let us apply this composite function to a set $C \subseteq Z$:

$$\begin{aligned}
(f^* \circ g^*)(C) &= f^*(g^*(C)) && \text{(by def. of composition)} \\
&= f^*(g^{-1}(C)) && \text{(by def. of } g^*\text{)} \\
&= f^{-1}(g^{-1}(C)) && \text{(by def. of } f^*\text{)}
\end{aligned}$$

From the previous section, we know that $f^{-1}(g^{-1}(C))$ is equal to $(g \circ f)^{-1}(C)$. And by our definition above, $(g \circ f)^{-1}(C)$ is just $(g \circ f)^*(C)$. Therefore, we have the elegant identity:

$$(g \circ f)^* = f^* \circ g^*$$

which can also be captured by the following commutative diagrams

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{P}(Y) \arrow[dl, "f^*"'] & \\
\mathcal{P}(X) & & \mathcal{P}(Z) \arrow[ul, "g^*"'] \arrow[ll, "(g\circ f)^*"]
\end{tikzcd}$$

This tells us that the "lifting" of a composition is the composition of the "lifted" functions, but with the order reversed. This reversal of order is a common and important theme in many areas of mathematics. ♢

Illustration 2.3.8 [CMI 2019 Part B] For a natural number n let Map(n) denote the set of all functions

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

Let $f \in \text{Map}(n)$. If for all $x \in \{1, \ldots, n\}$ we have $f(x) \neq x$, show that $f \circ f \neq f$.

Solution. We proceed by contradiction. Suppose, for the sake of argument, that $f \circ f = f$. Therefore

$$f(f(x)) = f(x) \text{ for all } x \in \{1, \ldots, n\}$$

Let y be an arbitrary element in the image of f. By definition, this means there exists some x such that $y = f(x)$. Substituting this into our identity, we get:

$$f(y) = f(f(x)) = f(x) = y$$

This implies that every element y in the image of f is a *fixed point* (i.e., it maps to itself). However, the problem statement explicitly assumes that $f(x) \neq x$ for all x. This means f possesses *no* fixed points. We have arrived at a contradiction: the image must contain fixed points for the composition identity to hold, yet the function is fixed-point free. Thus, we must conclude that $f \circ f \neq f$. ■

Exercise 2.3.1. Let $f : X \to Y$ be a function. Let A and B be subsets of X. Show that if $A \subseteq B$, then $f(A) \subseteq f(B)$.

Exercise 2.3.2. Let $f : X \to Y$ be a function. Let P and Q be subsets of Y. Show that if $P \subseteq Q$, then $f^{-1}(P) \subseteq f^{-1}(Q)$.

Exercise 2.3.3. Let $f : X \to Y$ be a function. Let P and Q be subsets of the *image* of f (i.e., $P, Q \subseteq \text{Image}(f)$). Show that if $f^{-1}(P) = f^{-1}(Q)$, then it must be that $P = Q$. (Think about why the condition that P and Q are in the image is important here).

Exercise 2.3.4. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Show that the image of their composition is a subset of the image of the second function. That is, show that $\text{Image}(g \circ f) \subseteq \text{Image}(g)$.

2.4 CARTESIAN PRODUCT

2.4.1 Motivating examples. The *Cartesian product* is a fundamental operation in set theory that allows us to combine sets to create a new, more complex set of ordered elements. Let us see two examples which will lead us to a formal definition.

a) *Study schedule.* Suppose we want to create a study schedule for the week. You have a set of available weekdays and a set of subjects to study. Let

$$D = \{\text{Mon, Tue, Wed, Thu, Fri}\}, \quad S = \{\text{Physics, Chemistry, Math}\}$$

To create a comprehensive list of all possible single-day, single-subject study sessions, we would need to pair every day with every subject. This would look like:

- (Mon, Physics), (Mon, Chemistry), (Mon, Math)
- (Tue, Physics), (Tue, Chemistry), (Tue, Math)
- ...and so on for all five days.

This systematic process of creating all possible ordered pairs from two sets is precisely what the Cartesian product does.

b) *Choosing an outfit.* Let us say we are picking an outfit from our wardrobe. You have a set of shirts and a set of pants.

$$S = \{\text{Red, Blue, White}\}, \quad P = \{\text{Black, Grey}\}$$

The set of all possible shirt-and-pants combinations is found by pairing each shirt with each pair of pants:

- (Red, Black), (Red, Grey)
- (Blue, Black), (Blue, Grey)
- (White, Black), (White, Grey)

There are $3 \times 2 = 6$ possible outfits. This set of all possible ordered pairs is the Cartesian product $S \times P$.

Definition. Let X and Y be non-empty sets. The **Cartesian product** of X and Y, denoted $X \times Y$, is the set of all possible **ordered pairs** (x, y) where the first element x comes from X and the second element y comes from Y. Formally:

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

The term "ordered" is critical. Unless $x = y$ and $X = Y$, the pair (x, y) is different from the pair (y, x). If $X \times X$ is taken, it's often abbreviated as X^2.

Example 2.4.2 (The Cartesian Plane)**.** The most famous example is the Cartesian plane from geometry. It is the set of all ordered pairs of real numbers, $\mathbb{R} \times \mathbb{R}$, which we denote as $\mathbb{R}^2$. Every point on the plane, like $(1, 3)$, is an element of this Cartesian product.

Definition. We can extend this idea to any number of sets. Let $X_1, X_2, \ldots, X_n$ be non-empty sets. Their Cartesian product is the set of all possible ordered n-tuples $(x_1, x_2, \ldots, x_n)$.

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) : x_i \in X_i \text{ for } i = 1, \ldots, n\}$$

This is also written using product notation: $\prod_{i=1}^{n} X_i$.

2.4.3 Natural projections. Once we have a Cartesian product, we often want to get back to the original component sets. We do this using functions called **projections**. For a product $X \times Y$, there are two natural projection maps:

- The projection onto X: $\pi_X : X \times Y \to X$, defined by $\pi_X((x, y)) = x$.
- The projection onto Y: $\pi_Y : X \times Y \to Y$, defined by $\pi_Y((x, y)) = y$.

These functions simply "pick out" the first or second component of the ordered pair. For the Cartesian plane $\mathbb{R}^2$, the projections $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$ give us the x-coordinate and y-coordinate of a point, respectively. This can be generalized to the Cartesian product of n-sets. Let $X_1, \ldots, X_n$ be n-sets and let $X = \prod_{i=1}^{n} X_i$. the i-th projection function $\pi_i : X \to X_i$ is defined as

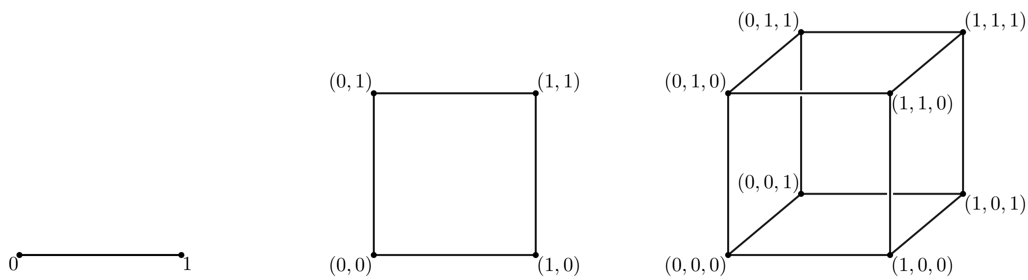
$$\pi_i(x_1, \ldots, x_n) = x_i$$

for all $(x_1, \ldots, x_n)$ in X. $\diamondsuit$

Example 2.4.4. A very important example of a Cartesian product in computer science, logic, and information theory is the Boolean hypercube. We start with the simplest non-trivial set, the Boolean set $B = \{0, 1\}$. The n-th Boolean hypercube is defined as the n-fold Cartesian product of this set with itself, which we denote by $\{0, 1\}^n$.

$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}}$$

The elements of this set are all possible n-tuples of 0s and 1s, which can be thought of as binary strings of length n.



- $n = 1$. Here $\{0, 1\}^1 = \{0, 1\}$. This can be visualized as a line segment with two points at its ends.
- $n = 2$: Here $\{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. These four points are the vertices of a square in the Cartesian plane.
- $n = 3$: Here $\{0, 1\}^3$ has $2^3 = 8$ elements, from $(0, 0, 0)$ to $(1, 1, 1)$. These eight points are the vertices of a cube in three-dimensional space.

For $n \geq 4$, we can no longer visualize these "hypercubes" easily, but they are fundamental structures in higher-dimensional geometry and computing.

2.4.5 A note on cardinality. A straightforward but important property follows from the multiplication principle of counting: if X and Y are finite sets, then the size (cardinality) of their Cartesian product is the product of their sizes.

$$|X \times Y| = |X| \times |Y|$$

More generally, if $X_1, \ldots, X_n$ are finite sets, then

$$\left| \prod_{i=1}^{n} X_i \right| = \prod_{i=1}^{n} |X_i|$$

2.4.6 More Formally... (Optional.) In mathematics, we try to build complex objects from simpler ones. The "ordered pair" can itself be defined in terms of sets and functions, rather than being taken as a fundamental concept. Here is one way to do it.

How can we capture the idea of an ordered pair (x, y) using only a function? The key features we need are "what the elements are" and "which one comes first". We can use a function whose domain is an index set, like $\{1, 2\}$, to keep track of the order.

We can define the ordered pair (x, y) to be a function $f : \{1, 2\} \to X \cup Y$ that satisfies:

- $f(1) = x$ (The first element is x)
- $f(2) = y$ (The second element is y)

Under this interpretation, the Cartesian product $X \times Y$ is no longer a primitive notion, but is defined as the set of all such functions:

$$X \times Y := \{f \in \text{Maps}(\{1, 2\}, X \cup Y) : f(1) \in X \text{ and } f(2) \in Y\}$$

This approach is more abstract, but it shows how core mathematical concepts can be constructed from a very small set of foundational ideas (sets and functions). $\diamondsuit$

2.4.7 The Universal Property (Optional.) This is a powerful and abstract idea from a field of mathematics called Category Theory. It defines an object not by what it's made of, but by how it relates to everything else. The *Universal Property of the Cartesian Product* gives the most essential characterization of what a product is.

Suppose we have a set X and a set Y. The Cartesian product $X \times Y$ comes equipped with its two projection maps,

$$\pi_X : X \times Y \to X \quad \text{and} \quad \pi_Y : X \times Y \to Y$$

Now, take *any* other set A and *any* pair of functions mapping from A into X and Y, say

$$f : A \to X \quad \text{and} \quad g : A \to Y$$

The universal property states that there exists a **unique function** $\varphi : A \to X \times Y$ such that the whole system works together harmoniously.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dotted, "\varphi"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Specifically, if we first use φ and then project, we get back our original functions f and g. Explicitly, the map $\varphi : A \to X \times Y$ is defined as $\varphi(a) = (f(a), g(a))$ for all $a \in A$. This relationship is usually shown in a **commutative diagram** as above. This diagram "commutes" (or works) if following different paths from one corner to another yields the same result.

- Going from A to X: The direct path is f. The indirect path is going through $X \times Y$ via φ and then projecting via π_X. For the diagram to commute, we must have $f = \pi_X \circ \varphi$.
- Going from A to Y: The direct path is g. The indirect path is via φ and then π_Y. For the diagram to commute, we must have $g = \pi_Y \circ \varphi$.

The property guarantees that for any choice of A, f, g, there is exactly one φ that makes this work. ♢

Exercise 2.4.1. Let the sets be $X = \{-2, 2\}$, $Y = \{0, 4\}$, and $Z = \{-3, 0, 3\}$. Evaluate the following Cartesian products:

1. $X \times Y$
2. $X \times Z$
3. $Z \times Y \times Y$

Exercise 2.4.2. Let A, B, and C be any three non-empty sets. Show that $A \subseteq B$ if and only if $A \times C \subseteq B \times C$.

Exercise 2.4.3. Let A, B, and C be sets. Show that the following distributive properties hold for the Cartesian product:

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Exercise 2.4.4. Can you think of a natural map (a function) from the set of maps from $X \times Y$ to Z, denoted $\mathrm{Maps}(X \times Y, Z)$, to the set of maps $\mathrm{Maps}(X, \mathrm{Maps}(Y, Z))$? This is a more advanced question related to a concept known as Currying in computer science and mathematics.

2.5 REAL VALUED FUNCTIONS

Real-valued functions are everywhere. The temperature at a given location on Earth at a given time is a real number—so temperature is a function from (location, time) pairs to

$\mathbb{R}$. The height of a person is a function from the set of people to $\mathbb{R}$. The price of a stock at the close of each trading day is a function from dates to $\mathbb{R}$. In physics, the position of a particle moving along a line is a function from time to $\mathbb{R}$. In all of these, the output is a real number, even though the inputs may come from very different kinds of sets.

Definition. A **real-valued function** is simply a function whose codomain is the set of real numbers, $\mathbb{R}$, or a subset of $\mathbb{R}$. The domain can be any set.

Example 2.5.1. Consider the set S of students in a class. We can define a function, let's call it μ, that maps each student to their marks in an exam.

$$\mu : S \to \mathbb{R}$$

Here, the set of students is the domain, and the codomain is the set of real numbers (since marks can be represented by numbers).

Example 2.5.2. A very useful type of real-valued function is the **indicator function** (also known as a characteristic function). Let X be a set and let A be a subset of X ($A \subseteq X$). The indicator function of A, denoted by 1_A, is a function from X to the real numbers.

$$1_A : X \to \mathbb{R}$$

It is defined as follows: for any element $x \in X$, the function $1_A(x)$ gives a value of 1 if x belongs to the subset A, and 0 otherwise.

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The indicator function essentially "indicates" whether an element is a member of a particular subset.

2.5.3 Some algebraic operations on real-valued functions. When we have two real-valued functions that share the same domain, we can combine them in several algebraic ways to create new functions. Let X be a non-empty set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be two functions.

i) (Sum.) The sum of f and g is a new function $f + g : X \to \mathbb{R}$. For any $x \in X$, its value is the sum of the individual function values.

$$(f + g)(x) = f(x) + g(x)$$

ii) (Difference.) Similarly, the difference of f and g is the function $f - g : X \to \mathbb{R}$.

$$(f - g)(x) = f(x) - g(x)$$

iii) (Product.) The product of f and g is the function $fg : X \to \mathbb{R}$.

$$(fg)(x) = f(x)g(x)$$

iv) (Maximum.) The maximum of f and g is the function $\max\{f, g\} : X \to \mathbb{R}$. For each x, it takes the larger of the two values $f(x)$ and $g(x)$.

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}$$

v) (Minimum.) The minimum of f and g is the function $\min\{f, g\} : X \to \mathbb{R}$. It takes the smaller of the two values.

$$\min\{f, g\}(x) = \min\{f(x), g(x)\}$$

Example 2.5.4.

a) *Total Marks.* Let X be the set of students in a class. Let $h : X \to \mathbb{R}$ be the function giving each student's marks in History, and $g : X \to \mathbb{R}$ be the function for marks in Geography. The total marks function, $t : X \to \mathbb{R}$, is the sum of these two functions: $t = h + g$. For a student $s \in X$, their total marks are $t(s) = (h + g)(s) = h(s) + g(s)$.

b) *Best Score.* Let $h_1 : X \to \mathbb{R}$ and $h_2 : X \to \mathbb{R}$ be the functions representing the marks of students in two different History exams. To find the best score for each student, we can use the maximum function. The function for the best score would be $\max\{h_1, h_2\}$, where for any student s, the score is $\max\{h_1(s), h_2(s)\}$.

2.5.5 Distributivity property. These operations on functions obey familiar algebraic laws, such as the distributive property. Let X be a non-empty set and let $f, g, h : X \to \mathbb{R}$ be three functions. Then, the following holds:

$$f(g + h) = fg + fh$$

This means that multiplying the function f by the sum of functions g and h gives the same result as summing the products fg and fh. A similar property holds for subtraction: $f(g - h) = fg - fh$.

Let us walk through the proof for the distributive property. To show that the two functions $f(g + h)$ and $fg + fh$ are equal, we must show that they produce the same output for every input x in their common domain X. Let x be an arbitrary element of X.

$$\begin{aligned}
[f(g + h)](x) &= f(x) \cdot (g + h)(x) && \text{(by definition of the product of functions)} \\
&= f(x) \cdot (g(x) + h(x)) && \text{(by definition of the sum of functions)} \\
&= f(x)g(x) + f(x)h(x) && \text{(by the distributive property of real numbers)} \\
&= (fg)(x) + (fh)(x) && \text{(by definition of the product of functions)} \\
&= (fg + fh)(x) && \text{(by definition of the sum of functions)}
\end{aligned}$$

Since $[f(g + h)](x) = (fg + fh)(x)$ for all $x \in X$, we have proven that $f(g + h) = fg + fh$. ♢

2.5.6 Important observations on indicator functions. Indicator functions have some very interesting properties when combined with the algebraic operations we just defined. The proofs of these properties are straightforward and are left to the reader. Let X be a non-empty set and let A, B be subsets of X. Then

a) The product of the indicator functions for A and B is equal to the indicator function of their intersection:

$$1_A \cdot 1_B = 1_{A \cap B}$$

b) The indicator function of the union of two sets is the maximum of their individual indicator functions.

$$1_{A \cup B} = \max\{1_A, 1_B\}$$

c) For the complement of a set A in X, denoted $A^c = X \setminus A$, the indicator function is given by:

$$1_{A^c} = 1 - 1_A$$

Here, '1' represents the constant function that maps every element of X to the number 1.

d) If the set X is finite, then for any subset $A \subseteq X$, the number of elements in A (its cardinality, $|A|$) can be found by summing the values of its indicator function over all elements of X.

$$|A| = \sum_{x \in X} 1_A(x)$$

This works because we add 1 for every element inside A and 0 for every element outside A, effectively just counting the elements in A.

2.6 THE INCLUSION-EXCLUSION PRINCIPLE

2.6.1 Illustrating the key point. Let us explore a foundational result that builds towards the general Inclusion-Exclusion Principle. Let X be a finite set, with two subsets A and B. We want to find a formula for the number of elements in neither A nor B, which is the cardinality of the set $A^c \cap B^c$. The formula is:

$$|A^c \cap B^c| = |X| - |A| - |B| + |A \cap B|$$

We will demonstrate this with two different proofs.

Proof 1: Using De Morgan's Laws. By De Morgan's laws, the intersection of complements is the complement of the union: $A^c \cap B^c = (A \cup B)^c$. Therefore, the cardinality is

$$|A^c \cap B^c| = |(A \cup B)^c| = |X| - |A \cup B|$$

Now using the fact that $|A \cup B| = |A| + |B| - |A \cap B|$, the desired result is clear.

Proof 2: Using Indicator Functions. This second proof uses the algebraic properties of indicator functions and provides a blueprint for the more general case. We have

$$\begin{aligned} |A^c \cap B^c| = \sum_{x \in X} 1_{A^c \cap B^c}(x) &= \sum_{x \in X} (1_{A^c} \cdot 1_{B^c})(x) \\ &= \sum_{x \in X} 1_{A^c}(x) \cdot 1_{B^c}(x) = \sum_{x \in X} (1 - 1_A(x))(1 - 1_B(x)) \end{aligned}$$

Therefore

$$\begin{aligned}
|A^c \cap B^c| &= \sum_{x\in X}(1 - 1_A(x) - 1_B(x) + 1_A(x)1_B(x)) \\
&= \sum_{x\in X}(1 - 1_A(x) - 1_B(x) + 1_{A\cap B}(x)) \\
&= \sum_{x\in X} 1 - \sum_{x\in X} 1_A(x) - \sum_{x\in X} 1_B(x) + \sum_{x\in X} 1_{A\cap B}(x) \\
&= |X| - |A| - |B| + |A \cap B|
\end{aligned}$$

Both proofs yield the same result, but the second method scales up elegantly to prove the general principle. ♢

2.6.2 The Inclusion-Exclusion Principle. The Inclusion-Exclusion Principle is a powerful counting technique that generalizes the simple formula for the union of two or three sets. It provides a way to find the number of elements in the union or intersection of multiple sets. We see two versions below. Each can be derived from the other.

> **Inclusion-Exclusion (Version 1.)** Let X be a finite set and let $A_1, A_2, \dots, A_n$ be subsets of X. The number of elements in the union of these n sets, namely
>
> $$\left|\bigcup_{i=1}^{n} A_i\right|$$
>
> is given by:
>
> $$\sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n+1}|A_1\cap\cdots\cap A_n|$$

In a more compact form using summation notation:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}|A_{i_1}\cap\cdots\cap A_{i_r}|$$

The formula works by adding the sizes of all the individual sets, then subtracting the sizes of all pairwise intersections, then adding back the sizes of all three-way intersections, and so on, alternating the sign at each step. It can be formally derived from the following version, the proof of which is what will occupy us for the rest of this section.

> **Inclusion-Exclusion (Version 2.)** This version tells us the number of elements that are in none of the sets A_i.
>
> $$\left|\bigcap_{i=1}^{n} A_i^c\right| = |X| - \sum_{i=1}^{n}|A_i| + \sum_{1\le i<j\le n}|A_i\cap A_j| - \cdots + (-1)^n|A_1\cap\cdots\cap A_n|$$

Using summation notation:

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = |X| + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} |A_{i_1}\cap\cdots\cap A_{i_r}|$$

To get Version 1 from Version 2, we just use

$$\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i\right)^c \qquad \Rightarrow \qquad \left|\bigcap_{i=1}^{n} A_i^c\right| = |X| - \left|\bigcup_{i=1}^{n} A_i\right|$$

Before we get into the proof, we need an elementary algebra lemma. $\diamondsuit$

2.6.3 Preparation: An Algebraic Identity. The proof of the Inclusion-Exclusion Principle relies on a general algebraic identity. Let $x_1, x_2, \ldots, x_n$ be real numbers. Consider the product:

$$P_n = (1-x_1)(1-x_2)\cdots(1-x_n)$$

When expanded, this product is equal to:

$$P_n = 1 - \sum_{1\le i\le n} x_i + \sum_{1\le i<j\le n} x_i x_j - \sum_{1\le i<j<k\le n} x_i x_j x_k + \cdots + (-1)^n x_1 x_2\cdots x_n$$

This can be written compactly as:

$$\prod_{i=1}^{n}(1-x_i) = 1 + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}x_{i_2}\cdots x_{i_r}$$

For example, for $n=2$:

$$(1-x_1)(1-x_2) = 1 - x_1 - x_2 + x_1x_2$$

And for $n=3$:

$$\begin{aligned}
(1-x_1)(1-x_2)(1-x_3) &= [(1-x_1)(1-x_2)](1-x_3)\\
&= [1-x_1-x_2+x_1x_2](1-x_3)\\
&= 1(1-x_1-x_2+x_1x_2) - x_3(1-x_1-x_2+x_1x_2)\\
&= 1-x_1-x_2+x_1x_2-x_3+x_1x_3+x_2x_3-x_1x_2x_3\\
&= 1-(x_1+x_2+x_3)+(x_1x_2+x_1x_3+x_2x_3)-x_1x_2x_3
\end{aligned}$$

This pattern holds for any n. We will now prove it formally. We prove the identity by mathematical induction.

$$\prod_{i=1}^{n}(1-x_i) = 1 + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}x_{i_2}\cdots x_{i_r}$$

The base case of the induction is when $n=1$. This holds trivially. Assume the formula is true for some integer $n\ge 1$. We need to prove the formula for $n+1$. Let us consider the product for $n+1$ numbers.

$$\prod_{i=1}^{n+1}(1-x_i) = \left(\prod_{i=1}^{n}(1-x_i)\right)(1-x_{n+1})$$

Now, we apply the inductive hypothesis to the first part of the product to the write the right hand side of the above as:

$$\left(1+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r}\right)(1-x_{n+1})$$

Let us expand this expression by distributing $(1-x_{n+1})$:

$$=\left(1+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r}\right)-x_{n+1}\left(1+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r}\right)$$
$$=\left(1+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r}\right)-\left(x_{n+1}+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} (x_{i_1}\cdots x_{i_r})x_{n+1}\right)$$

Let us analyze the terms. The term $\sum_{1\le i_1<\cdots<i_r\le n+1} x_{i_1}\cdots x_{i_r}$ consists of two types of products:

- Those that do not contain x_{n+1}. These are just products of r terms from $\{x_1,\ldots,x_n\}$.
- Those that do contain x_{n+1}. These are formed by taking a product of $r-1$ terms from $\{x_1,\ldots,x_n\}$ and multiplying by x_{n+1}.

Combining the terms from our expansion: For a given $r\in\{2,\ldots,n\}$, the terms with products of r variables are:

$$(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r} \quad \text{and} \quad -(-1)^{r-1}\sum_{1\le i_1<\cdots<i_{r-1}\le n}(x_{i_1}\cdots x_{i_{r-1}})x_{n+1}$$

Since $-(-1)^{r-1}=(-1)^r$, their sum is:

$$(-1)^r\left(\sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r}+\sum_{1\le i_1<\cdots<i_{r-1}\le n}(x_{i_1}\cdots x_{i_{r-1}})x_{n+1}\right)$$

This combined sum is precisely $(-1)^r\sum_{1\le i_1<\cdots<i_r\le n+1} x_{i_1}\cdots x_{i_r}$. By carefully grouping all terms, we reconstruct the formula for $n+1$:

$$1+\sum_{r=1}^{n+1}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n+1} x_{i_1}\cdots x_{i_r}$$

This completes the induction, and the proof of the algebraic identity is finished. ♢

2.6.4 Proof of the Inclusion-Exclusion Principle (Version 2.) We can now prove the Inclusion-Exclusion Principle (Version 2) using indicator functions and the algebraic identity we just proved. We want to prove:

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = |X|+\sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} |A_{i_1}\cap\cdots\cap A_{i_r}|$$

Proof. We have

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = \sum_{x\in X} 1_{\bigcap_{i=1}^{n} A_i^c}(x) = \sum_{x\in X}\prod_{i=1}^{n} 1_{A_i^c}(x) = \sum_{x\in X}\prod_{i=1}^{n}(1 - 1_{A_i}(x))$$

At this point, for each $x \in X$, the product inside the summation is of the form $\prod_{i=1}^{n}(1-y_i)$ where $y_i = 1_{A_i}(x)$. We can apply the algebraic identity we proved earlier to this product to get

$$\begin{aligned}\left|\bigcap_{i=1}^{n} A_i^c\right| &= \sum_{x\in X}\left(1 + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} 1_{A_{i_1}}(x)\cdots 1_{A_{i_r}}(x)\right)\\ &= \sum_{x\in X}\left(1 + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} 1_{A_{i_1}\cap\cdots\cap A_{i_r}}(x)\right)\end{aligned}$$

Now, we can swap the order of summation (since everything is finite) and get

$$\begin{aligned}\left|\bigcap_{i=1}^{n} A_i^c\right| &= \sum_{x\in X} 1 + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n}\sum_{x\in X} 1_{A_{i_1}\cap\cdots\cap A_{i_r}}(x)\\ &= |X| + \sum_{r=1}^{n}(-1)^r \sum_{1\le i_1<\cdots<i_r\le n} |A_{i_1}\cap\cdots\cap A_{i_r}|\end{aligned}$$

This completes the proof of the Inclusion-Exclusion Principle. ♢

2.7 INJECTIVE FUNCTIONS

Consider the function that assigns to each student in a class their roll number. No two students share the same roll number—so distinct students map to distinct numbers. This is exactly what it means for a function to be *injective*: different inputs always produce different outputs. Notice that injectivity says nothing about whether every possible output is hit—there may be roll numbers or ID numbers that are not assigned to anyone. The focus is solely on the guarantee that no two inputs collide.

Definition. Let $f : X \to Y$ be a function. We say that f is **injective** if it maps distinct elements in the domain to distinct elements in the codomain. That is, for any two elements $x, y \in X$, if $x \neq y$, then $f(x) \neq f(y)$. An injective function is also called a **one-to-one function**.

For the purpose of writing proofs, it is often more convenient to use the contrapositive form of the definition:

$$\text{A function } f \text{ is injective if} \quad (f(x) = f(y) \implies x = y)$$

In words, if two outputs are the same, then their corresponding inputs must have been the same.

Example 2.7.1.

a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n + 1$. To check if it's injective, we assume $f(m) = f(n)$ for some integers m and n.

$$f(m) = f(n) \qquad \Rightarrow \qquad m + 1 = n + 1 \qquad \Rightarrow \qquad m = n.$$

Since $f(m) = f(n)$ implies $m = n$, the function is injective.

b) *A non-injective function.* Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n^2$. This function is *not* injective. We can show this with a counterexample. Let $m = -1$ and $n = 1$. Then $m \neq n$. However,

$$f(-1) = (-1)^2 = 1 \quad \text{and} \quad f(1) = 1^2 = 1.$$

So, $f(-1) = f(1)$, but $-1 \neq 1$. We have found two different inputs that map to the same output. Therefore, f is not injective. (Note: If the domain were restricted to the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, then $g(n) = n^2$ would be injective.)

c) *A function on power sets.* Let X be a non-empty set and let $\mathcal{P}(X)$ be its power set. Consider the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$f(A) = X \setminus A$$

for all $A \in \mathcal{P}(X)$. In words, the function f maps a subset A of X to its complement in X. Let us check for injectivity. Suppose $f(A) = f(B)$ for two subsets $A, B \subseteq X$.

$$X \setminus A = X \setminus B$$

This means the complements are equal. Taking the complement of both sides gives us

$$X \setminus (X \setminus A) = X \setminus (X \setminus B)$$

which simplifies to $A = B$. Thus, f is injective.

2.7.2 A note on cardinality. If $f : X \to Y$ is an injective map between two *finite* sets, then the number of elements in the domain must be less than or equal to the number of elements in the codomain, i.e., $|X| \leq |Y|$. This is because each element of X must map to a unique element of Y. This is a trivial but powerful remark. $\diamondsuit$

2.7.3 An injection from $\mathbb{Z}$ to $\mathbb{N}$. It might seem that the set of integers,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

is "twice as large" as the set of non-negative integers $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. So it might appeal to our intuition that there should not exist any injection from $\mathbb{Z}$ to $\mathbb{N}$, so then it seems like $\mathbb{Z}$ can be "put inside" a seemingly smaller set $\mathbb{N}_0$. However, this intuition is ill-founded, as we demonstrate. Consider the function $f : \mathbb{Z} \to \mathbb{N}_0$ defined as:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$

This function maps non-negative integers to even numbers ($0 \mapsto 0, 1 \mapsto 2, 2 \mapsto 4, \dots$) and negative integers to odd numbers ($-1 \mapsto 1, -2 \mapsto 3, -3 \mapsto 5, \dots$).

We claim that this function f is injective. We want to show that if $f(m) = f(n)$, then we must have $m = n$. An important observation is that if $n \geq 0$, $f(n)$ is an even number. If $n < 0$, $f(n)$ is an odd number. Suppose $f(m) = f(n)$. Since an even number cannot equal an odd number, m and n must have the same sign (or both be zero). This leaves us with two cases.

- *Case 1:* $m \geq 0$ and $n \geq 0$. In this case, $f(m) = 2m$ and $f(n) = 2n$. The equation $f(m) = f(n)$ becomes $2m = 2n$, which implies $m = n$.
- *Case 2:* $m < 0$ and $n < 0$. In this case, $f(m) = -2m - 1$ and $f(n) = -2n - 1$. The equation $f(m) = f(n)$ becomes $-2m - 1 = -2n - 1$, which simplifies to $-2m = -2n$, and thus $m = n$.

In both possible cases, $f(m) = f(n)$ leads to $m = n$. Therefore, the function f is injective. ♢

2.7.4 Injections and compositions. How does injectivity behave with respect to function composition? We discuss two facts in this regard.

a) The composition of two injections is an injection. More precisely, let $f : X \to Y$ and $g : Y \to Z$ be injective functions. Then the composition $g \circ f : X \to Z$ is also an injection.
b) If a composition is injective, the first function must be injective. Explicitly, let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ is injective, then f must be injective.

Proof (a): Suppose $(g \circ f)(x) = (g \circ f)(y)$ for some $x, y \in X$. By definition of composition, this means $g(f(x)) = g(f(y))$. Since g is injective, we can conclude that its inputs must be equal: $f(x) = f(y)$. Now, since f is injective, we can conclude that its inputs must be equal: $x = y$. We have shown that $(g \circ f)(x) = (g \circ f)(y) \implies x = y$, so $g \circ f$ is injective.

Proof (b): We will prove this by contrapositive. The contrapositive statement is: "If f is not injective, then $g \circ f$ is not injective." Assume f is not injective. This means there exist two distinct elements $x, y \in X$ (so $x \neq y$) such that $f(x) = f(y)$. Since $f(x)$ and $f(y)$ are the same element in Y, applying the function g to them must yield the same result:

$$g(f(x)) = g(f(y))$$

This is the same as saying $(g \circ f)(x) = (g \circ f)(y)$. Since we started with $x \neq y$ and found that $(g \circ f)(x) = (g \circ f)(y)$, the function $g \circ f$ is not injective. This completes the proof. ♢

2.7.5 An interesting injection. There is an elegant connection between the power set of a set X and the set of functions from X to $\mathbb{R}$. Let X be a non-empty set. Let $\mathcal{P}(X)$ be its power set and let $\mathrm{Maps}(X, \mathbb{R})$ be the set of all functions from X to $\mathbb{R}$. Consider the map

$$\Phi : \mathcal{P}(X) \to \mathrm{Maps}(X, \mathbb{R})$$

defined by:

$$\Phi(A) = 1_A$$

where 1_A is the indicator function of the set A. This map takes a subset A and gives back a specific real-valued function, its indicator function. We will show that this map Φ is an injection.

Proof: Suppose $\Phi(A) = \Phi(B)$ for some $A, B \in \mathcal{P}(X)$. We want to show that this implies $A = B$. The condition $\Phi(A) = \Phi(B)$ means that the functions are identical: $1_A = 1_B$. This means that for every element $x \in X$, their outputs must be the same: $1_A(x) = 1_B(x)$.

To show that $A = B$, we can argue by contradiction. Assume that $A \neq B$. If the sets are not equal, then one must contain an element that the other does not. Without loss of generality, let's say there is an element $a \in A$ such that $a \notin B$. Now let's evaluate the indicator functions at this element a:

- Since $a \in A$, by definition $1_A(a) = 1$.
- Since $a \notin B$, by definition $1_B(a) = 0$.

But our initial assumption was that $1_A(x) = 1_B(x)$ for all x, which must include a. This leads to the conclusion that $1_A(a) = 1_B(a)$, which means $1 = 0$. This is a clear contradiction. Therefore, our assumption that $A \neq B$ must be false. It must be that $A = B$. This proves that Φ is an injective map. ♢

2.7.6 Partitions and fibers. Let us recall the definition of a 'fiber.' Let $f : X \to Y$ be a function. For any element $y \in Y$ in the codomain, we can look at all the elements in the domain that map to it. This set is called the **fiber** above y, and it is simply the preimage of the set $\{y\}$.

$$\text{Fiber above } y = f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}$$

We often use the shorthand $f^{-1}(y)$. Note that every element of X belongs to exactly one fiber (specifically, the fiber above $f(x)$). We can denote the set of these fibers (the partition) by X/f. This is a non-standard notation. The fiber containing a specific element x is denoted by $[x]_f$. Thus

$$[x]_f = f^{-1}(f(x))$$

The map $\pi : X \to X/f$ defined by $\pi(x) = [x]_f$ will be referred to as the **natural projection.** ♢

2.7.7 Set-theoretic quotienting. When we consider a function $f : X \to Y$, we can think of it as a process that takes an input from set X and produces an output in set Y. However, not all functions behave in the same way regarding the information they preserve. A function can "lose information" when it is not injective. This happens when it maps multiple, distinct elements from the domain X to the very same element in the codomain Y. For instance, if $f(x_1) = y$ and $f(x_2) = y$ for some $x_1 \neq x_2$, and someone tells us the output was y, we have lost the information needed to determine whether the original input was x_1 or x_2. The function has conflated these two distinct inputs. The goal of set-theoretic quotienting is to carefully dissect the function f to understand and isolate this information loss. We want to "factor out" the injective part of the function, which represents its essential, information-preserving core. This deconstruction is achieved in two conceptual steps, embodied by two functions: the projection $\pi : X \to X/f$ and an injective map $\bar{f}$. Let us get into the details of this latter map.

Let $f : X \to Y$ be a function and let X/f be the set of its non-empty fibers. We can define a new map $\bar{f} : X/f \to Y$ as follows:

$$\bar{f}([x]_f) = f(x) \quad \text{for any } x \in X.$$

This map takes an entire fiber (which is a set) and maps it to the single value in Y that all elements of that fiber map to under f. This map $\bar{f}$ has two important properties:

- $\bar{f}$ is an injective function.

  Proof. To prove $\bar{f}$ is injective, we must show that if $\bar{f}([x]_f) = \bar{f}([x']_f)$, then $[x]_f = [x']_f$. By the definition of $\bar{f}$, the condition $\bar{f}([x]_f) = \bar{f}([x']_f)$ means that $f(x) = f(x')$. By the definition of a fiber, $[x]_f = f^{-1}(f(x))$. If $f(x) = f(x')$, then any element in the fiber of x' must also be in the fiber of x, and vice-versa. Therefore, the fibers must be identical: $[x]_f = [x']_f$. This proves that $\bar{f}$ is injective.

- f can be "factored through" π and $\bar{f}$. More precisely, $f = \bar{f} \circ \pi$, that is, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ X/f \arrow[ur, "\bar{f}"] & \end{tikzcd}$$

  Proof. We want to show that $f = \bar{f} \circ \pi$. We check this by applying the composition to an arbitrary element $x \in X$.

$$(\bar{f} \circ \pi)(x) = \bar{f}(\pi(x)) = \bar{f}([x]_f) = f(x)$$

  Since $(\bar{f} \circ \pi)(x) = f(x)$ for all $x \in X$, the functions are equal.

This decomposition shows that any function can be seen as a projection onto its fibers, followed by an injection from the set of fibers into the original codomain. ♢

Exercise 2.7.1. Give an example of functions $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f$ is injective, but g is not.

Exercise 2.7.2. Let $f : X \to Y$ be a function. Show that f is injective if and only if each non-empty fiber of f has a size 1.

Exercise 2.7.3. Let X and Y be sets. Let $\pi : X \times Y \to X$ be the projection map defined as $\pi(x, y) = x$ for all $(x, y) \in X \times Y$. Describe the fibers of π.

Exercise 2.7.4. Let $P = \{n^2 : n \in \mathbb{Z}\}$. Does there exist an injective map from $\mathbb{Z}$ to P?

Exercise 2.7.5. Let F be a finite set. Show that there is an injective map from $\mathbb{N} \times F$ to $\mathbb{N}$.

2.8 SURJECTIVE FUNCTIONS

Consider a classroom where the teacher must assign each student to one of five project groups: Group 1 through Group 5. If the teacher ensures that every group has at least one member, then the assignment function (from students to groups) is *surjective*: every element of the codomain is hit. Another example: the function that maps each day of the year to the day of the week (Monday through Sunday) is surjective, because every day of the week occurs at least once in any year. By contrast, if a vending machine has 20 buttons but only 15 of them dispense a drink (the other 5 are broken), then the "button-to-drink" function is *not* surjective—some elements of the codomain are never reached.

Definition. A function $f : X \to Y$ is said to be **surjective** (or **onto**) if the image of f is equal to the codomain Y. In other words, for every element y in the codomain Y, there exists at least one element x in the domain X such that $f(x) = y$. We can write this formally as:

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y$$

Thus a function is surjective if it "hits" every single element in the codomain. No element in Y is left out.

Example 2.8.1. Let us consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 1$ for all $n \in \mathbb{Z}$. Is this function surjective? To determine this, we need to check if for any integer y in the codomain $\mathbb{Z}$, we can find an integer x in the domain $\mathbb{Z}$ such that $f(x) = y$. Let us take an arbitrary integer $y \in \mathbb{Z}$. We want to find an x such that:

$$f(x) = y, \quad \text{that is} \quad x + 1 = y$$

Solving for x, we get $x = y - 1$. Since y is an integer, $y - 1$ is also an integer. So, for any integer y, we can choose $x = y - 1$, which is in the domain $\mathbb{Z}$. Let us check: $f(y - 1) = (y - 1) + 1 = y$. Since we found a suitable x for any arbitrary y, the function f is indeed surjective.

Example 2.8.2. Now, let's modify the previous example slightly. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n+1$ for all $n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. Is this function surjective?

The codomain is the set of all integers, $\mathbb{Z}$. Let us see if every element in $\mathbb{Z}$ is in the image of f. The domain is the set of natural numbers. The image of f is the set $\{f(1), f(2), f(3), \dots\} = \{2, 3, 4, \dots\}$. Let us pick an element from the codomain, say $y = 0 \in \mathbb{Z}$. Is there an $x \in \mathbb{N}$ such that $f(x) = 0$?

$$x + 1 = 0 \qquad \Rightarrow \qquad x = -1$$

However, $x = -1$ is not in the domain $\mathbb{N}$. Therefore, there is no element in the domain that maps to 0. This means the function is not surjective.

Example 2.8.3. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$. Is this function surjective?

The codomain is $\mathbb{Z}$. Let us examine the image of f. The image consists of the squares of integers: $\{0, 1, 4, 9, 16, \dots\}$. Let us pick an element from the codomain that is not a perfect square, for example, $y = 2$. Is there an integer x such that $f(x) = 2$?

$$x^2 = 2$$

This would mean $x = \sqrt{2}$ or $x = -\sqrt{2}$. Neither of these are integers. So, there is no x in the domain $\mathbb{Z}$ that maps to 2. Therefore, the function is not surjective.

2.8.4 A note on finite sets. Let X and Y be finite sets and let $f : X \to Y$ be a surjective map. What can we say about the sizes of these sets, denoted by $|X|$ and $|Y|$? Since every element in Y must be mapped to by at least one element in X, the domain X must have at least as many elements as the codomain Y. Therefore, we can conclude that:

$$|X| \geq |Y|$$

Example 2.8.5. Let X be a non-empty set. Consider the function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $f(A) = X \setminus A$ for all $A \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.

Let us prove that this function is surjective. We need to show that for any set $B \in \mathcal{P}(X)$, there exists a set $A \in \mathcal{P}(X)$ such that $f(A) = B$. Let B be an arbitrary subset of X (i.e., $B \in \mathcal{P}(X)$). We want to find an A such that $X \setminus A = B$. Let us try choosing $A = X \setminus B$. Since B is a subset of X, $X \setminus B$ is also a subset of X, so $A = X \setminus B$ is in $\mathcal{P}(X)$. Now let's compute $f(A)$:

$$f(A) = f(X \setminus B) = X \setminus (X \setminus B)$$

The complement of the complement of a set is the set itself. So, $X \setminus (X \setminus B) = B$. We have found a set A (namely, $X \setminus B$) that maps to B. Since B was arbitrary, this holds for all sets in $\mathcal{P}(X)$. Thus, f is surjective.

2.8.6 An interesting surjection. This discussion is just a mirror of Point 2.7.3. We show that there exists a surjective function from the set of non-negative integers

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$ to the set of all integers $\mathbb{Z}$. Let us define a function $f : \mathbb{N}_0 \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We leave the justification of the surjectivity of f as an exercise to the reader. $\diamondsuit$

2.8.7 Surjections and composition. Let us discuss how the notion of surjectivity interacts with the operation of composition. As with injectivity, we discuss two facts in this regard.

a) Let $f : X \to Y$ and $g : Y \to Z$ be surjective functions. Then the composition $g \circ f : X \to Z$ is also surjective.

b) Let $f : X \to Y$ and $g : Y \to Z$ be functions. If their composition $g \circ f : X \to Z$ is surjective, then g must be surjective.

Proof (a): To prove that $g \circ f$ is surjective, we must show that for any arbitrary element $z \in Z$, there exists an element $x \in X$ such that $(g \circ f)(x) = z$.

Let $z \in Z$ be an arbitrary element. Since $g : Y \to Z$ is surjective, we know that there exists an element $y \in Y$ such that $g(y) = z$. Now, since $f : X \to Y$ is surjective, for this particular $y \in Y$, we know there exists an element $x \in X$ such that $f(x) = y$. Now we can compose the functions:

$$(g \circ f)(x) = g(f(x))$$

Substituting $f(x) = y$, we get:

$$g(f(x)) = g(y)$$

And we know that $g(y) = z$. Therefore:

$$(g \circ f)(x) = z$$

We have successfully found an $x \in X$ that maps to our arbitrary $z \in Z$. Thus, $g \circ f$ is surjective.

Proof (b): To prove that g is surjective, we must show that for any arbitrary element $z \in Z$, there exists an element $y \in Y$ such that $g(y) = z$.

Let $z \in Z$ be an arbitrary element. We are given that $g \circ f : X \to Z$ is surjective. This means that for our chosen z, there must exist an element $x \in X$ such that $(g \circ f)(x) = z$. By definition of composition, this means $g(f(x)) = z$.

Let us define $y = f(x)$. Since f is a function from X to Y, this element y must be in Y. Now we can substitute y back into our equation: $g(y) = z$. So, for our arbitrary $z \in Z$, we have found an element $y \in Y$ (namely, $y = f(x)$) that maps to z. Therefore, g is surjective. Note that this does not imply that f is surjective. $\diamondsuit$

2.8.8 A "functor" (Optional.) Let $f : X \to Y$ be a surjective function. Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ be the power sets of X and Y. We define a function

$$f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$$

as follows. For any subset $B \subseteq Y$, we define $f^*(B) = f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. We show that if f is surjective, then f^* is injective.

Proof: To prove that f^* is injective, we must show that if $f^*(B_1) = f^*(B_2)$ for some $B_1, B_2 \in\subseteq Y$, then it must be that $B_1 = B_2$. Assume $f^*(B_1) = f^*(B_2)$. This means $f^{-1}(B_1) = f^{-1}(B_2)$. To show $B_1 = B_2$, we need to show $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$. We will only show the first part, as the second is identical and is left as an exercise.

Let y_1 be an arbitrary element of B_1. Our goal is to show that y_1 must also be in B_2. Since $f : X \to Y$ is surjective, and $y_1 \in Y$, there must exist some $x_1 \in X$ such that $f(x_1) = y_1$. Since $f(x_1) = y_1$ and $y_1 \in B_1$, it follows that $f(x_1) \in B_1$. By the definition of the preimage, if $f(x_1) \in B_1$, then $x_1 \in f^{-1}(B_1)$. From our initial assumption, we know $f^{-1}(B_1) = f^{-1}(B_2)$. So, x_1 must also be in $f^{-1}(B_2)$. By definition of the preimage again, if $x_1 \in f^{-1}(B_2)$, then $f(x_1)$ must be in B_2. Since $f(x_1) = y_1$, this means $y_1 \in B_2$.

Since we started with an arbitrary $y_1 \in B_1$ and showed it must be in B_2, we have proven that $B_1 \subseteq B_2$. The argument for $B_2 \subseteq B_1$ is symmetric. Therefore, $B_1 = B_2$, and the function f^* is injective. ♢

2.8.9 Connection between injectivity and surjectivity. For non-empty sets X and Y, the following two statements are equivalent:

a) There exists an injective function from X to Y.

b) There exists a surjective function from Y to X.

This is a very important result that connects the two main properties of functions we have studied. It essentially says that if you can embed X into Y without elements of X collapsing, you can map Y onto X covering all of X. We give the formal details.

(a)⇒(b): Let $f : X \to Y$ be an injective function. We want to construct a surjective function from Y to X. Choose an element x_0 in X arbitrarily. Define a map $g : Y \to X$ as follows. If $y \in \text{Image}(f)$, then define $g(y) = x$, where x is the unique element in X such that $f(x) = y$. If $y \notin \text{Image}(f)$, then define $g(y)$ to be x_0. The reader is invited to check that g is a surjection.

(b) ⇒ (a): Let $g : Y \to X$ be a surjection. We want to construct an injection from X to Y. For each $x \in X$, pick an element y_x in the fiber of g above x. This can be done since the surjectivity of g ensures that no fiber of g is empty. Define a function $f : X \to Y$ as $f(x) = y_x$ for all $x \in X$. The reader is encouraged to check that f is indeed an injection. ♢

2.8.10 A simple fact about counting. Let X and Y be finite sets, and let $f : X \to Y$ be a surjective function. If we assume that each fiber of f has the same size, say k, then we can establish a direct relationship between the sizes of the sets. So, if $|f^{-1}(y)| = k$ for all $y \in Y$, then the total number of elements in X is the number of elements in Y multiplied by the size of each fiber.

$$|X| = k \cdot |Y|$$

This is because the fibers partition the set X into $|Y|$ disjoint subsets, each of size k. ♢

2.8.11 Cantor. Cantor's famous theorem states that there is no surjection from a given set to its power set. More precisely:

> **Theorem.** Let X be a non-empty set. Then there is no surjection from X to $\mathcal{P}(X)$.

This has the profound consequence that a power set is always "larger" than the set itself.

Proof. Suppose, for the sake of contradiction, that there exists a surjective function $f : X \to \mathcal{P}(X)$. Since f is a function from X to $\mathcal{P}(X)$, for any element $x \in X$, $f(x)$ is a subset of X. This allows us to ask a peculiar question: is the element x a member of the set $f(x)$?

Let us construct a special subset of X. Let us call it S. We define S to be the set of all elements x in X that are *not* members of their own image.

$$S = \{x \in X \mid x \notin f(x)\}$$

Since S is a collection of elements from X, S is a subset of X, which means $S \in \mathcal{P}(X)$. Since f was assumed to be surjective, every subset of X is in the image of f. In particular, there must exist some element $x^* \in X$ such that $f(x^*) = S$. Now we ask: is this element x^* in the set S? We have two possibilities, and both will lead to a contradiction.

- *Case 1:* Assume $x^* \in S$.

  By the very definition of S, if an element is in S, it must satisfy the condition $x^* \notin f(x^*)$. But we know $f(x^*) = S$. So, if $x^* \in S$, it must be that $x^* \notin S$. This is a contradiction.

- *Case 2:* Assume $x^* \notin S$.

  By the definition of S, if an element x^* is not in S, it must fail the condition for membership. The condition is $x^* \notin f(x^*)$. The negation of this condition is $x^* \in f(x^*)$. But again, we know $f(x^*) = S$. So, if $x^* \notin S$, it must be that $x^* \in S$. This is also a contradiction.

Since both possibilities lead to a logical contradiction, our initial assumption must be false. Therefore, there cannot be a surjective function from a set X to its power set $\mathcal{P}(X)$. ♢

Exercise 2.8.1. Find an example of an injection $f : A \to B$ and a surjection $g : B \to C$ such that $g \circ f$ is neither injective nor surjective.

Exercise 2.8.2. Find an example of a surjective function $f : A \to B$ and a function $g : B \to C$ such that $g \circ f$ is neither injective nor surjective.

Exercise 2.8.3. Let X be a finite set. Does there exist a function $f : X \to X$ that is injective but not surjective? Or surjective but not injective? What if X were infinite?

Exercise 2.8.4. Decide if the following functions are injective, surjective, or both (bijective).

- $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$.
- $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + n$.
- $f : \mathbb{Q} \setminus \{3\} \to \mathbb{Q}$ defined by $f(x) = \frac{2x+3}{x-3}$.

Exercise 2.8.5. **[CMI 2021 Part B]** Let f be a function from domain S to codomain T. Let g be another function from domain T to codomain U. For each of the blanks below choose a single letter corresponding to one of the four options listed underneath. (It is not necessary that each choice is used exactly once.)

- If $g \circ f$ is one-to-one then f and g ____________.
- If $g \circ f$ is onto then f and g ____________.

(a) must be one-to-one and must be onto respectively.

(b) must be one-to-one but need not be onto respectively.

(c) need not be one-to-one but must be onto respectively.

(d) need not be one-to-one and need not be onto respectively.

Exercise 2.8.6. **[CMI 2024 Part A]** Suppose f is a function whose domain is X and codomain is Y. It is given that $|X| > 1$ and $|Y| > 1$. No other information is known about X, Y and f.

For the statement S mentioned in the (a), (b), (c) and (d) below, answer which of the statements 1-12 apply.

a) $S =$ For each x in X, there exists y in Y such that $f(x) = y$.
b) $S =$ For each y in Y, there exists x in X such that $f(x) = y$.
c) $S =$ There exists a unique x in X such that for each y in Y it is true that $f(x) = y$.
d) $S =$ There exists a unique y in Y such that for each x in X it is true that $f(x) = y$.

1. S is always true.
2. S is always false.
3. S is true if and only if f is one-to-one.
4. If S is true then f is one-to-one but the converse is false.
5. If f is one-to-one then S is true but the converse is false.
6. S is true if and only if f is onto.
7. If S is true then f is onto but the converse is false.
8. If f is onto then S is true but the converse is false.
9. S is true if and only if f is a constant function.
10. If S is true then f is a constant function but the converse is false.
11. If f is a constant function then S is true but the converse is false.
12. None of the above.

Exercise 2.8.7. [CMI 2013 Part A] For sets A and B, let $f : A \to B$ and $g : B \to A$ be functions such that $f(g(x)) = x$ for each x. For each statement below, write whether it is TRUE or FALSE.

a) The function f must be one-to-one.
b) The function f must be onto.
c) The function g must be one-to-one.
d) The function g must be onto.

2.9 BIJECTIVE FUNCTIONS

Consider a dance class where every student is paired with exactly one partner, and every partner is paired with exactly one student. The pairing function is both injective (no two students share a partner) and surjective (no partner is left out). This is a *bijection*—a perfect one-to-one correspondence. Another example: the function that maps each seat in a fully occupied examination hall to the student sitting in it. Every seat has exactly one student (injective), and every student has a seat (surjective). A bijection is, in effect, a relabelling: it tells us that the two sets are "the same size" and can be perfectly matched up.

Definition. A function $f : X \to Y$ is said to be **bijective** if it is both injective and surjective. A bijective function is also called a **one-to-one correspondence**. This name is very descriptive because a bijection pairs up every element of X with exactly one element of Y, and every element of Y is paired with exactly one element of X.

If there is a bijection $f : X \to Y$, we can think of the set Y as a perfect "copy" or "relabeling" of the set X. For finite sets, this has a direct consequence on their sizes. If $f : X \to Y$ is a bijection and X and Y are finite, then they must have the same number of elements.

$$|X| = |Y|$$

We use the notation $X \cong Y$ to say that there exists a bijective function from X to Y.

Example 2.9.1. Let us revisit the function from the previous section, $f : \mathbb{N}_0 \to \mathbb{Z}$ defined as:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

We have already shown this function is surjective. Let us check if it is injective. Suppose $f(n_1) = f(n_2)$.

- If $f(n_1) = f(n_2) \geq 0$, then both n_1 and n_2 must be even. So $n_1/2 = n_2/2$, which implies $n_1 = n_2$.
- If $f(n_1) = f(n_2) < 0$, then both n_1 and n_2 must be odd. So $-(n_1 + 1)/2 = -(n_2 + 1)/2$, which implies $n_1 + 1 = n_2 + 1$, and thus $n_1 = n_2$.

An odd number cannot map to the same value as an even number, since evens map to non-negative integers and odds map to negative integers. Thus, the function is injective. Since it is both injective and surjective, it is a bijection. This shows that $\mathbb{N}_0 \cong \mathbb{Z}$.

Example 2.9.2. Let X be a set of size $n \geq 1$. For any integer k with $0 \leq k \leq n$, let $\binom{X}{k}$ denote the set of all subsets of X that have size k. Let us define a function

$$f : \binom{X}{k} \to \binom{X}{n-k}$$

as

$$f(S) = X \setminus S$$

for $S \in \binom{X}{k}$. Let us show f is a bijection.

Injectivity: Suppose $f(S_1) = f(S_2)$ for some $S_1, S_2 \in \binom{X}{k}$. This means $X \setminus S_1 = X \setminus S_2$. Taking the complement of both sides gives $S_1 = S_2$. So, f is injective.

Surjectivity: Let T be an arbitrary set in the codomain $\binom{X}{n-k}$. We need to find a set $S \in \binom{X}{k}$ such that $f(S) = T$. Let us choose $S = X \setminus T$. The size of S is

$$|X| - |T| = n - (n-k) = k$$

So S is in the domain $\binom{X}{k}$. Now,

$$f(S) = f(X \setminus T) = X \setminus (X \setminus T) = T$$

Since we found a suitable S for any T, f is surjective.

Example 2.9.3. Let $n \geq 1$ be an integer. Let D_n be the set of positive divisors of n. For example, $D_6 = \{1, 2, 3, 6\}$. Define a function $f : D_n \to D_n$ by $f(d) = n/d$. Let us show this is a bijection.

Injectivity: Suppose $f(d_1) = f(d_2)$ for $d_1, d_2 \in D_n$. Then $n/d_1 = n/d_2$, which implies $d_1 = d_2$. So f is injective.

Surjectivity: Let $d' \in D_n$ be an arbitrary element in the codomain. We need to find a $d \in D_n$ such that $f(d) = d'$. Let us choose $d = n/d'$. Since d' is a divisor of n, $d = n/d'$ is also an integer and a divisor of n. So $d \in D_n$. Then

$$f(d) = f(n/d') = n/(n/d') = d'$$

So f is surjective.

An interesting consequence is that if we sum over the elements of D_n, we can replace each element with its image under f. For example,

$$\sum_{d \in D_n} d = \sum_{d \in D_n} f(d) = \sum_{d|n} \frac{n}{d}$$

Example 2.9.4. To further nail down the notion of a bijection, let us look at two geometric examples. The reader not familiar with the Cartesian plane may skip these for now.

a) *Stereographic projection.* Consider the circle of radius 1 centered at $(0,1)$ in the plane. This circle passes through the origin and has its "north pole" at $N = (0,2)$. Remove the north pole to get a *punctured circle*

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1\} \setminus \{N\}$$

We claim there is a bijection $f : C \to \mathbb{R}$, where, by a slight abuse of notation, we are using $\mathbb{R}$ to denote the x-axis.

The idea is simple: given a point P on C, draw the line through N and P. Since $P \neq N$, this line is not vertical (check this!), so it meets the x-axis at exactly one point. Call that point $f(P)$. This map is injective because two different points on the circle give two different lines through N, which hit the x-axis in different places. It is surjective because for any point Q on the x-axis, the line through N and Q meets the circle at exactly one point other than N—that point is the preimage of Q. Figure 2.7 illustrates this construction.

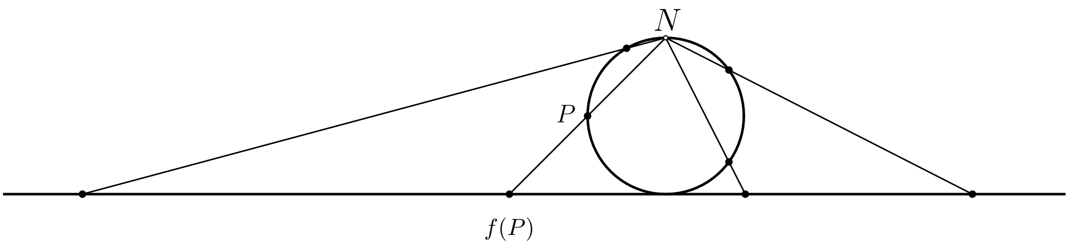


Figure 2.7: Stereographic projection: each point P on the punctured circle maps to the intersection of the line NP with the x-axis.

b) *Circle to lines through the origin.* Here is another example. Consider again the circle of radius 1 centered at $(0,1)$. This circle passes through the origin $O = (0,0)$. Let

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1\}$$

and let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$ that pass through the origin.

For each point $P \in C$ with $P \neq O$, the line through O and P is well-defined and belongs to $\mathcal{L}$. What about the point O itself? We cannot draw "the line through O and O," but there is a natural candidate: the *tangent* to the circle at O, which is nothing by the x-axis. So we define $g : C \to \mathcal{L}$ by

$$g(P) = \begin{cases} \text{the line through } O \text{ and } P & \text{if } P \neq O, \\ \text{the tangent to } C \text{ at } O & \text{if } P = O. \end{cases}$$

This map is a bijection. It is injective: if two *distinct* points $P_1, P_2 \in C$, both different from O, gave the same line ℓ, then ℓ would meet the circle in three points (P_1, P_2, and O), which is impossible for a line and a circle. And no $P \neq O$ maps to the x-axis either, because the x-axis meets the circle only at O (substitute $y = 0$ into $x^2 + (y-1)^2 = 1$ to check). It is surjective: any line ℓ through O either meets the circle at a second point $P \neq O$—in which case $g(P) = \ell$—or is tangent to the circle at O—in which case $g(O) = \ell$. Figure 2.8 shows the construction.

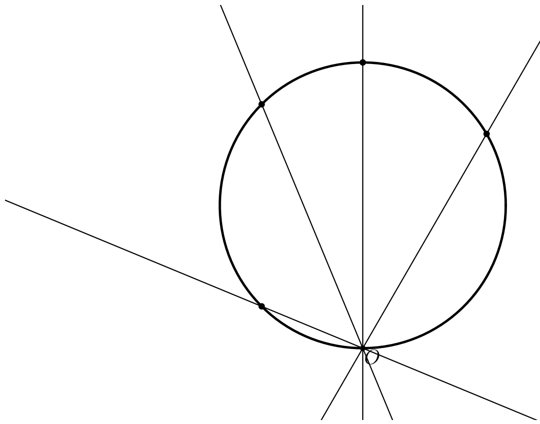


Figure 2.8: Each point P on the punctured circle determines a unique line through the origin.

2.9.5 Number of odd and even size subsets. The following result is usually proved using the binomial theorem. But here we give a more direct proof.

Lemma. Let X be a set of size n. Then the number of subsets of X of even size is same as the number of subsets of X of odd size.

Proof. Let $x \in X$ be an arbitrary element of X. Define a map $f : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$f(S) = S\Delta\{x\}$$

In words, we put x in S if it is not present in S, and take away x if it is present in S. This is a beautiful map. It is easy to verify that $f \circ f$ is the identity map on $\mathcal{P}(X)$, and it follows that f is a bijection. Further, f takes an odd sized subset to an even sized subset. The desired conclusion follows. ♢

2.9.6 Composition of bijections. Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Then their composition $g \circ f : X \to Z$ is also a bijection. To prove this, we need to show that $g \circ f$ is both injective and surjective. But we have already seen that the composition of two injections is an injection and the composition of two surjections is again a surjection. Thus $g \circ f$ is both an injection and a surjection. ♢

2.9.7 Inverse of a bijective function. One of the most important properties of a bijection is that it is "reversible." This leads to the concept of an inverse function.

Definition. The **inverse** of f, denoted $f^{-1} : Y \to X$, is defined as follows: For any

$y \in Y$, $f^{-1}(y)$ is the unique element $x \in X$ such that $f(x) = y$.

Let $f : X \to Y$ be a bijection. Why must f be a bijection for this to work?

- If f were not surjective, there would be some $y \in Y$ with no corresponding x. We wouldn't know what to define $f^{-1}(y)$ as.
- If f were not injective, there would be some $y \in Y$ that is the image of multiple x's (say x_1 and x_2). We wouldn't know whether to define $f^{-1}(y)$ as x_1 or x_2.

A bijection guarantees that for every $y \in Y$, there is exactly one $x \in X$ that maps to it, so the inverse is well-defined. Bijective functions are therefore also called **invertible functions**. Here are two quick examples.

a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(n) = n + 1$. This is a bijection. To find its inverse $f^{-1} : \mathbb{Z} \to \mathbb{Z}$, we set $y = f(x)$, so $y = x + 1$. Solving for x gives $x = y - 1$. So, the inverse function is $f^{-1}(y) = y - 1$, or using n as the variable, $f^{-1}(n) = n - 1$.

b) Let $f : D_n \to D_n$ be the divisor function $f(d) = n/d$. To find the inverse, we set $y = f(x)$, so $y = n/x$. Solving for x gives $x = n/y$. The inverse function is $f^{-1}(y) = n/y$. In this case, the function is its own inverse: $f = f^{-1}$.

Lemma. Let $f : X \to Y$ be a bijection. A function $g : Y \to X$ is the inverse of f if and only if

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y$$

Here id_X is the identity function on X, that is, $\mathrm{id}_X(x) = x$ for all $x \in X$.

Proof. This is an "if and only if" statement, so we must prove two directions.

($\Rightarrow$) Assume $g = f^{-1}$. We must show the compositions result in the identity functions.

- Consider $g \circ f : X \to X$. For any $x \in X$, let $y = f(x)$. By definition of the inverse, $g(y) = f^{-1}(y) = x$. So, $(g \circ f)(x) = g(f(x)) = g(y) = x$. Since this holds for all $x \in X$, we have $g \circ f = \mathrm{id}_X$.
- Consider $f \circ g : Y \to Y$. For any $y \in Y$, let $x = g(y) = f^{-1}(y)$. By definition of the inverse, this means $f(x) = y$. So, $(f \circ g)(y) = f(g(y)) = f(x) = y$. Since this holds for all $y \in Y$, we have $f \circ g = \mathrm{id}_Y$.

($\Leftarrow$) Assume $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. We must show that g is the inverse of f. We need to show that for any $y \in Y$, $g(y)$ is the unique element $x \in X$ such that $f(x) = y$. Let $y \in Y$. Let $x = g(y)$. Applying f to both sides gives

$$f(x) = f(g(y)) = (f \circ g)(y)$$

Since $f \circ g = \mathrm{id}_Y$, we have $f(x) = \mathrm{id}_Y(y) = y$. So we have found an x that maps to y. Is this x unique? Suppose there is another element $x' \in X$ such that $f(x') = y$. Then $g(f(x')) = g(y) = x$. But since $g \circ f = \mathrm{id}_X$, we have $g(f(x')) = \mathrm{id}_X(x') = x'$. Therefore, $x' = x$. The element is unique. This shows that g fits the definition of f^{-1}. ■ ◇

2.9.8 A note of caution. It is important not to confuse the notation for the inverse function, f^{-1}, with the notation for the preimage of a set. If $f : X \to Y$ is any function (not necessarily invertible) and $B \subseteq Y$, the preimage of B is written as:

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

This notation is used even when the inverse function f^{-1} does not exist. If f is a bijection, then the preimage of a single-element set, $f^{-1}(\{y\})$, is a set containing the single element $f^{-1}(y)$. ♢

2.9.9 An interesting puzzle. We seek to show that there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that for any proper, non-empty subset $S \subset \mathbb{N}$ (that is, $S \neq \emptyset$ and $S \neq \mathbb{N}$), the image of the set is not equal to the set itself, i.e., $f(S) \neq S$.

The solution involves a clever use of bijections to transfer a simple structure from the integers $\mathbb{Z}$ to the natural numbers $\mathbb{N}$. Let $\varphi : \mathbb{N} \to \mathbb{Z}$ be a bijection. We know that such a bijection exists. Using this bijection, we can "transfer" the problem from $\mathbb{N}$ to $\mathbb{Z}$.

Define a function $g : \mathbb{Z} \to \mathbb{Z}$ by $g(n) = n + 1$. This is a simple "shift" map. For example, if T is finite, $g(T)$ has the same number of elements but is shifted. If T is infinite, like the set of even numbers, $g(T)$ is the set of odd numbers. For any proper, non-empty subset $T \subset \mathbb{Z}$, it's clear that $g(T) \neq T$.

We now construct the desired function f by composing the functions we have $f = \varphi^{-1} \circ g \circ \varphi$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbb{N} \\ \mathbb{Z} \arrow[r, "g"] & \mathbb{Z} \arrow[u, "\varphi^{-1}"'] \end{tikzcd}$$

This function f is a composition of three bijections (φ, g, and φ^{-1}), so f itself is a bijection. Now, let S be a proper, non-empty subset of $\mathbb{N}$. Let $T = \varphi(S)$ be its image in $\mathbb{Z}$. Since φ is a bijection, T is a proper, non-empty subset of $\mathbb{Z}$. We know that $g(T) \neq T$. The image of S under f is $f(S) = (\varphi^{-1} \circ g \circ \varphi)(S) = \varphi^{-1}(g(\varphi(S))) = \varphi^{-1}(g(T))$. Since $g(T) \neq T$ and φ^{-1} is a bijection, it must be that $\varphi^{-1}(g(T)) \neq \varphi^{-1}(T)$. And since $\varphi^{-1}(T) = S$, we have $f(S) \neq S$. This completes the argument. ♢

2.9.10 A mundane exercise (Optional.) Let X, X', Y and Y' be non-empty sets. Assume there are bijections between X and X' and between Y and Y'. In notation, we can express this as

$$X \cong X' \quad \text{and} \quad Y \cong Y'$$

We want to show that the set of all functions from X to Y, denoted $\mathrm{Maps}(X, Y)$, is in bijection with the set of all functions from X' to Y', i.e., $\mathrm{Maps}(X, Y) \cong \mathrm{Maps}(X', Y')$.

This is a mundane fact because it is entirely obvious. Since $X \cong X'$ and $Y \cong Y'$, we can think of X' and Y' as copies of X and Y respectively. Thus $\mathrm{Maps}(X', Y')$ ought to be a copy of $\mathrm{Maps}(X, Y)$. Below we establish this formally.

Let $\varphi : X \to X'$ and $\psi : Y \to Y'$ be the given bijections. We need to construct a bijection $\alpha : \text{Maps}(X, Y) \to \text{Maps}(X', Y')$. For any function $f \in \text{Maps}(X, Y)$, we define its image $\alpha(f)$ to be the function $\psi \circ f \circ \varphi^{-1}$.

$$\begin{tikzcd} X' \arrow[r, "\alpha(f)"] \arrow[d, "\varphi^{-1}"] & Y' \\ X \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd}$$

This new function maps from X' to Y', so it is in $\text{Maps}(X', Y')$. We show α is a bijection.

Injectivity: Suppose $\alpha(f) = \alpha(g)$ for two functions $f, g \in Maps(X, Y)$.

$$\psi \circ f \circ \varphi^{-1} = \psi \circ g \circ \varphi^{-1}$$

We can compose on the left with ψ^{-1} and on the right with φ:

$$\psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi = \psi^{-1} \circ (\psi \circ g \circ \varphi^{-1}) \circ \varphi$$

Using associativity and the fact that $\psi^{-1} \circ \psi = \text{id}_Y$ and $\varphi^{-1} \circ \varphi = \text{id}_X$:

$$(\text{id}_Y \circ f) \circ \text{id}_X = (\text{id}_Y \circ g) \circ \text{id}_X$$

This simplifies to $f = g$. Thus, α is injective.

Surjectivity: Let f' be an arbitrary function in $\text{Maps}(X', Y')$. We need to find a function $f \in \text{Maps}(X, Y)$ such that $\alpha(f) = f'$. Let us construct a candidate for f: let $f = \psi^{-1} \circ f' \circ \varphi$. This function maps from X to Y, so it is in $\text{Maps}(X, Y)$. Now let's apply α to this f:

$$\begin{aligned} \alpha(f) = \psi \circ f \circ \varphi^{-1} &= \psi \circ (\psi^{-1} \circ f' \circ \varphi) \circ \varphi^{-1} \\ &= (\psi \circ \psi^{-1}) \circ f' \circ (\varphi \circ \varphi^{-1}) = \text{id}_{Y'} \circ f' \circ \text{id}_{X'} = f' \end{aligned}$$

We have found a function f that maps to f', so α is surjective. $\diamondsuit$

2.9.11 Two natural bijections (optional.) Let X, Y and Z be non-empty sets. There exist "natural" bijections that are fundamental in many areas of mathematics.

a) *Product of Codomains:* There is a natural bijection between

$$\text{Maps}(X, Y \times Z) \quad \text{and} \quad \text{Maps}(X, Y) \times \text{Maps}(X, Z)$$

We do not give too many details. But here is the key idea. A function $h : X \to Y \times Z$ can be seen as a pair of functions (f, g), where $f : X \to Y$ and $g : X \to Z$. The relationship is given by $h(x) = (f(x), g(x))$. The functions f and g are simply the compositions of h with the projection maps π_Y and π_Z. This correspondence is a bijection.

b) *Product of Domains (Currying):* There is a natural bijection between

$$\text{Maps}(X \times Y, Z) \quad \text{and} \quad \text{Maps}(X, \text{Maps}(Y, Z))$$

This important bijection, often called "currying", says that a function of two variables, $h(x, y)$, can be thought of as a function of one variable, x, that returns a function of the second variable, y. We encourage the reader to mull over it and write a proof of this beautiful statement.

Exercise 2.9.1. [UGA 2015] Let $d_1, d_2, \ldots, d_k$ be all the factors of a positive integer n including 1 and n. If $d_1 + d_2 + \ldots + d_k = 72$, then

$$\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}$$

is (a) $\frac{k^2}{72}$ (b) $\frac{72}{k}$ (c) $\frac{72}{n}$ (d) none of the above.

Exercise 2.9.2. [UGA 2020] The number of subsets of $\{1, 2, 3, \ldots, 10\}$ having an odd number of elements is

(a) 1024 (b) 512 (c) 256 (d) 50 .

2.10 COMBINATORIAL APPLICATIONS (OPTIONAL)

Theorem 2.10.1 ***Number of Injective Functions.*** Let X and Y be finite sets with $|X| = k$ and $|Y| = n$. Assume $n \geq k$. Let $\text{Inj}(X, Y)$ denote the set of all the injective functions from X to Y. Then

$$|\operatorname{Inj}(X, Y)| = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = {}^nP_k$$

Clearly, $|\operatorname{Inj}(X, Y)|$ depends only on $|X|$ and $|Y|$ and not on the details of X and Y.

Intuitive Explanation: Let us build an injective function. Let $X = \{x_1, \ldots, x_k\}$. For $f(x_1)$, we have n choices in Y. For $f(x_2)$, we can't reuse the value of $f(x_1)$, so we have $n-1$ choices left. For $f(x_3)$, we have $n-2$ choices left. ... For $f(x_k)$, we have $n-(k-1) = n-k+1$ choices left. By the multiplication principle, the total number of ways is the product of the number of choices at each step.

Rigorous Proof: We proceed by setting up a recurrence relation. Let $I(n, k)$ be the number of injective functions from a set of size k to a set of size n. Let X and Y be sets of size k and n respectively. Let $x \in X$ be an arbitrary fixed element in X. For each $y \in Y$, let $\text{Inj}_y(X, Y)$ denote the set of all the injective maps from X to Y *which take* x

to y. Note that $\mathrm{Inj}(X,Y)$ is just the union of all the $\mathrm{Inj}_y(X,Y)$. In symbols

$$\mathrm{Inj}(X,Y) = \bigcup_{y\in Y} \mathrm{Inj}_y(X,Y)$$

Also, $\mathrm{Inj}_{y_1}(X,Y)$ and $\mathrm{Inj}_{y_2}(X,Y)$ are disjoint if $y_1 \neq y_2$. Thus, from the above we have

$$|\,\mathrm{Inj}(X,Y)| = \sum_{y\in Y} \mathrm{Inj}_y(X,Y) \tag{*}$$

For any $y \in Y$ we construct a bijection between $\mathrm{Inj}_y(X,Y)$ and $\mathrm{Inj}(X \setminus \{x\}, Y \setminus \{y\})$.[1] Fix $y \in Y$. For each $f \in \mathrm{Inj}_y(X,Y)$, define a map $\tilde{f} : X \setminus \{x\} \to Y \setminus \{y\}$ by restricting f to $X \setminus \{x\}$. Explicitly

$$\tilde{f}(x') = f(x')$$

for all $x' \in X \setminus \{x\}$. Define a map $\varphi : \mathrm{Inj}_y(X,Y) \to \mathrm{Inj}(X \setminus \{x\}, Y \setminus \{y\})$ as $\varphi(f) = \tilde{f}$ for all $f \in \mathrm{Inj}_y(X,Y)$. The reader is invited to show that φ is a bijection. Therefore

$$|\,\mathrm{Inj}_y(X,Y)| = I(k-1, n-1)$$

for all $y \in Y$. Using (*) we get

$$I(n,k) = n \cdot I(n-1, k-1)$$

We can unroll this recurrence to get

$$I(n,k) = n \cdot (n-1) \cdot I(n-2, k-2) = \cdots = n(n-1)\cdots(n-k+1) \cdot I(n-k, 0)$$

There is only one function from the empty set (size 0), the empty function, and it is trivially injective. So $I(n-k,0) = 1$. This gives the final formula

$$I(n,k) = n(n-1)\cdots(n-k+1) = {}^nP_k$$

and we are done.

Corollary 2.10.2. Let X and Y be sets of the same size, $|X| = |Y| = n$. A function $f : X \to Y$ is injective if and only if it is surjective. Therefore, the set of bijections $\mathrm{Bij}(X,Y)$ is the same as the set of injections $\mathrm{Inj}(X,Y)$. The number of bijections is:

$$|\,\mathrm{Bij}(X,Y)| = {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

2.10.3 Binomial coefficients. Let X be a set of size n. Let $0 \le k \le n$ and let $\binom{X}{k}$ denote the set of all the subsets of X of size k. The size of $\binom{X}{k}$, evidently, depends only on $|X|$ (and k) and does not depend on the details of X. We denote this size as $\binom{n}{k}$ and this is pronounced as "n choose k." By Point 2.9.2 we see that

$$\binom{n}{k} = \binom{n}{n-k}$$

which is a well-known combinatorial identity. ♢

[1] The existence of such a bijection should be intuitively obvious.

Theorem 2.10.4. Let $n \geq 1$ and $0 \leq k \leq n$. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{{}^nP_k}{k!}$$

Proof. Let $X = \{1, \ldots, k\}$ and $Y = \{1, \ldots, n\}$. We know $|\operatorname{Inj}(X, Y)| = {}^nP_k$. Let us define a function

$$\Psi : \operatorname{Inj}(X, Y) \to \binom{Y}{k}$$

as

$$\Psi(f) = \operatorname{Image}(f)$$

for all $f \in \operatorname{Inj}(X, Y)$. Since any injective function from a k-element set has an image of size k, this function is well-defined. Note that Ψ is surjective: Any k-element subset $A \subseteq Y$ can be obtained as the image of an element of $\operatorname{Inj}(X, Y)$. Indeed, we can simply define a bijection from X to A, which gives an injection from X to Y.

Now, let's consider the fibers of Ψ. A fiber of Ψ corresponding to a set $A \in \binom{Y}{k}$ is the set of all injective functions from X to Y that have A as their image. This is just the set of all bijections from X to A. Since $|X| = |A| = k$, the number of such bijections is $k!$. So, every fiber of the surjective map Ψ has size $k!$. Now Point 2.8.10 yields that

$$|\operatorname{Inj}(X, Y)| = \left|\binom{Y}{k}\right| \cdot k!$$

Substituting the known values:

$${}^nP_k = \binom{n}{k} \cdot k!$$

Rearranging gives the desired formula. ∎

Lemma 2.10.5 ***Number of Surjective Functions.*** Let X and Y be finite sets with $|X| = n$ and $|Y| = k$. Assume $n \geq k$. Let $\operatorname{Sur}(X, Y)$ denote the set of all the surjective functions from X to Y. Then

$$|\operatorname{Sur}(X, Y)| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Proof. Idea: We start with the set of all possible functions from X to Y, which is k^n. From this, we subtract the functions that are *not* surjective. A function is not surjective if it misses at least one element of Y. Let $Y = \{y_1, \ldots, y_k\}$. Let A_i be the set of functions that do not have y_i in their image. We want to find the size of the set of functions that are not in any A_i, which is $|(A_1 \cup \cdots \cup A_k)^c|$. By PIE, this is $|\operatorname{Maps}(X, Y)| - |A_1 \cup \cdots \cup A_k|$. The formula expands to:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots + (-1)^{k-1}\binom{k}{k-1}(1)^n$$

This is precisely the summation formula given above.

Rigorous Proof: A function is not surjective if its image misses at least one element of the codomain Y. Let $Y = \{y_1, \ldots, y_k\}$. For each $i \in \{1, \ldots, k\}$, let A_i be the set of functions $f : X \to Y$ such that y_i is not in the image of f. The set of all non-surjective functions is $A_1 \cup A_2 \cup \cdots \cup A_k$. We want to find the size of the complement of this set.

$$|\operatorname{Sur}(X,Y)| = |\operatorname{Maps}(X,Y)| - |A_1 \cup A_2 \cup \cdots \cup A_k|$$

By the Principle of Inclusion-Exclusion:

$$|A_1 \cup \cdots \cup A_k| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_l| - \ldots$$

The size of a set like $A_{i_1} \cap \cdots \cap A_{i_j}$ is the number of functions that miss all of the j elements $\{y_{i_1}, \ldots, y_{i_j}\}$. These are functions from X to a set of size $k - j$. The number of such functions is $(k-j)^n$. There are $\binom{k}{j}$ ways to choose which j elements to miss. So the j-th term in the sum is $\binom{k}{j}(k-j)^n$. Putting it all together:

$$|\operatorname{Sur}(X,Y)| = k^n - \left[\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \ldots\right]$$

This is equivalent to the summation formula stated above. ■

Example 2.10.6. Let us use the formula for $|\operatorname{Sur}(X,Y)|$ where $|X| = n$ and $|Y| = k$.

- $k = 1$: $|\operatorname{Sur}(X,Y)| = \binom{1}{0}(1-0)^n = 1$. This makes sense; there is only one function to a set with one element, and it must be surjective (if X is non-empty).
- $k = 2$: $|\operatorname{Sur}(X,Y)| = \binom{2}{0}(2-0)^n - \binom{2}{1}(2-1)^n = 2^n - 2$. This is the total number of functions (2^n) minus the two constant functions that miss one of the elements.
- $k = 3$: $|\operatorname{Sur}(X,Y)| = \binom{3}{0}(3-0)^n - \binom{3}{1}(3-1)^n + \binom{3}{2}(3-2)^n = 3^n - 3 \cdot 2^n + 3$.

2.11 CHALLENGING PROBLEMS

Problem 2.11.1. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Show that the set $\{n \in \mathbb{N} : \ \sigma(n) \geq n\}$ is infinite.

Problem 2.11.2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be function. Show that there are infinitely many integers x such that the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by

$$g(x) = f(x) + cx$$

for all $x \in Z$, is not a bijection. Hint: Target the failure or injectivity of g.

Problem 2.11.3. [CMI 2011 Part B] A function g from a set X to itself satisfies $g^m = g^n$ for positive integers m and n with $n > m$. Here g^n stands for $g \circ g \circ \cdots \circ g$ (n times). Show that g is injective if and only if g is surjective.

Problem 2.11.4. Let n be a positive integer and $X = \{1, \ldots, n\}$. Let $f : X \to X$ be a bijection such that $f(k) - k$ takes the same value for all k. Show that $f(k) = k$ for all k.

Problem 2.11.5. [CMI 2019 Part B] For a natural number n denote by $\text{Map}(n)$ the set of all functions

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

Count the number of functions $f \in \text{Map}(n)$ such that $f \circ f = f$.

2.12 SOLUTIONS TO CHALLENGING PROBLEMS

Solution ***to Problem 2.11.1.*** Assume on the contrary that the set $\{n \in \mathbb{N} : \sigma(n) \geq n\}$ is finite. Then there is a natural number N such that $\sigma(n) \leq n - 1$ for all $n > N$. Let

$$M = \max\{\sigma(n) : \ 1 \leq n \leq N\}$$

Let $S = \{1, \ldots, M\}$. We claim that $\sigma(k) \in S$ for all $k \in S$. Indeed, if $\sigma(k) \notin S$ for some $k \in S$, then $\sigma(k) > M$, and hence, in particular, $\sigma(k) > k$. Thus k cannot exceed N. But then, by definition of M, we have $\sigma(k) \leq M$ and hence $\sigma(k) \in S$. This proved the claim. Therefore $\sigma(M + 1)$ cannot lie in S, giving $\sigma(M + 1) \geq M + 1$. Noting that $M + 1 > N$, we get a contradiction. ■

Solution ***to Problem 2.11.3.*** First suppose that g is injective. We show that g is surjective. Let x in X be arbitrary. We have

$$g^m(x) = g^n(x) \qquad \Rightarrow \qquad g(g^{m-1}(x)) = g(g^{n-1}(x)) \qquad \Rightarrow \qquad g^{m-1}(x) = g^{n-1}(x)$$

Continuing this way, we obtain

$$x = g^{n-m}(x)$$

But since $n > m$, the above gives that $x = g(g^{n-m-1}(x))$ and hence x is in the image of g. This shows surjectivity of g.

Now assume that g is surjective. We want to show that g is injective. Suppose that $g(x) = g(y)$ for some $x, y \in X$. We will show that $x = y$ to finish the proof. Since g is surjective, so is g^m. So we can find a and b in X such that $g^m(a) = x$ and $g^n(b) = y$. Now

$$\begin{aligned} g(x) = g(y) \qquad &\Rightarrow \qquad g(g^m(a)) = g(g^m(b)) \\ &\Rightarrow \qquad g^{m+1}(a) = g^{m+1}(b) \\ &\Rightarrow \qquad g^{n-m-1}(g^{m+1}(a)) = g^{n-m-1}(g^{m+1}(b)) \\ &\Rightarrow \qquad g^n(a) = g^n(b) \\ &\Rightarrow \qquad g^m(a) = g^m(b) \end{aligned}$$

where we have used the fact that $g^n = g^m$. But since $x = g^m(a)$ and $y = g^m(b)$, we deduce that $x = y$ and we are done. ■

Solution ***to Problem 2.11.4.*** Say $f(k) - k = c$ for all k. Then

$$\sum_{k \in X} (f(k) - k) = \sum_{k \in X} c = cn \qquad \Rightarrow \qquad \sum_{k \in X} f(k) - \sum_{k \in X} k = cn$$

But note that since $f : X \to X$ is a bijection, we have

$$\sum_{k \in X} f(k) = \sum_{k \in X} k$$

So we deduce that $cn = 0$, and hence $c = 0$. This shows that $f(k) = k$ for all k. ■

Solution ***to Problem 2.11.5***. We are looking for the number of functions f such that $f(f(x)) = f(x)$ for all x. As we established in part Illustration 2.3.8, this condition implies that every element in the image of f acts as a fixed point. In fact, the converse is also true: if every element in the image of f is a fixed point of f, then f satisfies $f \circ f = f$. We leave the proof of the converse direction to the reader.

Now let $\mathcal{F}$ be the set of all the elements f in $\mathrm{Map}(n)$ that satisfy $f \circ f = f$. We partition $\mathcal{F}$ as follows: For each $A \subseteq \{1, \ldots, n\}$, let $\mathcal{F}_A$ be the set of all those elements in $\mathcal{F}$ whose image is exactly A. It is clear that

$$|\mathcal{F}| = \sum_{A \subseteq \{1,\ldots,n\}} |\mathcal{F}_A| \tag{*}$$

Let us find the value of $|\mathcal{F}_A|$ for a fixed $A \subseteq \{1, \ldots, n\}$. Any element in f in $\mathcal{F}_A$ must satisfy

1. For every element $a \in A$, we must have $f(a) = a$.
2. For every element $x \notin A$, the output $f(x)$ must be some element in A (otherwise the image would be larger than A).

Conversely, any $f \in \mathrm{Map}(n)$ that satisfies the above two is necessarily in $\mathcal{F}_A$. So to construct an element in $\mathcal{F}_A$, we map the elements inside A to themselves, and the $n - |A|$ elements outside A, each can be mapped to any of the $|A|$ elements in A. Thus we have

$$|\mathcal{F}_A| = |A|^{n-|A|}$$

which by (*) yields

$$|\mathcal{F}| = \sum_{A \subseteq \{1,\ldots,n\}} |A|^{n-|A|} = \sum_{k=1}^{n} \left[\sum_{\substack{A \subseteq \{1,\ldots,n\} \\ |A|=k}} |A|^{n-|A|} \right] = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}$$

Note: This sum is a known expansion related to forests of rooted trees on n labeled vertices, often associated with Cayley's formula. ■

CHAPTER 3

RELATIONS

3.1 BASICS

In our daily lives, we constantly work with the idea of relationships. For example, we can talk about the relationship between a person and their mother, a city and the country it's in, or a number and another number that is smaller than it. These are all intuitive ideas. The goal in mathematics is to take such an intuitive concept and give it a precise, unambiguous definition. How can we capture the essence of a "relationship" in a formal way?

Let us think about what all these examples have in common. Each one connects an object from one collection (a set) to an object from another collection (which could even be the same collection).

- The "is the mother of" relation connects people to other people.
- The "is the capital of" relation connects cities (like Paris) to countries (like France).
- The "is less than" relation connects numbers (like 3) to other numbers (like 5).

In each case, the relationship can be completely described by listing every single pair of objects for which the relationship holds true. For instance, the "is the capital of" relation is fully defined by the set of pairs

$$\{(\text{Paris, France}), (\text{London, UK}), (\text{New Delhi, India}), \dots\}$$

This observation is the key to the formal definition. If we want to define a relation between two sets, say X and Y, the first step is to consider all possible pairings of elements, where the first element of the pair comes from X and the second comes from Y. This giant collection of all possible pairs is precisely the **Cartesian product**, denoted $X \times Y$.

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Now, any specific relationship we want to define is just a way of selecting a specific group of these pairs—namely, the ones for which the relationship is true. And what is a "selection" or a "group" of elements from a larger set? It is simply a **subset**. This leads us directly to the formal definition.

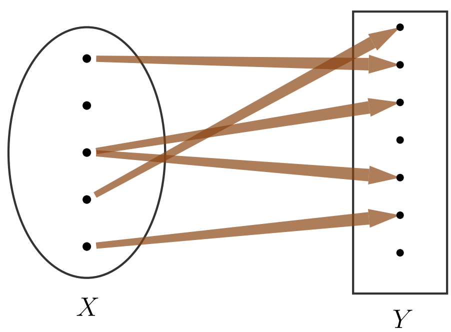
Definition. A **relation** from a set X to a set Y is a subset of the Cartesian product $X \times Y$.

3.1.1 Notation. Let R be a relation from a set X to a set Y. Let $x \in X$ and $y \in Y$. An alternate way of writing $(x, y) \in R$ is

$$xRy$$

We can read this as "x is related to y under R." ♢

3.1.2 Visualizing a relation. Just like we can visualize a function via a "dot and arrow" diagram, we can do the same with relations. If R is a relation from X to Y, then we draw two sets of dots, one for X and one for Y. We draw an arrow from the dot corresponding to $x \in X$ to the dot corresponding to $y \in Y$ if and only if $(x, y) \in R$.



Note crucially that some of the dots corresponding to elements in X may not feature any arrow associated with them. Also, the dot corresponding to an element in X may have multiple arrows associated to it. The same comments hold for Y. ♢

Example 3.1.3. Let us make this concrete with an example. Consider two sets, X and Y:

$$X = \{1, 4, 9\}$$

$$Y = \{2, 4, 6, 8, 10\}$$

Now, let's define a relation R from X to Y with the rule "x is greater than y". Formally, we write this as:

$$R = \{(x, y) \in X \times Y : x > y\}$$

To build the set R, we systematically check every possible pair (x, y) from $X \times Y$ to see if it satisfies the condition $x > y$.

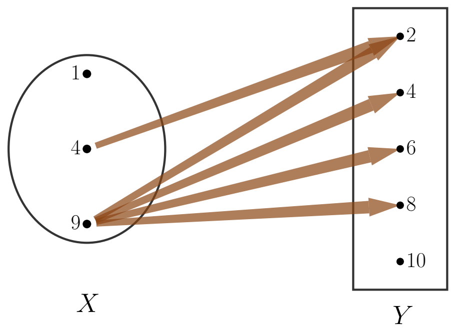
- Let us start with $x = 1$. Are there any elements y in the set Y for which $1 > y$? No, all elements in Y are greater than 1.

- Now, let's take $x = 4$. We look for elements in Y that are less than 4. The only such element is $y = 2$. Thus, the pair $(4, 2)$ is in our relation R. We can write $4R2$.
- Finally, for $x = 9$. Which elements in Y are less than 9? These are $2, 4, 6$, and 8. So, the pairs $(9, 2), (9, 4), (9, 6)$, and $(9, 8)$ are all included in R.

So, our relation R is the set of these specific ordered pairs:

$$R = \{(4, 2), (9, 2), (9, 4), (9, 6), (9, 8)\}$$

This relationship can be visualized with a diagram, where we draw the sets X and Y and then draw an arrow from an element $x \in X$ to an element $y \in Y$ if and only if $(x, y) \in R$.



3.1.4 Relation on a set. While relations between two different sets are common, it is often the case that we are interested in relationships between elements of the *same* set. This is a crucial special case.

Definition. Let X be a set. A relation **on** X is simply a relation from the set X to itself. In other words, a relation on X is a subset of the Cartesian product $X \times X$.

When we have a relation on a set, then, when depicting the relation by means of a "dot and arrow" diagram, we need not draw two sets of dots. We may show only one set of dots (to depict the set X) and we draw an arrow from x_1 to x_2 if and only $(x_1, x_2) \in R$. ♢

Example 3.1.5. Let us consider the set X containing all integers from 2 to 17, inclusive.

$$X = \{2, 3, 4, \ldots, 16, 17\}$$

We can define a "divides" relation, let's call it D, on this set X. Let us specify the rule for our relation D as follows: a pair (a, b) from $X \times X$ is in D if a divides b evenly, and as an additional condition, a must be strictly less than b.

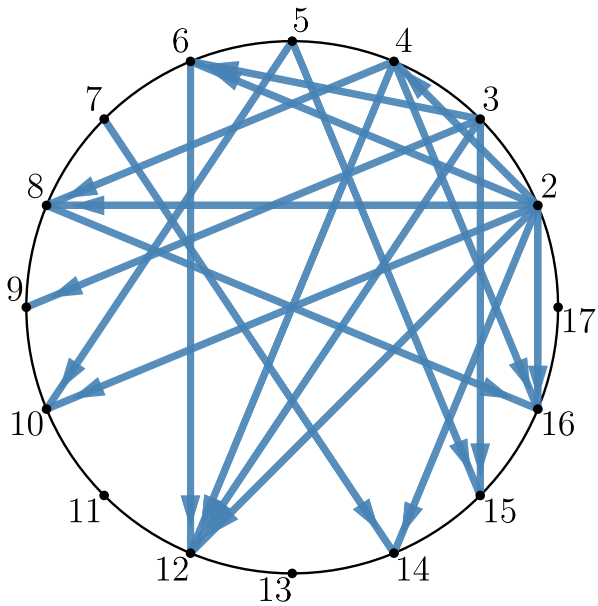
$$D = \{(a, b) \in X \times X : a \text{ divides } b, \text{ and } a < b\}$$

Let us find some elements that belong to this relation D:

- The pair $(2, 4)$ is in D because 2 divides 4 and $2 < 4$.
- The pair $(2, 6)$ is in D because 2 divides 6 and $2 < 6$.
- The pair $(3, 9)$ is in D because 3 divides 9 and $3 < 9$.
- The pair $(4, 16)$ is in D because 4 divides 16 and $4 < 16$.

Conversely, let's look at some pairs that are *not* in D.

- The pair $(4, 2)$ is not in D. While 2 divides 4, our relation requires the first element to divide the second. 4 does not divide 2. Furthermore, the condition $a < b$ is not met.
- The pair $(3, 7)$ is not in D because 3 does not divide 7.
- The pair $(5, 5)$ is not in D. Even though 5 divides 5, our relation includes the strict inequality $a < b$, which is not satisfied here.



The diagram above captures the relation R. Since the relation R is a relation from X to itself, we have not drawn two sets of dots, but rather only one set of dots for X, as no confusion can arise.

Relations are a fundamental and unifying concept in mathematics. They appear in many different contexts. Here are a few more examples to showcase their versatility.

Example 3.1.6 (Divisibility on integers)**.** This is a natural extension of our previous example, but now on infinite sets. Let $X = \mathbb{Z} \setminus \{0\}$ (the set of all non-zero integers) and let $Y = \mathbb{Z}$ (the set of all integers). We can define the general divisibility relation R as:

$$R = \{(x, y) \in X \times Y : x \text{ divides } y\}$$

For example, $(3, 9) \in R$, $(-2, 8) \in R$, and $(5, 0) \in R$ (since any non-zero integer divides 0). However, $(3, 10) \notin R$.

Example 3.1.7 (Relative primality.)**.** Let the set be $X = \mathbb{N}$, the set of natural numbers $\{1, 2, 3, \dots\}$. We can define a relation R on X where two numbers are related if they are "relatively prime" (or "coprime"), meaning they share no common prime factors.

$$R = \{(a, b) \in X \times X : \text{there is no prime number that divides both } a \text{ and } b\}$$

For example, $(8, 9) \in R$. The prime factors of 8 are just $\{2\}$, and the prime factors of 9 are just $\{3\}$. Since there is no overlap, they are relatively prime. In contrast, $(6, 9) \notin R$ because the prime number 3 is a factor of both 6 and 9.

We can note a couple of interesting properties of this relation:

- The relation is symmetric: if $(a, b) \in R$, then it must also be that $(b, a) \in R$. The definition of sharing common factors does not depend on the order of a and b.
- For any natural number $b \in \mathbb{N}$, the pair $(1, b)$ is always in R. This is because the number 1 has no prime factors, so it is impossible for it to share a prime factor with any other number.

Example 3.1.8 (Every function is a relation.)**.** This is a key insight. Every function can be understood as a special kind of relation. Let $f : X \to Y$ be a function. The **graph** of this function is the set of all pairs $(x, f(x))$ where x is in the domain X. We can define a relation R_f that is precisely this graph.

$$R_f = \{(x, y) \in X \times Y : y = f(x)\}$$

What makes a relation a function? It is a special property: for every element $x \in X$, there is *exactly one* pair (x, y) in the relation R_f. This uniqueness is the defining characteristic of a function. Most relations are not functions because an element x can be related to multiple elements y.

Example 3.1.9 (Points and lines in the plane)**.** Relations are not just for numbers. They can be defined between sets of geometric objects. Let P be the set of all points in the Euclidean plane. Let L be the set of all lines in the Euclidean plane. We can define an "incidence" relation R that connects points to the lines they lie on.

$$R = \{(p, l) \in P \times L : \text{the line } l \text{ passes through the point } p\}$$

So, the statement $(p, l) \in R$ is just a formal way of saying that the point p lies on the line l.

We can connect this relation to a fundamental axiom of geometry: If we take any two distinct points $p_1, p_2 \in P$ (with $p_1 \neq p_2$), then there exists a *unique* line $l \in L$ such that both $(p_1, l) \in R$ and $(p_2, l) \in R$. This is the formal relational way of stating that two distinct points uniquely determine a line.

3.1.10 Number of relations. Let X and Y be finite sets with $|X| = m$ and $|Y| = n$. The set of all the relations from X to Y is nothing but $\mathcal{P}(X \times Y)$—the power set of $X \times Y$. Thus the number of relations from X to Y is 2^{mn}. $\diamondsuit$

Exercise 3.1.1. Let $R = \{(1,3),(4,2),(2,4),(2,3),(3,1)\}$ be a relation on the set $A = \{1,2,3,4\}$. The relation R is :

(a) a function (b) reflexive (c) not symmetric (d) transitive

Exercise 3.1.2. Consider the following relations on the set of natural numbers $\mathbb{N}$:

a) $R_1 = \{(a,b) : a \leq b\}$
b) $R_2 = \{(a,b) : a = b\}$
c) $R_3 = \{(a,b) : a > b\}$
d) $R_4 = \{(a,b) : a + b \leq 3\}$

For each of these relations, determine which of the ordered pairs $(1,1),(1,3),(2,4)$, and $(2,1)$ it contains.

3.2 EQUIVALENCE RELATIONS

Definition. We say that a relation R on a set X is **reflexive** if for every element $x \in X$, the pair (x,x) is in R. In our shorthand notation, this means xRx for all $x \in X$.

Example 3.2.1 (Inequality.)**.** Let our set be $X = \mathbb{N}$, the natural numbers. Consider the relation

$$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} : a \leq b\}$$

This relation is reflexive because for any natural number a, it is always true that $a \leq a$. Therefore, the pair (a,a) is in R for all $a \in \mathbb{N}$.

Example 3.2.2 (Divisibility.)**.** Let $X = \mathbb{N}$ again. Consider the divisibility relation

$$D = \{(a,b) \in \mathbb{N} \times \mathbb{N} : a \text{ divides } b\}$$

This relation is also reflexive. For any natural number $a \in \mathbb{N}$, it is true that a divides itself (since $a = 1 \cdot a$). Thus, $(a,a) \in D$ for all $a \in \mathbb{N}$.

Example 3.2.3 (A non-reflexive relation: Relative primality.)**.** Let $X = \mathbb{N}$ and consider the relation of relative primality,

$$R = \{(a,b) : \gcd(a,b) = 1\}$$

This relation is *not* reflexive. To show this, we only need to find a single counterexample. Consider the number $2 \in \mathbb{N}$. The pair $(2,2)$ is not in R because $\gcd(2,2) = 2 \neq 1$. In fact, $(n,n) \notin R$ for any $n > 1$.

3.2.4 Number of reflexive relations. Let X be a finite set with n elements. How many different reflexive relations can we define on X? A relation on X is a subset of $X \times X$. The set $X \times X$ contains n^2 ordered pairs. For a relation R to be reflexive, it *must* contain all the "diagonal" pairs (x, x) for every $x \in X$. There are n such pairs. So, these n pairs are fixed; they must be in our subset. The remaining pairs are the "off-diagonal" ones, (x, y) where $x \neq y$. There are $n^2 - n$ of these. For each of these off-diagonal pairs, we have two choices: it can either be in the relation or not. Since there are $n^2 - n$ such pairs, the total number of ways to make these choices is 2^{n^2-n}. This is the total number of reflexive relations on a set of size n. ♢

Definition. We say a relation R on a set X is **symmetric** if for any pair of elements $a, b \in X$, *if* (a, b) is in R, then (b, a) must also be in R. In other words, if aRb, then it must be that bRa.

Example 3.2.5 (Unit circle.)**.** Let our set be $X = \mathbb{R}$, the set of all real numbers. Define a relation

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

The reader familiar with the basics of coordinate geometry will immediately see that this relation describes points on the unit circle (with origin as the center) in the Cartesian plane. This relation is symmetric. To see why, suppose that a pair (x, y) is in R. This means, by definition, that $x^2 + y^2 = 1$. Since addition is commutative, we can swap the terms to get $y^2 + x^2 = 1$. But this is precisely the condition for the pair (y, x) to be in R. So, if $(x, y) \in R$, then $(y, x) \in R$.

Example 3.2.6 (Non-example: Divisibility.)**.** Let $X = \mathbb{N}$ and consider the divisibility relation $D = \{(a, b) : a \text{ divides } b\}$. This relation is not symmetric. For a counterexample, consider the pair $(2, 4)$. We know $(2, 4) \in D$ because 2 divides 4. However, the reverse pair $(4, 2)$ is not in D, because 4 does not divide 2.

3.2.7 Number of symmetric relations. Let X be a set with n elements. To count the number of symmetric relations, we can again think about which pairs from $X \times X$ we can choose. Let us partition the n^2 pairs into two types:

- *Diagonal pairs:* (x, x). There are n of these. For a symmetric relation, there are no restrictions on these pairs. A pair (x, x) can be in the relation or not. This gives 2^n choices for the diagonal elements.
- *Off-diagonal pairs:* (x, y) where $x \neq y$. These pairs can be grouped into sets of the form $\{(x, y), (y, x)\}$. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ such sets. For the relation to be symmetric, for each such set, we have two choices: either we include *both* pairs in our relation, or we include *neither*. We cannot include just one. So, for each of the $\binom{n}{2}$ sets of pairs, we have 2 choices. This gives $2^{\binom{n}{2}}$ choices for the off-diagonal elements.

The total number of symmetric relations is the product of these choices:

$$2^n \times 2^{\binom{n}{2}} = 2^{n+\frac{n(n-1)}{2}} = 2^{\frac{2n+n^2-n}{2}} = 2^{\frac{n(n+1)}{2}}$$ ♢

Definition. A relation R on a set X is **transitive** if for any elements $a, b, c \in X$, whenever $(a, b) \in R$ and $(b, c) \in R$, it must also be that $(a, c) \in R$.

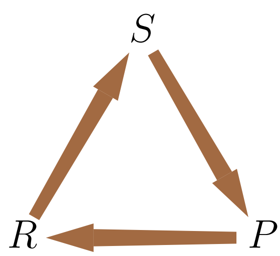
Example 3.2.8 (Divisibility)**.** Let $X = \mathbb{N}$ and D be the divisibility relation, that is,

$$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} : \ a \text{ divides } b\}$$

Then D is transitive. To prove this, suppose $(a, b) \in D$ and $(b, c) \in D$. By definition, this means a divides b and b divides c. This implies there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 b$. Substituting the first equation into the second gives $c = k_2(k_1 a) = (k_1 k_2)a$. Since $k_1 k_2$ is an integer, this shows that a divides c. Therefore, $(a, c) \in D$.

Example 3.2.9 (Non-example: Rock-Paper-Scissors)**.** Let $X = \{\text{rock, paper, scissors}\}$. Define the "defeats" relation

$$D = \{(\text{paper, rock}), (\text{scissors, paper}), (\text{rock, scissors})\}$$



This relation is not transitive. For instance, we have (scissors, paper) $\in D$ and (paper, rock) $\in D$. For D to be transitive, we would need (scissors, rock) to be in D. But it is not; in fact, the reverse is true.

3.2.10 Number of transitive relations. Unlike for reflexive and symmetric relations, there is no simple, "good" formula for the number of transitive relations on a set of size n. This is a well-known difficult combinatorial problem. ♢

We now come to a very important type of relation that combines all three properties we have just discussed. Equivalence relations are fundamental because they capture the essence of what it means for different objects to be "the same" in some particular respect.

Definition. A relation R on a set X is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Example 3.2.11 (Equality Modulo n)**.** Let our set be $X = \mathbb{Z}$, the set of all integers. Fix a natural number $n > 1$. We define a relation R_n on $\mathbb{Z}$ as follows:

$$(x, y) \in R_n \quad \text{if and only if} \quad n \text{ divides } (x - y).$$

This is a very famous equivalence relation. Let us prove it by checking the three properties.

- *Reflexive:* For any integer $x \in \mathbb{Z}$, we have $x - x = 0$. Since n divides 0 (as $0 = 0 \cdot n$), we have $(x, x) \in R_n$. So, R_n is reflexive.
- *Symmetric:* Suppose $(x, y) \in R_n$. This means n divides $(x - y)$. So, $x - y = kn$ for some integer k. Multiplying by -1, we get $y - x = (-k)n$.

  Since $-k$ is also an integer, this shows that n divides $(y - x)$. Therefore, $(y, x) \in R_n$. So, R_n is symmetric.
- *Transitive:* Suppose $(x, y) \in R_n$ and $(y, z) \in R_n$. This means n divides $(x - y)$ and n divides $(y - z)$. So, there exist integers k and l such that $x - y = kn$ and $y - z = ln$. If we add these two equations, we get $(x - y) + (y - z) = kn + ln$, which simplifies to $x - z = (k + l)n$. Since $k + l$ is an integer, this shows that n divides $(x - z)$. Therefore, $(x, z) \in R_n$. So, R_n is transitive.

Since R_n is reflexive, symmetric, and transitive, it is an equivalence relation.

Example 3.2.12 (Parallel Lines)**.** Let X be the set of all lines in the Euclidean plane. Let R be the relation defined by $(l, m) \in R$ if line l is parallel to line m. (We consider a line to be parallel to itself). This is an equivalence relation.

- *Reflexive:* Any line l is parallel to itself.
- *Symmetric:* If line l is parallel to line m, then line m is parallel to line l.
- *Transitive:* If line l is parallel to line m, and line m is parallel to line p, then line l is parallel to line p.

Example 3.2.13 (The Fiber Relation)**.** This is a powerful way to generate equivalence relations from functions. Let $f : X \to Y$ be any function from a set X to a set Y. We can define a relation on the domain X based on the values of the function. The **fiber relation**, let's call it R_f, on the set X is defined as follows: two elements $a, b \in X$ are related if they map to the same element in Y.

$$aR_f b \quad \text{if and only if} \quad f(a) = f(b).$$

We claim that for any function f, the relation R_f is an equivalence relation. To this end we must show that R_f is reflexive, symmetric, and transitive.

- *Reflexive:* Let a be any element in X. Since equality is reflexive, it is always true that $f(a) = f(a)$. By the definition of R_f, this means $(a, a) \in R_f$. Thus, R_f is reflexive.
- *Symmetric:* Let $a, b \in X$ and suppose $(a, b) \in R_f$. This means $f(a) = f(b)$. Since equality is symmetric, this implies $f(b) = f(a)$. By definition, this means $(b, a) \in R_f$. Thus, R_f is symmetric.
- *Transitive:* Let $a, b, c \in X$ and suppose $(a, b) \in R_f$ and $(b, c) \in R_f$. By definition, this means $f(a) = f(b)$ and $f(b) = f(c)$. Since equality is transitive, this implies $f(a) = f(c)$. By definition, this means $(a, c) \in R_f$. Thus, R_f is transitive.

Since R_f satisfies all three properties, it is an equivalence relation. This is a very general and useful result. It tells us that any function naturally "partitions" its domain into sets of elements that are considered "equivalent" because they share the same output value.

Exercise 3.2.1. Let S be a set and let X be the power set of S (the set of all subsets of S). Let a relation R on X be defined as $R = \{(A, B) \in X \times X : A \subseteq B\}$. Is this relation i) reflexive, ii) symmetric, iii) transitive?

Exercise 3.2.2. Let T be the set of all triangles in the Euclidean plane. Let C be a relation on T defined such that $(T_1, T_2) \in C$ if and only if triangle T_1 is congruent to triangle T_2. Show that C is an equivalence relation.

Exercise 3.2.3. Let X be a set. A relation R on X is called **anti-symmetric** if for any $a, b \in X$, whenever $(a, b) \in R$ and $(b, a) \in R$, it must be that $a = b$. If X is a set of size n, how many anti-symmetric relations are there on X?

Exercise 3.2.4. Let X be the set of all the maps from $\mathbb{N}$ to $\{0, 1\}$. In other words, $X = \text{Maps}(\mathbb{N}, \{0, 1\})$. Define a relation $\sim$ on X as follows: for f and g in X, we write $f \sim g$ if

$$\text{there exists } N \geq 1 \text{ such that } f(n) = g(n) \text{ for all } n \geq N$$

Show that $\sim$ is an equivalence relation.

Exercise 3.2.5. Define a relation $\sim$ on $\mathbb{R} \setminus \mathbb{Q}$ by writing $a \sim b$ if a/b is a rational number. Show that $\sim$ is an equivalence relation.

Exercise 3.2.6. **[CMI 2020 Part B]** This exercise is optional if the reader is not yet initiated in the subject of enumerative combinatorics. Let S and T be non-empty sets.

- A relation R on S is said to be **antisymmetric** if it satisfies the following condition: if (a, b) is in R, then (b, a) must NOT be in R.
- We say that a relation from S to T has **no isolated elements** if each s in S is related to some t in T and if for each t in T, some s in S is related to t.
- We say that a relation R from $\{1, 2, \ldots k\}$ to $\{1, 2, \ldots n\}$ is **non-crossing** if the following never happens: (i, x) and (j, y) are both in R with $i < j$ but $x > y$.

i) For $S = \{1, 2, \ldots, k\}$, how many antisymmetric relations are there from S to S ?

ii) Write a recurrence equation for $f(k, n) =$ the number of non-crossing relations from $\{1, 2, \ldots k\}$ to $\{1, 2, \ldots n\}$ that have no isolated elements in either set. (See below for the definitions of the two underlined terms and their visual meaning. Drawing pictures may be useful.) Your recurrence should have only a fixed number of terms on the RHS.

iii) Using your recurrence in (ii) or otherwise, find a formula for $f(3, n)$.

Visual meaning: one can visualise a relation R very similarly to a function. List 1 to k as dots arranged vertically in increasing order on the left and similarly list 1 to n on the right.

For each (s,t) in R, draw a straight line segment from s on the left to t on the right. In the situation one wants to avoid for non-crossing relations, the segments connecting i with x and j with y would cross. Having no isolated elements also has an obvious visual meaning.

3.3 EQUIVALENCE RELATIONS AND PARTITIONS

Definition. Let X be a set. A **partition** of X is a collection π of non-empty subsets of X (called "blocks" or "parts") such that:

a) The union of all the subsets in π is the entire set X. (This means every element of X belongs to at least one block of the partition).

b) Any two distinct subsets in π are disjoint. That is, if $A, B \in \pi$ and $A \neq B$, then their intersection $A \cap B = \emptyset$. (This means every element of X belongs to *exactly one* block).

Example 3.3.1 (Even and Odd Integers)**.** Let $X = \mathbb{Z}$, the set of integers. Let E be the set of all even integers and O be the set of all odd integers. The collection $\pi = \{E, O\}$ is a partition of $\mathbb{Z}$.

Example 3.3.2 (The Trivial Partition)**.** For any non-empty set X, we can form the **trivial partition** by putting each element into its own separate block.

$$\pi = \{\{x\} : x \in X\}$$

This is a valid partition because the union of all these singleton sets is clearly X, and any two distinct singleton sets are disjoint.

3.3.3 From partitions to relations. Here is where the concepts begin to connect. Every partition gives rise to an equivalence relation in a very natural way. Let π be a partition of a set X. We can define a relation $\sim_\pi$ on X with the following rule:

$$a \sim_\pi b \quad \text{if and only if} \quad a \text{ and } b \text{ are in the same block of } \pi.$$

Let us prove that this is always an equivalence relation.

- *Reflexive:* For any $a \in X$, a is in the same block as itself. So $a \sim_\pi a$.
- *Symmetric:* If $a \sim_\pi b$, then a and b are in the same block. This also means b and a are in the same block. So $b \sim_\pi a$.
- *Transitive:* If $a \sim_\pi b$ and $b \sim_\pi c$, then a and b are in some block $A \in \pi$, and b and c are in some block $B \in \pi$. Since b is in both A and B, the blocks are not disjoint ($A \cap B \neq \emptyset$). By the definition of a partition, this means the blocks must be the same, $A = B$. Therefore, a, b, and c are all in the same block, which means $a \sim_\pi c$.

This is a crucial idea: partitions naturally give rise to equivalence relations. We will see below how we can also go in the reverse direction. ♢

Definition. Let $\sim$ be an equivalence relation on a set X. For any element $x \in X$, the **equivalence class** of x, denoted $[x]_\sim$ or simply $[x]$, is the set of all elements in X that are related to x.

$$[x]_\sim = \{y \in X : x \sim y\}$$

Since every equivalence relation is reflexive, we know that $x \sim x$, which means that x is always an element of its own equivalence class, i.e., $x \in [x]$.

Example 3.3.4 (Concentric Circles)**.** Let $X = \mathbb{R}^2$, the set of all points in the plane. Define a relation $\sim$ on X as follows: for two points p and q, we say $p \sim q$ if and only if they are the same distance from the origin $(0,0)$. This is an equivalence relation. What is the equivalence class of the point $(1,0)$?

$$[(1,0)]_\sim = \{p \in \mathbb{R}^2 : p \sim (1,0)\} = \{p \in \mathbb{R}^2 : d(p,0) = d((1,0),0)\}$$

The distance of $(1,0)$ from the origin is 1. So, the equivalence class of $(1,0)$ is the set of all points in the plane that are at a distance of 1 from the origin. This is precisely the unit circle. Each equivalence class is a circle centered at the origin.

Example 3.3.5 (Remainder modulo 4)**.** Let $X = \mathbb{Z}$ and let $\sim$ be the relation of congruence modulo 4, i.e., $a \sim b$ if 4 divides $a - b$. What is the equivalence class of 0?

$$[0]_\sim = \{n \in \mathbb{Z} : 0 \sim n\} = \{n \in \mathbb{Z} : 4 \text{ divides } (0-n)\} = \{n \in \mathbb{Z} : 4 \text{ divides } n\}$$

This is the set of all integer multiples of 4: $\{\ldots, -8, -4, 0, 4, 8, \ldots\}$. Notice that $[0]_\sim = [4]_\sim = [-8]_\sim$, etc.

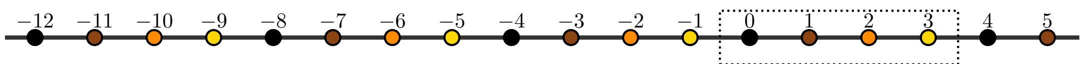


Figure 3.1: The equivalence classes are color coded. For instance, the equivalence class of 1 is shown in brown.

3.3.6 From equivalence relations to partitions. Equivalence classes have two critical properties that lead directly to the connection with partitions. Let $\sim$ be an equivalence relation on a non-empty set X. For any two elements $x, y \in X$:

a) Two elements are related if and only if their equivalence classes are identical.

$$x \sim y \iff [x] = [y]$$

b) Two equivalence classes are either identical or they are completely disjoint.

$$[x] \cap [y] \neq \emptyset \iff [x] = [y]$$

Thus the set of all the equivalence classes partition the set X. Let us prove these statements carefully.

Proof of (a). This is a bi-conditional, so we must prove two directions. First suppose that $x \sim y$. We want to show that the sets $[x]$ and $[y]$ are equal. To do this, we show that $[x] \subseteq [y]$ and $[y] \subseteq [x]$. Let z be an arbitrary element in $[x]$. By definition, this means $x \sim z$. We were given that $x \sim y$. By symmetry, this means $y \sim x$. Now we have $y \sim x$ and $x \sim z$. By transitivity, this implies $y \sim z$. But this is the definition of $z \in [y]$. Since z was an arbitrary element of $[x]$, we have shown $[x] \subseteq [y]$. The proof for $[y] \subseteq [x]$ is identical. Since we have shown inclusion in both directions, we conclude $[x] = [y]$.

Now assume $[x] = [y]$. We want to show $x \sim y$. We know that an element is always in its own equivalence class, so $y \in [y]$. Since we assumed $[x] = [y]$, this means y must also be in $[x]$. By the definition of the class $[x]$, this means $x \sim y$.

Proof of (b). First assume $[x] = [y]$. Then their intersection is just $[x]$. Since $x \in [x]$, the class is non-empty, so the intersection is non-empty. This direction is straightforward.

For the other direction, assume $[x] \cap [y] \neq \emptyset$. This means there is at least one element, let's call it z, that is in both classes. So, $z \in [x]$ and $z \in [y]$. By definition, this means $x \sim z$ and $y \sim z$. From $y \sim z$, by symmetry we get $z \sim y$. Now we have $x \sim z$ and $z \sim y$. By transitivity, this implies $x \sim y$. But we just proved in property (a) that if $x \sim y$, then it must be that $[x] = [y]$. This completes the proof. $\diamondsuit$

3.3.7 The Fundamental Connection. We have now established a beautiful correspondence.

- We saw that any partition π of a set X defines an equivalence relation $\sim_\pi$.
- We just proved that any equivalence relation $\sim$ on a set X defines a partition of X, namely the set of its equivalence classes.

This is not a coincidence. There is a perfect one-to-one correspondence between the set of all equivalence relations on X and the set of all partitions of X. We make this formal. Let $\mathcal{E}$ be the set of all equivalence relations on X, and let $\mathcal{P}$ be the set of all partitions of X. We can define two functions:

$$\begin{aligned} \psi : \mathcal{E} \to \mathcal{P} \quad &\text{where} \quad \psi(\sim) = \{[x]_\sim : x \in X\} \\ \phi : \mathcal{P} \to \mathcal{E} \quad &\text{where} \quad \phi(\pi) = \sim_\pi \end{aligned}$$

In words, the map ψ takes an equivalence relation and gives back its corresponding partition of equivalence classes. The map ϕ takes a partition and gives back the equivalence relation where elements are related if they are in the same block.

These two maps are inverses of each other. For any partition π, if we first apply ϕ to get the relation $\sim_\pi$, and then apply ψ to that relation, we get back our original partition π.

$$\psi(\phi(\pi)) = \pi$$

Similarly, for any equivalence relation $\sim$, if we first find its partition of equivalence classes $\psi(\sim)$, and then find the relation defined by that partition, we get back our original relation $\sim$.

$$\phi(\psi(\sim)) = \sim$$

This establishes that equivalence relations and partitions are, in a deep sense, two different ways of looking at the exact same underlying mathematical structure. $\diamondsuit$

Part II

Real Functions

CHAPTER 4

GRAPHS

4.1 THE CARTESIAN PLANE

4.1.1 Historical Overview. For centuries, the worlds of geometry and algebra existed in parallel. Geometry, pioneered by the ancient Greeks like Euclid, was the study of shapes, lines, and spaces. It was a visual and intuitive discipline. Algebra, with its roots in the work of Persian mathematicians like Al-Khwarizmi, was the abstract study of symbols and the rules for manipulating them. It was a language of logic and computation. The two fields were powerful, but separate.

The grand unification came in the 17th century, primarily from the mind of the French philosopher and mathematician René Descartes. While lying in bed and observing a fly crawling on the ceiling, Descartes (so the story goes) realized he could describe the fly's position at any moment by using just two numbers—its perpendicular distances from two walls. This was a profound insight. It meant that a geometric position could be uniquely converted into a pair of algebraic numbers.

This idea, which he published in his 1637 work *La Géométrie*, laid the foundation for what we now call Cartesian geometry or analytic geometry. It created a bridge between algebra and geometry, allowing us to use algebraic equations to describe geometric shapes and to visualize algebraic relationships as graphs. The French mathematician Pierre de Fermat had independently developed similar ideas, but Descartes' publication secured his legacy. The 'Cartesian Plane' is named in his honor. $\diamondsuit$

Definition. The **Cartesian Plane** is the set of all possible ordered pairs of real numbers, denoted by $\mathbb{R}^2$ (or $\mathbb{R} \times \mathbb{R}$). Each ordered pair (x, y) is called a **point** in the plane.

- The first number, x, is called the **abscissa** or the **x-coordinate**.
- The second number, y, is called the **ordinate** or the **y-coordinate**.

4.1.2 Visualizing the Cartesian plane. It is important to understand how this notion connects with geometry. Before we draw our axes, we can imagine a perfectly flat, infinite sheet of paper. This is the geometric world, often called the *Euclidean Plane*, which the reader must have made contact with in the course of his/her mathematical education. It's filled with points and shapes, but those points don't have coordinates; they don't have numerical addresses.

The moment we choose a point to be our **origin** and draw two perpendicular **axes**, we lay a grid over this entire geometric world. It is this act of imposing a coordinate system that gives every point a unique address (x, y). Therefore, *the Euclidean plane can be thought of as the Cartesian plane* once we have fixed a coordinate system onto it. This powerful idea is the bridge that allows us to describe geometry using algebra. Once we have chosen an origin and drawn our two perpendicular lines, one of the lines may be thought of as "horizontal" and the other "vertical."

- The horizontal line is called the x**-axis**. More precisely, the x-axis is the set
$$\{(x, y) \in \mathbb{R}^2 : \; y = 0\}$$
- The vertical line is called the y**-axis**. Formally, it is the set
$$\{(x, y) \in \mathbb{R}^2 : \; x = 0\}$$
- The point of intersection of the two axes is $(0, 0)$ and is called the **origin**.

These axes divide the plane into four infinite regions called **quadrants**, which are numbered counter-clockwise starting from the top right.

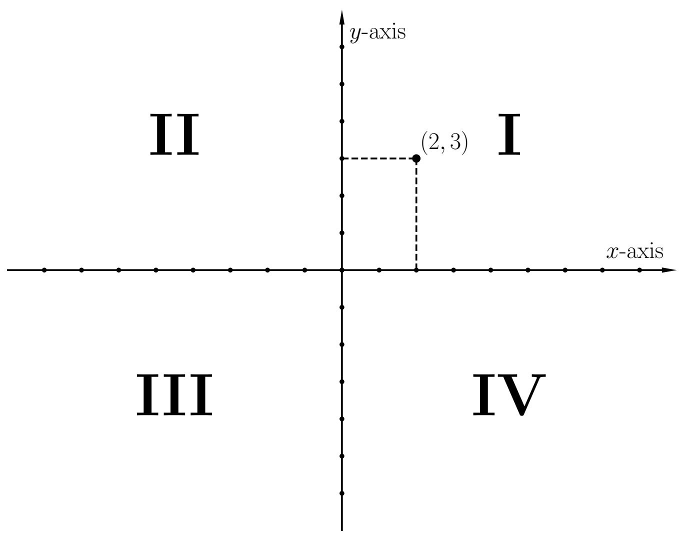
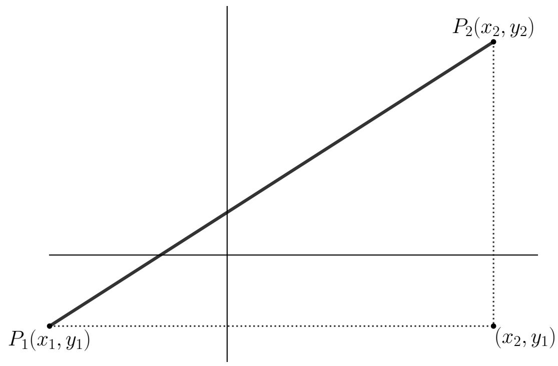


Figure 4.1: The Cartesian Plane with its essential features.

Formally, the first quadrant is defined as
$$\{(x, y) \in \mathbb{R}^2 : \; x, y > 0\}$$
The reader is invited to write down the description of the other three quadrants. $\diamondsuit$

4.1.3 Distance formula. A natural first question to ask is how to find the distance between two points in the Cartesian plane in terms of the coordinates of the two points. First we introduce a notation: For a real number x, we write $|x|$ to mean x if $x \geq 0$ and $-x$ if $x < 0$. In other words, $|x|$ is the magnitude of x, and is read as the *absolute value* of x or as *modulus* of x. Note that $|x|^2$ is just x^2.

Coming back to the question about distance. Let's say we have two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.

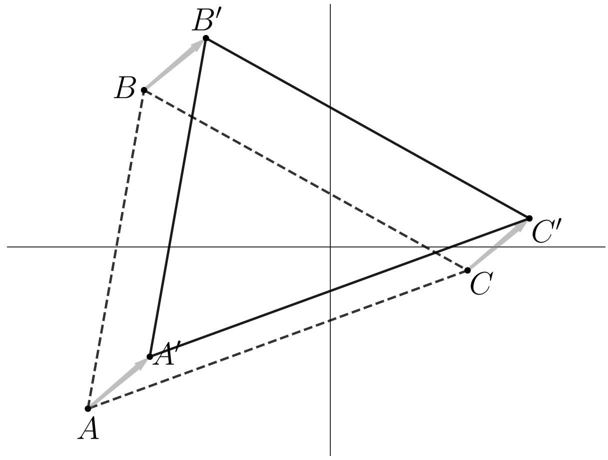


We can form a right-angled triangle with the horizontal and vertical lines passing through these points. The length of the horizontal side is $|x_2 - x_1|$ and the length of the vertical side is $|y_2 - y_1|$. By the Pythagorean theorem, the square of the distance d (the hypotenuse) is the sum of the squares of the other two sides.

Lemma. The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Cartesian plane is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

4.1.4 Translations. Suppose we have a subset S of the Cartesian plane which we want to "shift" one unit to the right. How could we do this? Clearly, all we need to do is shift *each point* in S by a unit to the right. A moment's thought shows to the reader that if P is a point with coordinates (a, b), then the point Q obtained by shifting P one unit to the right is nothing but the point with coordinates $(a + 1, b)$. Of course, we can imagine shifting in any direction and by any magnitude we please. To convey the geometric idea, below we show an example of a translation of a triangle.



We now formalize this below.

Definition. Let h and k be real numbers. The **translation** of the Cartesian plane by (h, k) is the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\tau(x, y) = (x + h, y + k)$$

for all $(x, y) \in \mathbb{R}^2$.

A positive h moves the plane to the right, a negative h to the left. A positive k moves it up, and a negative k moves it down. $\diamondsuit$

Exercise 4.1.1. Plot the following points on a Cartesian plane and identify their quadrants: $A(2, 5)$, $B(-3, 4)$, $C(-1, -1)$, $D(5, -2)$, $E(0, 4)$, $F(-3, 0)$.

Exercise 4.1.2. Find the distance between the following pairs of points:

1. $(1, 2)$ and $(4, 6)$
2. $(-2, 3)$ and $(3, -9)$
3. (a, b) and $(-a, -b)$

Exercise 4.1.3. A point $P(x, y)$ is equidistant from $A(5, 1)$ and $B(-1, 5)$. Show that $3x - 2y = 0$.

Exercise 4.1.4. What is the geometric shape formed by the set of all points (x, y) whose distance from the origin is exactly 5? Write an algebraic equation for this shape.

Exercise 4.1.5. The point $P(3,4)$ is translated by the vector (h,k) to the new point $P'(-1,7)$. What is the translation vector (h,k)?

Exercise 4.1.6. Consider a triangle with vertices $A(1,1)$, $B(4,1)$, and $C(4,5)$.

1. Calculate the lengths of all three sides.
2. The entire triangle is translated by the vector $(-5,-2)$. Find the coordinates of the new vertices A', B', and C'.
3. Show that the lengths of the sides of triangle $A'B'C'$ are identical to the lengths of the sides of triangle ABC. What does this tell you about translations?

4.2 SOME STANDARD FUNCTIONS AND GRAPHS

4.2.1 Graph of a function. Consider a function whose domain is a subset of $\mathbb{R}$ and whose target is $\mathbb{R}$. The rule which defines the function can be complex and may not allow us to work with it at an intuitive level. This is where the idea of a **graph** can be extremely useful. It's a way to turn the abstract rule of a function into a concrete geometric object that we can see and analyze. We can "see" the whole function in one go. We first give the formal definition.

Definition. Let $f : S \to \mathbb{R}$ be a function, where S is a subset of $\mathbb{R}$. The **graph** of f is defined as

$$\Gamma_f = \{(x, f(x)) : \ x \in S\}$$

So the graph of f is simply the collection of all points (x,y) in the Cartesian plane such that x is in the domain S and y is the output you get when you apply the function to x, that is, $y = f(x)$.

For example, if we have the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ for all $x \in \mathbb{R}$, its graph is the set of all points (x, x^2). The real magic of a graph is its visual intuition. By looking at the shape of the graph, we can instantly understand key properties of the function.

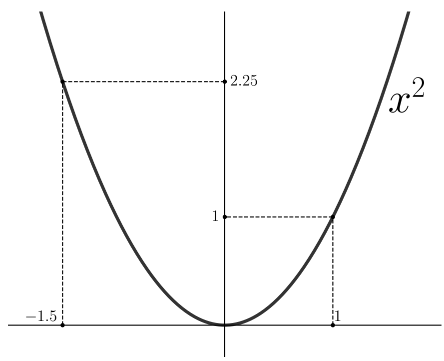


Figure 4.2: The graph of the "squaring-function." Notice that the graph reflects, in its "convex shape", the fact that the function grows faster as the value of x increases.

In short, a graph translates the algebraic properties of a function into a geometric story that is often much easier to follow. But, like any tool, graphs have their limitations.

- *Complexity of the domain.* This visual approach works well for functions $f : S \to \mathbb{R}$ where S is a sufficiently well-behaved set, like an interval or a union of finitely many intervals. If S is a complicated set, like the set of all the rationals, it is not possible to draw a remotely accurate plot of f, simply because there is no way to draw the rationals in any satisfactory way.
- *Scale and Scope.* The part of the graph we see depends entirely on the viewing window. Zoom in too much, and one might miss the big picture. Zoom out too far, and one might smooth over important local details, like rapid wiggles or sharp turns. A poorly chosen scale can give a very misleading impression of the function's behavior.
- *Lack of rigour.* At the end, graphs are a coarse pictorial representation of "real mathematical phenomenon." They frequently help us see the truth, which can then be proven analytically, even though this latter task is not always easy. One tool (which we will neither assume not employ) that often helps us convert graphical insight into formal mathematics is calculus. However, the goal of this part of the manuscript is not to be overly fastidious with the rigour, but to appreciate the power of the visual facility that graphs endow us with.

Despite these limitations, the graph remains one of the most fundamental and useful tools in all of mathematics for understanding the relationship between an input and an output. In this section we describe some interesting real valued functions whose domain is a subset of reals and discuss the sketch of their graphs.[1] We will also learn how many concepts associated to a function manifest in the language of graphs. ♢

4.2.2 The modulus function. The modulus function models the distance between two points on a line. The distance from home to a shop 5 km east is the same as the distance to a park 5 km west. The modulus function cares only about the distance (5), not the direction (+ or −). Here is a formal definition.

Definition. The **modulus function**, also known as the **absolute value function**, is the map $|\cdot| : \mathbb{R} \to \mathbb{R}$ defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

This function gives the magnitude of a number, regardless of its sign. Its graph, shown in Figure 4.3 is a distinctive 'V' shape, with the vertex at the origin, as shown above. The justification for this is easy. Suppose $a > 0$. Then the point on the graph corresponding to a is (a, a). The distance of this point from the x and y-axes is the same (namely, a). The line joining this point with the origin is then easily seen to be at an angle of 45° with the positive direction of the x-axis. This justifies the "right arm" of the graph.

[1]Some of these functions will be studied in more detail in separate chapters dedicated to them.

Similarly we can argue for the left arm. We will study the modulus function in more detail in Chapter 7. ♢

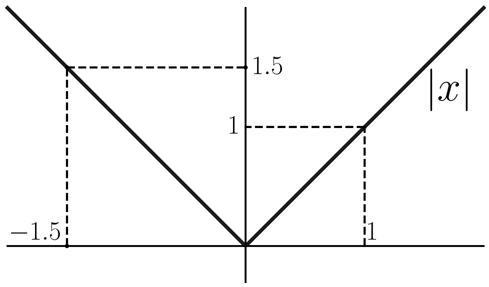


Figure 4.3: Graph of the modulus function.

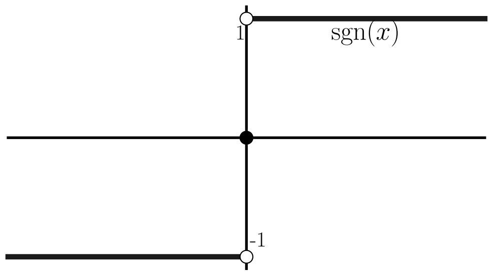


Figure 4.4: Graph of the signum function.

4.2.3 The signum function. Think of a switch for the direction of a motor. It can be moving forward (+1), backward (−1), or be stationary (0). The *signum function* models this.

Definition. The **signum function** $\text{sgn} : \mathbb{R} \to \mathbb{R}$ can be used to model this. It is defined as

$$\text{sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$$

In words, the signum function is 1 for positive numbers, −1 for negative numbers, and 0 for zero. Its graph, shown in Figure 4.4, consists of three pieces: a point at the origin, and two horizontal rays at $y = 1$ and $y = -1$. The reason for this is self-explanatory. More on the signum function in Chapter 8. ♢

4.2.4 Indicator functions. Let A be a subset of $\mathbb{R}$. The **indicator** of A, denoted $1_A(x)$, is the ultimate "yes/no" function. It equals 1 if the input x is in the set A, and 0 otherwise. Formally

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

For a real world scenario, imagine a special offer that is valid only during particular times in a day, say 3 to 4 pm and 6 to 7 pm. Clearly, the indicator function can be used to model this phenomenon.

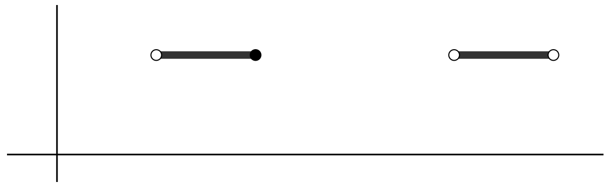


Figure 4.5: Graph of the indicator function of the set $\{x : 1 < x \leq 2\} \cup \{x : 4 < x < 5\}$

Its graph, shown in Figure 4.5, is a horizontal line at $y = 0$, with a segment "lifted" to $y = 1$ over the set A. ♢

4.2.5 The floor function. If one is 25.75 years old, one's age in completed years is 25. We remain 25 until we hit our 26th birthday. This motivates the notion of the "floor value" of a real number.

> *Definition.* If x is any real number, there is a unique integer n such that
>
> $$n \leq x < n + 1$$
>
> The **floor value** of x, written $\lfloor x \rfloor$, is defined as this unique integer n. The **floor function**, also called the **step function**, is defined as the function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{R}$ which takes x to $\lfloor x \rfloor$.

In words, the floor function outputs the greatest integer less than or equal to the input.

The graph of the floor function is shown in Figure 4.6 on the left. Note that $\lfloor n \rfloor = n$ for all integers n. In fact, $\lfloor n + \alpha \rfloor = n$ for all $0 \leq \alpha < 1$. This is why we see the regions where the graph is constant and then suddenly jumps. ♢

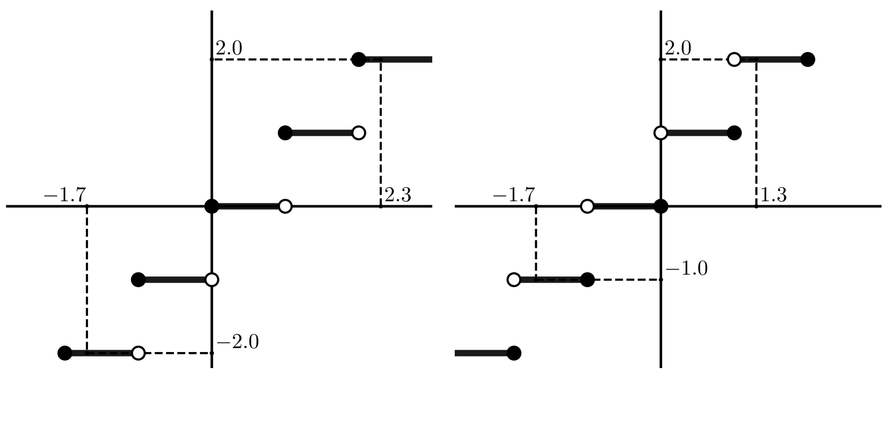


Figure 4.6: The graphs of the floor function (left) and the ceiling function (right).

4.2.6 The ceiling function. Imagine a parking garage that charges by the hour. If we park for 15 minutes (0.25 hours), we are charged for the full first hour. If we park for 2 hours and 1 minute, we pay for 3 hours. To model this phenomenon we define the *ceiling function*.

Definition. For any real number x, the **ceiling value** of x as the unique integer n such that

$$n - 1 < x \leq n$$

The **ceiling function** is defined as the function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{R}$ which takes $x \in \mathbb{R}$ to $\lceil x \rceil$.

In words, the ceiling function outputs the smallest integer greater than or equal to the input. The graph of the ceiling function is much like that of the floor function and is shown in Figure 4.6 on the right. The reader is encouraged to convince himself/herself of the accuracy of the above diagram. $\diamondsuit$

4.2.7 The fractional part function. Think of a stopwatch that displays seconds and hundredths of a second. At time t, the whole number of seconds is $\lfloor t \rfloor$, while the hundredths of a second represent what we will call the *fractional part* of t, and is written as $\{t\}$. This fractional part continually increases from .00 up to .99, and just as it's about to reach the next whole second, it resets to .00. To model this we define the *fractional part function*.

Definition. The function $\{\cdot\} : \mathbb{R} \to \mathbb{R}$ defined as

$$\{x\} = x - \lfloor x \rfloor$$

for all $x \in \mathbb{R}$, is called the **fractional part function**. This function isolates the "decimal part" of a number. The number $\{x\}$ is called the **fractional part** of x.

It is unfortunate that the above is the standard notation for this function. The number $\{x\}$ can easily be confused with the set which comprises exactly one element, namely x. Its graph is a series of diagonal lines rising from 0 to 1, creating a "sawtooth" pattern, shown in Figure 4.7.

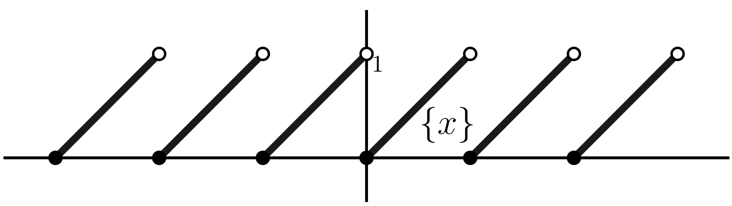


Figure 4.7: Graph of the fractional part function.

The floor and the fractional part functions will be discussed in much more depth in Chapter 10. $\diamondsuit$

4.2.8 Linear functions. Let's try to understand how the fare of a taxi is calculated. There's a fixed starting charge, say c, and a cost per kilometer, say m. The total fare is a linear function of the distance travelled. Thus the cost of travelling x kilometers is $mx + c$.

In general, if m and c are any real numbers (not necessarily positive), we can define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = mx + c$ for all x. Any such function is called a **linear function**. Its graph is always a straight line. In Figure 4.8) on the left we have shown the graphs for c=0 and various values of m, and on the right we have shown the graphs with a fixed value of m and various values of c.

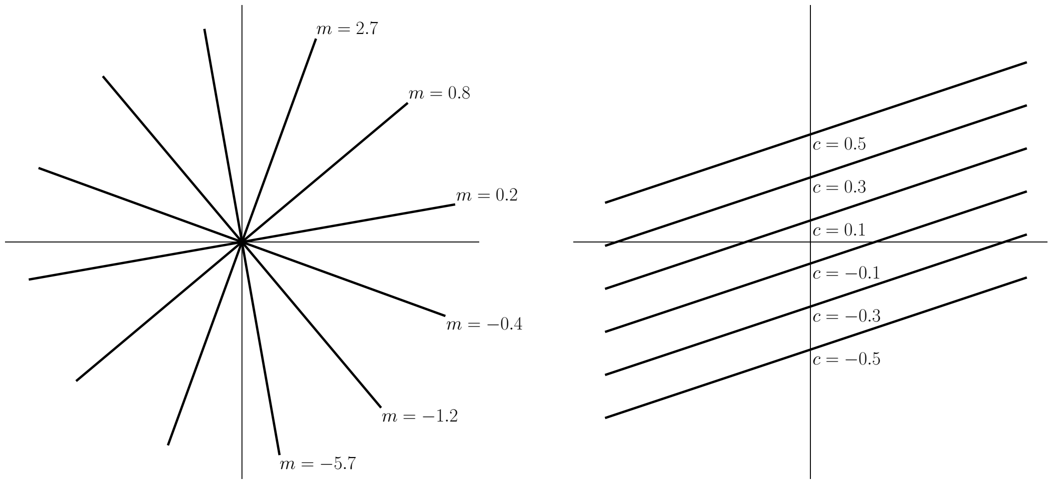


Figure 4.8

The parameter m is called the *slope* (steepness), and c is called the y-intercept (where it crosses the vertical axis). Note that this function is bijective if and only if $m \neq 0$. When $m = 0$ then this is a constant function and hence its image has then size 1. More discussion on linear functions can be found in Chapter 5 ♢

4.2.9 Polynomials. Polynomials generalize linear functions. Let n be a non-negative integer and $a_0, a_1, \ldots, a_n$ be real numbers. Define a function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{for all } x \in \mathbb{R}$$

Any such function is called a **polynomial**. Note that each constant function is a polynomial. In particular, the function which takes the value 0 everywhere is also a polynomial. Polynomials are immensely important in mathematics because they are simple to compute and, remarkably, can be used to approximate much more complex functions (a key idea in calculus known as Taylor series). The path of a thrown ball under gravity follows a parabolic trajectory, the graph of a degree-2 polynomial. A degree-3 (cubic) polynomial, with its characteristic "S" shape, can model things that rise, inflect, and then continue to rise, such as the relationship between the side length of a box and its volume, or some basic economic models of growth that temporarily level off. The graph of polynomials are particularly well-behaved. In the Figure 4.9, we show an example. ♢

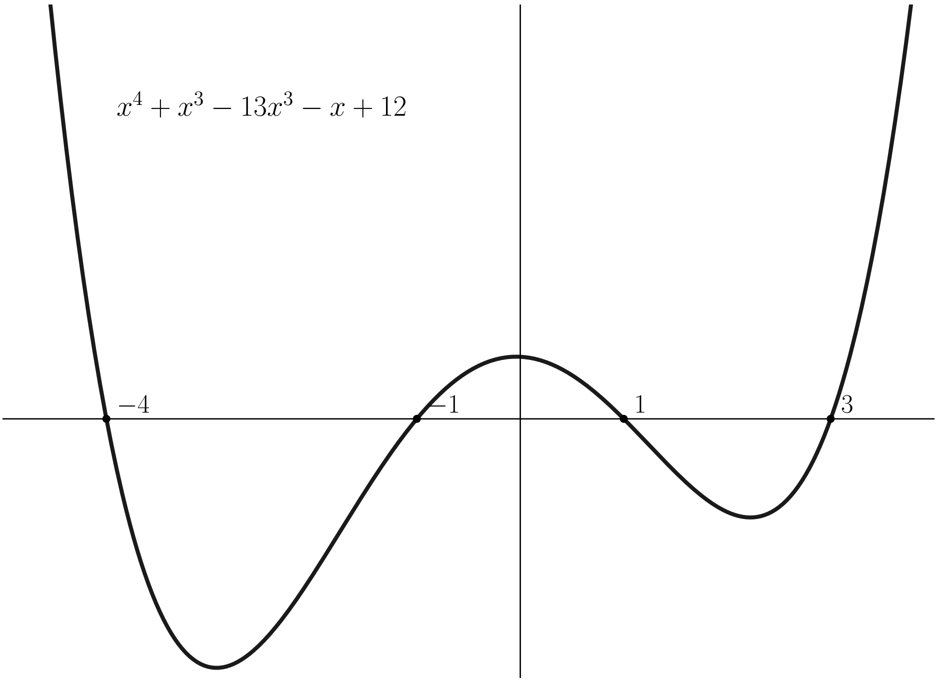


Figure 4.9

4.2.10 Surds. Let n be a positive integer and consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$f(x) = x^n \text{ for all } x \in \mathbb{R}$$

It is a fact that this function is a bijection. Thus this function is invertible, and the inverse is denoted as $\sqrt[n]{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, or, simply as $\sqrt[n]{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, since doing this does not add or subtract any content and is more practical. In words, if $x \geq 0$, then $\sqrt[n]{x}$ is the unique non-negative real number which when raised to the power n yields x. These functions will be referred to as **surds**. When $n = 2$ it is called the **square root function**.

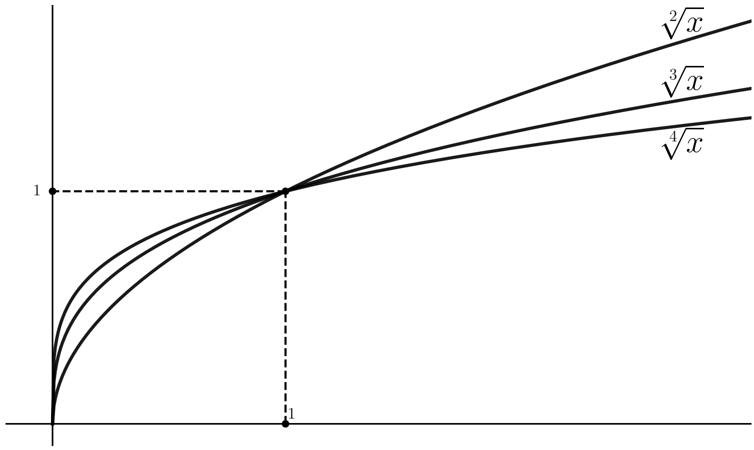


Figure 4.10: Three different surd functions.

It should be noted that the square root function takes only non-negative values. Thus $\sqrt{6}$ is a positive real; it is not a negative number. $\diamondsuit$

4.2.11 Rational functions. Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two polynomials, where Q is not identically zero. It is a fact (that we would not be proving here) that the set of points where any nonzero polynomial takes the value 0 is finite. In symbols, the set $\rho_Q = \{x \in \mathbb{R} : \ Q(x) = 0\}$ is finite. The elements of ρ_Q are called the **roots** of Q. Now we can define a function $f : \mathbb{R} \setminus \rho_Q \to \mathbb{R}$ as

$$f(x) = \frac{P(x)}{Q(x)}$$

for all $x \in \mathbb{R}$. Any such function is called a **rational function**. Clearly, every polynomial is a rational function, since we can take Q as the constant function 1. The graphs of rational functions are fascinating and can be more complex than polynomials.

They are known for features called *asymptotes*—lines that the graph approaches but never touches. These occur where the denominator is zero (vertical asymptotes) or as x approaches infinity (horizontal or slant asymptotes).

To see a real world example, consider the average cost to produce an item. Suppose there is a large fixed set cost (e.g., for a factory) and then a smaller cost per item. The

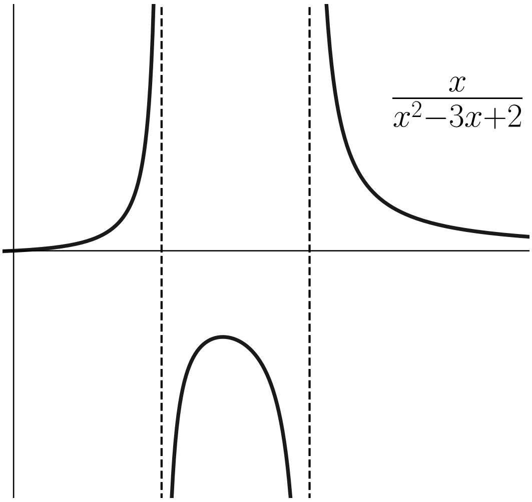


Figure 4.11: Graph of the function given by the expression $\dfrac{x}{x^2-3x+2}$. The vertical lines are the "asymptotes."

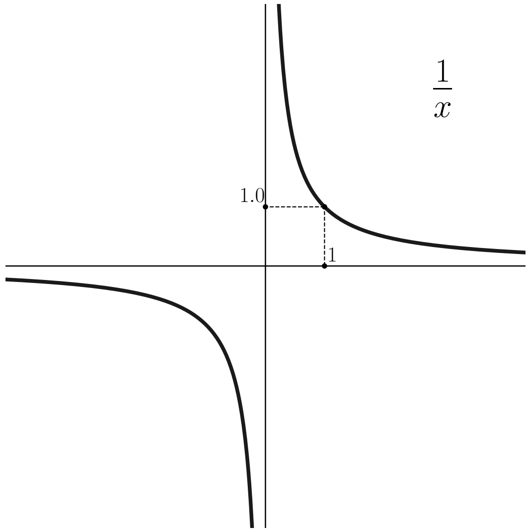


Figure 4.12: The graph of the function given by $1/x$. The shape is called a "rectangular hyperbola."

average cost per item is given by

$$\text{Average Cost}(x) = \frac{\text{Fixed Cost} + (\text{Cost per item} \times x)}{x}$$

When you produce very few items (x is small), the average cost is huge because the fixed cost dominates. As you produce more and more items (x becomes large), the average cost gets closer and closer to the per-item cost, which acts as a horizontal asymptote.

A better example is modeling the concentration of a (psychedelic) drug in the bloodstream over time. A simple model might look like

$$C(t) = \frac{At}{k+t^2}$$

where t is time after administration of the drug. Initially, concentration rises as the drug is absorbed (the At term dominates). But over time, the body metabolizes and eliminates the drug, a process reflected in the t^2 term in the denominator, causing the concentration to peak and then fall, approaching zero. The constant k relates to absorption and distribution rates. $\diamondsuit$

4.2.12 Exponential function. Let $a > 0$. We define a function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = a^x$$

for all $x \in \mathbb{R}$.[2] Figure 4.13 shows the graphs of a few exponential functions. It is characterized by its rate of growth. For $a > 1$, it starts slow and then increases incredibly

[2]The serious reader will question what is meant by things like $2^{\sqrt{2}}$, or even the meaning of $2^{3/4}$. A proper discussion of these requires us to undertake a serious study of real numbers, something that is done usually at the college level in a course on real analysis.

rapidly. If $a \neq 1$, it is a fact that this map is injective and has its image as the set of all the positive reals. Compound interest on a bank deposit or unchecked population

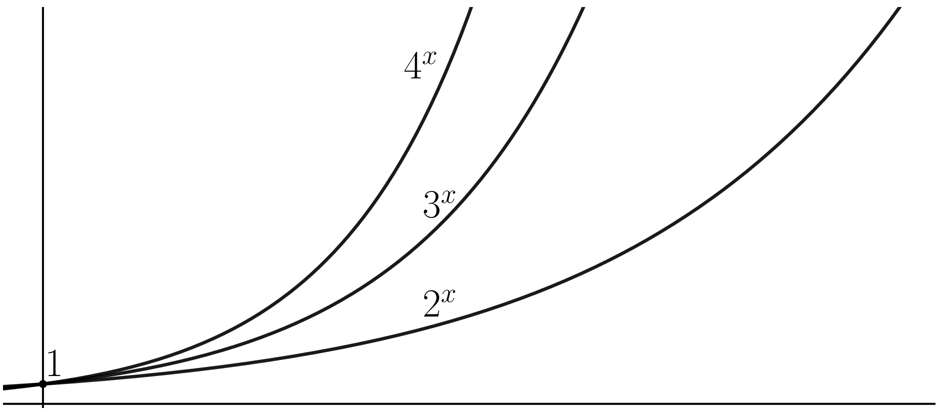


Figure 4.13: Graphs of some exponential functions.

growth. The amount of growth depends on the current amount, leading to an acceleration in growth over time. It should be noted that a full appreciation for the properties of exponential growth, particularly concerning the natural base e, including the definition of the number e, emerges through the lens of calculus (real analysis). $\diamondsuit$

4.2.13 Logarithmic function. Let $a > 0, a \neq 1$. As mentioned earlier the map $f : \mathbb{R} \to \mathbb{R}_{>0}$ defined as

$$f(x) = a^x \text{ for all } x \in \mathbb{R}$$

is a bijection. Thus the inverse of this map is defined, and is denoted as $\log_a : \mathbb{R}_{>0} \to \mathbb{R}$. This function is called the **logarithm with base** a. Its graph shows rapid initial growth that dramatically slows down. A few examples are shown in Figure 4.14. The Richter

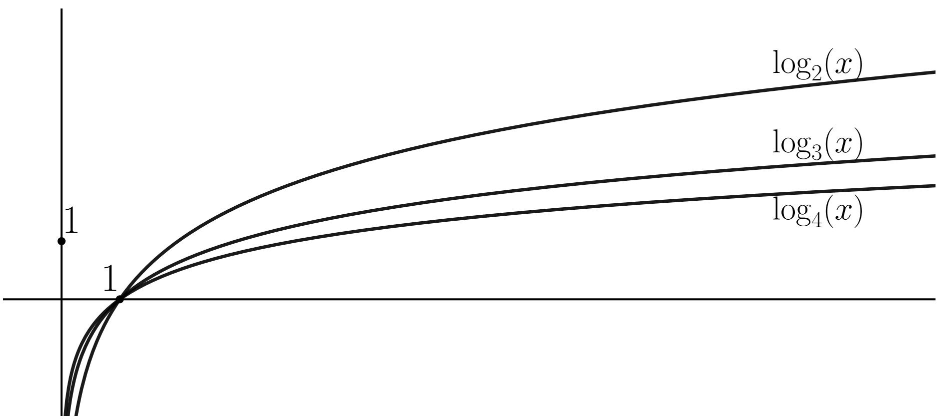


Figure 4.14: Graphs of logarithmic functions with a few different bases.

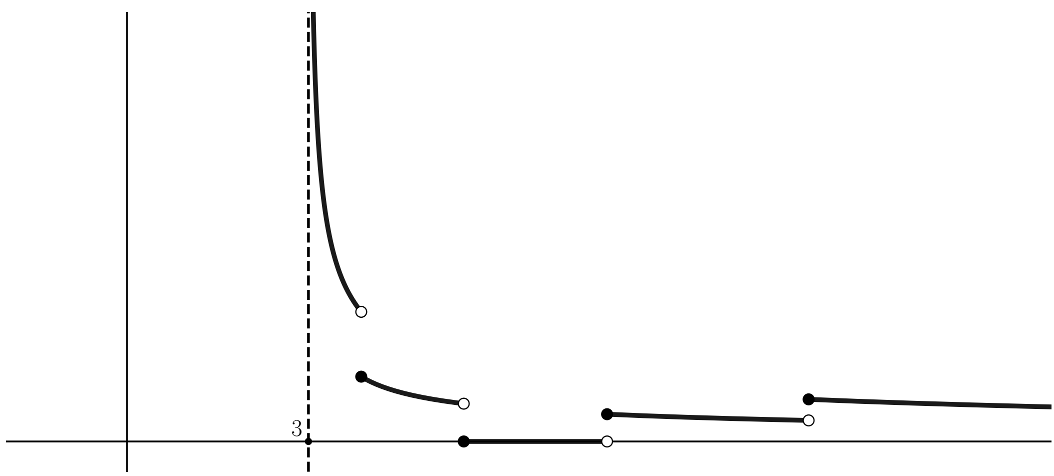
scale for earthquakes. An increase from magnitude 5 to 6 doesn't feel like a small step; it represents a tenfold increase in measured amplitude. The logarithm helps us manage and represent these huge ranges in a more comprehensible way. As with its inverse, the true power of the logarithmic function is fully understood when studied with calculus, which clarifies its role in describing rates of change. ♢

4.2.14 A complicated function. The true power of these elementary functions is that they are building blocks. Through operations like addition, multiplication, and especially composition (plugging one function into another), we can construct functions of incredible complexity to model the real world. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \frac{|\lfloor \log_2(x^2+1) \rfloor - 5|}{\sqrt{x-3}}$$

for all x in $\mathbb{R}$. Let us understand it by breaking it down.

- We start with a polynomial, $x^2 + 1$.
- We feed that into a logarithm, $\log_2(\cdot)$.
- We take the floor of the result, creating a step function.
- We subtract 5 and take the absolute value, creating V-shapes on top of the steps.
- The denominator is a surd function, $\sqrt{x-3}$, which tells us the domain of the entire function must be $x > 3$.
- Finally, we divide the two parts, creating a rational structure with a vertical asymptote at $x = 3$.



By understanding the behavior of each piece, we can begin to predict the shape and properties (domain, discontinuities, asymptotes) of the combined, complex function. ♢

4.2.15 Piecewise defined functions. Many times a function is defined "piecewise." This means that the behaviour of the function is governed by different formulas or rules on different parts of its domain. This is not really a new notion. A function was not restricted to be governed by *any* neat expression, let alone being governed by a single expression throughout its domain. Nevertheless, this is a very useful way of constructing complex functions by stitching together simpler ones, such as linear or quadratic functions, over specific intervals.

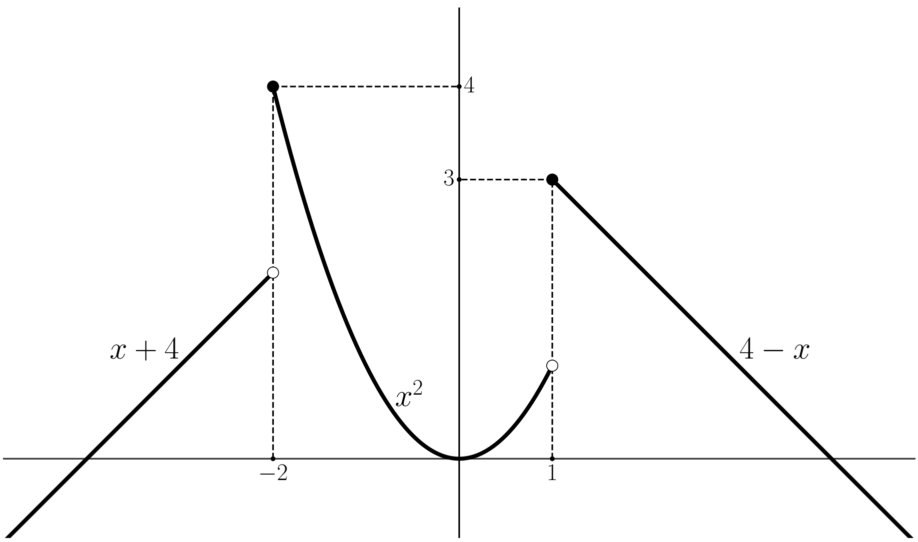
We generally express piecewise functions using a large brace to group the different rules, listing the specific sub-domain for each rule to the right. The general format looks like this:

$$f(x) = \begin{cases} \text{Rule 1} & \text{if } x \in \text{Interval 1} \\ \text{Rule 2} & \text{if } x \in \text{Interval 2} \\ \vdots & \vdots \end{cases}$$

Let us take a concrete example. Consider the function defined as

$$f(x) = \begin{cases} x+4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x < 1 \\ 4-x & \text{if } x \ge 1 \end{cases}$$

The graph of the function is shown below



4.3 MANIPULATING GRAPHS

In the previous section, we introduced a gallery of fundamental functions and their corresponding graphs. Now, we will explore the dynamic relationship between algebraic manipulations of a function and the geometric transformations of its graph. Understanding this connection provides a powerful visual toolkit for analyzing functions, solving equations, and reasoning about complex mathematical relationships without getting lost in symbols.

4.3.1 Translation of the domain. When we alter the input to a function, we are effectively shifting its graph horizontally. Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a new function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = f(x - c), \text{ for all } x \in \mathbb{R}$$

where c is a constant. A natural question arises. *What is the relationship between the graph of f and the graph of g?* Recall that Γ_f and Γ_g denote the graphs of f and g respectively. Note that

$$\begin{array}{llll} (x,y) \in \Gamma_f & \iff & y = f(x) & \\ & \iff & y = f((x+c) - c) & \\ & \iff & y = g(x+c) & \iff \quad (x+c, y) \in \Gamma_g \end{array}$$

So we see that (x, y) is on the graph of f if and only if $(x + c, y)$ is on the graph of g. Thus Γ_g is the "horizontal shift by c" of Γ_f. To say this formally, we define a map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\tau(x, y) = (x + c, y)$$

for all $(x, y) \in \mathbb{R}^2$. Then $\Gamma_g = \tau(\Gamma_f)$. We emphasize that

- If $c > 0$, the graph shifts to the *right* by c units.
- If $c < 0$, the graph shifts to the *left* by $|c|$ units.

This can seem counter-intuitive at first. Why does subtracting a positive c shift the graph to the right? One may think of it this way: for the new function g to produce the same output that f produced at $x = 0$, we need its input, $x - c$, to be zero. This happens when $x = c$. So, the point that was at the origin for f is now at $x = c$ for g.

As an example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = x^2, \text{ for all } x \in \mathbb{R}$$

Defined the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = f(x - 3) = (x - 3)^2, \text{ for all } x \in \mathbb{R}$$

The graphs of f and g are shown in Figure 4.15.
It is easy to argue that the image of f and g are the same: If $y \in \text{Image}(f)$, then there is $x \in \mathbb{R}$ such that $y = f(x)$, and hence $y = f((x+c)-c) = g(x+c)$, showing $y \in \text{Image}(g)$. This shows that $\text{Image}(f) \subseteq \text{Image}(g)$, and the reverse containment can be established similarly. However, this is also clear from the graphical interpretation. A point y is in the image of f if and only if the horizontal line passing through $(0, y)$ meets the graph of f at at least one point. Since the graph of g is just a horizontal shift of the graph of f, a horizontal line that meets the graph of f also meets the graph of g, and conversely.

We make one final remark. The above discussion readily generalizes to the setting where f has only a subset S of $\mathbb{R}$ as its domain, rather than the whole of $\mathbb{R}$. In this general setting, the domain of g would be $S - c = \{x - c : \ x \in S\}$. We invite the first-time reader to redo all the details for this general setting. $\diamondsuit$

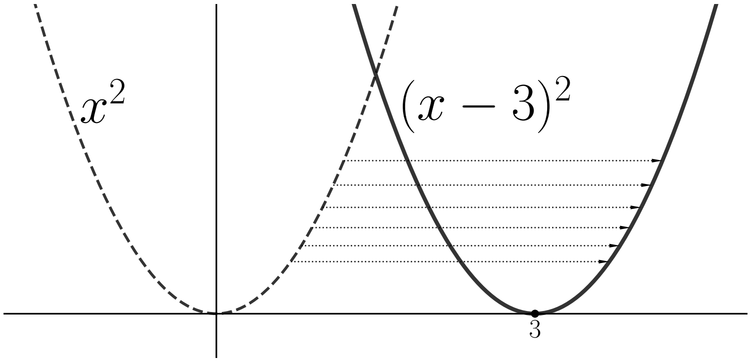


Figure 4.15

4.3.2 Translation of the target. This is more direct than a domain translation. Given a function $f : S \to \mathbb{R}$ where S is a subset of $\mathbb{R}$ and a real number c, we can define a new function $f : S \to \mathbb{R}$ as

$$g(x) = f(x) + c$$

The graph of g is obtained by shifting the graph of f vertically:

- If $c > 0$, the graph shifts *up* by c units.
- If $c < 0$, the graph shifts *down* by $|c|$ units.

The reader is invited to write formal proofs of these. For a concrete example, define a function $g : \mathbb{R}_{>0} \to \mathbb{R}$ as

$$g(x) = \log_2(x) + 3, \text{ for all } x \in \mathbb{R}$$

The shape of Γ_g is identical to that of Γ_f, where $f : \mathbb{R}_{>0} \to \mathbb{R}$ is the function defined as

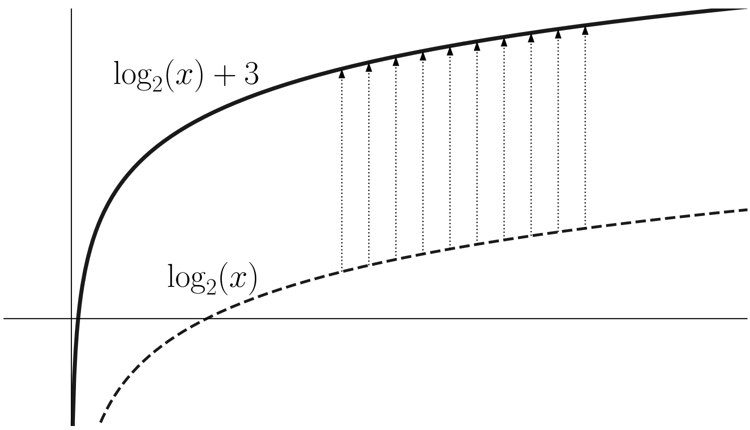


Figure 4.16

$f(x) = \log_2(x)$ for all $x \in \mathbb{R}$, but every point on the graph has been moved 3 units up. This is shown in Figure 4.16. ♢

4.3.3 Scaling of the domain. Let $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$ be a function, and c be a nonzero real number. What happens if we multiply the input by a constant? Let $S_c = \{x/c : x \in S\}$ and define a function $g : S_c \to \mathbb{R}$ as

$$g(x) = f(cx)$$

for all $x \in \mathbb{R}$.

- If $|c| > 1$, the graph is compressed horizontally by a factor of c.
- If $0 < |c| < 1$, the graph is stretched horizontally by a factor of $1/c$.

The following formally establishes the relationship between the graphs of f and g.

$$\begin{aligned} (x,y) \in \Gamma_f \quad &\iff \quad y = f(x) \\ &\iff \quad y = f(c \cdot \frac{x}{c}) \\ &\iff \quad y = g(x/c) \quad \iff \quad (x/c, y) \in \Gamma_g \end{aligned}$$

Let us see an example. Recall that $(-1,1)$ denotes the set $\{x \in \mathbb{R} : -1 < x < 1\}$ and let $f : (-1,1) \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{1 - x^2}, \text{ for all } x \in \mathbb{R}$$

whose graph is a semicircle of radius 1. To quickly see why this is, observe that if (x,y) is a point on the graph then $x^2 + y^2 = 1$, and hence the distance of the point (x,y) from the origin is 1. Now define the function $g : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ as

$$g(x) = f(2x) = \sqrt{1 - (2x)^2} = \sqrt{1 - 4x^2}, \text{ for all } x \in (-\frac{1}{2}, \frac{1}{2})$$

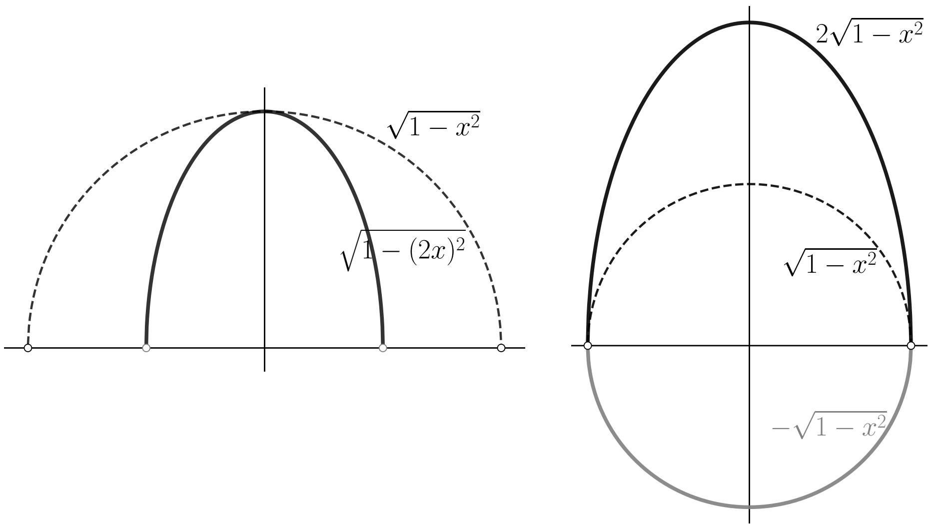
The graphs of f and g are shown in Figure 4.17a and the "compression" spoken of earlier is visible. The semicircle has been squeezed. $\diamondsuit$

4.3.4 Scaling of the target: Vertical axis stretching. Just as we can scale the input (domain), we can also scale the output (codomain). This is achieved by multiplying the entire function by a constant. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ be a function. Let c be a real number and define a function $g : S \to \mathbb{R}$ as

$$g(x) = c \cdot f(x)$$

for all $x \in \mathbb{R}$. This transformation stretches or compresses the graph vertically.

- If $|c| > 1$, the graph is stretched vertically by a factor of $|c|$.
- If $0 < |c| < 1$, the graph is compressed vertically by a factor of $|c|$.
- If $c < 0$, the graph is also reflected across the x-axis in addition to being stretched or compressed.



(a) (b)

Figure 4.17

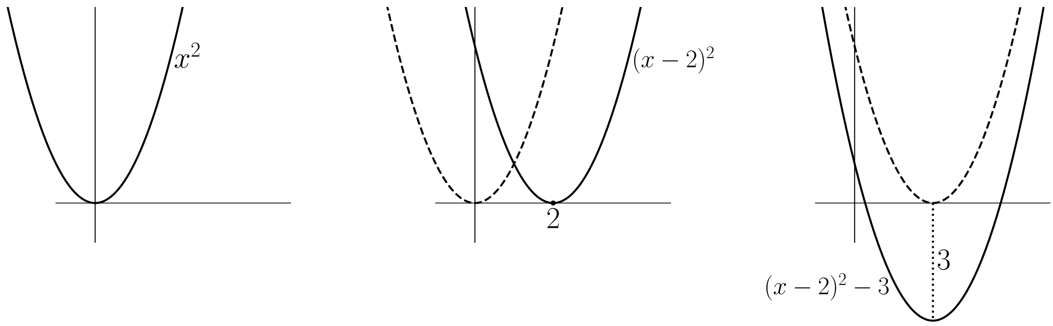
Let's return to our semicircle example. We have $f : (0,1) \to \mathbb{R}$ defined as $f(x) = \sqrt{1-x^2}$. The function $g : (0,1) \to \mathbb{R}$ defined as $g(x) = 2f(x) = 2\sqrt{1-x^2}$ stretches the graph vertically by a factor of 2, turning the semicircle into a semi-ellipse that is taller. The function $h : (0,1) \to \mathbb{R}$ given by $h(x) = -f(x) = -\sqrt{1-x^2}$ simply reflects the semicircle across the x-axis, creating a downward-opening semicircle. This is shown in Figure 4.17b ♢

4.3.5 Drawing graphs of quadratic polynomials. The above discussed principles allow us to easily sketch the graphs of *quadratic polynomials* by employing a simple algebraic trick. A **quadratic polynomial** is a function $f : \mathbb{R} \to \mathbb{R}$ such that there exist real numbers a, b and c, such that $a \neq 0$ and

$$f(x) = ax^2 + bx + c$$

for all $x \in \mathbb{R}$. We can obtain a full understanding of a quadratic polynomial from its graph. Let us see how to sketch them by means of an example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2 - 4x + 1$. The key idea is to "make a perfect square" by noticing that

$$f(x) = (x^2 - 4x) + 1 = \left[(x-2)^2 - 4\right] + 1 = (x-2)^2 - 3$$



This is a general method and works for all quadratic polynomials. ♢

4.3.6 Composition: Sophisticated domain morphing. Let S and T be subsets of $\mathbb{R}$ and $f : S \to \mathbb{R}$ and $g : T \to \mathbb{R}$ be functions such that $\text{Image}(f) \subseteq T$. We can form the composite function $g \circ f : S \to T$. This function can be seen as a (possibly complicated) transformation of the domain of g. The original function $f(x)$ "pre-processes" the input x, and its output becomes the input for the outer function $g(x)$.

Let us see an example. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^2 \quad \text{and} \quad g(x) = 2^x$$

The composition $h = g \circ f$ takes an input x, squares it (making it non-negative), and

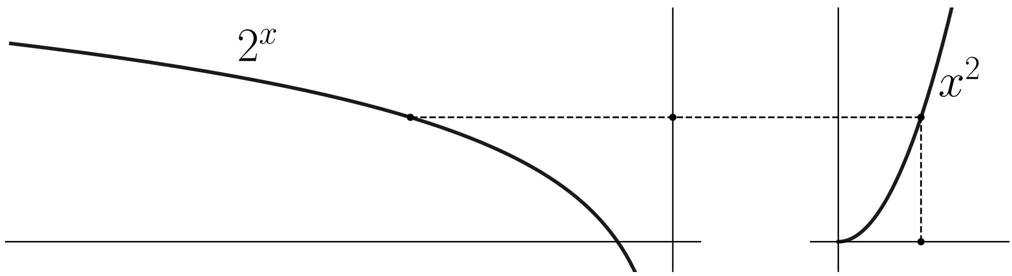


Figure 4.18

then calculates 2 to that power. The resulting graph is a symmetric "bell" shape, much steeper than the original parabola. It combines the symmetric nature of f with the rapid growth of g. See Figure 4.18. ♢

4.3.7 Inverses. Let A and B be subsets of $\mathbb{R}$ and $f : A \to B$ be a function. If f is bijective, then we can define the inverse function $f^{-1} : B \to A$ of f. The graphs of f and f^{-1} have a beautiful relationship: they are reflections of each other across the line "$y = x$", by which we mean the set of points $L = \{(x, y) \in \mathbb{R}^2 : \ x = y\}$. Heuristically, this is because finding the inverse is equivalent to swapping the roles of input and output. Let us show a more formal proof. We have

$$\begin{aligned} (a,b) \in \Gamma_f \qquad &\iff \qquad b = f(a) \\ &\iff \qquad a = f^{-1}(b) \qquad \iff \qquad (b,a) \in \Gamma_{f^{-1}} \end{aligned}$$

In words, If the point (a, b) is on the graph of f (meaning $f(a) = b$), then the point (b, a) must be on the graph of f^{-1} (since $f^{-1}(b) = a$), and conversely. Using this, and using the simple geometric fact that the reflection of the point (x, y) in the line L is (y, x), our assertion is established.

Let $f : \mathbb{R} \to \mathbb{R}_{>0}$ and $g : \mathbb{R}_{>0} \to \mathbb{R}$ be defined as

$$f(x) = e^x \quad \text{and} \quad g(x) = \ln(x)$$

We know that f and g are inverses of each other. The graphs in Figure 4.19 depict the

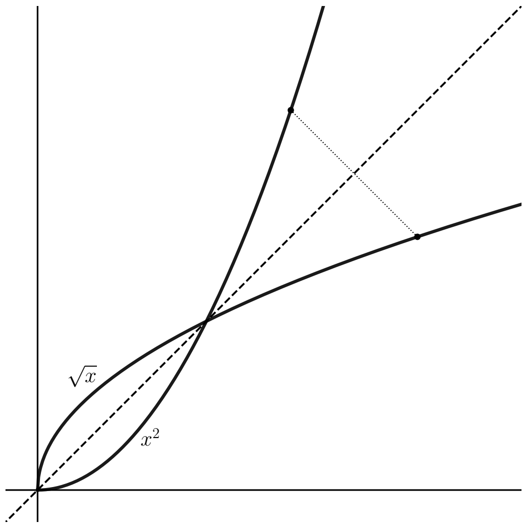


Figure 4.19

"reflection phenomenon." ♢

Add some illustrations.

Exercise 4.3.1. Sketch the graphs of the following functions. For each, clearly indicate the domain, any intercepts, and asymptotes.

(a) $3 + 2\ln(1 - x)$

(b) $e^{-x} + 1$

(c) $-\ln(x - 1)$

(d) $2^{x-1} - 3$

(e) $\ln|x|$

(f) $\frac{1}{1+e^{-x}}$

Exercise 4.3.2. Sketch the graphs of the following functions.

(a) $\frac{2x-3}{x+1}$

(b) $\frac{x^2-1}{x^2+1}$

Exercise 4.3.3. Sketch the graphs of the following functions involving radicals. Pay special attention to the domain.

(a) $\sqrt{4 - x^2}$
(b) $\sqrt{x+2} - 1$

Exercise 4.3.4. Sketch the graphs of the following functions involving the floor function and fractional part.

(a) $f(x) = \lfloor x/2 \rfloor$
(b) $g(x) = \{x\} - \frac{1}{2}$
(c) $h(x) = x - \lfloor x \rfloor^2$
(d) $k(x) = \lfloor \sqrt{x} \rfloor$
(e) $\ell(x) = \sqrt{\lfloor x \rfloor}$

Exercise 4.3.5. Sketch the graphs of the following functions involving absolute values.

(a) $f(x) = |x^2 - 4|$
(b) $g(x) = |x - 1| + |x + 2|$

4.4 SOME USES OF GRAPHS

In this section, we demonstrate how graphs afford crucial understanding across a diverse array of situations. Furthermore, we aim to convey the indispensability of graphical insight—a perspective that a purely analytic or algebraic approach can never fully replace, owing to the unique qualitative information captured by the visual nature of graphs. We do not, however, claim to illustrate every possible application of graphs.

4.4.1 Image in terms of graphs. Many set theoretic concepts find geometric meaning in the language of graphs. Recall that the **image** (or range) of a function $f : X \to \mathbb{R}$ is the set of all possible output values. We can determine the image of a function directly from its graph. It is a simple but powerful observation that a point y is in the image of f if and only if the horizontal line passing through $(0, y)$ intersects the graph of f at least once. In other words, the image is the set of all y-values "hit" by the graph.

Consider the function $f(x) = x^2$. If we draw a horizontal line $y = c$, it will intersect the parabola only if $c \geq 0$. The line $y = 4$ intersects the graph at two points, $x = 2$ and $x = -2$. The line $y = 0$ intersects at one point, $x = 0$. The line $y = -1$ never intersects the graph. Therefore, by observing all possible horizontal lines, we can conclude that the image of $f(x) = x^2$ is the set $[0, \infty)$. ♢

4.4.2 Injectivity and the horizontal line test. The concept of injectivity also has a famous graphical transcription. A function f is injective if and only if every horizontal line intersects the graph of f at *at most one point.* This is famously known as the *horizontal line test.* If one can find even one horizontal line that cuts the graph in two or more places, the function is not injective. This is because multiple x-values are being mapped to the same y-value.

The function $x \mapsto x^3 : \mathbb{R} \to \mathbb{R}$ is injective. Any horizontal line one draws will intersect its graph exactly once. In contrast, $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$ is not injective, because the line $y = 4$ (and any other line $y = c$ for $c > 0$) intersects the graph at two distinct points. $\diamondsuit$

4.4.3 Solving equations: Finding points of intersection. A major advantage of graphs is their ability to visualize the solutions to an equation. Any equation is of the form

$$f(x) = g(x)$$

where f and g are functions defined on a common domain, say X (which, in our current setting, is a subset of $\mathbb{R}$). By **solutions** to this equation we mean the set of points $x \in X$ for which the above equation indeed holds. The key, though straightforward, observation is that the solutions are precisely the x-coordinates of the points where the graphs of f and g intersect. The graphical approach does not help us explicitly write down the solutions (and this may not be possible even with analytical tools) but frequently allows us to answer qualitative questions like finding the *number* of solutions. Let us illustrate this point via an example.

Suppose we want to find the number of solutions of the equation $2^x = x + 2$. We can graph the exponential curve $y = 2^x$ and the line $y = x + 2$. A quick sketch, shown in Figure 4.20, reveals that the two graphs intersect at two points.

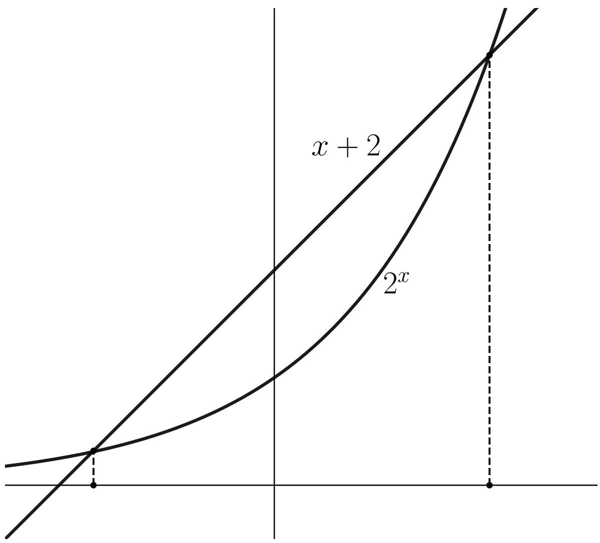


Figure 4.20

This tells us there are exactly two solutions. While finding their exact values may be difficult, we immediately know how many solutions to look for. $\diamondsuit$

4.4.4 Solving inequalities: Which function is on top. Graphs can help us study inequalities. The solutions to $f(x) > g(x)$ is the set of all x-values for which the graph of $f(x)$ lies *above* the graph of $g(x)$. This point is best conveyed via an example.

To solve $x^2 > x + 2$, that is, to find all real numbers x such that $x^2 > x + 2$, we graph the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 2$$

for all $x \in \mathbb{R}$. We find they intersect at $x = -1$ and $x = 2$. By observing the graph, we

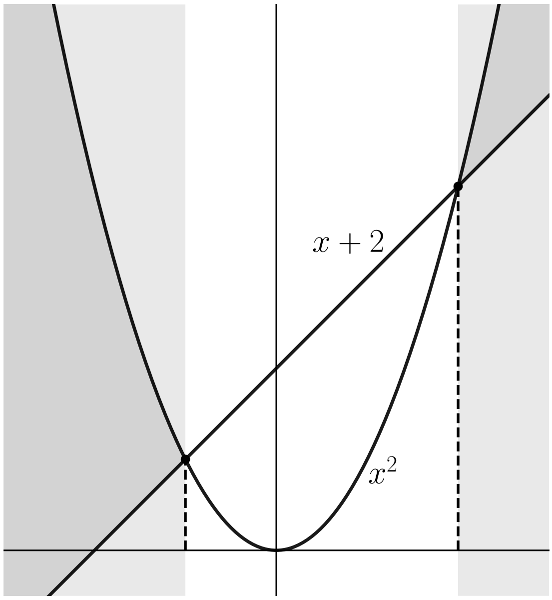


Figure 4.21

can see that the parabola (the curve in black) is above the line when $x < -1$ or when $x > 2$. Thus, the solution is $(-\infty, -1) \cup (2, \infty)$. See Figure 4.21. ♢

4.4.5 The maximum and minimum of two functions. Given two functions $f, g : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, we can define two new functions

$$h(x) = \max\{f(x), g(x)\} \quad \text{and} \quad m(x) = \min\{f(x), g(x)\}$$

Their graphs have a simple, intuitive construction.

- The graph of $\max\{f(x), g(x)\}$ is the "upper envelope" of the two individual graphs. At any given x, you simply choose the higher of the two y-values.
- The graph of $\min\{f(x), g(x)\}$ is the "lower envelope" of the two graphs.

This will be clear from the following example.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 2$$

for all $x \in \mathbb{R}$. In Figure 4.22, the graphs of f and g are shown in dashed, and the graph of $\max\{f, g\}$ is shown in solid. ♢

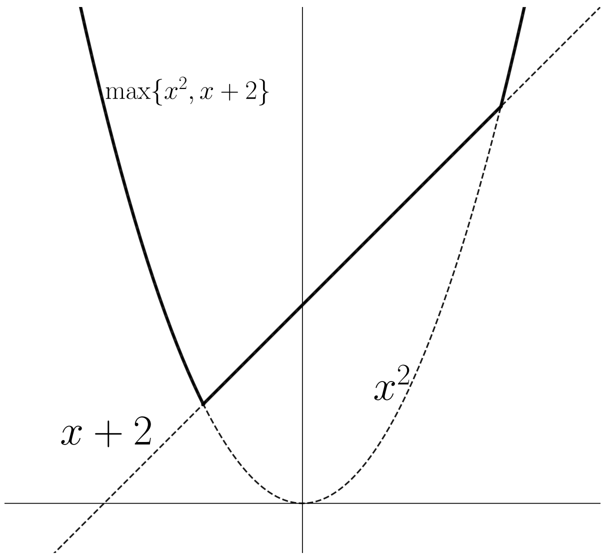


Figure 4.22

4.4.6 Visualizing dynamics: Recurrence relations. Consider a sequence of number $x_0, x_1, x_2, \ldots$ as per the following rule.

$$x_{n+1} = \sqrt{2 + x_n}, \text{ for all } n \geq 0$$

with the initializing value as $x_0 = 0$. What can we say about this sequence? Does it grow without bound? Or does it "converge" to a particular value? We see how graphs can *show* us exactly what happens! First we interpret the problem *dynamically.* Define the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$f(x) = \sqrt{2 + x}$$

for all $x \in \mathbb{R}$. Note that

$$f(x_0) = x_1, \quad f(x_1) = x_2, \quad f(x_2) = x_2, \quad \ldots, \quad f(x_n) = x_{n+1}, \quad \ldots$$

Thus the points of the given sequence are obtained by iterative applications of the function f to the *seed* that is x_0. We plot the graph of f as well as line "$y = x$", or, more precisely, the line

$$L = \{(x, y) : \; x = y\}$$

Starting at $x_0 = 0$, we move up to the curve, then across to the line, then up, then across. The cobweb diagram clearly shows the sequence increasing and converging to the point where $\sqrt{2 + x} = x$, which is $x = 2$.
The definition of the points $P_1, P_2, P_3, \ldots$ and $Q_1, Q_2, Q_3, \ldots$ should be clear from Figure 4.23. Note that the y-coordinate of P_1 is $f(0) = x_1$, which is same as the x-coordinate of Q_1. Next, the y-coordinate of P_2 is $f(\sqrt{2}) = \sqrt{2 + \sqrt{2}} = x_2$, which is same as the x-coordinate of Q_2. Thus, reasoning this way, we see that the x-coordinate of Q_n is x_n. From the diagram it is clear that the point Q_n moves closer and closer to the intersection of the two curves, namely the point $(2, 2)$. Therefore, we see that x_n becomes closer and

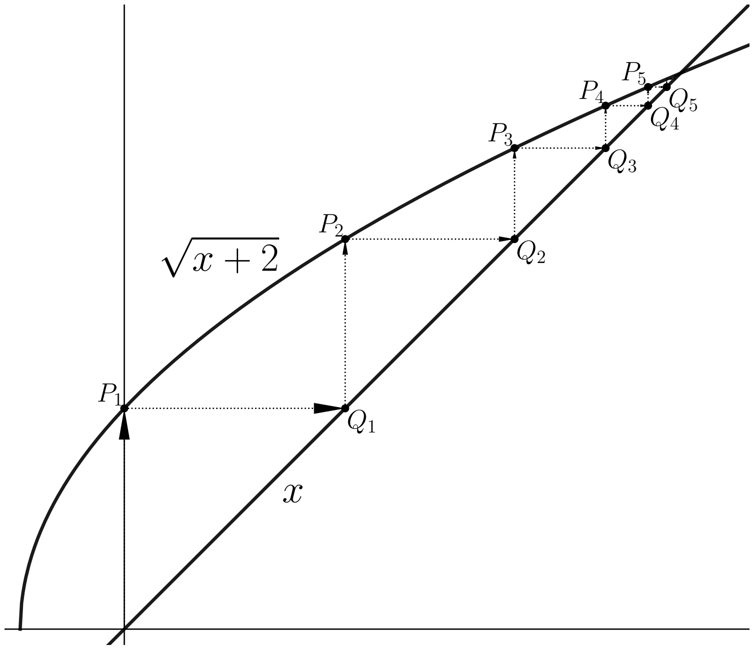


Figure 4.23

closer to 2 as n increases, but always remains less than 2. This result was not obvious from the definitions, and the graphical interpretation really shined a light on it.

This is a general technique and is not restricted to our special example. We record the above discussion in the general setting. Let $f : S \to S$ be a function, where $S \subseteq \mathbb{R}$. Consider a sequence $x_0, x_1, x_2, \ldots$ defined as

$$x_{n+1} = f(x_n), \text{ for all } n \geq 0$$

where x_0 is some given value. This is a simple *dynamical system* where the next value in a sequence is a function of the previous value. To see where the sequence goes, we can use a graphical method called a **cobweb plot**.

a) Plot the function $y = f(x)$ and the line $y = x$ on the same axes.

b) Start at an initial value x_0 on the x-axis.

c) Move vertically to the curve $y = f(x)$. The y-coordinate of this point is $f(x_0)$, which is x_1.

d) Move horizontally from this point to the line $y = x$. The x-coordinate of this new point is also x_1.

e) Repeat: move vertically to the curve $y = f(x)$ to find x_2, then horizontally to the line $y = x$, and so on.

The sequence of points on the x-axis, $x_0, x_1, x_2, \ldots$, shows the "trajectory" of the system. ♢

4.5 A COMMON FALLACY

Suppose we want to find all the solutions of the following equation.

$$|x-5| = 2x+2 \tag{*}$$

Here is one way to do it. We first "remove" the modulus sign by squaring both sides. This is a standard algebraic move, relying on the fact that $|a|^2 = a^2$.

$$\begin{aligned}(|x-5|)^2 = (2x+2)^2 \quad &\Rightarrow \quad (x-5)^2 = (2x+2)^2\\ &\Rightarrow \quad x^2 - 10x + 25 = 4x^2 + 8x + 4\\ &\Rightarrow \quad 0 = x^2 + 6x - 7 \quad \Rightarrow \quad 0 = (x+7)(x-1)\end{aligned}$$

So we conclude that 7 and 1 are the solutions of the original equation. But let us check these results against (*). Setting $x = 1$ we get that the left hand side and the right hand side are

$$|1-5| = |-4| = 4 \quad \text{and} \quad 2(1)+2 = 4$$

respectively. This is as expected. However, setting $x = 7$ we see that the left hand side and the right hand side are

$$|-7-5| = |-12| = 12 \quad \text{and} \quad 2(-7)+2 = -12$$

These two values do not match!

The value $x = -7$ is an *extraneous solution.* It emerged from the algebra, yet it fails the original problem. What is the source of this apparent contradiction?

4.5.1 The core fallacy: Implication versus equivalence. Why did the algebra produce a liar? The issue lies in the logical direction of the steps taken. As we manipulate at the given equation and morph it into others, we implicitly assume that the solution set of any intermediate equation is exactly the same as that of the predecessor. But this is where the error lies. As we morph the equations, the solution set of each successive equation is at least as large as that of its predecessor, and, in some steps, can be strictly larger. A simple and rather trite example of this is the following: Suppose we have the equation $x = 1$. Clearly, it has only one solution, namely 1. But if we square both the sides, we get the equation $x^2 = 1$, which has two solutions.

The proper reasoning during any manipulation is this: Suppose we have an equation E of the form $f(x) = g(x)$ whose solution set is S. This means that each x in the set S satisfies the equation $f(x) = g(x)$. Now if we "square both the sides" (or perform any other operation), we get the equation $f(x)^2 = g(x)^2$. So what we have really said is that if x satisfies $f(x) = g(x)$, then x also satisfies $f(x)^2 = g(x)^2$. The confusion arises because we confuse the direction of implication. We implicitly think that $f(x)^2 = g(x)^2$ implies $f(x) = g(x)$. Thus, if T is the solution set of $f(x)^2 = g(x)^2$, then all we can say is that $S \subseteq T$, and not that $S = T$.

The correct way to view the solution process is not as "finding the answer," but as "trapping the answer." Solving an equation is a process of narrowing down the universe of real numbers to a small finite set of *candidates.* Let S be the set of true solutions. Let

C be the set of numbers found at the end of your algebraic (or any) manipulation. Then all we can say just yet is that

$$S \subseteq C$$

The manipulation guarantees that the true answer lies inside your final set C. It does *not* guarantee that every member of C is a true answer. So here is the correct workflow:

a) *Derivation (Finding Candidates):* Perform operations (squaring, multiplying, factoring etc) to arrive at a simple set of roots. Acknowledge that you are merely finding a set of suspects, C.

b) *The Filter (Verification):* This is not an optional "check your work" step; it is a logical necessity. You must substitute each candidate back into the original equation to see if it belongs to S.

c) *Conclusion:* Report only the survivors of the filter.

4.5.2 Double Implication. Fortunately, not every algebraic maneuver expands the solution set. Many standard operations are *reversible*, meaning they preserve the truth value of the equation in both directions. In logical terms, these steps constitute an equivalence, denoted by $\iff$.

Consider the operation of adding a constant c to both sides of an equality $f(x) = g(x)$. Since addition has a definitive inverse (subtraction), we can confidently say:

$$f(x) = g(x) \qquad \iff \qquad f(x) + c = g(x) + c.$$

Because the implication holds both ways, no solutions are created and none are lost. More generally, applying any *bijective* (one-to-one and onto) function to both sides of an equation preserves the solution set exactly. Common examples of such safe operations include (but are certainly not limited to):

- Multiplying or dividing by a non-zero constant.
- Cubing both sides (since the map $x \mapsto x^3$ is a bijection on $\mathbb{R}$).
- Taking the exponential (since $x \mapsto e^x$ is injective).

In these cases, the set of candidates C we derive is identical to the set of true solutions S. The danger of extraneous solutions arises solely when we apply non-injective functions—such as squaring (which maps both x and $-x$ to x^2) or multiplying by an expression involving x that might be zero. These actions collapse distinct values into the same output, muddying the logical trail back to the original statement. ♢

4.5.3 Summary. Just because an equation looks friendly at the end of a derivation does not mean it is trustworthy. Algebraic manipulation frequently expands the solution set, creating "ghosts" or extraneous solutions. In the modulus example, the squaring operation expanded the solution set to include the solution for $-(x - 5) = 2x + 2$, which was not the question asked. The only defense against this is the rigorous distinction between a *candidate* and a *solution*. ♢

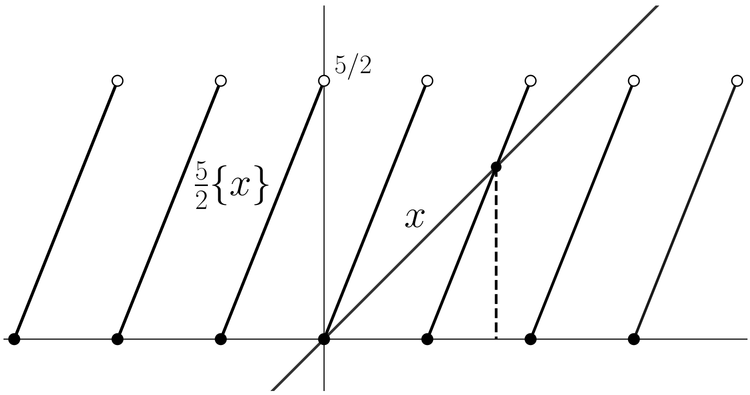
4.6 SOME SOLVED EXAMPLES

Illustration 4.6.1 Find all the real numbers x which satisfy $4\{x\} = x + \lfloor x \rfloor$.

Solution. Note that for any real number x we have

$$4\{x\} = x + \lfloor x \rfloor \iff 5\{x\} = x + (\lfloor x \rfloor + \{x\}) \iff 5\{x\} = x + x \iff \frac{5}{2}\{x\} = x$$

This manipulation puts the given equation in the form $f(x) = g(x)$, where f and g are functions whose graph we can easily draw. Here $f : \mathbb{R} \to \mathbb{R}$ is the function defined as $f(x) = \frac{5}{2}\{x\}$ for all x and $g : \mathbb{R} \to \mathbb{R}$ is defined as $g(x) = x$ for all x. Thus the graph of f is a vertically stretched version of the graph of the fractional part function, as explained in Point 4.3.4. We get the following plot.



It is immediate from the above graph that there are exactly two solutions to the given equation, one of which is $x = 0$, and the other can also be found using simple geometry. Illuminated by this, let us now solve the equation analytically. Suppose x is such that $\frac{5}{2}\{x\} = x$. Then clearly x cannot be negative. Also, we have

$$\left|\frac{5}{2}\{x\}\right| = |x| \quad\Rightarrow\quad \frac{5}{2}|\{x\}| = |x| \quad\Rightarrow\quad \frac{5}{2}|x| > |x| \quad\Rightarrow\quad \frac{5}{2} > |x|$$

So each solution of this equation is in the range $[0, \frac{5}{2})$. Let x be a solution of the equation. We make three cases.

- *Case 1:* $0 \le x < 1$.

  Clearly $x = 0$ is a solution. Let us see if there can be another solution in this region. Say $0 < x < 1$. Then

  $$\frac{5}{2}\{x\} = x \quad\Rightarrow\quad \frac{5}{2}x = x \quad\Rightarrow\quad \frac{5}{2} = 1$$

  which is a contradiction. So there is only one solution in $[0, 1)$, namely 0.

- *Case 2:* $1 \leq x < 2$.

  Here we have

  $$\frac{5}{2}\{x\} = x \qquad \Rightarrow \qquad \frac{5}{2}(x-1) = x \qquad \Rightarrow \qquad x = \frac{5}{3}$$

  Indeed, $5/3$ lies in $[1, 2)$ and satisfies the given equation. So there is exactly one solution in the range $[1, 2)$, namely $5/3$.

- *Case 3:* $2 \leq x < 5/2$.

  Here we have

  $$\frac{5}{2}\{x\} = x \qquad \Rightarrow \qquad \frac{5}{2}(x-2) = x \qquad \Rightarrow \qquad x = \frac{10}{3}$$

  But $10/3$ does not lie in the range $[2, 5/2)$ so there is no solution here.

Therefore, there are exactly two solutions to the given equation, and they are 0 and $5/3$. ■

Illustration 4.6.2 Find all the real numbers x which satisfy $\lfloor x \rfloor - 1 + x^2 \geq 0$.

Solution. The given inequality is equivalent to

$$\lfloor x \rfloor \geq 1 - x^2 \tag{*}$$

This is a simple but crucial observation since we understand the graphs of both the functions defined by the two expressions above, while that was not the case for $\lfloor x \rfloor - 1 + x^2$. We may read Equation (*) as $f(x) \geq g(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is the floor function and $g : \mathbb{R} \to \mathbb{R}$ is the function defined as $g(x) = 1 - x^2$. We will use what we learnt in Point 4.4.4.

We know how to draw the graph of f. How do we draw the graph of g? We first think of $1 - x^2$ as $-(x^2 - 1)$. So the graph of g is obtained by reflecting the graph of the function given by $x^2 - 1$ about the x-axis (see Point 4.3.4). Now the graph of $x^2 - 1$ is a vertical translate of the graph of x^2 by one unit, as explained in Point 4.3.2. Putting this all together, we have in Figure 4.24 the following plot, which immediately shows that the set of all the x satisfying the given inequality is of the form $(-\infty, a] \cup [b, \infty)$. The values of a and b can be calculated with little effort from the picture above. We proceed to supply an analytical solution based on the intuition just gained. Suppose x satisfies the inequality:

$$\lfloor x \rfloor \geq 1 - x^2$$

- *Case 1:* $x \geq 1$. A moment's thought shows to the reader that every x with $x \geq 1$ satisfies the inequality at hand. Thus each such x is in the solution set.

- *Case 2:* $0 \leq x < 1$.

  Here $\lfloor x \rfloor = 0$. Thus

  $$\lfloor x \rfloor \geq 1 - x^2 \qquad \Rightarrow \qquad 0 \geq 1 - x^2 \qquad \Rightarrow \qquad x^2 \geq 1 \qquad \Rightarrow \qquad x \notin [0, 1)$$

  So there is no solution to the inequality in the region $[0, 1)$.

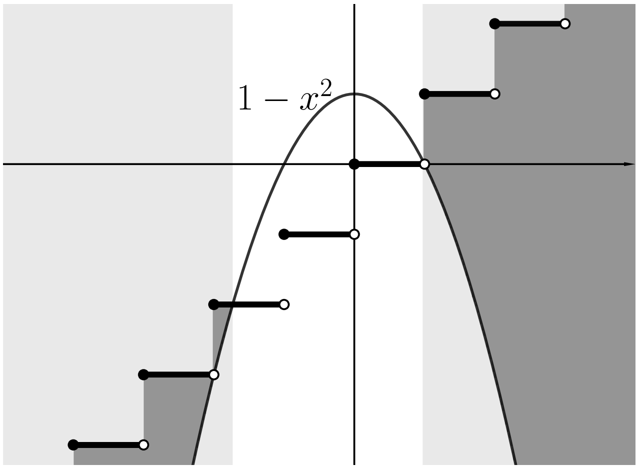


Figure 4.24

- *Case 3:* $-1 \leq x < 0$.

  Here $\lfloor x \rfloor = -1$. We have

  $$\lfloor x \rfloor \geq 1 - x^2 \qquad \Rightarrow \qquad -1 \geq 1 - x^2 \qquad \Rightarrow \qquad x^2 \geq 2$$

  This means $x \geq \sqrt{2}$ or $x \leq -\sqrt{2}$. But then $x \notin [-1, 0)$. Thus there are again no solutions in this region.
- *Case 4:* $-2 \leq x < -1$.

  Here $\lfloor x \rfloor = -2$. We have

  $$\lfloor x \rfloor \geq 1 - x^2 \qquad \Rightarrow \qquad -2 \geq 1 - x^2 \qquad \Rightarrow \qquad x^2 \geq 3$$

  This means $x \geq \sqrt{3}$ or $x \leq -\sqrt{3}$. Thus x must lie in the intersection of

  $$[-2, -1) \quad \text{and} \quad (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$$

  Since $-\sqrt{3} \approx -1.732$, the intersection is the interval $[-2, -\sqrt{3}]$. A simple check shows that each point in $[-2, -\sqrt{3}]$ is indeed a solution to our inequality. So this is part of the solution set.
- *Case 5:* $x < -2$.

  Fix $x < -2$. We will show that $\lfloor x \rfloor \geq 1 - x^2$. Let n be such that $-(n+1) \leq x < -n$, where $n \geq 2$. Then $\lfloor x \rfloor = -(n+1)$. From $-(n+1) \leq x < -n$ we get

  $$n^2 \leq x^2 \leq (n+1)^2 \qquad \Rightarrow \qquad 1 - (n+1)^2 \leq 1 - x^2 \leq 1 - n^2$$

  But if $n \geq 2$ then $1 - n^2 \leq -(n+1)$. This is because

  $$1 - n^2 \leq -(n+1) \qquad \Longleftrightarrow \qquad 0 < n^2 - n - 2 \qquad \Longleftrightarrow \qquad 0 < (n-1)(n-2)$$

  which is true. Thus

  $$1 - x^2 \leq \lfloor x \rfloor$$

  if $x \leq -2$. Thus our inequality indeed holds, and hence $(-\infty, 2]$ is part of the solution set.

Combining the results of the above cases, we see that the set of all the solutions to the given inequality is

$$(-\infty, -\sqrt{3}] \cup [1, \infty)$$

■

Exercise 4.6.1. Find the number of solutions of $\lfloor x \rfloor = \{x\}$.

Exercise 4.6.2. Using a graph, find the solution set for the inequality $|x-2| < \frac{x}{2}$. Further, justify your answer with a detailed analytical solution.

Exercise 4.6.3. Find all real numbers x such that $-1 \leq \lfloor x \rfloor - x^2 + 4 \leq 2$.

Exercise 4.6.4. Given the graph of a function $f(x)$, describe the sequence of transformations needed to obtain the graph of $g(x) = -2f(x+3) - 1$.

Exercise 4.6.5. Based on a rough graphical sketch, how many solutions does the equation $\log(x) = \frac{1}{x}$ have?

Exercise 4.6.6. Sketch the graph of $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \min\{x^2, 4\}$.

Exercise 4.6.7. A function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \lceil x \rceil - \lfloor x \rfloor$$

for all $x \in \mathbb{R}$. What are the possible output values of this function? Can you sketch its graph?

4.7 FINDING DOMAIN AND RANGE

4.7.1 An important remark. Many questions pertaining to functions at the high school level present a complicated expression—usually featuring the symbol x embedded within a chaotic mix of standard functions—and ask the reader to find the domain. Given our previous discussion on functions, this framing is technically meaningless; a function is defined with its domain, so asking to find it retrospectively makes little sense. However, we will continue to use this language below to align ourselves with tradition. Consequently, when we ask "what is the domain of $\sqrt{x-2}$" we are effectively asking, "what is the set of all real numbers x such that the expression $\sqrt{x-2}$ yields a valid real number?"

On that note, when asked to find the range of an expression featuring x (or y, or some other variable), what is being asked is to find the set of all the real values that the expression can take when x is varied over the domain (in the sense discussed above) of the expression. ♢

Illustration 4.7.2 Find the domain and range of

$$a)\ \frac{9}{x-6}, \quad b)\ \frac{3x+1}{4x+2}, \quad c)\ \frac{x}{x}$$

Solution. a) This is a rational function. For the function to be defined, the denominator cannot be zero. Therefore, the domain is the set of all real numbers except 6. In symbols,

$$\text{Domain} = \mathbb{R} \setminus \{6\}$$

To find the range, we want to find the set of all the real y such that there is an $x \neq 6$ satisfying

$$y = \frac{9}{x-6} \tag{*}$$

Rearranging the equation to isolate x, and remembering that $x \neq 6$, we have

$$y(x-6) = 9 \quad \Longleftrightarrow \quad x - 6 = \frac{9}{y} \quad \Longleftrightarrow \quad x = \frac{9}{y} + 6$$

Thus we can find an $x \neq 0$ satisfying (*) as long as $y \neq 0$. Therefore the range is the set of all the nonzero real numbers.

(b) Again, this is a rational function. For the function to be defined, the denominator cannot be zero. The only value of x that makes the denominator vanish is $-1/2$ and thus this is the only point missing from the domain of the given expression. In symbols,

$$\text{Domain} = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$$

To find the range, we want to find the set of all real y such that there is an $x \neq -\frac{1}{2}$ satisfying

$$y = \frac{3x+1}{4x+2}$$

Rearranging the equation to isolate x in terms of y, while remembering that $x \neq -1/2$

$$y(4x+2) = 3x+1 \quad \Longleftrightarrow \quad 4xy + 2y = 3x + 1 \quad \Longleftrightarrow \quad x = \frac{1-2y}{4y-3}$$

Thus we can find a valid x satisfying the equation as long as the denominator is not zero. Therefore, $4y - 3 \neq 0$, which implies $y \neq \frac{3}{4}$. The range is the set of all real numbers except $\frac{3}{4}$.

$$\text{Range} = \mathbb{R} \setminus \{3/4\}$$

(c) The expression is defined if and only if the denominator is not zero. Therefore, the domain is the set of all real numbers except 0. For any x in the domain (where $x \neq 0$), the fraction simplifies to 1. Thus the range is $\{1\}$. ■

Before we proceed let us look at an alternate way to find the range in part (a) and (b) of the above problem. We will employ a "divide and conquer" approach. For part (a), define the function $f : \mathbb{R} \setminus \{6\} \to \mathbb{R}$ as

$$f(x) = \frac{9}{x-6}$$

We can express f as the composition of two functions as follows. Define $g : \mathbb{R} \setminus \{6\} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = 1/(x-6) \quad \text{and} \quad h(x) = 9x$$

for all x. Then $f = h \circ g$. We want to find the range (or image) of f, which is same as the range of $h \circ g$. But

$$\text{Image}(h \circ g) = h(g(\mathbb{R} \setminus \{6\}))$$

What is $g(\mathbb{R} \setminus \{6\})$? It is nothing but the image of g, and, by Point 4.4.1, it is the same as the set of all the real numbers such that the horizontal line corresponding to them meets the graph of g somewhere. But the graph of g is just a horizontal translate (Point 4.3.1) of the graph of the reciprocal function ($x \mapsto 1/x$), and hence the image of g is same as the image of the reciprocal function. Thus the image of g is $\mathbb{R} \setminus \{0\}$. Now h is nothing but the "scale by 9" function, and hence the image of f is same as the set obtained by scaling each output of g by 9, which is, again, all the nonzero reals.

For part (b) define $f : \mathbb{R} \setminus \{-1/2\} \to \mathbb{R}$ as

$$f(x) = \frac{3x+1}{4x+2}$$

Observe that

$$f(x) = \frac{(3x+3/2)-1/2}{4x+2} = \frac{3}{4} - \frac{1}{8} \cdot \frac{1}{x+1/2}$$

Thus, the graph of f is a vertical translate (by 3/4) of the graph of the function given by $1/(8(x+1/2))$. The image of the latter is same as all non-negative reals (by an argument similar to the one given for part (a)), and hence by Point 4.3.2 we can now deduce that the image of f is nothing but $\mathbb{R} \setminus \{3/4\}$.

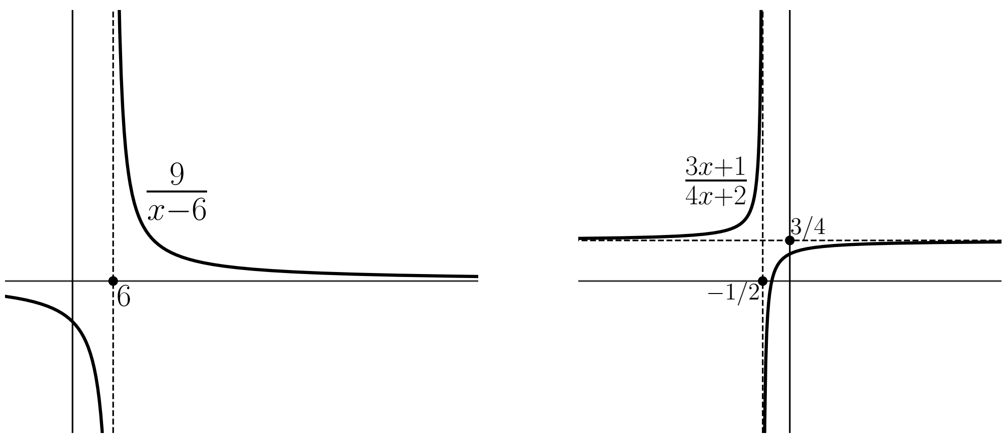


Illustration 4.7.3 Find the domain and range of

$$\frac{x^2-9x}{x^2-81}$$

Solution. The function is undefined where the denominator is zero. The denominator is

$$x^2-81=(x-9)(x+9)$$

and hence it vanished precisely when x is in $\{-9,9\}$. Therefore, the domain is all real numbers except 9 and -9.

$$\text{Domain}=\mathbb{R}\setminus\{9,-9\}$$

We simplify the expression (let call it $f(x)$) for values within the domain (where $x\neq 9$ and $x\neq -9$):

$$f(x)=\frac{x(x-9)}{(x+9)(x-9)}=\frac{x}{x+9}$$

To find the range, we need to find the set of all the real numbers y such that $y=\frac{x}{x+9}$ while x varies in the domain of $f(x)$. Solving for x in terms of y we get

$$\begin{aligned} y(x+9)=x \quad &\iff \quad xy+9y=x \\ &\iff \quad 9y=x-xy \quad \iff \quad x=\frac{9y}{1-y} \end{aligned}$$

From this relation, we see that $y\neq 1$. Additionally, we must check the specific point excluded from the domain that was "cancelled out" during simplification ($x=9$). If x were 9, y would be:

$$y=\frac{9}{9+9}=\frac{9}{18}=\frac{1}{2}$$

Since x cannot be 9, y cannot be $\frac{1}{2}$. The restriction $x\neq -9$ corresponds to the vertical asymptote which is covered by $y\neq 1$. Thus, the range is all real numbers excluding 1 and $\frac{1}{2}$.

$$\text{Range}=\mathbb{R}\setminus\left\{1,\frac{1}{2}\right\}$$

The reader is invited to arrive at the same conclusion graphically. ■

Illustration 4.7.4 Find the range of

$$\frac{\sqrt{x+4}}{x-4}$$

Solution. Here we give an intuitive yet informal solution (which can be made rigorous using calculus). A proper solution will be seen in Illustration 6.1.3 It would be convenient to change variables. Write t in place of $x-4$. As x varies over all real numbers, so does t. Thus we may reformulate the given question to finding the range of

$$\frac{\sqrt{(t+4)+4}}{t}=\frac{\sqrt{t+8}}{t}$$

Of course, we need the radicand to be non-negative and the denominator to be nonzero, and hence t ranges only over $[-8, \infty) \setminus \{0\}$. The above expression can also be written as

$$\sqrt{\frac{t+8}{t^2}} \text{ if } t > 0 \quad \text{and} \quad -\sqrt{\frac{t+8}{t^2}} \text{ if } t \in [-8, 0)$$

Now if t is a very small positive quantity, then the above yields a very large positive value for our function. As t grows larger and larger, the expression will take arbitrarily small positive values (since t^2 grows faster than $8 + t$). Therefore, as t ranges over positive reals, the function takes all positive values.

Similarly, if t is a negative quantity of small magnitude, then the above yields a very large negative quantity. As t reaches -8, this value drops to 0. Thus as t ranges over $[-8, 0)$, the function takes all non-positive values.

Summarizing, the range of the function is all of $\mathbb{R}$. ■

Exercise 4.7.1. Find the domain and range of a function $f(x) = (2x - 1)/(x + 4)$.

Exercise 4.7.2. Find domain and range:

$$a)\ \frac{5}{\sqrt{x-3}}, \quad b)\ \frac{2x+1}{\sqrt{5-x}}, \quad c)\ \frac{\sqrt{x-4}}{\sqrt{x-6}}$$

4.8 SKETCHING RELATIONS

Just as a function can be visualized as a curve on the Cartesian plane, so too can a relation. This graphical representation transforms a relation from an abstract set of ordered pairs into a geometric shape or region, allowing us to use our visual intuition to understand its properties. While the graph of a function must obey the strict rule that every input has exactly one output (the "vertical line test"), the graph of a relation has no such constraint. This freedom allows relations to describe a much wider universe of geometric forms: circles, ellipses, enclosed regions, and intricate patterns that a single function could never capture. We begin by enunciating the definition of a relation in the narrow context that we need it.

4.8.1 Relation from $S \subseteq \mathbb{R}$ to $T \subseteq \mathbb{R}$. Let S and T be subsets of the real numbers $\mathbb{R}$. Recall that a *relation from S to T* is a subset of the Cartesian product $S \times T$. Also recall that a *relation on S* is just a relation from S to S. If R is a relation from S to T, then we may depict this relation by plotting all the points of $\mathbb{R}^2$ that are there in R. Note that if $f : S \to T$ is a function, then the graph of f is nothing but the relation

$$\Gamma_f = \{(x, f(x)) : \ x \in S\}$$

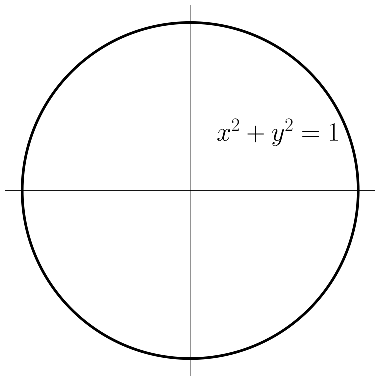
Example 4.8.2 (The Unit Circle: A Simple and Fundamental Example)**.** Consider the relation R on the set of real numbers $\mathbb{R}$ defined by the equation:

$$x^2 + y^2 = 1$$

Formally, the relation is the set

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

From the distance formula seen in Point 4.1.3, we know this equation describes the set of all points (x, y) whose distance from the origin $(0, 0)$ is exactly 1, which is nothing but the circle of unit radius centered at the origin. We will refer to this as the **unit circle**. The sketch is shown below.



Notice that this sketch fails the vertical line test. For instance, the line $x = 0.5$ intersects the circle at two points, $y = \sqrt{0.75}$ and $y = -\sqrt{0.75}$. This is why this relation cannot be described by a single function $f : \mathbb{R} \to \mathbb{R}$.

4.8.3 Relation using level sets. A powerful way to define a relation is by using a function of two variables. Let S and T be subsets of $\mathbb{R}$, and consider a function $f : S \times T \to \mathbb{R}$.

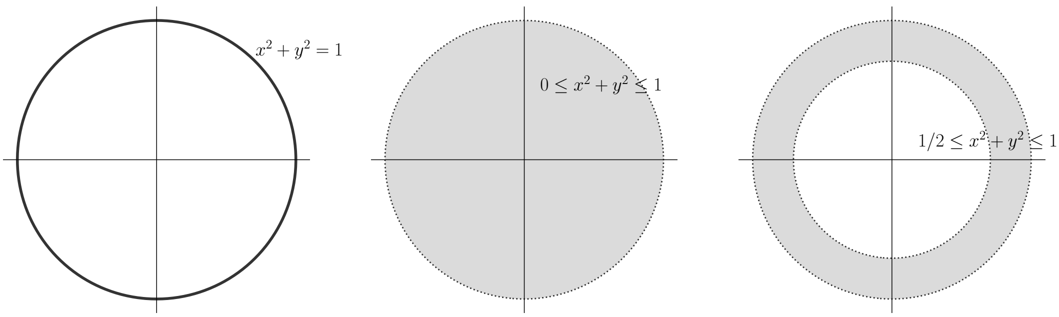
> *Definition.* For any constant $c \in \mathbb{R}$, the **level set** of f for the value c is defined as the set of all points (x, y) in the domain where the function's output is exactly c.

In other words, the level set of f for the value c is nothing but the fiber of f above c, that is, $f^{-1}(c)$. Since any level set of f is a subset of $S \times T$, it is also a relation from S to T. In symbols, this relation is

$$\{(x, y) \in S \times T : \; f(x, y) = c\}$$

Of course, we may take the pre-image of any subset of $\mathbb{R}$ under f and obtain a relation from S to T. Let us see how the unit circle discussed above can be recovered, as well as many more interesting relations, as a level set of a suitable function. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) = x^2 + y^2$.

- The unit circle is nothing but the level set of f for the value 1, or, succinctly, $f^{-1}(1)$.
- Different level sets of f produce concentric circles.
- Further, $f^{-1}([0,1])$ is the disc of unit radius centered at the origin, and $f^{-1}([1/2,1])$ is the annular region between the circle of radius 1/2 and the unit circle (both centered at the origin).



The above illustrates a powerful and robust way to generate relations. In fact, every relation from S to T can be obtained this way, in a rather trivial fashion. Suppose R is a relation from S to T. Then define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x) = \begin{cases} 1 \text{ if } x \in R \\ 0 \text{ otherwise} \end{cases}$$

Clearly the level set of f for the value 1 is nothing but the relation R. For this reason, for the rest of the section, we will use this "functional perspective" to illustrate the notions regarding relations on $\mathbb{R}$, or, more generally, from $S \subseteq \mathbb{R}$ to $T \subseteq \mathbb{R}$. $\diamondsuit$

We will now explore transformation principles, as we did for graphs of functions. The same principles of translation and scaling that we applied to the graphs of functions in Section 4.3 can be applied to the graphs of relations. This allows us to take a basic shape and move, stretch, or compress it to create a whole family of related graphs.

4.8.4 Translation. Recall that for any two real numbers h and k, we can define the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\tau(x,y) = (x+h, y+k)$$

for all $(x,y) \in \mathbb{R}^2$. Also recall that each translation is invertible.

Let us look at circle example and see how the sketch of our relation changes as we pre-compose the defining function by a translation. We have a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. The relation C (which we call the unit circle) is $f^{-1}(1)$. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined as $\tau(x,y) = (x-2, y-1)$ for all $(x,y) \in \mathbb{R}^2$. Clearly, τ is a translation.

Define $g = f \circ \tau$. Explicitly, this map is given by

$$g(x,y) = (x-2)^2 + (y-1)^2$$

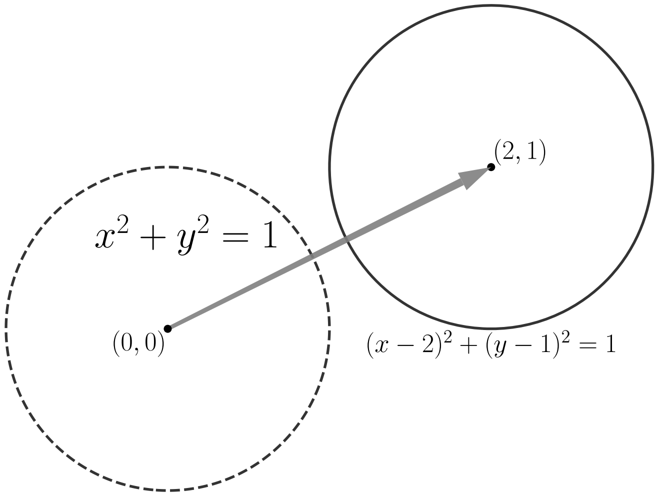


Figure 4.25: Example of translating a relation.

for all $(x, y) \in \mathbb{R}^2$. The level set $g^{-1}(1)$ is the unit circle with its center translated to the point $(2, 1)$. See Figure 4.25 In fact,

$$g^{-1}(1) = \tau^{-1}(f^{-1}(1))$$

Thus we see that pre-composing f by a translation τ ends up translating the level set of f for the value 1 by the inverse of the translation τ. ♢

4.8.5 Scaling. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = x^2 + y^2$$

and let $C = f^{-1}(1)$. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\sigma(x, y) = (x/2, 3y)$$

for all $(x, y) \in \mathbb{R}^2$. Note that σ is a bijection. What σ does is scale the x and the y-axes by factors 2 and 3 respectively. Define $g : \mathbb{R}^2 \to \mathbb{R}$ as

$$g(x, y) = f(\sigma(x, y)) = f(x/2, 3y)$$

The level set $g^{-1}(1)$ is a squished version (in both x and y-directions) of f^{-1}. This is shown in Figure 4.26 ♢

Example 4.8.6 (A More Exotic Relation.)**.** Our framework allows for the natural creation of much richer relations. Only our imagination is the limit. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \lfloor x^2 \rfloor - \lfloor 2y^2 \rfloor$$

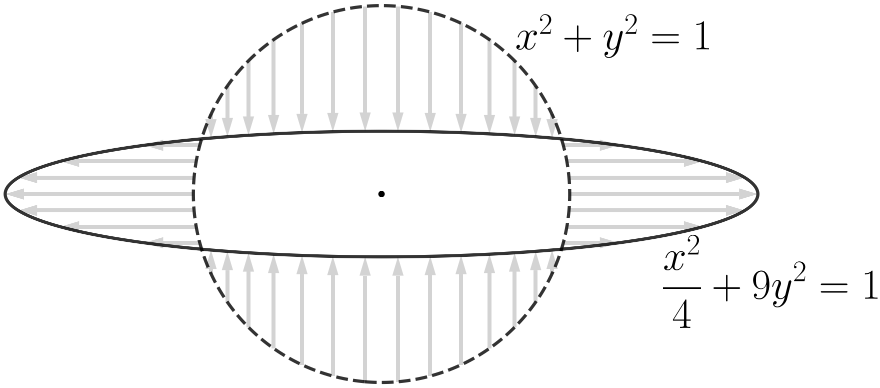
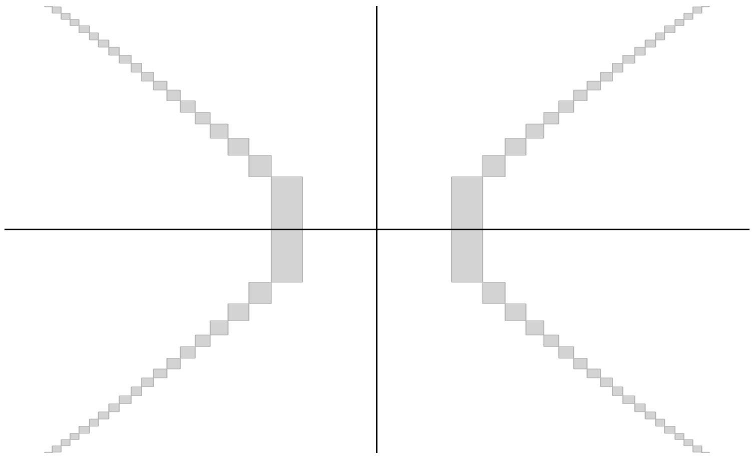


Figure 4.26: Example of scaling a relation.



The level set of f for the value 1 is shown above.

Illustration 4.8.7 What is the number of integral value(s) of k for which the subsets of $\mathbb{R}^2$ given by

$$y = \sqrt{-x^2 - 2x} \quad \text{and} \quad x + y - k = 0$$

have exactly 2 distinct points in common.

Solution. The set of points (x, y) in $\mathbb{R}^2$ which satisfy $y = \sqrt{-x^2 - 2x}$ are precisely those which satisfy

$$-x^2 - 2x \geq 0, \quad y \geq 0 \quad \text{and} \quad y^2 = -x^2 - 2x$$

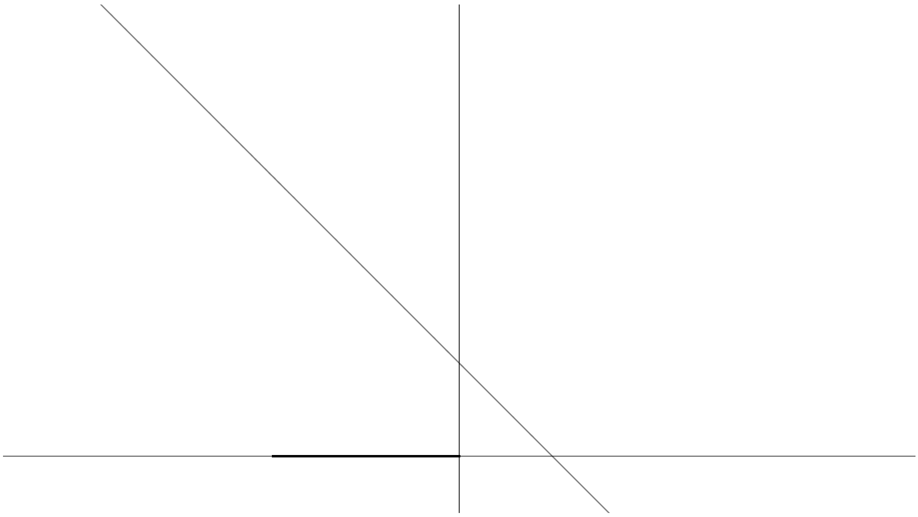
which is same as the set of points (x, y) satisfying

$$y \geq 0 \quad \text{and} \quad y^2 + x^2 + 2x = 0$$

which in turn is same as the set of points (x, y) satisfying

$$(x + 1)^2 + y^2 = 1 \quad \text{and} \quad y \geq 0$$

Using Point 4.8.4 we know that this is the "upper half" of the circle with radius 1 and $(-1, 0)$ as the center. Also, for any k, the set of point (x, y) satisfying $x + y - k = 0$ is the line inclined at an angle of $135°$ from the positive direction of x-axis and passing through $(k, 0)$. We can answer the question with this geometric understanding. The main idea is extant in the figure below.



For exactly two different points in common, the value of k should be chosen such that the line corresponding to k intersects the semicircle in exactly two distinct points. From the diagram, it is clear that the set of such values of k is the interval $(0, \sqrt{2} - 1)$. ■

CHAPTER 5

LINEAR FUNCTIONS

In our exploration of functions, we have seen various shapes and forms, from the gentle curve of a parabola to the sharp steps of the floor function. We now turn our attention to the simplest, yet arguably one of the most important, families of functions: the linear functions. Their name gives away their most defining characteristic—their graph is always a straight line.

Life is full of linear relationships. The cost of filling a car with gasoline is a linear function of the number of litres you pump. The distance a car travels at a constant speed is a linear function of time. Understanding these functions provides the foundation for modeling a vast array of phenomena in science, economics, and engineering. In this chapter, we will dissect the anatomy of a straight line, exploring how two simple numbers, the slope and the intercept, completely define its position and steepness in the Cartesian plane.

5.1 SLOPE, INTERCEPT, AND GRAPH

5.1.1 Definitions. We start by recalling the definition of a linear function we saw earlier.[1]

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is called a **linear function** if there exist real numbers m and c such that

$$f(x) = mx + c$$

for all $x \in \mathbb{R}$.

The constant m is called the **slope** or **gradient** of the function. It determines the steepness and direction of the line. The constant c is called the y**-intercept**. It is the value of the function when $x = 0$, and it determines where the line crosses the vertical y-axis. ◊

[1]The definition we give is actually of *affine* linear functions, and affine linear relations.

Example 5.1.2. The simplest (non-trivial) examples of linear functions are the maps $f, g : \mathbb{R} \to \mathbb{R}$ defined as
$$f(x) = x \quad \text{and} \quad g(x) = -x$$
for all $x \in \mathbb{R}$. Let's try to draw the graph of f. We have

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : \ y = f(x)\} = \{(x, y) \in \mathbb{R}^2 : \ x = y\}$$

So the graph of f consists of all the points in the Cartesian plane which have the same x and y-coordinates. A simple geometric consideration shows that this is exactly the line passing through the origin at an angle of $45°$ from the positive direction of the x-axis. Indeed, the origin surely lies on the graph of f. Further, if $a > 0$, and $P = (a, a)$, $Q = (a, 0)$, then the triangle formed by the origin O, the point P, and the point Q is a right angled triangle, with the right angle at Q. Finally, noting that $PQ = OQ$ shows that $\angle POQ = 45°$. Conversely, if we pick any point on this line, it also lies on the graph. The reader is invited to justify this.

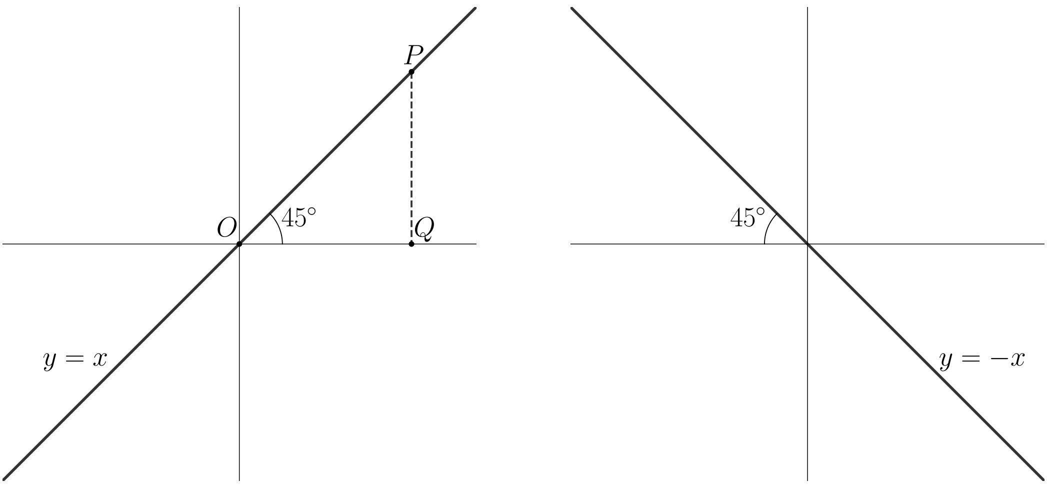


Figure 5.1: Figure showing lines with slope 1 and -1 respectively.

A similar geometric argument shows that the graph of g is the line passing through the origin, inclined at an angle $45°$ with the *negative* direction of the x-axis.

5.1.3 The graph of a linear function. We show that

The graph of any linear function is a straight line.

Let us first discuss the case when the intercept is 0. Let m be a real number and $f : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f(x) = mx$$

for all x. The graph of f is a line passing through the points $O(0, 0)$ and $P(1, m)$. Instead of giving a formal argument, let us show the truth of this in a simple example when $m = 2$. (We leave the reader to fill in the formal details.) We know from Example

5.1.2 that the graph of the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x$ for all x is a straight line passing through $O(0, 0)$ and $A(1, 1)$. By Point 4.3.4 we know that the graph of f is obtained by "stretching by a factor of 2" in the vertical direction. This is shown in Figure 5.2.

Similarly, if m is fixed, the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = mx + c$$

for various values of c is a translate of the graph corresponding to $c = 0$. This is an instance of Point 4.3.2. One example is depicted in Figure 5.3 where we have $m = 1/2$ and $c = 1$. Of course, the translate of straight line is again a straight line, and thus, from this discussion it is clear that, the graph of any linear function is a straight line. $\diamondsuit$

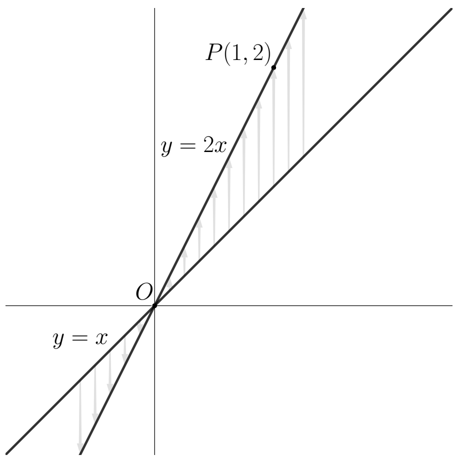


Figure 5.2: Graph of "$y = 2x$" from the graph of "$y = x$."

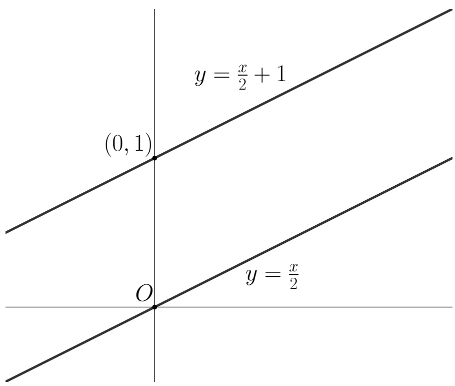


Figure 5.3: Graph of "$y = \frac{x}{2} + 1$" from the graph of "$y = \frac{x}{2}$."

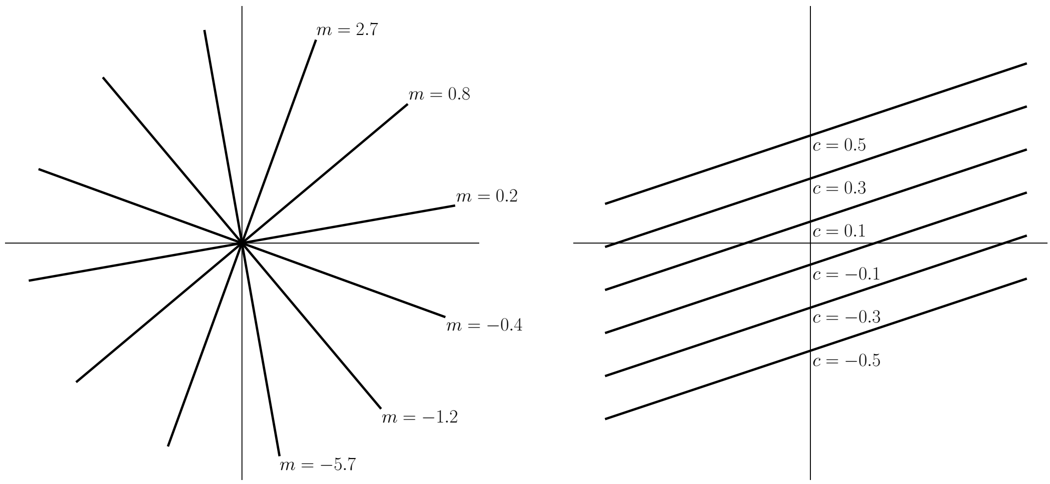


Figure 5.4: On the left: lines with intercept 0 but various slopes. On the right: lines with a fixed slope but varying intercepts.

5.1.4 Geometric interpretation of slope. The slope of a non-vertical line is the ratio of the vertical change (the "rise") to the horizontal change (the "run") between any two distinct points on the line. The reader familiar with a little bit of trigonometry will recognize this as $\tan(\theta)$ in the following diagram.

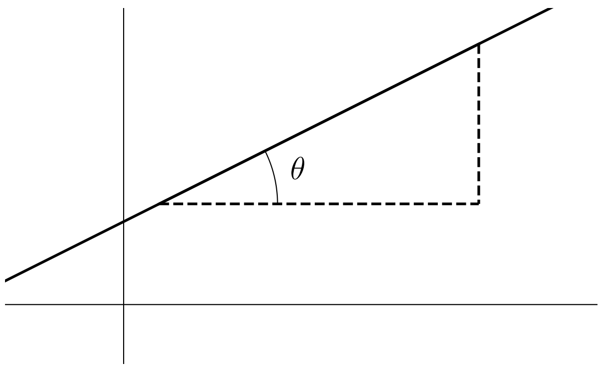


Illustration 5.1.5 Let

$$S = \{x - y : \; x, y \in \mathbb{R}, \; x^2 + y^2 = 1\}$$

Then the maximum number in the set S is

(a) 1 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $1 + \sqrt{2}$.

Solution. Let us first state the problem in words. Consider all the pairs of real numbers (x, y) such that $x^2 + y^2 = 1$, and for each such pair record the difference $y - x$. The question is *what is the maximum number one records*? Let us first ask if 0 is in S. This is same as asking if there is a point $(x, y) \in \mathbb{R}^2$ such that $x = y$ and also $x^2 + y^2 = 1$. In other words, we are asking if the sets

$$L = \{(x, y) \in \mathbb{R}^2 : \; x - y = 0\} \quad \text{and} \quad C = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 = 1\}$$

intersect. Thinking in purely algebraic terms answers this question in the affirmative. But we unlock a completely different mode of thinking once we interpret the two sets geometrically. The set L is the line passing through the origin making an angle of 45° with the positive direction of the x-axis. The set C is the circle centered at the origin and has radius 1 (the standard unit circle). Clearly these two intersect (at two distinct points).

We can use this way of thinking to solve the problem at hand without getting into any (tedious) calculations. The problem can be recast as follows. *What is the largest value of c such that the sets*

$$L_c = \{(x, y) \in \mathbb{R}^2 : \; x - y = c\} \quad \textit{and} \quad C = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 = 1\}$$

intersect. But what is L_c geometrically? Rewriting L_c as $\{(x, y) \in \mathbb{R}^2 : \; y = x - c\}$, it

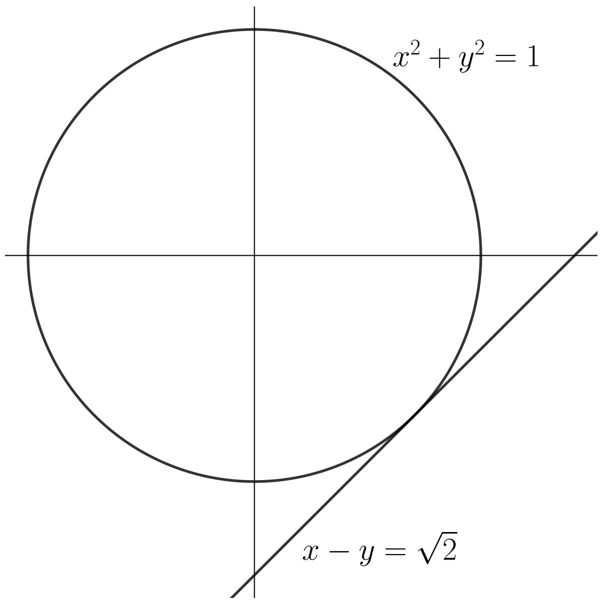


Figure 5.5

is clear that L_c is just a downward translate of L by c units. Thus we are asking for the largest c such that translating L by c units in the downward direction still gives a line that intersects C. Figure 5.5 below illustrates that the largest value of c corresponds to one of the two tangents to C which are parallel to L. A simple geometric exercise shows that the line L needs to be translated by $\sqrt{2}$ units downward to make it tangent to C. ■

Illustration 5.1.6 Let

$$f(x) = \begin{cases} 2x+3 \text{ if } x \leq 1 \\ a^2x+1 \text{ if } x > 1 \end{cases}$$

If the range of f is all of $\mathbb{R}$, then number of integral values a may take is:

(a) 2 (b) 3 (c) 4 (d) 5 [(c)]

Solution. From Figure 5.6 it is clear that the range of a for which f is surjective is the set $(-2, 2) \setminus \{0\}$. There are exactly two integers in it, namely 1 and -1. The reader is invited to fill in the formal details.

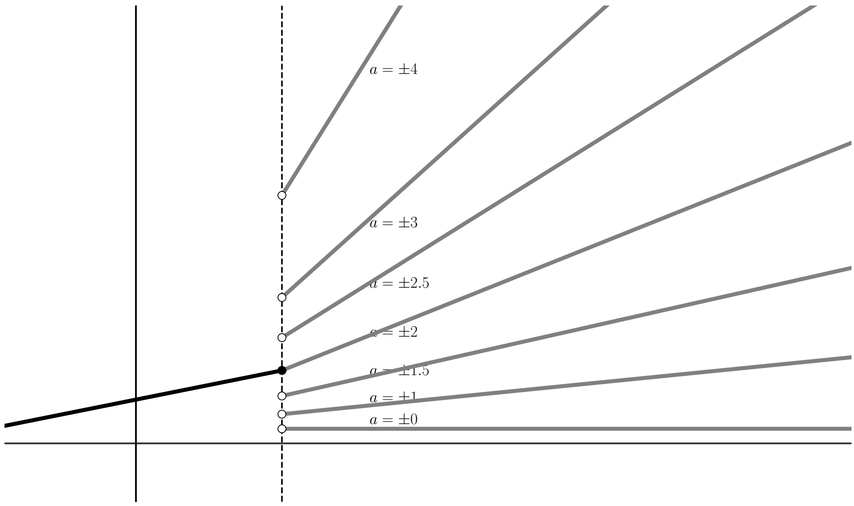


Figure 5.6

■

Exercise 5.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a linear function. Show that f is a bijection if and only if the slope of f is nonzero. Also show that, if f is bijective, then the inverse of f is also linear, and find the inverse explicitly in terms of the slope and intercept of f.

Exercise 5.1.2. Let a be a real number. The number of distinct solutions (x, y) of the system of equations $(x-a)^2 + y^2 = 1$ and $x^2 = y^2$, can only be

(a) $0, 1, 2, 3, 4$ or 5 (b) 0, 1 or 3 (c) $0, 1, 2$ or 4 (d) $0, 2, 3,$ or 4

Exercise 5.1.3. Let $P = \{(x, y) : x + 1 \geqslant y, x \geqslant -1, y \geqslant 2x\}$. Then the minimum value of $(x + y)$ where (x, y) varies over the set P is

(a) -1 (b) -3 (c) 3 (d) 0

Exercise 5.1.4. Show that whenever $2x + 4y = 1$, we also have $x^2 + y^2 \geq \frac{1}{20}$.

5.2 LINEAR RELATIONS

5.2.1 The Special Case of Vertical Lines. We have seen that the graphs of linear functions are straight lines. From the discussion above, it should also be clear that every line in the Cartesian plane is the graph of a suitable linear function *except* for the vertical lines. The reason is simply that the graph of *any* function cannot contain more than one point on any vertical line. This should cause some discontentment. However, we can give a unified treatment where this exception does not occur by moving to the more general framework of *relations*.

Definition. A relation R on $\mathbb{R}$ is said to be a **linear relation** if there exist real numbers a, b and c such that not both of a and b are 0, and

$$R = \{(x, y) \in \mathbb{R}^2 : \; ax + by + c = 0\}$$

In the language of level sets, we can define a linear relation as a level set of any function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\varphi(x, y) = ax + by$$

where a and b are fixed real numbers, not both 0.

Example 5.2.2. The most basic non-trivial examples of linear relations are

$$R = \{(x, y) \in \mathbb{R}^2 : \; x + y = 0\} \quad \text{and} \quad S = \{(x, y) \in \mathbb{R}^2 : x - y = 0\}$$

The graphs of these are exactly the same as the one's shown in Figure 5.1. In fact, if $b \neq 0$, the relation R is same as the graph of a suitable linear function, namely the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \frac{-c - ax}{b}$$

for all $x \in \mathbb{R}$.

To get a vertical line, we just need to choose $b = 0$ (and $a \neq 0$). For then we would get a relation

$$R = \{(x, y) \in \mathbb{R}^2 : \; ax + c = 0\} = \{(x, y) \in \mathbb{R}^2 : \; x = -c/a\} = \{(-c/a, y) \in \mathbb{R}^2 : \; y \in \mathbb{R}\}$$

which is clearly the set of all the points whose x-coordinate is $-c/a$. Thus it is the vertical line passing through the point $(-c/a, 0)$.

From the discussion above, it should be clear to the reader that every straight line in $\mathbb{R}^2$ is just a linear relation and conversely.

5.3 FINDING THE EQUATION OF A LINE

We have seen above that every linear relation is a straight line in the Cartesian plane. We have also remarked that every line in the Cartesian plane is a linear relation. A natural question is how to find the linear relation corresponding to a given line.

5.3.1 Two-point form. The most natural way of describing a line is to mention two distinct points through which the line passes.

Example 5.3.2. Let us find the description of the line ℓ passing through $(1,5)$ and $(3,1)$. What this means is to find a, b and c such that $\ell = \{(x,y) \in \mathbb{R}^2 : \ ax+by+c=0\}$. Alternatively, since ℓ is not a vertical line, one can find m and c such that the graph of the linear function with slope m and intercept c is the same as ℓ. We do the latter. Let $f : \mathbb{R} \to \mathbb{R}$ be such a linear function. Then

$$\begin{aligned} f(1)=5,\ f(3)=1 \qquad &\Rightarrow \qquad m+c=5,\ 3m+c=1 \\ &\Rightarrow \qquad m=\frac{1-5}{3-1}=-2,\ c=7 \end{aligned}$$

So the function f is given by $f(x) = -2x+7$ for all $x \in \mathbb{R}$.

We generalize the above. Suppose $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two distinct points in $\mathbb{R}^2$ and ℓ is the line passing through them. How do we describe ℓ in terms of P_1 and P_2? We make two cases

a) *Case 1:* $x_1 = x_2$.

In this case ℓ is a vertical line, and a moment's thought convinces the reader that ℓ is nothing but the relation $\{(x,y) \in \mathbb{R}^2 : \ x = x_1\}$.

b) *Case 2:* $x_1 \neq x_2$.

In this case ℓ is not a vertical line and hence is same as the graph of a suitable linear function. Let $f : \mathbb{R} \to \mathbb{R}$ be a linear function with slope m and intercept c, such that the graph of f is ℓ. Then we have

$$f(x_1)=y_1 \quad \Rightarrow \quad mx_1+c=y_1 \qquad \text{and} \qquad f(x_2)=y_2 \quad \Rightarrow \quad mx_2+c=y_2$$

Subtracting the two equations we get

$$m(x_2-x_1)=y_2-y_1 \qquad \Rightarrow \qquad m=\frac{y_2-y_1}{x_2-x_1}$$

and

$$c = y_1 - mx_1 = y_1 - \frac{y_2-y_1}{x_2-x_1}x_1 = \frac{y_1x_2-y_1x_1-x_1y_2+x_1y_1}{x_2-x_1}$$

So we have found m and c in terms of P_1 and P_2. $\diamondsuit$

5.3.3 Other ways to prescribe a line. There are many ways to uniquely describe a line in the Cartesian plane. We would not be discussing them here and a treatment of those can be found in any text on coordinate geometry on straight lines. $\diamondsuit$

Exercise 5.3.1. For each of the following, write the equation of the line in the form $y = mx + c$ and state its slope and y-intercept.

a) The line passing through $(-2, 0)$ and $(2, 8)$.
b) The line with slope -3 passing through $(1, 1)$.

Exercise 5.3.2. The temperature conversion from Celsius (C) to Fahrenheit (F) is a linear function. We know that water freezes at $0°C$ ($32°F$) and boils at $100°C$ ($212°F$).

1. Find the linear function F that converts Celsius to Fahrenheit.
2. What is the slope of this function and what does it represent?
3. Find the inverse function C. Is there a temperature at which the Celsius and Fahrenheit readings are the same?

CHAPTER 6

QUADRATIC POLYNOMIALS

When we first embark on our mathematical journey, we are introduced to the linear equation. It is a world of constant change, where for every step forward, we rise (or fall) by a fixed amount. If we buy one loaf of bread for a rupee, we expect two to cost two rupees. If we walk for an hour, we travel a certain distance; walk for two, and we expect to travel double that distance. It is comfortable and predictable. However, the moment we step outside the artificial constructs of simple commerce or steady walking, the universe reveals a more complex nature. Nature abhors a straight line. Rivers meander, coastlines fracture, and orbits curve. The straight line is a useful approximation for short distances, but it fails to capture the dynamic reality of the physical world. To understand the true shape of things—from the arch of a bridge to the flight of an arrow—we must leave the safety of linearity.

The first "quadratic problems" were problems of the earth itself: the measurement of land. Consider the dilemma of an ancient surveyor. If a farmer pays tax on a square field with a side length of 10 units, and he wishes to double his production area, he cannot simply double the side of the field to 20. If he did, he would quadruple the area, not double it. This is the first profound realization of non-linearity: the output outpaces the input. To manage an empire of fields, granaries, and city walls, ancient civilizations needed to understand the relationship between a single dimension (length, x) and a two-dimensional space (area, x^2). The quadratic polynomial was born of the necessity to divide the earth fairly. It was the mathematical tool required to navigate a world that has both length and width.

If the ancients gave us the geometry of squares, it was Galileo Galilei who showed us that nature itself moves in parabolas. Until the 17th century, the motion of falling bodies was a mystery, often described by Aristotle as a linear "desire" to return to the earth. Galileo changed the world by rolling bronze balls down inclined planes. He discovered a stunning truth: nature counts time in squares. As an object falls, it does not merely travel a distance proportional to time (t); it travels a distance proportional to the square of time (t^2). Gravity is an engine of acceleration. It does not pull steadily; it pulls harder and harder the longer you fall.

Because of this, any object thrown in a gravitational field—a stone, a spear, or a baseball—traces a specific curve called a parabola. This shape is the visual manifestation of a quadratic equation etched against the sky. To describe the trajectory of a projectile, linearity is helpless. The path curves, slows, stops at a peak, and accelerates downward. We study quadratics because they are the native language of gravity.

6.1 ROOTS

6.1.1 Why we solve for zero. Recall from Point 4.3.5 that a **quadratic polynomial** is a function $f : \mathbb{R} \to \mathbb{R}$ such that there exist real numbers a, b and c, where $a \neq 0$, such that

$$f(x) = ax^2 + bx + c$$

for all $x \in \mathbb{R}$. We call the values of x for which $f(x) = 0$ the **roots** of the equation. Why do we bother with them? Physically, if $f(t)$ represents the height of a projectile at time t, the roots tell us when the object sits on the ground. If $f(x)$ represents profit based on production cost x, the roots define our break-even points. To find these roots, we must solve:

$$ax^2 + bx + c = 0$$

6.1.2 Deriving the ancient formula. While factoring is an elegant method when the numbers are kind, we need not guess these values. We can extract them through a process of algebraic surgery known as "completing the square." Let us retrace the steps of the ancient mathematicians. We begin with our equation:

$$ax^2 + bx + c = 0$$

First, we banish the constant term to the other side and divide by a to isolate the x terms:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now comes the stroke of insight. We wish to turn the left side into a perfect square, $(x + k)^2 = x^2 + 2kx + k^2$. Matching terms, we see that $2k = \frac{b}{a}$, implying $k = \frac{b}{2a}$. Therefore, we add $k^2 = \left(\frac{b}{2a}\right)^2$ to *both* sides to maintain balance:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

The left side is now a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

We find a common denominator for the right side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Finally, we take the square root of both sides (remembering the $\pm$):

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Isolating x gives us the legendary Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, we must pause here to consider the constraints of the real number system. The operation of taking a square root, in the realm of real functions, is strictly forbidden for negative numbers. Therefore, the validity of the formula derived above hinges entirely on the expression living under the radical:

$$\Delta = b^2 - 4ac$$

We call this quantity the **discriminant**, for it discriminates between the types of roots the quadratic possesses.

- If $\Delta > 0$, we have a positive number under the root, yielding two distinct real solutions.
- If $\Delta = 0$, the square root vanishes, collapsing the $\pm$ term.
- If $\Delta < 0$, we encounter a problem. There is no real number which, when squared, produces a negative result.

It is worth mentioning that mathematicians, unsatisfied with the idea of "no solution," expanded the number system to include the *imaginary unit* i (where $i^2 = -1$). In that broader context of *complex numbers*, negative discriminants yield complex solutions. However, for our current study of real-valued functions, we simply say that no real roots exist. $\diamondsuit$

Illustration 6.1.3 Find the domain and range of

$$f(x) = \frac{\sqrt{x+4}}{x-4}$$

Solution. We have seen an intuitive solution for this in Illustration 4.7.4. Here we give an analytical proof. The expression inside the square root must be non-negative. Thus $x + 4 \geq 0$ giving $x \geq -4$. Also, we cannot divide by zero, so must have $x - 4 \neq 0 \implies x \neq 4$. Combining these, the domain is the interval $[-4, \infty)$ with the point 4 removed. So the domain of the given expression is

$$D_f = [-4, 4) \cup (4, \infty)$$

Finding the range is slightly more involved. We wish to find the set of all the real numbers y such that $y = f(x)$ for some x in D_f. It is helpful to define a substitution to simplify the algebra. Let $u = \sqrt{x+4}$. Rearranging for x, we get $x = u^2 - 4$. Substituting this into our expression for y:

$$y = \frac{u}{(u^2-4)-4} = \frac{u}{u^2-8}$$

As x varies over D_f, u varies over $[0,\infty) \setminus \{\sqrt{8}\}$. So the original problem of finding the range is equivalent to determining the values y can take as u varies over $[0,\infty) \setminus \{\sqrt{8}\}$. Keeping in mind the range of u, we have

$$y = \frac{u}{u^2-8} \qquad \Longleftrightarrow \qquad y(u^2-8) = u \qquad \Longleftrightarrow \qquad yu^2 - u - 8y = 0$$

For a specific real number y to be in the range, there must exist a real non-negative u that satisfies this equation. We calculate the discriminant of this quadratic in terms of u, denoted Δ_u:

$$\Delta_u = (-1)^2 - 4(y)(-8y) = 1 + 32y^2$$

Note that $y = 0$ is obtained when $y = 0$. So we focus on nonzero y. For any nonzero real number y we have $y^2 > 0$, and thus $\Delta_u = 1 + 32y^2 \geq 1 > 0$. This implies that for *any* nonzero real value y, there are always real solutions for u.[1] However, we have a restriction: we require $u \geq 0$. Do we necessarily get a non-negative u for a given nonzero value of y? The two values of u are

$$\frac{1-\sqrt{1+32y^2}}{2y} \quad \text{and} \quad \frac{1+\sqrt{1+32y^2}}{2y}$$

If y is positive then the second expression is also non-negative. If y is negative then the first expression is non-negative.[2] This guarantees that a positive solution for u exists for every non-zero y, and we have already seen that $u = 0$ for $y = 0$. Since we can find a valid u (and consequently a valid x) for any real y, the function covers the entire real line.

$$R_f = \mathbb{R}$$

and we are done. ■

Exercise 6.1.1. Find the domain and range of

$$a)\ \frac{1}{x^2-x-6}, \quad b)\ \frac{x-3}{x^2+9x-22}$$

6.2 VIETA'S FORMULAS

6.2.1 Factorizing using roots. While the Quadratic Formula is a powerful brute-force tool for finding roots, there exists a more elegant relationship between the roots and the

[1] The nonzero-ness of y was needed so that the expression $yu^2 - u - 8y$ is a quadratic polynomial.

[2] A cleaner way to see this is by considering the product of the two values of u, which is -8, and hence one value must be positive and the other negative.

structure of the polynomial itself. This relationship allows us to deconstruct a quadratic into simpler, linear building blocks. Let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic polynomial given by $f(x) = ax^2 + bx + c$.

Suppose we know that a real number α is a root of this polynomial, meaning $f(\alpha) = 0$. This implies that $a\alpha^2 + b\alpha + c = 0$. We claim that $(x - \alpha)$ must be a "factor" of the polynomial. To see this, write the variable x as $(x - \alpha) + \alpha$. Now, let us evaluate $f(x)$ using this form:

$$f(x) = a\left((x-\alpha)+\alpha\right)^2 + b\left((x-\alpha)+\alpha\right) + c$$

Expanding the squared term and distributing the coefficients, we get:

$$f(x) = a\left[(x-\alpha)^2 + 2\alpha(x-\alpha) + \alpha^2\right] + b(x-\alpha) + b\alpha + c$$

Now, we group the terms by powers of $(x - \alpha)$:

$$f(x) = a(x-\alpha)^2 + [2a\alpha + b](x-\alpha) + [a\alpha^2 + b\alpha + c]$$

Behold the last term in the brackets: $a\alpha^2 + b\alpha + c$. This is precisely $f(\alpha)$, which we know is zero. Thus, the constant term vanishes, leaving us with:

$$f(x) = (x-\alpha)\left[a(x-\alpha) + (2a\alpha + b)\right]$$

We have successfully factored out $(x - \alpha)$. Inside the square brackets remains a linear expression. If we pull out the leading coefficient a, we are left with a term that must correspond to the second root, β. Ultimately, if α and β are the roots, the polynomial yields to the form:

$$f(x) = a(x-\alpha)(x-\beta)$$

6.2.2 The Formula. François Viète (1540–1603) discovered that roots are not random numbers; they are intimately tied to the coefficients.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic polynomial given by $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Assume $\Delta_f \geq 0$. If α and β are the real roots of f (with possible repetition), then:

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Proof. By Point 6.2.1 we know that

$$f(x) = a(x-\alpha)(x-\beta)$$

for all x. Thus

$$(ax^2 + bx + c) - a(x-\alpha)(x-\beta) = 0$$

for all $x \in \mathbb{R}$. Simplifying the above, we get

$$(b + a(\alpha+\beta))x + (c - a\alpha\beta) = 0$$

for all x. This forces that

$$\alpha + \beta = -\frac{b}{a}, \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

and we are done. ■ ♢

Illustration 6.2.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic polynomial given by

$$f(x) = x^2 - bx + c$$

for all x. Assume that b is an odd positive integer, that $b + c = 35$, and that f has two prime numbers as roots. Find the values of b and c.

Solution. Let the two roots of the polynomial be p and q. Since the leading coefficient is 1, Vieta's formulas provide us with two equations:

$$p + q = b \quad \text{and} \quad pq = c$$

We are given a peculiar piece of information: b is an *odd* integer. Since $b = p + q$, this means the sum of our two prime roots is odd. Recall that the sum of two integers is odd if and only if one is even and the other is odd. Therefore, one of our roots must be an even number. There is only one even prime number in existence: 2. Thus, without loss of generality, let $p = 2$. Now we can rewrite our variables in terms of the remaining unknown root, q:

$$b = 2 + q \quad \text{and} \quad c = 2q$$

We are given the constraint $b + c = 35$. Substituting our expressions for b and c:

$$(2 + q) + 2q = 35 \qquad \Rightarrow \qquad q = 11$$

We must verify our assumptions. Is $q = 11$ a prime number? Yes. Now we can determine b and c:

$$b = 2 + 11 = 13 \quad \text{and} \quad c = 2 \times 11 = 22$$

and we are done. ∎

Exercise 6.2.1. Consider the quadratic polynomial $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = x^2 - px + q$$

Assume that the roots of f are prime numbers. If $p + q = 11$ and $a = p^2 + q^2$ then find the value of $f(a)$ where a is an odd positive integer.

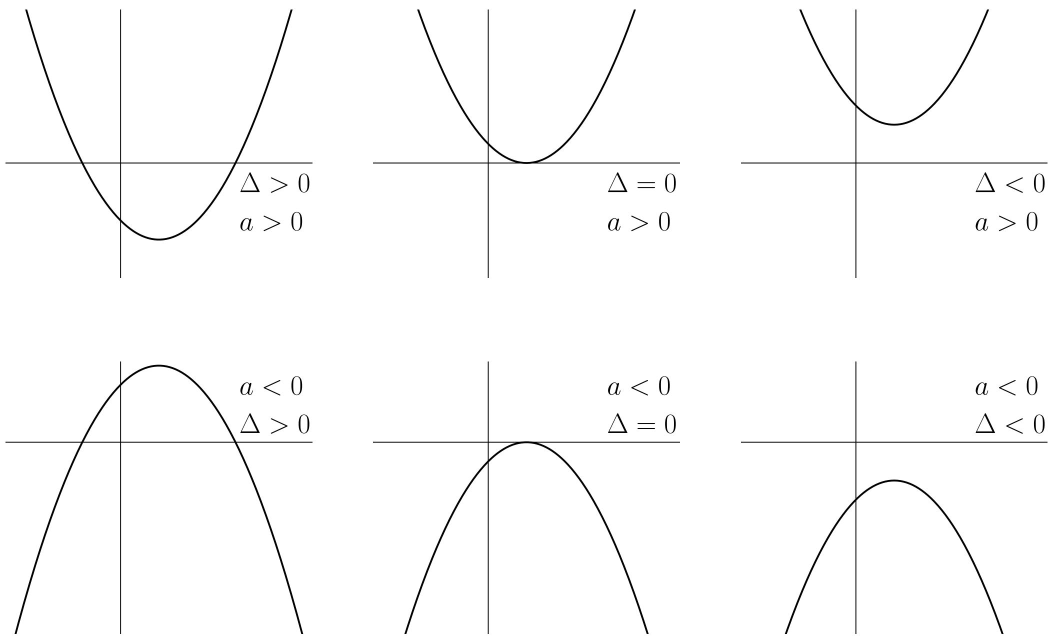
6.3 SIX TYPES OF GRAPHS

When we graph a quadratic polynomial given by $ax^2 + bx + c$, we always obtain a parabola. However, its orientation and position relative to the x-axis depend on two parameters: the sign of the leading coefficient a and the sign of the discriminant Δ. We can categorize these into six distinct cases:

Orientation: If $a > 0$, the parabola opens upward (like a cup). If $a < 0$, it opens downward (like a frown).

Intersection: This is determined by the discriminant, and the various curves are shown in the figure below.

- If $\Delta > 0$, the formula yields two distinct real numbers. The graph cuts the x-axis twice.
- If $\Delta = 0$, the square root vanishes. We have one repeated root. The graph touches the x-axis at exactly one point (the vertex).
- If $\Delta < 0$, the square root is imaginary. The graph never touches the x-axis (it floats entirely above or below it).



6.3.1 Signs and roots: a geometric analysis. We have already classified parabolas based on their leading coefficient and discriminant. Now, let us sharpen our tools. By simply observing the position of a parabola relative to the coordinate axes, we can deduce powerful algebraic truths about the signs of the polynomial and the location of its roots. We restrict our discussion to the case where the leading coefficient is positive ($a > 0$), meaning the parabola opens upward. The logic for $a < 0$ is a mirror image of this discussion. We do not intend to prepare an exhaustive list of all possible scenarios or questions that can be formed around this idea. Neither do we intend to have the reader memorize any of the following facts. The aim is to gain an intuitive understanding which will allow the reader to make such deductions in real time.

a) *Where is the polynomial negative?* A quadratic polynomial $f(x) = ax^2 + bx + c$ (with $a > 0$) takes negative values precisely when its graph dips below the x-axis, which happens if and only if the quadratic has two distinct roots, which in turn happens if and only if the discriminant is positive. The parabola cuts the x-axis at two points, say α and β with $\alpha < \beta$. The graph lies below the axis *between* these roots. Thus, $f(x) < 0$ if and only if $x \in (\alpha, \beta)$. Outside this interval, the function is positive.

Example. Consider $f(x) = x^2 - 5x + 6$. The roots are found by factoring: $(x-2)(x-3) = 0$, so $\alpha = 2$ and $\beta = 3$. Since the parabola opens up, it must dip below the axis between the roots.

$$f(x) < 0 \iff 2 < x < 3$$

If we instead check $g(x) = x^2 + 4$, the discriminant is $0^2 - 4(1)(4) = -16 < 0$. The graph floats above the axis. There is no x for which $g(x) < 0$.

b) *When are both roots positive?* For the roots to be positive, the entire "dip" of the parabola must occur to the right of the y-axis. This requires three conditions to be met simultaneously:

- Real Roots exist: $\Delta \geq 0$.
- Product of roots is positive: The roots must have the same sign. From Vieta's formulas, $\alpha\beta = c/a$. Since $a > 0$, we need $c > 0$. Geometrically, this means the y-intercept $f(0) = c$ is positive.
- Sum of roots is positive: To ensure the roots are both positive (rather than both negative), their sum must be positive. $\alpha + \beta = -b/a > 0$. Since $a > 0$, this implies $b < 0$. Geometrically, the vertex $-b/2a$ must be on the positive x-axis.

Example. Consider $f(x) = x^2 - 4x + 3$. The discriminant is $(-4)^2 - 4(1)(3) = 4 > 0$. Thus this has two distinct roots. The product of the roots is $c/a = 3/1 = 3 > 0$, and hence the roots have the same sign. The sum of the roots is $-b/a = -(-4)/1 = 4 > 0$.

c) *When are both roots negative?* This is the mirror case. The parabola must intersect the x-axis entirely to the left of the origin.

- Real Roots exist: $\Delta \geq 0$.
- Product is positive: $c/a > 0$ (implying $c > 0$). The roots must have the same sign.
- Sum is negative: $\alpha + \beta = -b/a < 0$. Since $a > 0$, this implies $b > 0$. The vertex must be on the negative x-axis.

Example. Consider $f(x) = x^2 + 5x + 6$.

- $\Delta = 25 - 24 = 1 > 0$.
- Product $6/1 = 6 > 0$.
- Sum $-5/1 = -5 < 0$.

The conditions are met. Factoring confirms the roots are -2 and -3, both negative.

d) *When do roots have opposite signs?* For one root to be negative and the other positive, the parabola must straddle the y-axis. It must cut the x-axis once to the left of the origin and once to the right. This happens if and only if the product of the roots is negative.

$$\alpha\beta = \frac{c}{a} < 0$$

Since we assume $a > 0$, this simplifies to just $c < 0$. Note that we don't need to check Δ here. If $c/a < 0$, then $ac < 0$, which means $-4ac > 0$. Since $b^2 \geq 0$, the discriminant $\Delta = b^2 - 4ac$ is automatically positive. The geometry confirms this: if a parabola opens upward ($a > 0$) and crosses the y-axis below the origin ($c < 0$), it *must* cross the x-axis twice.

Example. Consider $f(x) = x^2 + x - 2$. The constant term $c = -2$ is negative. This immediately tells us the roots have opposite signs. Indeed, $(x + 2)(x - 1) = 0$ gives roots -2 and 1.

Illustration 6.3.2 Determine the set of all the a such that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = x^2 + (a - 1)x + 16$$

takes only positive values.

Solution. We are asked to find a such that $f(x) > 0$ for all real x. Let us analyze the geometry of this condition. First, observe the leading coefficient of x^2 is 1, which is positive. This guarantees that the parabola opens upward. For an upward-opening parabola to remain strictly positive, it must never touch or cross the x-axis, which translates to the algebraic condition that the quadratic has no real roots. Therefore, the discriminant Δ must be strictly negative. Substituting the coefficients from our function:

$$(a - 1)^2 - 4(1)(16) < 0 \qquad \iff \qquad (a - 1)^2 < 64$$

Thus, the set of all possible values for a is the interval $(-7, 9)$. ■

Illustration 6.3.3 Let a be a real number. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = 4x^2 + ax + (a - 3)$$

What are all possible values of a such that f takes a negative value for at least one negative x.

Solution. The leading coefficient is 4, so the parabola opens upward. For such a function to take a negative value at all, it must dip below the x-axis. This implies it must have two distinct real roots. Let these roots be α and β, with $\alpha \leq \beta$. The function takes negative values precisely in the interval between the roots, (α, β).

The problem demands that this interval (α, β) overlaps with the set of negative numbers $(-\infty, 0)$. This occurs if the smaller root α is negative ($\alpha < 0$). We can break this down into two mutually exclusive cases based on the behavior of the roots:

Case 1: The roots have opposite signs ($\alpha < 0 < \beta$). Geometrically, this means the parabola crosses the x-axis once on the left of the origin and once on the right. This happens if and only if $f(0) < 0$.

$$f(0) = a - 3 < 0 \implies a < 3$$

Case 2: Both roots are non-positive ($\alpha \le \beta < 0$). For this to happen, three conditions must be met simultaneously:

a) The roots must exist ($\Delta \ge 0$). In fact, for f to become strictly negative, we need distinct roots, so $\Delta > 0$.

$$\begin{aligned} \Delta = a^2 - 16(a-3) = a^2 - 16a + 48 > 0 \quad &\Rightarrow \quad (a-4)(a-12) > 0 \\ &\Rightarrow \quad a \in (-\infty, 4) \cup (12, \infty) \end{aligned}$$

b) The vertex (which lies midway between the roots) must be on the negative side:

$$\frac{-a}{2(4)} < 0 \quad \iff \quad -a < 0 \quad \iff \quad a > 0$$

c) The y-intercept must be non-negative (since the parabola cuts the axis to the left of the origin and opens up, it crosses the y-axis on its way up).

$$f(0) = a - 3 \ge 0 \quad \iff \quad a \ge 3$$

Taking the intersection of these three conditions for Case 2:

$$a \in ((-\infty, 4) \cup (12, \infty)) \cap (0, \infty) \cap [3, \infty)$$

This simplifies to:

$$a \in [3, 4) \cup (12, \infty)$$

Combining Case 1 and Case 2

$$a \in (-\infty, 3] \cup (3, 4) \cup (12, \infty) = (-\infty, 4) \cup (12, \infty)$$

and we are done. ■

Illustration 6.3.4 Let k be a real number and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = -2x^2 + kx + k^2 + 5$$

What is the set of all possible values of k such that f has two distinct roots and only one root lies between 0 and 2?

Solution. First, observe the shape of the graph. The leading coefficient is -2, so the parabola opens downward. Let's check if real roots always exist. The constant term $c = k^2 + 5$ is always positive. The leading coefficient $a = -2$ is negative. Since a and c have opposite signs, the product of the roots $P = c/a$ is negative. This guarantees that one root is negative and the other is positive. Thus, distinct real roots always exist for any real k. Let the roots be $\alpha < 0 < \beta$.

The problem states that exactly one root lies in the interval $(0, 2)$. Since α is negative, it can never be in $(0, 2)$. Therefore, the condition simplifies to:

$$0 < \beta < 2$$

We already know $\beta > 0$. So we only need to ensure $\beta < 2$. Visualize the downward-opening parabola. It starts from $-\infty$, crosses the x-axis at α (negative), rises to a peak, crosses the y-axis at $f(0) = k^2 + 5$ (positive), and then crosses the x-axis again at β (positive) before plunging to $-\infty$. Since $f(0)$ is positive (above the axis), for the root β to occur *before* the point $x = 2$, the graph must have already crossed the x-axis and become negative by the time it reaches 2. Therefore, the condition is simply:

$$f(2) < 0$$

This is same as

$$-2(2)^2 + k(2) + k^2 + 5 < 0 \qquad \Longleftrightarrow \qquad k^2 + 2k - 3 < 0$$

Factoring the quadratic in k, we get the above is equivalent to

$$(k+3)(k-1) < 0 \qquad \Longleftrightarrow \qquad k \in (-3, 1)$$

and we are done. ■

Illustration 6.3.5 Find the domain and range of

$$f(x) = \frac{x^2 + 5x + 25}{x + 3}$$

Solution. The only points missing from the domain are those where the denominator vanishes, which is -3. So the domain is $D_f = \mathbb{R} \setminus \{-3\}$. To find the range, we need to find the set of all the y such that $y = f(x)$ for some x in D_f. Keeping in mind that $x \notin D_f$, we have We solve for x in terms of y:

$$\begin{aligned} y = \frac{2x^2 + 10x + 50}{x + 3} \qquad &\Longleftrightarrow \qquad y(x+3) = 2x^2 + 10x + 50 \\ &\Longleftrightarrow \qquad 2x^2 + (10 - y)x + (50 - 3y) = 0 \end{aligned}$$

So the question becomes this: for which values of y does this quadratic expression has a root different from -3? But -3 cannot be a root of this quadratic, so the question is simply this: for which values of y does this quadratic have a root? For a root to exists, the discriminant must be non-negative:

$$\Delta = (10 - y)^2 - 4(2)(50 - 3y) \geq 0 \qquad \Longleftrightarrow \qquad y^2 + 4y - 300 \geq 0$$

To solve this inequality, we find the roots of $y^2 + 4y - 300 = 0$:

$$y = \frac{-4 \pm \sqrt{16 - 4(1)(-300)}}{2} = \frac{-4 \pm \sqrt{1216}}{2} = -2 \pm 4\sqrt{19}$$

Since the inequality is ≥ 0, the valid values for y are those outside the roots:

$$y \in (-\infty, -2 - 4\sqrt{19}] \cup [-2 + 4\sqrt{19}, \infty)$$

and this is the range. ■

Exercise 6.3.1. Let a be a nonzero real number. Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = ax^2 - (3 + 2a)x + 6$$

What is the set of all the possible values for a such that f is positive for exactly 3 distinct negative integral inputs.

Exercise 6.3.2. Find the domain and range of

$$\frac{x^3 - 125}{x^2 - 2x - 15}$$

Exercise 6.3.3. Find the domain and range of

$$\frac{3x^2 + 3x - 4}{3 + 3x - 4x^2}$$

6.4 MINIMUM VALUE

Let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic polynomial given by $f(x) = ax^2 + bx + c$. A glance at the graph of any quadratic reveals a singular feature: a turning point. If the parabola opens upward ($a > 0$), there is a definitive bottom, a minimum value. If it opens downward ($a < 0$), there is a definitive top, a maximum value. Finding this extremum does not require the machinery of calculus. We need only exploit the symmetry inherent in the curve. Every parabola is symmetric about a vertical line passing through its vertex. The roots (if they exist) are equidistant from this line. Recall from the Quadratic Formula that the roots are:

$$\frac{-b \pm \sqrt{\Delta}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$$

The term $-\frac{b}{2a}$ sits precisely in the middle. Even if the roots are not real, this axis of symmetry remains at:

$$x = -\frac{b}{2a}$$

This can be seen formally by using the equation

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Since the vertex lies on this axis, the minimum (or maximum) value of the function occurs exactly at this x-coordinate. Substituting this back into the function gives us the extremal value:

$$\begin{aligned} f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c &= a\frac{b^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + c = c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} = -\frac{\Delta}{4a} \end{aligned}$$

This allows us to describe the **range** of a quadratic polynomial purely in terms of its coefficients:

- If $a > 0$, the function has a minimum value of $-\frac{\Delta}{4a}$. The range is $\left[-\frac{\Delta}{4a}, \infty\right)$.
- If $a < 0$, the function has a maximum value of $-\frac{\Delta}{4a}$. The range is $\left(-\infty, -\frac{\Delta}{4a}\right]$.

Illustration 6.4.1 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = 2x^2 - 3x + 2$$

for all x. What is $f([0, 2])$?

Solution. We are asked to find the image of the interval $[0, 2]$ under the function f. Since the domain is restricted, we cannot simply say the minimum is at the vertex and the maximum is infinity. We must check where the vertex lies relative to our interval. First, find the x-coordinate of the vertex:

$$x_v = -\frac{b}{2a} = -\frac{-3}{2(2)} = \frac{3}{4}$$

Does this lie inside our interval $[0, 2]$? Yes, $0 < 0.75 < 2$. Since $a = 2 > 0$, the parabola opens upward, meaning the vertex represents the global minimum of the function. Because the vertex is inside our interval, the minimum value of f on $[0, 2]$ is the value at the vertex:

$$f\left(\frac{3}{4}\right) = 2\left(\frac{9}{16}\right) - 3\left(\frac{3}{4}\right) + 2 = \frac{9}{8} - \frac{18}{8} + \frac{16}{8} = \frac{7}{8}$$

Now, because the parabola is monotonic on either side of the vertex, the maximum value must occur at one of the endpoints of the interval. We evaluate both:

$$f(0) = 2(0)^2 - 3(0) + 2 = 2 \quad \text{and} \quad f(2) = 2(2)^2 - 3(2) + 2 = 8 - 6 + 2 = 4$$

Comparing the endpoints, the maximum value is 4. Thus, as x travels from 0 to 2, $f(x)$ dips down to $7/8$ and rises back up to 4.

$$f([0, 2]) = \left[\frac{7}{8}, 4\right]$$

and we are done. ■

Illustration 6.4.2 Let $f : [0, 5] \to [0, 5]$ be an invertible function defined by $f(x) = ax^2 + bx + c$, where a, b and c are nonzero real numbers. Then one of the roots of the equation $cx^2 + bx + a = 0$ is :

(a) a (b) b (c) c (d) $a + b + c$

Solution. Since f is a bijection, it is in particular an injection. Thus the line of symmetry of the full parabola given by the quadratic polynomial $x \mapsto ax^2 + bx + c$ should not pass through the segment $(0, 5)$ in the x-axis. In fact, this is also a sufficient condition for the injectivity of f. Now there are two possibilities, either f is increasing or it is decreasing. In the former case, we must have $f(0) = 0, f(5) = 5$, and in the latter case we must have $f(0) = 5$ and $f(5) = 0$. These two cases can be packaged into one single statement by saying that 0 and 5 are the roots of the quadratic polynomial given by $(x - f(0))(x - f(5))$, which by Vieta's theorem yields

$$0 + 5 = f(0) + f(5) = c + (25a + 5b + c) \quad \text{and} \quad 0 \times 5 = f(0)g(5) = c(25a + 5b + c)$$

Since c is nonzero, we obtain from the second equation that $25a + 5b + c = 0$, which when used in the first equation yields that $c = 5$. Therefore

$$25a + 5b + c = 0 \qquad \Rightarrow \qquad c^2a + cb + c = 0$$

which shows that a is a root of the quadratic given by $cx^2 + bx + a$. ■

Exercise 6.4.1. Let $f : (-\infty, 2] \longrightarrow \mathbb{R}$ be defined as

$$f(x) = x(4 - x)$$

for all x. Show that f injective and its image is $(-\infty, 4]$. Thus we obtain a bijection $f : (-\infty, 2] \to (-\infty, 4]$. Find the inverse of this map.

Exercise 6.4.2. Find the set of all the values of k such that the least value of the expression $x^2 + 2kx + k^2 + 3k$ for x lying in $[0, 2]$ is 4.

Exercise 6.4.3. Find the least value of $2x^2 + 3y^2$ under the constraint that $x + y = 2$ and that x and y are both non-negative.

CHAPTER 7

THE MODULUS FUNCTION

We have seen the modulus function before. In this chapter we take a deeper look. Here we go beyond the basic definition to explore its properties and its role in solving complex equations and inequalities. The goal is to develop a robust intuition for how to handle this function in challenging scenarios.

7.1 ELEMENTARY PROPERTIES

7.1.1 Definition. We begin by re-stating the formal definition.

Definition. For any real number x, the **modulus** or **absolute value** of x, denoted by $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The **modulus function** (or the **absolute value function**) is defined as the map $|\cdot| : \mathbb{R} \to \mathbb{R}$ which takes a real number x and maps it to $|x|$.

Geometrically, $|x|$ represents the distance of the number x from the origin (0) on the number line. Similarly, $|x - a|$ represents the distance between the points x and a. The graph of the modulus function is shown in Figure 7.1 below

7.1.2 Properties of the modulus function. For any $x, y \in \mathbb{R}$:

a) $|x| \geq 0$.

b) $|x| = |-x|$.

c) $x^2 = |x|^2$.

d) $|xy| = |x||y|$.

e) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ for $y \neq 0$.

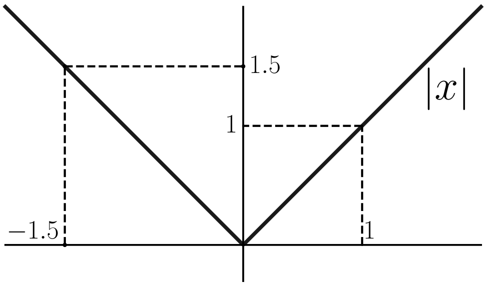


Figure 7.1

The proofs of these are a straightforward matter of unravelling the definitions. ♢

7.1.3 Max and min. Let x and y be real numbers. Then

$$\max\{x, y\} = \frac{x + y + |x - y|}{2} \quad \text{and} \quad \min\{x, y\} = \frac{x + y - |x - y|}{2}$$

The proof is straightforward. Assume without loss of generality that $x \geq y$, so that $\max\{x, y\} = x$ and $\min\{x, y\} = y$. Now

$$\frac{x + y + |x - y|}{2} = \frac{x + y + x - y}{2} = x \quad \text{and} \quad \frac{x + y + |x - y|}{2} = \frac{x + y + y - x}{2} = y$$

and we are done. ♢

Exercise 7.1.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$|f(x) - f(y)| = |x - y|$$

for all real numbers x and y. (Such a function is called an isometry.)

7.2 TRIANGLE INEQUALITY

A most important property about the modulus function is the triangle inequality, which captures the geometric essence of distances.

Theorem 7.2.1 ***The Triangle Inequality.*** For any real numbers x and y, we have:

$$|x + y| \leq |x| + |y|$$

Equality holds if and only if x and y have the same sign.

The proof of the above is a straightforward exercise and we leave it to the reader. A useful variant of the triangle inequality says that for any two real numbers x and y we have

$$|x - y| \geq ||x| - |y||$$

This can be derived by writing $x = (x - y) + y$ and applying the standard triangle inequality.

Theorem 7.2.2 ***Hlwaka's Inequality (Optional).*** Prove that for any real numbers a, b, c, the following inequality holds:

$$|a| + |b| + |c| + |a + b + c| \geq |a + b| + |b + c| + |c + a| \tag{*}$$

Proof. This is not obvious. We give a standard proof. Let the left-hand side be L and the right-hand side be R. Since the absolute value function is always non-negative, both L and R are non-negative. Therefore, the inequality $L \geq R$ is equivalent to $L^2 \geq R^2$. We will now compute both squares. The square of the left-hand side is:

$$\begin{aligned} L^2 &= (|x| + |y| + |z| + |x + y + z|)^2 \\ &= |x|^2 + |y|^2 + |z|^2 + |x + y + z|^2 + 2(|x||y| + |x||z| + |y||z|) \\ &\qquad + 2(|x||x + y + z| + |y||x + y + z| + |z||x + y + z|) \end{aligned}$$

Using $|a|^2 = a^2$ and $|a||b| = |ab|$, we get

$$\begin{aligned} L^2 &= x^2 + y^2 + z^2 + (x + y + z)^2 \\ &\quad + 2(|xy| + |xz| + |yz|) + 2(|x(x + y + z)| + |y(x + y + z)| + |z(x + y + z)|) \\ &= x^2 + y^2 + z^2 + (x^2 + y^2 + z^2 + 2xy + 2xz + 2yz) \\ &\quad + 2(|xy| + |xz| + |yz|) + 2(|x^2 + xy + xz| + |y^2 + xy + yz| + |z^2 + xz + yz|) \\ &= 2(x^2 + y^2 + z^2 + xy + xz + yz) \\ &\quad + 2(|xy| + |xz| + |yz|) + 2(|x^2 + xy + xz| + |y^2 + xy + yz| + |z^2 + xz + yz|) \end{aligned}$$

Next, we compute the square of the right-hand side:

$$\begin{aligned} R^2 &= (|x + y| + |y + z| + |z + x|)^2 \\ &= |x + y|^2 + |y + z|^2 + |z + x|^2 + 2(|(x + y)(y + z)| \\ &\qquad + |(y + z)(z + x)| + |(z + x)(x + y)|) \end{aligned}$$

Expanding this expression we get

$$\begin{aligned} R^2 &= (x + y)^2 + (y + z)^2 + (z + x)^2 \\ &\quad + 2(|xy + xz + y^2 + yz| + |yz + yx + z^2 + zx| + |zx + zy + x^2 + xy|) \\ &= (x^2 + 2xy + y^2) + (y^2 + 2yz + z^2) + (z^2 + 2zx + x^2) \\ &\quad + 2(|y(x + y + z) + xz| + |z(x + y + z) + xy| + |x(x + y + z) + yz|) \\ &= 2(x^2 + y^2 + z^2 + xy + xz + yz) \\ &\quad + 2(|y(x + y + z) + xz| + |z(x + y + z) + xy| + |x(x + y + z) + yz|) \end{aligned}$$

Now, the inequality $L^2 \geq R^2$ is true if and only if $L^2 - R^2 \geq 0$. By cancelling the common terms, we are reduced to proving that

$$\begin{aligned}&|xy| + |xz| + |yz| + |x^2 + xy + xz| + |y^2 + xy + yz| + |z^2 + xz + yz| \\ &\geq |y(x + y + z) + xz| + |z(x + y + z) + xy| + |x(x + y + z) + yz|\end{aligned}$$

But since

$$\begin{aligned}y(x + y + z) + xz &= y^2 + xy + yz + xz \\ z(x + y + z) + xy &= z^2 + xz + yz + xy \\ x(x + y + z) + yz &= x^2 + xy + xz + yz\end{aligned}$$

our goal is

$$\begin{aligned}&|xy| + |yz| + |zx| + |x(x + y + z)| + |y(x + y + z)| + |z(x + y + z)| \\ &\geq |y(x + y + z) + xz| + |z(x + y + z) + xy| + |x(x + y + z) + yz|\end{aligned}$$

This final inequality is a direct result of the standard triangle inequality and we leave the justification to the reader. ■

Exercise 7.2.1. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$|a_1| + \cdots + |a_n| \geq |a_1 + \cdots + a_n|$$

7.3 EQUATIONS AND INEQUALITIES INVOLVING MODULUS

Before we begin, we remind the reader of the message in Section 4.5.

Illustration 7.3.1 Find the number of solutions of the equation

$$|x - 3| + |x + 5| = 7x$$

Solution. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined as

$$f(x) = |x - 3| + |x + 5| \quad \text{and} \quad g(x) = 7x$$

for all $x \in \mathbb{R}$. We want to find the number of all the x such that $f(x) = g(x)$. Graphically, this corresponds to find the number of points of intersections in the graphs of f and g. Figure 7.2 readily shows that there is exactly one point of intersection of the two graphs. Before we get into proving this analytically, let us comment on the nature of the graph of f. We can think of the expression $|x - 3| + |x + 5|$ as the sum of the distance between x and 3 and the distance between x and -5. When x is between -5 and 3, this sum is always 8. This is what we see in the graph of f—the portion of the graph above the interval $[-5, 3]$ is horizontal. When $x < -5$, we have

$$|x-3| + |x+5| = 3 - x - 5 - x = -2 - 2x$$

This is reflected in the graph of f as the "left arm" of the graph is (part of) a straight line. Similarly we justify the right arm. For an analytical proof, we make three cases. First, suppose $x < -5$. Then

$$|x-3|+|x+5| = 7x \quad \Rightarrow \quad -2-2x = 7x \quad \Rightarrow \quad -2 = 9x \quad \Rightarrow \quad x = \frac{-2}{9}$$

which is not smaller than -5. So there is no solution in this region. If $-5 \leq x \leq 3$, then we have

$$|x-3|+|x+5| = 7x \quad \Rightarrow \quad 3-x+x+5 = 7x \quad \Rightarrow \quad 8 = 7x \quad \Rightarrow \quad x = \frac{8}{7}$$

which indeed lies in the region $[-5, 3]$ and does satisfy our equation. So there is exactly one solution in $[-5, 3]$. By a similar analysis, we can show that there is no solution if $x > 3$. ∎

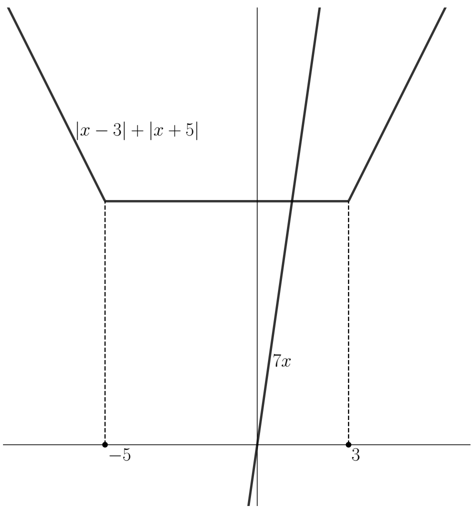


Figure 7.2

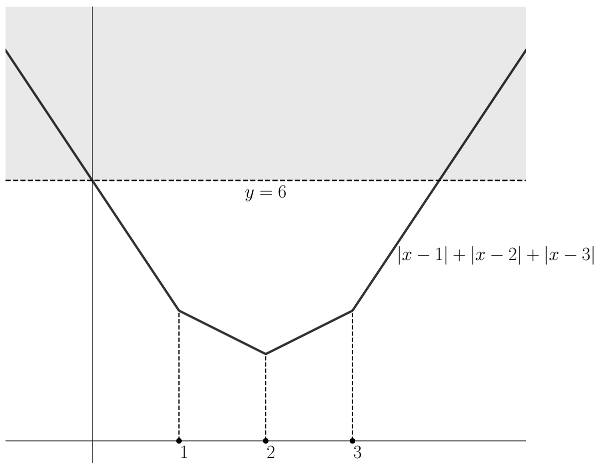


Figure 7.3

Illustration 7.3.2 Find all the x that satisfy $|x-1| + |x-2| + |x-3| \geq 6$.

Solution. We develop a graphical understanding. See Figure 7.3. The shaded region shows all the points (x, y) such that $y \geq 6$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x-1| + |x-2| + |x-3|$ for all x. The problem asks to find all the x such that $f(x) > 6$. This is same as asking for all the x such that the corresponding point on the graph of f, namely $(x, f(x))$ lies in the shaded region. The diagram suggests that this region is the union of two intervals, one being $(-\infty, 0]$ and the other being $[a, \infty)$, for a certain value of a.

Let us also comment on the nature of the graph of f. Note that f "changes" behaviour at the point $1, 2$ and 3. What are these numbers? They are precisely the numbers that feature in the description of the function f. This is typical of such functions, and the reader should start getting a feel for this by now. For an analytical proof, the reader is invited to do a case analysis. ■

Illustration 7.3.3 Sketch the relation $\{(x, y) \in \mathbb{R}^2 : \ |x + y| = 1\}$.

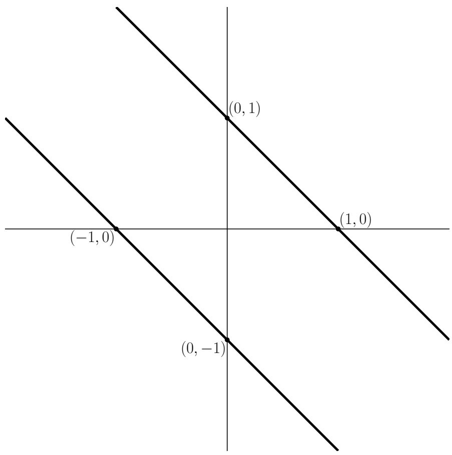
Solution. Note that for any real number a, we have

$$|a| = 1 \qquad \Longleftrightarrow \qquad a = 1 \text{ or } a = -1$$

Thus

$$\{(x, y) \in \mathbb{R}^2 : \ |x + y| = 1\} = \{(x, y) \in \mathbb{R}^2 : \ x + y = 1\} \cup \{(x, y) \in \mathbb{R}^2 : \ x + y = -1\}$$

Since we know how to sketch linear relations, we can now sketch the original relation.



■

Exercise 7.3.1. The number of real values of x satisfying the equation $3|x - 2| + |1 - 5x| + 4|3x + 1| = 13$ is :

(a) 1 (b) 4 (c) 2 (d) 3

Exercise 7.3.2. Find the minimum value of the function $f(x) = |x-1|+|x-3|+|x-8|$. For which value of x is this minimum achieved? Can you generalize this result?

Exercise 7.3.3. Let $f : [-9, 9] \to \mathbb{R}$ be defined as

$$f(x) = 10 - |x - 10|$$

for all $x \in [-9, 9]$. What are the maximum and the minimum values of f.

Exercise 7.3.4. Solve the equation $|x^2 - 4| + |x^2 - 9| = 5$.

Exercise 7.3.5. Let α denote a real number. The range of values of $|\alpha - 4|$ such that $|\alpha - 1| + |\alpha + 3| \leq 8$ is

(a) $(0, 7)$ (b) $(1, 8)$ (c) $[1, 9]$ (d) $[2, 5]$

Exercise 7.3.6. The area of the region in the plane $\mathbb{R}^2$ given by points (x, y) satisfying $|y| \leq 1$ and $x^2 + y^2 \leq 2$ is

(a) $\pi + 1$ (b) $2\pi - 2$ (c) $\pi + 2$ (d) $2\pi - 1$.

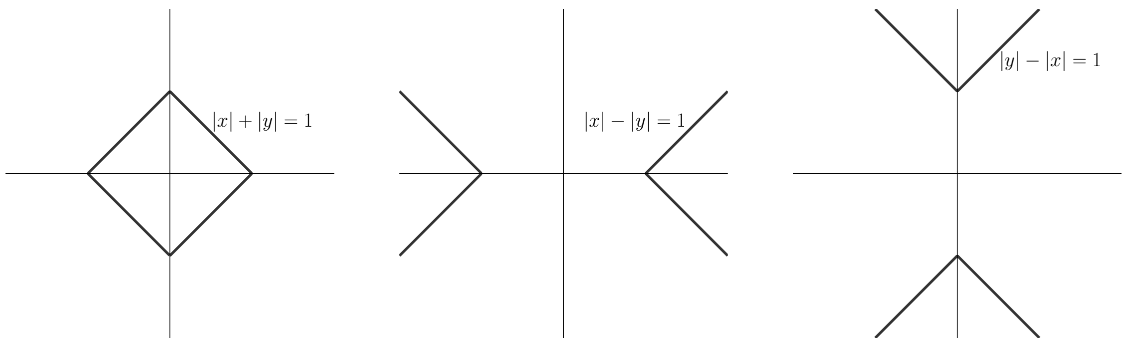
Exercise 7.3.7. What is the minimum value of the function $|x - 3| + |x + 2| + |x + 1| + |x|$ for real x ?

(a) 3 (b) 5 (c) 6 (d) 8

7.4 DIAMOND RELATION

7.4.1 Diamond relation. We sketch three interesting relations arising out of the modulus functions, namely

$$\{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\}, \quad \{(x, y) \in \mathbb{R}^2 : |x| - |y| = 1\}, \quad \{(x, y) \in \mathbb{R}^2 : |y| - |x| = 1\},$$



We will refer to the first of these as the **closed diamond** relation and the other two as *open diamonds*. This terminology is not standard. Of course, one can create variants by translation, scaling etc. ♢

Illustration 7.4.2 Describe the set of all the ordered pairs $(x, y) \in \mathbb{R}^2$ such that

$$|x-1|+|y-1|=|x+1|+|y+1|$$

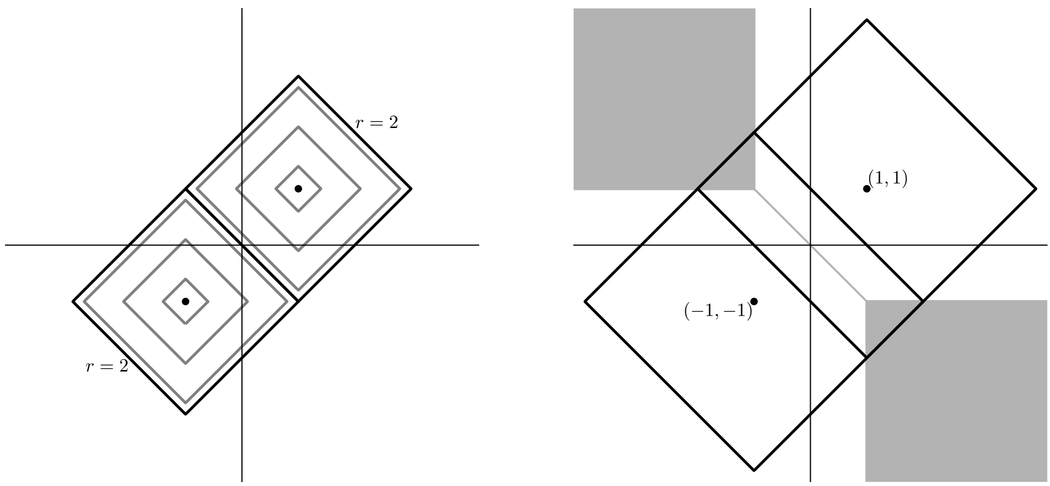
Solution. For any real number r let S_r be the set of all the points $(x, y) \in \mathbb{R}^2$ such that

$$|x-1|+|y-1|=|x+1|+|y+1|=r$$

Then the required set is nothing but the union of all the S_r as r ranges over all real numbers. This simple step is extremely useful. It is clear that if r is negative then S_r is empty. Suppose r is a small positive number. What is S_r? It is the intersection of the regions given by the equations

$$|x-1|+|y-1|=r \quad \text{and} \quad |x+1|+|y+1|=r$$

These regions are sketched on the left below for a few different value of r. We see that when $r < 2$, then the two regions do not intersect and hence correspondingly S_r is empty. However, for $r = 2$, the two regions intersection for the first time and, in fact, their intersection is a line segment passing through the origin as shown in the figure.



As we further increase the value of r beyond 2, we see that parts of quadrants start accruing into our region of solutions. We see this in the right side of the image above, where the two regions are sketched for $r = 3$. Thus S_3 is the union of the two line segments formed by intersecting the two squares shown. It should now be clear that as we continuously increase the value of r from 2 onward, the region S_r will evolve to sweep out the two gray boxes shown. Thus S is nothing but the gray region shown above (which includes the line segment passing through the origin spoken of above).

Let us now validate the geometric conclusion by means of an analytic proof. We can rearrange the equation to separate the variables:

$$|x-1|-|x+1|=|y+1|-|y-1| \tag{1}$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(t)=|t+1|-|t-1|$$

for all $t \in \mathbb{R}$. Equation (1) then becomes

$$-f(x) = f(y) \qquad \Longleftrightarrow \qquad f(x) + f(y) = 0$$

The reader must verify that

$$f(t) = \begin{cases} 2 & \text{if } t > 1 \\ 2t & \text{if } -1 \le t \le 1 \\ -2 & \text{if } t < -1 \end{cases}$$

Thus the image of f is the closed interval $[-2, 2]$. Now we solve $f(x) + f(y) = 0$. We proceed with a case analysis on the value of x:

i) If $x > 1$, then $f(x) = 2$. The equation becomes $2 + f(y) = 0$, so $f(y) = -2$. This occurs if and only if $y \le -1$. This gives us the region $\{(x, y) \mid x > 1, y \le -1\}$.

ii) If $x = 1$, then $f(x) = 2(1) = 2$. The equation is again $f(y) = -2$, which means $y \le -1$. This gives us the ray $\{(x, y) \mid x = 1, y \le -1\}$.

iii) If $-1 < x < 1$, then $f(x) = 2x$. The equation becomes $2x + f(y) = 0$, so $f(y) = -2x$. Since $-1 < x < 1$, we have $-2 < 2x < 2$, so $-2 < f(y) < 2$. This implies that we must be in the middle case for y as well, i.e., $-1 < y < 1$. In this range, $f(y) = 2y$. So the equation is $2y = -2x$, which simplifies to $y = -x$. This gives the line segment $\{(x, y) \mid y = -x, -1 < x < 1\}$.

iv) If $x = -1$, then $f(x) = 2(-1) = -2$. The equation becomes $-2 + f(y) = 0$, so $f(y) = 2$. This occurs if and only if $y \ge 1$. This gives the ray $\{(x, y) \mid x = -1, y \ge 1\}$.

v) If $x < -1$, then $f(x) = -2$. The equation is $f(y) = 2$, which means $y \ge 1$. This gives the region $\{(x, y) \mid x < -1, y \ge 1\}$.

Combining all these pieces, the set of all solutions to the given equation is the union of three sets:

- The closed quadrant: $Q_1 = \{(x, y) \in \mathbb{R}^2 \mid x \ge 1 \text{ and } y \le -1\}$
- The closed quadrant: $Q_2 = \{(x, y) \in \mathbb{R}^2 \mid x \le -1 \text{ and } y \ge 1\}$
- The line segment: $L = \{(x, y) \in \mathbb{R}^2 \mid y = -x \text{ for } -1 \le x \le 1\}$

This finishes the proof. ■

Exercise 7.4.1. How many ordered pair (x, y) satisfy the two equations

$$|x + y - 4| = 5 \quad \text{and} \quad |x - 3| + |y - 1| = 5?$$

Exercise 7.4.2. If $a, b \in \mathbb{R}$ are distinct numbers satisfying

$$|a - 1| + |b - 1| = |a| + |b| = |a + 1| + |b + 1|$$

then the minimum possible value of $|a - b|$ is : (a) 3 (b) 0 (c) 1 (d)

2

7.5 COMPOSITION AND MODULUS

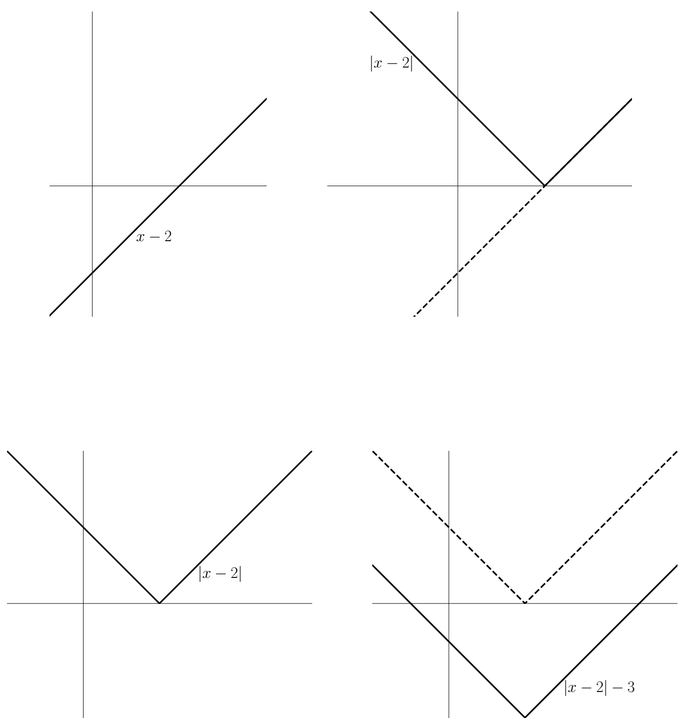
7.5.1 Post-composing with the modulus function. Let $f : S \to \mathbb{R}$ be a function defined on a subset S of $\mathbb{R}$. Let $g : S \to \mathbb{R}$ be another function defined as $g(x) = |f(x)|$ for all $x \in S$. Thus g is obtained from f by post-composing it with the modulus function. Can we understand g in terms of graphs? More precisely, if the graph of f was already drawn, can we use it to draw the graph of f?

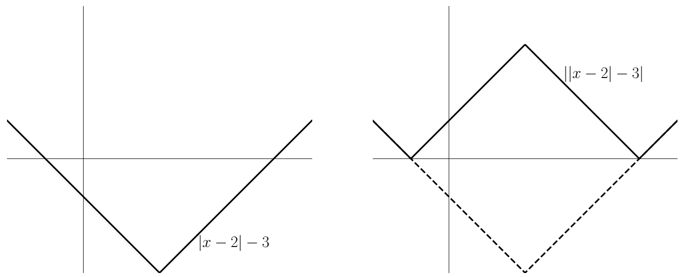
To obtain the graph of g we keep the portion of the graph of f that lies above or on the x-axis as it is. The portion of the graph that lies below the x-axis is reflected in the x-axis. The reason for this is self-explanatory and we encourage the reader to justify it. ♢

Illustration 7.5.2 Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = ||x - 2| - 3|$$

Solution. Let us do this step by step. The following graphs should make it clear how we draw the graph of f.





■

7.5.3 Pre-composing with the modulus function. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = f(|x|) \text{ for all } x \in \mathbb{R}$$

To obtain the graph of g from the graph of f, we discard the portion of the graph for $x < 0$ and replace it with a reflection of the portion for $x \geq 0$ in the y-axis. The resulting graph is symmetric about the y-axis. Again, we encourage the reader to justify this. ♢

Exercise 7.5.1. Let a be a real number. The equation $||x - 1| + a| = 4$ has

(a) 3 distinct real roots for unique value of a.

(b) 4 distinct real roots for $a \in (-\infty, -4)$

(c) 2 distinct real roots for $|a| < 4$

(d) no real roots for $a > 4$

7.6 MISCELLANEOUS PROBLEMS

Illustration 7.6.1 Let

$$f(x) = \frac{2|x| - 1}{x - 3}$$

What is the range of f? Also, if S is the set of all real number y such that $f^{-1}(y)$ has size 2, then explicitly describe S.

Solution. To understand its behavior and find its range, we break the definition into cases based on the sign of x. The domain is clearly $\mathbb{R} \setminus \{3\}$.

Case 1: $x \geq 0$ (and $x \neq 3$). Here $|x| = x$, so the function becomes:

$$f(x) = \frac{2x - 1}{x - 3} = \frac{2(x - 3) + 5}{x - 3} = 2 + \frac{5}{x - 3}$$

As x traverses the interval $[0, 3)$, the term $x - 3$ grows from -3 to negative number of arbitrarily small magnitude, so $\frac{5}{x-3}$ goes from $-5/3$ to $-\infty$. Thus, $f(x)$ takes values in

$$(-\infty, 2 - 5/3] = (-\infty, 1/3]$$

As x traverses $(3, \infty)$, the term $x - 3$ goes from arbitrarily small positive values to ∞, so $\frac{5}{x-3}$ goes from $+\infty$ to 0. Thus, f takes values in $(2, \infty)$. The range for this part is

$$(-\infty, 1/3] \cup (2, \infty)$$

Case 2: $x < 0$. Here $|x| = -x$, so the function becomes:

$$f(x) = \frac{-2x - 1}{x - 3} = \frac{-2(x - 3) - 7}{x - 3} = -2 - \frac{7}{x - 3}$$

As x varies over $(-\infty, 0)$, the denominator $x - 3$ varies over $(-\infty, -3)$. Consequently, the fraction $\frac{7}{x-3}$ varies over $(-7/3, 0)$. Therefore, the term $-\frac{7}{x-3}$ varies over $(0, 7/3)$. Adding -2 to this, the values of $f(x)$ range over

$$(-2, -2 + 7/3) = (-2, 1/3)$$

.

Conclusion on Range: Combining the images from both cases:

$$\text{Range} = (-\infty, 1/3] \cup (2, \infty) \cup (-2, 1/3) = (-\infty, 1/3] \cup (2, \infty)$$

Now we seek the set S of all y values such that $f^{-1}(y)$ has exactly 2 elements. This means the horizontal line at height y must intersect the graph of f twice. From our analysis:

- For $y \in (-\infty, -2]$, we only get solutions from Case 1 (since Case 2 range is $(-2, 1/3)$). So exactly 1 solution.
- For $y \in (-2, 1/3)$, we get one solution from Case 1 (specifically from $x \in [0, 3)$) and one solution from Case 2. So exactly 2 solutions.
- At $y = 1/3$, we get $x = 0$ from Case 1 (endpoint) and no solution from Case 2 (open interval). So exactly 1 solution.
- For $y \in (1/3, 2]$, there are no solutions.
- For $y \in (2, \infty)$, we get one solution from Case 1 ($x > 3$) and no solution from Case 2. So exactly 1 solution.

Thus, the set where the fiber size is 2 is $S = (-2, 1/3)$. The graph of the function is shown in Figure 7.4, against which all the conclusions drawn above can be tallied. ■

Illustration 7.6.2 A high voltage current is applied on the points (x, y) in the plane described by the relation

$$y + |y| - x - |x| = 0$$

On which of the following curves can a person move so that he remains safe ?

(a) $y = x^2$ (b) $y = |x|$ (c) $y = -\log_3 x$ (d) $y = 3 + |x|$

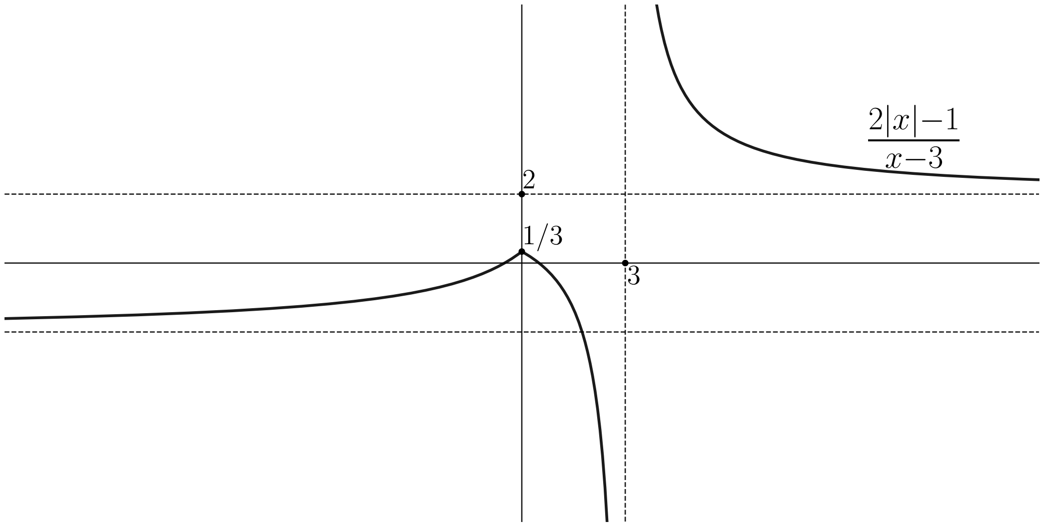


Figure 7.4

Solution. We first analyze the region defined by the equation $y + |y| - x - |x| = 0$, which can be rewritten as:

$$y + |y| = x + |x|$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(t) = t + |t|$ for all t. The equation of interest is simply $g(y) = g(x)$. Note that:

$$g(t) = \begin{cases} 2t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

We analyze this quadrant by quadrant:

1. *First Quadrant. ($x \geq 0, y \geq 0$):* Here $g(x) = 2x$ and $g(y) = 2y$. The equation becomes $2y = 2x \implies y = x$. This is the safe line in the first quadrant.
2. *Third Quadrant ($x < 0, y < 0$):* Here $g(x) = 0$ and $g(y) = 0$. The equation becomes $0 = 0$. This is always true. Thus, the entire third quadrant (including the negative axes) is safe.
3. *Second Quadrant ($x < 0, y \geq 0$):* Here $g(x) = 0$ and $g(y) = 2y$. The equation becomes $2y = 0 \implies y = 0$. This is the negative x-axis (already covered).
4. *Fourth Quadrant ($x \geq 0, y < 0$):* Here $g(x) = 2x$ and $g(y) = 0$. The equation becomes $0 = 2x$ and hence $x = 0$. This is the negative y-axis (already covered).

Thus, the safe region consists of the entire third quadrant plus the line $y = x$ in the first quadrant. Now we check the options to see which curve lies entirely outside the danger zone.

$$D = \{(x, y) : y + |y| = x + |x|\} = \text{3rd Quadrant} \cup \text{Line } y = x \text{ for } x \geq 0$$

- (a) $y = x^2$: Intersects $y = x$ at $(0, 0)$ and $(1, 1)$. Intersects danger zone.

- (b) $y = |x|$: Intersects $y = x$ for $x \geq 0$. Intersects danger zone.
- (c) $y = -\log_3 x$: Defined for $x > 0$. Crosses $y = x$ (e.g., somewhere between 0 and 1). Intersects danger zone.
- (d) $y = m + |x|$ with $m > 3$: For $x \geq 0$, $y = x + m$. Since $m > 0$, $x + m > x$, so $y > x$. This avoids the line $y = x$. For $x < 0$, $y = -x + m$. Since $m > 3$ and $-x > 0$, we have $y > 0$. This puts the points in the second quadrant. The danger zone in the second quadrant is only the axis $y = 0$ (if we consider boundaries) or empty. The danger zone is the 3rd quadrant and $y = x$ in 1st. Does $y = m + |x|$ touch the 3rd quadrant? No, because y is always positive. Does it touch $y = x$ in 1st quadrant? No, because $y = x + m$ is parallel to and above $y = x$.

Thus, curve (d) never touches the danger zone. We encourage the reader to come to the same conclusion by drawing the graphs of the relevant functions and relations. ■

Exercise 7.6.1. Let

$$f(x) = \left|\left|x^2 - 4x + 3\right| - 2\right|$$

Which of the following is/are correct?

(a) $f(x) = m$ has exactly two real solutions of different sign for all $m > 2$.

(b) $f(x) = m$ has exactly two real solutions for all $m \in (2, \infty) \cup \{0\}$.

(c) $f(x) = m$ has no solutions for all $m < 0$.

(d) $f(x) = m$ has four distinct real solution for all $m \in (0, 1)$. [(a, b, c, d)]

Exercise 7.6.2. The number of integers satisfying the equation

$$\left|x^2 + 5x\right| + \left|x - x^2\right| = |6x|$$

is (a) 3 (b) 5 (c) 7 (d) 9 [(c)]

CHAPTER 8

THE SIGNUM FUNCTION

In the previous chapter, we explored the modulus function, which gives the magnitude of a real number. We now turn to a related function that captures a different fundamental property: its sign. The signum function, from the Latin word for "sign," is a simple yet powerful tool. Its primary utility, which will be the main focus of this chapter, is in providing a systematic method for solving inequalities involving rational functions. This technique, often called the **wavy curve method** or the method of intervals, transforms the abstract algebraic problem of solving an inequality into a concrete visual task of analyzing a number line.

Beyond this practical application, the signum function offers deeper insights into the structure of functions and serves as a bridge to other important concepts in mathematics and engineering, such as the Heaviside step function.

8.1 BASICS

8.1.1 The signum function. The **signum function**, denoted by $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$, is defined as:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The graph of the signum function (shown in Figure 8.1) consists of two horizontal rays and a single point at the origin.

8.1.2 Properties of the signum function. The signum function has several elementary properties that follow directly from its definition. For any $x \in \mathbb{R}$:

a) The image of the signum function is $\{-1, 0, 1\}$.

b) For any two real numbers x and y we have $\operatorname{sgn}(xy) = \operatorname{sgn}(x) \cdot \operatorname{sgn}(y)$. In words, the sign of a product is the product of the signs.

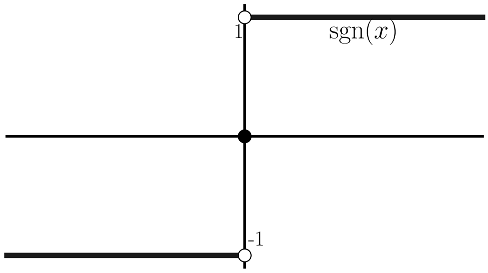


Figure 8.1

c) The above property directly leads to the following "upgradation." Let $f, g : X \to \mathbb{R}$ be functions defined on a set X. Then

$$\operatorname{sgn} \circ (fg) = (\operatorname{sgn} \circ f) \cdot (\operatorname{sgn} \circ g)$$

Recall that fg simply denotes the product of f and g as discussed in Section 2.5.

d) For $x \neq 0$ we have

$$\operatorname{sgn}(x) = \frac{x}{|x|}$$

The above can be restated by saying that for all real numbers x we have

$$x = |x| \cdot \operatorname{sgn}(x)$$

This identity shows a direct relationship between a number, its magnitude (modulus), and its sign.

8.2 THE WAVY CURVE METHOD

The primary application of the signum function is in determining the sign of an expression, which is the key to solving inequalities. The "wavy curve method" provides a graphical way to solve inequalities of the form

$$f(x) > 0, \quad f(x) < 0, \quad f(x) \geq 0 \quad \text{or} \quad f(x) \leq 0$$

where $f(x)$ is a rational function. The core idea is that a rational function f defined by

$$f(x) = \frac{P(x)}{Q(x)}$$

can only change its sign at the points where it is either zero (the roots of the numerator $P(x)$) or undefined (the roots of the denominator $Q(x)$).[1] These points are called the **critical points**. These critical points partition the number line into intervals, and within each interval, the sign of $f(x)$ remains constant.

[1]The reason for this is that $P(x)$ and $Q(x)$ are continuous functions. A full understanding of this can only emerge once the reader has undertaken the study of calculus. At this point, an intuitive understanding is sufficient.

Illustration 8.2.1 Solve the inequality

$$\frac{(x-1)(x+2)}{(x-3)} \geq 0$$

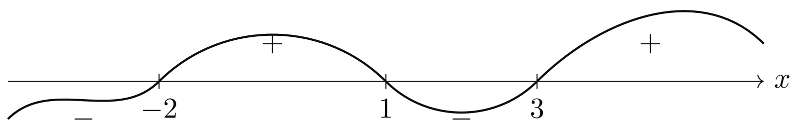
Solution. We have

$$\operatorname{sgn}\left(\frac{(x-1)(x+2)}{(x-3)}\right) = \frac{\operatorname{sgn}(x-1)\cdot\operatorname{sgn}(x+2)}{\operatorname{sgn}(x-3)} = \operatorname{sgn}(x+2)\cdot\operatorname{sgn}(x-1)\cdot\operatorname{sgn}(x-3)$$

for all $x \neq 3$, where we have used Point 8.1.2. Suppose we start out with a value of x which is much greater that 3 and slowly decrease the value. How would the sign of the given expression change? It will

1. first be positive.
2. as we cross three, the sign will flip to a negative, since the sign of only one of the three terms will flip.
3. between 1 and 3 the sign will remain negative.
4. as we cross 1, the sign will again flip and become positive.
5. between 1 and -2 the sign will remain positive.
6. finally, as we cross -2, the sign will flip for one last time and the expression will become negative and remain this way.

This can be conveniently depicted by a "wave" as shown. The wavy curve looks like this:



We want the regions where the function is ≥ 0. These are $[-2, 1]$ and $(3, \infty)$. We include -2 and 1 because they are roots of the numerator and the inequality is inclusive ($\geq$). We exclude 3 because it is a root of the denominator. The solution set is $[-2, 1] \cup (3, \infty)$. ■

Illustration 8.2.2 Solve the inequality

$$\frac{(x-2)^2(x+1)}{(x-4)^3(x+3)} \leq 0$$

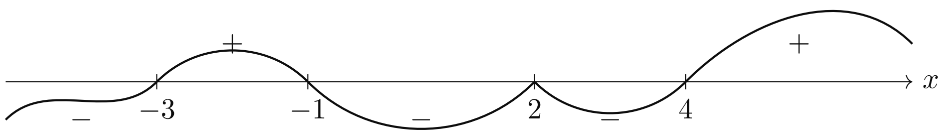
Solution. We have

$$\operatorname{sgn}\left(\frac{(x-2)^2(x+1)}{(x-4)^3(x+3)}\right) = \frac{\operatorname{sgn}((x-2)^2)\cdot\operatorname{sgn}(x+1)}{\operatorname{sgn}((x-4)^3)\cdot\operatorname{sgn}(x+3)}$$

If $x \neq 2, 4, -3$, then the above is the same as

$$\operatorname{sgn}(x+3)\cdot\operatorname{sgn}(x+1)\cdot\operatorname{sgn}(x-4)$$

Again, we used Point 8.1.2 in the above steps. We can now solve this just as in the previous illustration, but we need to keep in mind that at $x = 2$ the sign is 0. The wavy curve looks as follows



We want the regions where the function is ≤ 0. These are $(-\infty, -3)$, $(-1, 2)$, and $(2, 4)$. We must also include points where the function is exactly 0. These are the roots of the numerator: $x = -1$ and $x = 2$. Combining these, the solution set is

$$(-\infty, -3) \cup [-1, 4)$$

Notice that the interval becomes $[-1, 4)$ because we include -1 and 2, and the union of $[-1, 2]$ and $[2, 4)$ is simply $[-1, 4)$. We exclude -3 and 4 as they are roots of the denominator. ■

8.2.3 Steps for the wavy curve method. The above method can be codified as a general strategy, though it only serves to obscure the simplicity of the technique. Nevertheless, we risk spelling it out. To solve an inequality involving a rational function f:

1. **Standard Form**: Express the inequality in the form $f(x) \geq 0$ or $f(x) \leq 0$. Ensure one side is zero.
2. **Factorize**: Factorize the numerator and the denominator of $f(x)$ into linear factors of the form $(x - a)$. If you encounter irreducible quadratic factors (like $x^2 + 1$) that are always positive, they can be ignored as they do not affect the sign of the function.
3. **Find Critical Points**: Identify all the roots of the numerator and the denominator. These are the critical points.
4. **Plot on Number Line**: Mark all critical points on a number line. This divides the line into several intervals.
5. **Determine Sign in the Rightmost Interval**: Choose a test value greater than the largest critical point and determine the sign of $f(x)$. This sign will be the sign for the entire rightmost interval. A simpler way is to look at the signs of the coefficients of x in all factors. If the product of these signs is positive, the rightmost interval is positive; otherwise, it is negative.
6. **Determine Signs in Other Intervals (The "Wave")**: Moving from right to left across the number line:

   - If you cross a critical point corresponding to a factor with an **odd** power (e.g., $(x - a)^1, (x - a)^3$), the sign of $f(x)$ flips.
   - If you cross a critical point corresponding to a factor with an **even** power (e.g., $(x - a)^2, (x - a)^4$), the sign of $f(x)$ stays the same.

7. **Identify Solution Set**: Based on the original inequality (≥ 0 or ≤ 0), select the intervals that satisfy it. Be careful with the endpoints:

- For $\geq$ or $\leq$, include the roots of the numerator.
- Never include the roots of the denominator, as the function is undefined there (use open circles for them).

Exercise 8.2.1. Sketch the graph of the function $g(x) = x \cdot \text{sgn}(x)$. How does it relate to another function we have studied?

Exercise 8.2.2. Solve the following inequalities using the wavy curve method:

1. $(x + 1)(x - 2)(x + 4) < 0$
2. $\dfrac{x^2 - 4}{x^2 - 9} \geq 0$
3. $\dfrac{(x - 1)^3(x + 2)^2}{x(x - 4)} \leq 0$
4. $x^2 + 2x > 15$

Exercise 8.2.3. Sketch the graph of $f(x) = sgn(\sin(x))$ for $x \in [-2\pi, 2\pi]$.

Exercise 8.2.4. Find the solution set for the inequality

$$\frac{|x| - 1}{|x| - 2} \geq 0$$

(Hint: Let $y = |x|$ and solve for y first.)

Exercise 8.2.5. 8 The set of all real numbers x satisfying the inequality

$$x^3(x + 1)(x - 2) \geq 0$$

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(a) the interval $[2, \infty)$ (b) the interval $[0, \infty)$ (c) the interval $[-1, \infty)$ (d) none

CHAPTER 9

SQUARE ROOTS

The square root function is not merely an algebraic curiosity; it is a fundamental language used by nature to describe relationships of scale and energy. In physics, it governs the flow of time for falling objects—Galileo showed us that the time it takes for an object to fall a distance d is proportional to $\sqrt{d}$. In statistics and economics, the square root appears in the calculation of standard deviation, a crucial metric for measuring risk and volatility; often, risk does not grow linearly with time or size, but rather with the square root. In biology, metabolic rates and surface-area-to-volume ratios—which determine how animals regulate heat and how cells absorb nutrients—are deeply tied to power laws involving roots. To master surds is to master the mathematics of non-linear scaling, a skill essential for understanding a world where output is rarely a straight line of input.

9.1 EQUATIONS

Before we begin, we remind the reader of the message in Section 4.5.

Illustration 9.1.1 Solve for x:

$$\sqrt{25-x} = 2 - \sqrt{9+x}$$

Solution. Before performing algebraic manipulations, we must respect the domain constraints. For the square roots to define real numbers, the radicands must be non-negative:

$$25 - x \geq 0 \implies x \leq 25 \quad \text{and} \quad 9 + x \geq 0 \implies x \geq -9$$

Thus, our search for solutions is restricted to the interval $[-9, 25]$. To eliminate the radicals, we rearrange the terms to isolate one square root or balance the equation. Fix a particular solution S. It is helpful to make both sides positive to avoid sign errors during squaring. Let's rewrite the equation as:

$$\sqrt{25-x} + \sqrt{9+x} = 2$$

Squaring both sides yields:

$$(25 - x) + (9 + x) + 2\sqrt{(25 - x)(9 + x)} = 4$$

Simplifying the linear terms:

$$34 + 2\sqrt{(25 - x)(9 + x)} = 4$$

Isolating the radical term:

$$2\sqrt{(25 - x)(9 + x)} = 4 - 34 = -30$$

$$\sqrt{(25 - x)(9 + x)} = -15$$

At this point, we can stop. The principal square root function, by definition, yields non-negative values. The left-hand side is non-negative, while the right-hand side is negative (-15). This is a contradiction. So the existence of any solution of the original equation leads to an absurdity, having us conclude that the equation has no real solution. ■

Illustration 9.1.2 Solve for x:

$$(x + 4)(x + 1) - 3\sqrt{x^2 + 5x + 2} = 6$$

Solution. The given equation is equivalent to:

$$(x^2 + 5x + 4) - 3\sqrt{x^2 + 5x + 2} = 6 \tag{*}$$

Fix a particular x satisfying the above. Notice the recurring pattern $x^2 + 5x$. This suggests a substitution, namely $y = \sqrt{x^2 + 5x + 2}$. By definition of y, we deduce that

$$y^2 = x^2 + 5x + 2 \qquad \Rightarrow \qquad x^2 + 5x = y^2 - 2$$

Substituting this back into (*) we get:

$$(y^2 - 2 + 4) - 3y = 6 \qquad \Rightarrow \qquad y^2 - 3y + 2 = 6 \qquad \Rightarrow \qquad y^2 - 3y - 4 = 0$$

Factoring this quadratic in y:

$$(y - 4)(y + 1) = 0$$

This gives two *potential* values for y: 4 and -1. However, by definition of y, we have $y \geq 0$, and thus we discard $y = -1$. Let us explore the possibility $y = 4$. We have

$$\sqrt{x^2 + 5x + 2} = 4$$

Squaring both sides:

$$x^2 + 5x + 2 = 16 \qquad \Rightarrow \qquad x^2 + 5x - 14 = 0 \qquad \Rightarrow \qquad (x + 7)(x - 2) = 0$$

The last equation has two solutions: $x = -7$ and $x = 2$. A simple check reveals that both these values of x are indeed solutions of (*). The solution set is $\{-7, 2\}$. ■

Illustration 9.1.3

$$\frac{8}{\sqrt{10-2x}} - \sqrt{10-2x} = 2$$

Solution. Fix a solution x of the given equation. The structure immediately invites substitution. Let $u = \sqrt{10-2x}$. The equation becomes:

$$\frac{8}{u} - u = 2 \qquad \Rightarrow \qquad 8 - u^2 = 2u \qquad \Rightarrow \qquad u^2 + 2u - 8 = 0$$

Factoring the quadratic:

$$(u+4)(u-2) = 0$$

The possible values for u are -4 and 2. But u is non-negative by its definition, and thus we discard -4 and proceed with $u = 2$. Reverting to x we get:

$$\sqrt{10-2x} = 2 \qquad \Rightarrow \qquad 10 - 2x = 4 \qquad \Rightarrow \qquad x = 3$$

So we have shown that there is only one candidate for the solution of the original equation, and a simple check shows that 3 is indeed a solution. ∎

Illustration 9.1.4

$$\sqrt{6x - x^2 - 5} = 2x - 6$$

Solution. Solving the given equation is equivalent to finding the set of all the x such that

$$2x - 6 \geq 0 \quad \text{and} \quad 6x - x^2 - 5 = (2x-6)^2$$

This is simply because $\sqrt{\alpha} = \beta$ for two real numbers α and β if and only if $\beta \geq 0$ and $\alpha = \beta^2$. Now

$$2x - 6 \geq 0 \qquad \Longleftrightarrow \qquad x \geq 3$$

So we need to find all the $x \geq 3$ such that

$$6x - x^2 - 5 = (2x-6)^2$$

which is equivalent to

$$5x^2 - 30x + 41 = 0$$

Using the quadratic formula, the last equation has two solutions:

$$x = 3 \pm \frac{2\sqrt{5}}{5}$$

However, only one of these is at least 3 in value, and hence there is exactly one solution to the original equation, namely

$$3 + \frac{2}{\sqrt{5}}$$

and we are done. ∎

Illustration 9.1.5

$$\frac{2+\sqrt{19-2x}}{x}=1$$

Solution. Let x be a solution to the given equation. Then we get

$$2+\sqrt{19-2x}=x \qquad \Rightarrow \qquad 19-2x=(x-2)^2 \qquad \Rightarrow \qquad 19-2x=x^2-4x+4$$

Rearranging into standard form:

$$x^2-2x-15=0 \qquad \Rightarrow \qquad (x-5)(x+3)=0$$

So the candidates for x are 5 and -3. However, a simple check shows that -3 fails to satisfy the original equation, while 5 does. Thus, the only solution is 5. ■

Exercise 9.1.1.

$$\sqrt{2x-4}-\sqrt{x+5}=1$$

Exercise 9.1.2. Solve for x

$$\sqrt{x-2}+\sqrt{x+3}=-1$$

Exercise 9.1.3. Solve for x:

$$(x-2)(x-5)-2\sqrt{x^2-7x+16}=2$$

Exercise 9.1.4. Solve for x:

$$\sqrt{2x+3}-\frac{4}{\sqrt{2x+3}}=3$$

Exercise 9.1.5. Solve for x:

$$\sqrt{15-2x-x^2}=x+3$$

Exercise 9.1.6. Solve for x:

$$\frac{1+\sqrt{x^2-9}}{x}=1$$

9.2 INEQUALITIES

Illustration 9.2.1 Find all the x such that

$$\sqrt{5-x}\leq x+7$$

Solution. If α and β are two real numbers then $\sqrt{\alpha} \leq \beta$ if and only if $\alpha, \beta \geq 0$ and $\alpha \leq \beta^2$. Thus the given inequality is satisfied by exactly those x which satisfy

$$5 - x \geq 0, x + 7 \geq 0 \quad \text{and} \quad 5 - x \leq (x+7)^2$$

This is same as finding all the x in the range $(-7, 5]$ such that

$$\begin{aligned} 5 - x \leq (x+7)^2 \quad &\iff \quad x^2 + 15x + 44 \geq 0 \\ &\iff \quad (x+11)(x+4) \geq 0 \\ &\iff \quad x \in (-\infty, -11] \cup [-4, \infty) \end{aligned}$$

Therefore the set of all the solutions of the given inequality is

$$(-\infty, 5] \cap ((-\infty, -11] \cup [-4, \infty))$$

which is $[-4, 5]$. ■

Illustration 9.2.2 Find all the x such that

$$\sqrt{2x+5} + \sqrt{x-1} > 8$$

Solution. If x is a solution of the given inequality, then we must have

a) $2x + 5 \geq 0 \quad \Rightarrow \quad x \geq -2.5$, and
b) $x - 1 \geq 0 \quad \Rightarrow \quad x \geq 1$.

Combining the above, we see that each x solving the inequality must lie in $[1, \infty)$. Now let $f : [1, \infty) \to \mathbb{R}$ be defined as

$$f(x) = \sqrt{2x+5} + \sqrt{x-1}$$

Reframing the problem, we want to find all the x such that $f(x) > 0$. Note that $f(x)$ is the sum of two strictly increasing functions, so $f(x)$ itself is strictly increasing. We look for the point where equality holds: $f(x) = 8$. By inspection or educated guessing (looking for perfect squares), we test $x = 10$:

$$f(10) = \sqrt{2(10)+5} + \sqrt{10-1} = \sqrt{25} + \sqrt{9} = 5 + 3 = 8$$

Since $f(x)$ is strictly increasing, we have:

$$f(x) > 8 \quad \iff \quad x > 10$$

Thus, the solution set is $(10, \infty)$. ■

Illustration 9.2.3 Find all the x such that

$$\sqrt{2x - x^2} < 5 - x$$

Solution. If α and β are real numbers, then $\sqrt{\alpha} < \beta$ if and only if $\alpha, \beta \geq 0$ and $\alpha < \beta^2$. Thus x is a solution to the given inequality if and only if

$$2x - x^2 \geq 0, 5 - x \geq 0 \quad \text{and} \quad 2x - x^2 < (5-x)^2$$

The former two inequalities are equivalent to $x \in [0, 2]$. So we need to find all the x in $[0, 2]$ which satisfy

$$2x - x^2 < (5-x)^2 \qquad \Longleftrightarrow \qquad 2x^2 - 12x + 25 > 0$$

but the latter is true for all x, thanks to the fact that the discriminant of the quadratic is negative. Thus, the solution is the interval $[0, 2]$. ■

Illustration 9.2.4 Find all the x such that

$$x - 3\sqrt{x-3} - 1 > 0$$

Solution. If x is a solution to the given inequality then we must have $x - 3 \geq 0$, thanks to the surd, and thus all the solutions will lie in $[3, \infty)$. Now for any real number x we have

$$x - 3\sqrt{x-3} - 1 > 0 \qquad \Longleftrightarrow \qquad (x-3) - 3\sqrt{x-3} + 2 > 0$$

Let $f : [3, \infty) \to \mathbb{R}$ be defined as $f(x) = \sqrt{x-3}$. Thus we are looking for precisely those $x \in [3, \infty)$ such that

$$\begin{aligned} f(x)^2 - 3f(x) + 2 > 0 \qquad &\Longleftrightarrow \qquad (f(x) - 1)(f(x) - 2) > 0 \\ &\Longleftrightarrow \qquad f(x) \in (-\infty, 1) \cup (2, \infty) \end{aligned}$$

Now

$$f(x) < 1 \qquad \Longleftrightarrow \qquad \sqrt{x-3} < 1 \qquad \Longleftrightarrow \qquad 3 \leq x < 4$$

and

$$f(x) > 2 \qquad \Longleftrightarrow \qquad \sqrt{x-3} > 2 \qquad \Longleftrightarrow \qquad x > 7$$

So the set of all the solutions is $[3, 4) \cup (7, \infty)$. ■

Illustration 9.2.5 Find all the x such that

$$\frac{4 - \sqrt{x+1}}{1 - \sqrt{x+3}} \leq 3$$

Solution. Owing to the surds, we must have $x \geq -1$. Further, the denominator must not vanish. Thankfully, the condition $x \geq -1$ already guarantees this. Thus we are looking for all the x in $[-1, \infty)$ such that the given inequality is true. We want to cross multiply to make the inequality more amenable to analysis. For this we need to be careful about the sign of the denominator.

Sign Analysis: For any $x \geq -1$, we have $x + 3 \geq 2$, so $\sqrt{x+3} \geq \sqrt{2} > 1$. Therefore, the denominator $1 - \sqrt{x+3}$ is always negative. Therefore, for $x \in [-1, \infty)$ we have

When we multiply both sides by this negative denominator, the direction of the inequality flips:

$$\frac{4-\sqrt{x+1}}{1-\sqrt{x+3}} \leq 3 \qquad \iff \qquad 4-\sqrt{x+1} \geq 3(1-\sqrt{x+3})$$
$$\iff \qquad 1+3\sqrt{x+3} \geq \sqrt{x+1}$$

Observe the two sides. Since $x+3 > x+1$, we clearly have $\sqrt{x+3} > \sqrt{x+1}$. It follows immediately that:

$$3\sqrt{x+3}+1 > \sqrt{x+3} > \sqrt{x+1}$$

Thus the inequality holds true for all x in $[-1, \infty)$. The solution set is $[-1, \infty)$. ■

Exercise 9.2.1. Solve the following inequality:

$$3-x > 3\sqrt{1-x^2}$$

Exercise 9.2.2. Solve the following inequality:

$$\sqrt{8+2x-x^2} > 6-3x$$

Exercise 9.2.3. Solve the following inequality:

$$\frac{\sqrt{2x-1}}{x-2} < 1$$

Exercise 9.2.4. Solve the following inequality:

$$\sqrt{-x^2+6x-5} > 8-2x$$

Exercise 9.2.5. Solve the following inequality:

$$\frac{1-\sqrt{21-4x-x^2}}{x+1} \geq 0$$

Exercise 9.2.6. Solve the following inequality:

$$\sqrt{x-6}-\sqrt{10-x} \geq 1$$

Exercise 9.2.7. Solve the following inequality:

$$\sqrt{1-\frac{x+2}{x^2}} < \frac{2}{3}$$

Exercise 9.2.8. Solve the following inequality:

$$\frac{3}{\sqrt{2-x}} - \sqrt{2-x} < 2$$

Exercise 9.2.9. Solve the following inequality:

$$\frac{x^2 - 13x + 40}{\sqrt{19x - x^2 - 78}} \leq 0$$

Exercise 9.2.10. Solve the following inequality:

$$\sqrt{x^2 - 3x - 10} < 8 - x$$

Exercise 9.2.11. Solve the following inequality:

$$\sqrt{x+3} + \sqrt{x+15} < 6$$

9.3 DOMAIN AND RANGE

Illustration 9.3.1 Find the domain and range of

$$\sqrt{\frac{1}{2x^2 - 5x - 3}}$$

Solution. The domain is determined by two constraints: the expression under the square root must be non-negative and the denominator should not be zero. This is equivalent to

$$12x^2 - 5x - 3 > 0$$

which yields that the domain is

$$D = \left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{3}{4}, \infty\right)$$

Now for the range. Define $f : D \to \mathbb{R}, \rho : (0, \infty) \to \mathbb{R}$ and $\sqrt{\cdot} : [0, \infty) \to \mathbb{R}$ be defined as

$$f(x) = 2x^2 - 5x - 3, \quad \rho(x) = \frac{1}{x} \quad \text{and} \quad \sqrt{\cdot}(x) = \sqrt{x}$$

What we want is the image of $\sqrt{\cdot} \circ \rho \circ f$, which is nothing but

$$\sqrt{\cdot}(\rho(\text{Image}(f)))$$

The image of f is $(0, \infty)$, and thus $\rho(\text{Image}(f)) = (0, \infty)$. This in turn means that $\sqrt{\cdot}(\rho(\text{Image}(f))) = (0, \infty)$. Thus the range of the given expression is $(0, \infty)$. ■

Illustration 9.3.2 Find the domain and range of

$$\sqrt{\frac{x^2-7x+12}{x^2-2x-3}}$$

Solution. The domain consists of all the points where the denominator is nonzero and the expression in the surd is non-negative. Factorizing the numerator and the denominator, the given expression is same as

$$\sqrt{\frac{(x-3)(x-4)}{(x-3)(x+1)}}$$

The denominator vanishes exactly on $x = 3$ and -1. For $x \neq 3$, the term $(x-3)$ cancels, and the expression becomes:

$$\sqrt{\frac{x-4}{x+1}}$$

Using the wavy curve method (or common sense), this expression is nonzero if and only if x is in

$$(-\infty, -1) \cup [4, \infty)$$

The domain is the intersection of this set with $\mathbb{R} \setminus \{-1, 3\}$, which gives that the domain is

$$D = (-\infty, -1) \cup [4, \infty)$$

Now define $f : D \to \mathbb{R}$ as

$$f(x) = \frac{x-4}{x+1}$$

The range of the given expression is nothing but the square root of all the values in the image of f. So first we determine the image of f. We have

$$f(x) = \frac{x-4}{x+1} = \frac{(x+1)-5}{x+1} = 1 - \frac{5}{x+1}$$

- As x varies over $(-\infty, -1)$, the term $x+1$ ranges over $(-\infty, 0)$. Thus $\frac{5}{x+1}$ ranges over $(-\infty, 0)$, and hence $-\frac{5}{x+1}$ ranges over $(0, \infty)$. So $f(x)$ ranges over $(1, \infty)$.
- As x varies in $[4, \infty)$, the term $x+1$ ranges over $[5, \infty)$. Thus $\frac{5}{x+1}$ ranges over $(0, 1]$. Then $-\frac{5}{x+1}$ ranges over $[-1, 0)$. So $f(x)$ ranges over $[0, 1)$.

So the image of f is $[0, 1) \cup (1, \infty)$. Consequently, the range of the given expression is $[0, 1) \cup (1, \infty)$. ■

Illustration 9.3.3 Find the domain and range of

$$\sqrt{x-x^2} + \sqrt{3x-x^2-2}$$

Solution. For x to be in the domain, both radicands must be non-negative simultaneously. Thus

- $x - x^2 \geq 0 \iff x(1-x) \geq 0 \iff x \in [0,1]$, and
- $3x - x^2 - 2 \geq 0 \iff (x-1)(x-2) \leq 0 \iff x \in [1,2]$.

The domain is the intersection of these two sets:

$$[0,1] \cap [1,2] = \{1\}$$

The domain contains only the single element $\{1\}$. Since the domain has only one point, we simply evaluate the function at $x = 1$ to find the range:

$$\sqrt{1-1^2} + \sqrt{3(1) - 1^2 - 2} = \sqrt{0} + \sqrt{0} = 0$$

The range is $\{0\}$. ■

Exercise 9.3.1. Find the domain and range of

$$\sqrt{16 - x^2 - 6x}$$

Exercise 9.3.2. Find the domain and range of

$$\frac{\sqrt{x^2-1}}{x}$$

Exercise 9.3.3. Find the domain and range of

$$\sqrt{\frac{2x-1}{x^2-4}}$$

CHAPTER 10

GIF AND FPF

10.1 THE FLOOR FUNCTION

Let us recall the definition of the floor function (or the greatest integer function, or step function) and the fractional part function discussed in Section 4.2.

If x is any real number, there is a unique integer n such that

$$n \leq x < n + 1$$

The **floor value** of x, written $\lfloor x \rfloor$, is defined as this unique integer n. The **floor function**, also called the **step function**, is defined as the function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{R}$ which takes x to $\lfloor x \rfloor$.

A cousin of the floor function is the ceiling function, whose definition we also recall. For any real number x, the **ceiling value** of x, written $\lceil x \rceil$, is defined as the unique integer n such that

$$n - 1 < x \leq n$$

The **ceiling function** is defined as the function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{R}$ which takes $x \in \mathbb{R}$ to $\lceil x \rceil$. The graphs of the floor and the ceiling functions are shown in Figure 4.6.

10.1.1 Finding dividend. Imagine you have a collection of n indistinguishable items (say, coins) and you want to pack them into bags, where each bag holds exactly d coins.

- The division n/d gives you the precise number of bags you can fill, potentially resulting in a fraction (e.g., "5.7 bags").
- In the physical world, you cannot have "0.7 of a bag" if you are counting full sets. You only care about the number of *complete* bags.
- The floor function $\lfloor \cdot \rfloor$ is designed to act as the mathematical equivalent of "ignoring the loose coins" or chopping off the decimal part.

Therefore, the number of full bags (the quotient) is simply the integer part of the fraction n/d. This is also called the **dividend** when n is divided by d. Let us discuss this in a more abstract setting. Let n be an arbitrary integer and d be a positive integer. A simple and intuitive fact is that there exist unique integers q (dividend) and r (remainder) such that:

$$n = dq + r, \quad \text{where } 0 \leq r < d \tag{10.1}$$

We will express q in terms of n and d by using the floor function. To isolate q, we divide the entire equation by d:

$$\frac{n}{d} = \frac{dq + r}{d} = q + \frac{r}{d}$$

Since $0 \leq r < d$, dividing by the positive integer d gives:

$$0 \leq \frac{r}{d} < 1$$

Operating the floor function on both sides we get

$$\left\lfloor \frac{n}{d} \right\rfloor = \left\lfloor q + \frac{r}{d} \right\rfloor$$

Now since q is an integer and $0 \leq r/d < 1$, by definition of the floor function we have:

$$\left\lfloor \frac{n}{d} \right\rfloor = q$$

We thus conclude that the quotient when n is divided by d is given explicitly by $\lfloor n/d \rfloor$. ♢

10.1.2 Elementary properties of the floor function. Let x and y be real numbers. Then

a) If $x \leq y$ then $x - 1 < \lfloor x \rfloor \leq \lfloor y \rfloor \leq y$.

b) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ for all integers m.

c) For any positive integer n we have

$$\left\lfloor \frac{\lfloor x \rfloor}{d} \right\rfloor = \left\lfloor \frac{x}{d} \right\rfloor$$

d)

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$$

e) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. We prove these one by one.

a) Fix a real number x. To show that $x - 1 \leq \lfloor x \rfloor$, we make two cases. If x is an integer then the inequality is clear. So suppose that x is not an integer. But then observe that there must be an integer between $x - 1$ and x, and $\lfloor x \rfloor$ must (at least) be this integer. So we must have $x - 1 < \lfloor x \rfloor$. The rest of part (a) is left as an exercise.

b) Let $x = n + \alpha$ where n is an integer and $0 \le \alpha < 1$. In other words, $n = \lfloor x \rfloor$ and $\alpha = x - \lfloor x \rfloor$. Now

$$x + m = (n + \alpha) + m = (n + m) + \alpha \tag{1}$$

Thus $\lfloor x + m \rfloor = n + m$, giving the desired result once we write $\lfloor x \rfloor$ in place of n.

c) Let $x = a + \alpha$ where a is an integer and $0 \le \alpha < 1$. Thus $a = \lfloor x \rfloor$. Say

$$q = \left\lfloor \frac{\lfloor x \rfloor}{d} \right\rfloor = \left\lfloor \frac{a}{d} \right\rfloor \tag{2}$$

Thus q is the dividend when a is divided by d (see 10.1.1). So we can write $a = dq + r$ for some integer q and some $0 \le r < d$. Thus

$$\left\lfloor \frac{x}{d} \right\rfloor = \left\lfloor \frac{a + \alpha}{d} \right\rfloor = \left\lfloor \frac{dq + \alpha + r}{d} \right\rfloor = \left\lfloor q + \frac{r + \alpha}{d} \right\rfloor \tag{3}$$

But

$$\begin{aligned} 0 \le r < d,\ 0 \le \alpha < 1 \quad &\Rightarrow \quad 0 \le r \le q - 1,\ 0 \le \alpha < 1 \\ &\Rightarrow \quad 0 \le r + \alpha < (d - 1) + 1 \\ &\Rightarrow \quad 0 \le r + \alpha < d \\ &\Rightarrow \quad 0 \le \frac{r + \alpha}{d} < 1 \end{aligned} \tag{4}$$

Using this in (3) gives

$$\left\lfloor \frac{x}{d} \right\rfloor = q \tag{5}$$

From (2) and (5) we see that the desired equality holds.

d) We leave this as an exercise.

e) This relationship is best seen by writing x and y in terms of their integer and fractional parts. Let

$$x = n + \alpha \quad \text{and} \quad y = m + \beta$$

where $n = \lfloor x \rfloor$ and $m = \lfloor y \rfloor$. Thus $0 \le \alpha, \beta < 1$. Now let's look at $\lfloor x + y \rfloor$:

$$\lfloor x + y \rfloor = \lfloor (n + \alpha) + (m + \beta) \rfloor = \lfloor (n + m) + (\alpha + \beta) \rfloor$$

Since $n + m$ is an integer, we can pull it out of the floor function (see part (b)):

$$\lfloor x + y \rfloor = (n + m) + \lfloor \alpha + \beta \rfloor = (\lfloor x \rfloor + \lfloor y \rfloor) + \lfloor \alpha + \beta \rfloor \tag{*}$$

whence it is clear that $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor$. To argue for the other inequality, we note that all the action is in the term $\lfloor \alpha + \beta \rfloor$. Since $0 \le \alpha < 1$ and $0 \le \beta < 1$, the sum $\alpha + \beta$ must be in the range $0 \le \alpha + \beta < 2$ and hence

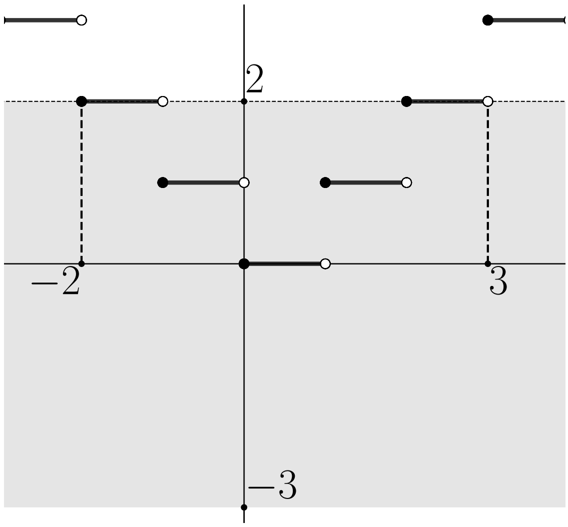
$$\lfloor \alpha + \beta \rfloor \le 1$$

Using this in (*) gives the desired result.

Illustration 10.1.3 Find all the real numbers x such that $|\lfloor x\rfloor|$ lies in $[-3,2]$.

(a) $[-3,2]$ (b) $[-2,3)$ (c) $[-3,3]$ (d) $[-2,3]$

Solution. Define the function $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=|\lfloor x\rfloor|$. We are interested in $f^{-1}([-3,2])$. Since f cannot take negative values, this is same as $f^{-1}((-\infty,2))$. The graph of f is shown below, along with the horizontal strip corresponding to $(-\infty,2)$ shown as the shaded region.



From the figure it is clear that $f^{-1}((-\infty,2))$ is $[-2,3)$. Option (b) is correct. The reader is encouraged to write an analytical proof. ■

Illustration 10.1.4 Let

$$f(n)=\frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor+\cdots+\left\lfloor\frac{n}{n}\right\rfloor\right)$$

Prove that $f(2n)>f(n)$.

Solution. Let $g(n)$ be defined as

$$g(n)=\left\lfloor\frac{n}{1}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor+\cdots+\left\lfloor\frac{n}{n}\right\rfloor$$

so that $f(n)=g(n)/n$. Note that

$$g(2n)=g(n)+\sum_{k=1}^{n}\left\lfloor\frac{2n}{2k-1}\right\rfloor$$

and hence

$$f(2n) - f(n) = \frac{1}{2n}g(2n) - \frac{1}{n}g(n) = \left[\frac{g(n)}{2n} + \frac{1}{2n}\sum_{k=1}^{n}\left\lfloor\frac{2n}{2k-1}\right\rfloor\right] - \frac{g(n)}{n}$$
$$= \frac{1}{2n}\sum_{k=1}^{n}\left\lfloor\frac{2n}{2k-1}\right\rfloor - \frac{g(n)}{2n}$$
$$= \frac{1}{2n}\sum_{k=1}^{n}\left(\left\lfloor\frac{2n}{2k-1}\right\rfloor - \left\lfloor\frac{n}{k}\right\rfloor\right)$$

But each term in the last summation above is non-negative since $2n/(2k-1) \geq n/k$. Further, the term corresponding to $k = 1$ is strictly positive. Thus $f(2n) - f(n) > 0$ and we are done. ■

Exercise 10.1.1. Draw the graph of the function $f : \mathbb{R} \setminus \mathbb{Z} \to \mathbb{R}$ defines as

$$f(x) = \frac{1}{\lceil x \rceil - \lfloor x \rfloor}$$

Exercise 10.1.2. Let

$$S_r = \sum_{r=1}^{n} r!$$

Show that

$$S_n - 7\left[\frac{S_n}{7}\right]$$

is a constant for $n > 6$.

Exercise 10.1.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \left\lfloor\frac{2x^2}{x^2+1}\right\rfloor$$

What is the size of the image of f?

Exercise 10.1.4. Let α and β be real numbers. Show that if $\lfloor\alpha\rfloor = \lfloor\beta\rfloor$, then $|\alpha - \beta| \leq 1$.

Exercise 10.1.5. Prove that

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor = \left\lfloor\frac{n^2}{4}\right\rfloor$$

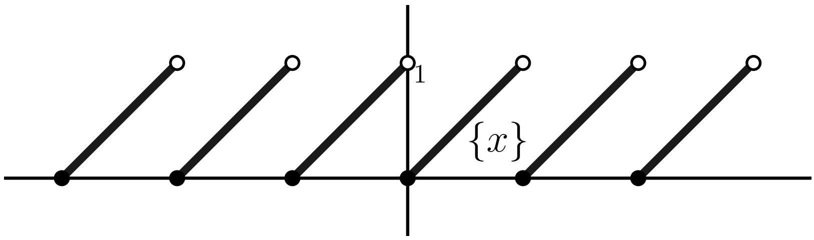
for all natural numbers n.

10.2 FRACTIONAL PART FUNCTION

10.2.1 Definition: Fractional part function. The function $\{\cdot\} : \mathbb{R} \to \mathbb{R}$ defined as

$$\{x\} = x - \lfloor x \rfloor$$

for all $x \in \mathbb{R}$, is called the **fractional part function**. This function isolates the "decimal part" of a number. The number $\{x\}$ is called the **fractional part** of x. It is unfortunate that the above is the standard notation for this function. The number $\{x\}$ can easily be confused with the set which comprises exactly one element, namely x.



10.2.2 A characterization of the fractional part function. Let x and y be real numbers. Then the following are equivalent

a) The fractional parts of x and y are the same, that is, $\{x\} = \{y\}$.

b) $x - y$ is an integer.

Proof. (a) $\Rightarrow$ b): Assume that the fractional parts of x and y are the same, that is, $\{x\} = \{y\}$. Thus

$$x - \lfloor x \rfloor = y - \lfloor y \rfloor$$

Our goal is to show that $x - y$ is an integer. Let's rearrange the equation above to isolate $x - y$:

$$x - y = \lfloor x \rfloor - \lfloor y \rfloor$$

The right hand side is clearly an integer, and hence so is the left hand side. Therefore, $x - y$ is equal to an integer.

(b) $\Rightarrow$ (a): Assume that $x - y$ is an integer, and let us call this integer k. This means we can write $x = y + k$. Our goal is to show that $\{x\} = \{y\}$. But

$$\begin{aligned}\{x\} = x - \lfloor x \rfloor &= y + k - \lfloor y + k \rfloor \\ &= y + k - (\lfloor y \rfloor + k) \\ &= y - \lfloor y \rfloor = \{y\}\end{aligned}$$

This finishes the proof. ♢

10.2.3 Elementary properties of the fractional part function. Let x and y be a real numbers. Then we have

a) $0 \leq \{x\} < 1$.
b) $\{x+m\} = \{x\}$ if m is an integer.
c) $\{-x\} = 1 - \{x\}$ if x is not an integer.
d) We have
$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$
if and only if $\{x\} + \{y\} < 1$.

Proof. Part (a) is clear and part (b) is immediately from 10.2.2. Let us give a proof of (c). We have

$$\{-x\} = -x - \lfloor -x \rfloor$$

We also have $x = \lfloor x \rfloor + \{x\}$, and hence $-x = -\lfloor x \rfloor - \{x\}$. Using this in the equation above, we get

$$\{-x\} = (-\lfloor x \rfloor - \{x\}) - \lfloor -\lfloor x \rfloor - \{x\} \rfloor$$

Since $-\lfloor x \rfloor$ is an integer, we can pull it out of the floor function:

$$\begin{aligned} \{-x\} = -\lfloor x \rfloor - \{x\} - (\lfloor -\{x\} \rfloor - \lfloor x \rfloor) \quad &\Rightarrow \quad \{-x\} = -\lfloor x \rfloor - \{x\} - \lfloor -\{x\} \rfloor + \lfloor x \rfloor \\ &\Rightarrow \quad \{-x\} = -\{x\} - \lfloor -\{x\} \rfloor \end{aligned}$$

Now, let's analyze the term $\lfloor -\{x\} \rfloor$. We are given that x is *not* an integer. This is crucial, because it means that $\{x\}$ is not 0. From property (a), we know $0 \leq \{x\} < 1$. Since x is not an integer, we can be more specific and write $0 < \{x\} < 1$. Multiplying by -1 reverses the inequalities:

$$-1 < -\{x\} < 0$$

What is the floor (the greatest integer less than or equal to) a number that is strictly between -1 and 0? It must be -1. Therefore, $\lfloor -\{x\} \rfloor = -1$. Substituting this back into our equation we get the desired result. The proof of part (d) is an exercise. $\diamondsuit$

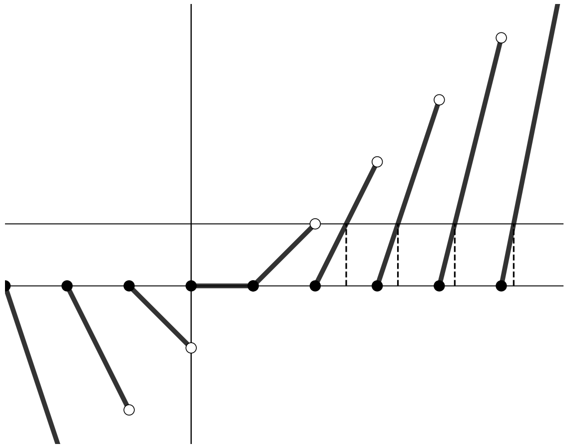
Illustration 10.2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \{x\}\lfloor x \rfloor$$

for all x. What is the preimage of 1 under f?

Solution. Based on the graph shown below, the preimage of 1 under f is precisely the set

$$S = \{n + \frac{1}{n} : \ n \geq 2\}$$



For an analytical proof, suppose x lies in the preimage of 1 under f. Say $n = \lfloor x \rfloor$ and $\alpha = \{x\}$. Then

$$f(x) = 1 \qquad \Rightarrow \qquad \lfloor x \rfloor \{x\} = 1 \qquad \Rightarrow \qquad n\alpha = 1 \qquad \Rightarrow \qquad \alpha = \frac{1}{n}$$

Also, since $n\alpha = 1$, we see that n cannot be negative, for then α would also be negative. Similarly, n cannot be 0 or 1. So n must be at least 2. So any element of $f^{-1}(1)$ lies in the set S defined above. The reverse containment is easily checked. So we have $S = f^{-1}(1)$ and we are done. ■

Exercise 10.2.1. Let α be an irrational number. Show that if m and n are distinct integers, then $\{m\alpha\}$ is also distinct from $\{n\alpha\}$.

Exercise 10.2.2. If $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined as

$$f(x) = \lfloor x \rfloor + \lfloor -x \rfloor \quad \text{and} \quad g(x) = \{x\}$$

for all x. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h = f \circ g$. Then which of the following is incorrect?

(a) f and h are identical functions.

(b) $f = g$ has no solution.

(c) $f(x) + h(x) > 0$ is not true for any x.

(d) $f - h$ is a periodic function.

Exercise 10.2.3. Let the solution set of the equation :

$$\sqrt{\left\lfloor x + \left\lfloor \frac{x}{2} \right\rfloor \right\rfloor} + \left\lfloor \sqrt{\{x\}} + \left\lfloor \frac{x}{3} \right\rfloor \right\rfloor = 3$$

is $[a, b)$. Find the product ab.

Exercise 10.2.4. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined as

$$f(x)=\{x\}+\{x+1\}+\{x+2\}+\ldots\ldots+\{x+99\}$$

then $\lfloor f(\sqrt{2})\rfloor$ is equal to : (a) 5050 (b) 4950 (c) 41 (d) 14 [(c)]

Exercise 10.2.5. The set of all the solutions of the inequality

$$\{x\}(\{x\}-1)(\{x\}+2)\geq 0$$

is: (a) $(-2,1)$ (b) $\mathbb{Z}$ (c) $[0,1)$ (d) $[-2,0)$ [(b)]

10.3 HERMITE'S IDENTITY

10.3.1 Hermite's Identity: Toy case. We show that for each real number x we have

$$\left\lfloor x+\frac{1}{2}\right\rfloor+\lfloor x\rfloor=\lfloor 2x\rfloor$$

Idea. Let us say a few words towards the intuition before seeing a proper proof. The identity relates the floor of a number x to the floor of $2x$. Think of x as a point on a number line. When we double x to get $2x$, we are stretching the number line by a factor of 2. If the fractional part of x is small (less than 0.5), doubling it will not push $2x$ past the next integer threshold relative to $2\lfloor x\rfloor$. If the fractional part of x is large (0.5 or more), doubling it *will* push $2x$ past the next integer, adding an extra 1 to the result. The term $\lfloor x+1/2\rfloor$ acts as a detector. It adds 1 to the sum exactly when the fractional part of x is ≥ 0.5, balancing the equation. Let us see the formal details.

Proof. We want to prove that for any real x we have

$$\lfloor x\rfloor+\left\lfloor x+\frac{1}{2}\right\rfloor=\lfloor 2x\rfloor$$

Let $x=n+\alpha$, where $n=\lfloor x\rfloor$ is the integer part and $0\leq\alpha<1$ is the fractional part. Substituting this into the expressions, we get that the left hand side is

$$\lfloor n+\alpha\rfloor+\left\lfloor(n+\alpha)+\frac{1}{2}\right\rfloor=n+n+\left\lfloor\alpha+\frac{1}{2}\right\rfloor=2n+\left\lfloor\alpha+\frac{1}{2}\right\rfloor$$

and the right hand side is

$$\lfloor 2(n+\alpha)\rfloor=\lfloor 2n+2\alpha\rfloor=2n+\lfloor 2\alpha\rfloor$$

The equality of these two can now be easily seen by making two cases, namely

$$0\leq\alpha<\frac{1}{2}\quad\text{and}\quad\frac{1}{2}\leq\alpha<1$$

We leave this for the reader to justify. ♢

10.3.2 Hermite's Identity. Let x be a real number and n be a natural number. Then

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor$$

Intuition. This is a generalization of the toy case. The interval $[0, 1)$ for the fractional part of x is divided into n sub-intervals of length $1/n$. The sum on the left "probes" the fractional part of x. As k increases, we add larger fractions k/n to x. Eventually, these additions cause the value inside the floor function to "tip over" into the next integer. The number of times this "tip over" happens corresponds exactly to the extra integer value gained when we multiply x by n in the expression $\lfloor nx \rfloor$.

Proof. Let $x = I + f$, where $I = \lfloor x \rfloor$ is an integer and $0 \leq f < 1$. We partition the interval $[0, 1)$ into n sub-intervals:

$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right)$$

The fractional part f must lie in exactly one of these intervals. Suppose f lies in the m-th interval (where m is an integer, $0 \leq m \leq n-1$):

$$\frac{m}{n} \leq f < \frac{m+1}{n} \qquad (*)$$

The right hand side is

$$\lfloor n(I + f) \rfloor = \lfloor nI + nf \rfloor = nI + \lfloor nf \rfloor$$

From (*) we know that

$$m \leq nf < m + 1$$

and hence Therefore, $\lfloor nf \rfloor = m$, giving that the right hand side is $nI + m$. Now let us look at the left hand side:

$$\sum_{k=0}^{n-1} \left\lfloor I + f + \frac{k}{n} \right\rfloor = \sum_{k=0}^{n-1} \left(I + \left\lfloor f + \frac{k}{n} \right\rfloor \right) = nI + \sum_{k=0}^{n-1} \left\lfloor f + \frac{k}{n} \right\rfloor$$

We need to evaluate the term $\lfloor f + \frac{k}{n} \rfloor$ for $k = 0, 1, \ldots, n-1$. Note that

$$0 \leq f + \frac{k}{n} < 2, \quad k = 0, 1, \ldots, n-1$$

Thus the value of $\lfloor f + \frac{k}{n} \rfloor$ is either 0 or 1, depending on whether $f + \frac{k}{n}$ is smaller than 1 or at least 1. Now the condition

$$f + \frac{k}{n} \geq 1$$

is equivalent to

$$f \geq 1 - \frac{k}{n} = \frac{n-k}{n}$$

Recalling that $\frac{m}{n} \leq f < \frac{m+1}{n}$, it is a matter of simple arithmetic to check that the above happens precisely when

$$k \in \{n - m, n - m + 1, \ldots, n - 1\}$$

The number of such values is $(n-1)-(n-m)+1=m$. Thus, exactly m terms in the summation are equal to 1, and the rest are 0. So the left hand side is also $nI+m$. This proves the identity. $\diamondsuit$

Illustration 10.3.3 If x satisfies the equation

$$\lfloor x+0.19\rfloor+\lfloor x+0.20\rfloor+\cdots+\lfloor x+0.91\rfloor=542$$

then find the value of $\lfloor 100x\rfloor$.

Proof. Let S be the given sum, that is

$$S=\sum_{k=19}^{91}\left\lfloor x+\frac{k}{100}\right\rfloor=542$$

The number of terms in this summation is 73. Write $x=I+f$, where $I=\lfloor x\rfloor$ and $f=\{x\}$. Since $19/100\le k/100\le 91/100$, every term in the sum, $\lfloor x+\frac{k}{100}\rfloor$, will be either I or $I+1$. We will get a handle on I by some estimates. We have

$$73I\le S\le 73(I+1)\qquad\Rightarrow\qquad I\le\frac{542}{73}\ \text{ and }\ \frac{469}{73}\le I\qquad\Rightarrow\qquad I=7$$

where we have used the fact that I is an integer to arrive as the final conclusion.

Now let us gain some understanding of the fractional part f of x. We have

$$S=\sum_{k=19}^{91}\left\lfloor 7+f+\frac{k}{100}\right\rfloor=\sum_{k=19}^{91}\left(7+\left\lfloor f+\frac{k}{100}\right\rfloor\right)=\left(\sum_{k=19}^{91}7\right)+\left(\sum_{k=19}^{91}\left\lfloor f+\frac{k}{100}\right\rfloor\right)$$

Thus

$$S=(73\times 7)+\sum_{k=19}^{91}\left\lfloor f+\frac{k}{100}\right\rfloor=511+\sum_{k=19}^{91}\left\lfloor f+\frac{k}{100}\right\rfloor$$

Using $S=542$, we get

$$542=511+\sum_{k=19}^{91}\left\lfloor f+\frac{k}{100}\right\rfloor\qquad\Rightarrow\qquad\sum_{k=19}^{91}\left\lfloor f+\frac{k}{100}\right\rfloor=31$$

Since $0\le f<1$ and $0.19\le k/100\le 0.91$, the value $f+k/100$ is always positive and less than 2. This means the term $\lfloor f+(k/100)\rfloor$ can only be 0 or 1. The sum being 31 means that 31 of the terms are equal to 1 and the remaining $73-31=42$ terms are equal to 0. Since $f+k/100$ increases as k increases, the *last* 31 terms must be 1, and the first 42 terms must be 0.

- The first 42 terms correspond to $k=19,20,\ldots,60$.
- The last 31 terms correspond to $k=61,62,\ldots,91$.

This gives us two conditions on f:

1. For $k = 60$ (the last term that is 0):

$$\left\lfloor f + \frac{60}{100} \right\rfloor = 0 \implies f + 0.60 < 1 \implies f < 0.40$$

2. For $k = 61$ (the first term that is 1):

$$\left\lfloor f + \frac{61}{100} \right\rfloor = 1 \implies f + 0.61 \geq 1 \implies f \geq 0.39$$

So, we have found that $0.39 \leq f < 0.40$.

We can now calculate $\lfloor 100x \rfloor$. We know $x = 7 + f$, and $0.39 \leq f < 0.40$. Thus

$$7.39 \leq x < 7.40 \quad \Rightarrow \quad 100 \times 7.39 \leq 100x < 100 \times 7.40 \quad \Rightarrow \quad 739 \leq 100x < 740$$

By the definition of the floor function, $\lfloor 100x \rfloor = 739$. ■

Exercise 10.3.1. Let

$$S = \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{1}{200} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{1}{100} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{3}{200} \right\rfloor + \cdots + \left\lfloor \frac{1}{4} + \frac{199}{200} \right\rfloor$$

then the value of S is ___________.

Exercise 10.3.2. Show that

$$\left\lfloor \frac{x-1}{2} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor$$

is identically equal to $\lfloor x - 1 \rfloor$.

10.4 EQUATIONS

Before we begin, we remind the reader of the message in Section 4.5.

Illustration 10.4.1 The number of positive integral values of x satisfying

$$\left\lfloor \frac{x}{9} \right\rfloor = \left\lfloor \frac{x}{11} \right\rfloor$$

is

(a) 21 (b) 22 (c) 23 (d) 24

Solution. Let x be a solution to the given equation and say

$$k = \left\lfloor \frac{x}{9} \right\rfloor = \left\lfloor \frac{x}{11} \right\rfloor$$

Since x is a positive integer, k must be a non-negative integer. From the definition of the floor function, we have the inequalities:

$$k \leq \frac{x}{9} < k+1 \qquad \Rightarrow \qquad 9k \leq x < 9k+9$$

$$k \leq \frac{x}{11} < k+1 \qquad \Rightarrow \qquad 11k \leq x < 11k+11$$

For a solution to exist for a specific k, the intervals $[9k, 9k+9)$ and $[11k, 11k+11)$ must overlap. The intersection of these two intervals is:

$$[\max(9k, 11k), \min(9k+9, 11k+11))$$

Since $k \geq 0$ and $11k \geq 9k$, the intersection is $[11k, 9k+9)$. For this interval to be non-empty, we require $11k < 9k+9$, which implies $2k < 9$, or $k < 4.5$. Thus, possible integer values for k are $0, 1, 2, 3, 4$. We now calculate the number of integers x for each k in $\{0, 1, 2, 3, 4\}$.

- $k = 0$: Interval $[0, 9)$. Positive integers are $\{1, 2, \ldots, 8\}$. (8 values)
- $k = 1$: Interval $[11, 18)$. Integers are $\{11, 12, \ldots, 17\}$. (7 values)
- $k = 2$: Interval $[22, 27)$. Integers are $\{22, 23, \ldots, 26\}$. (5 values)
- $k = 3$: Interval $[33, 36)$. Integers are $\{33, 34, 35\}$. (3 values)
- $k = 4$: Interval $[44, 45)$. Integer is $\{44\}$. (1 value)

Total number of values $= 8+7+5+3+1 = 24$. Option (d) is correct. ■

Illustration 10.4.2 The number of real values of x satisfying the equation

$$\left\lfloor \frac{2x+1}{3} \right\rfloor + \left\lfloor \frac{4x+5}{6} \right\rfloor = \frac{3x-1}{2}$$

are greater than or equal to (a) 7 (b) 8 (c) 9 (d) 10

Solution. Let us look at the left hand side. Writing

$$y = \frac{2x+1}{3}$$

we see that the left hand side is

$$\lfloor y \rfloor + \left\lfloor y + \frac{1}{2} \right\rfloor$$

which by Point 10.3.1 is same as $\lfloor 2y \rfloor$. So the original equation becomes:

$$\left\lfloor \frac{4x+2}{3} \right\rfloor = \frac{3x-1}{2}$$

Since the left hand side is an integer, the right hand side must also be an integer, say k. Thus

$$\frac{3x-1}{2}=k \qquad \Rightarrow \qquad 3x-1=2k \qquad \Rightarrow \qquad x=\frac{2k+1}{3}$$

Substitute x back into the simplified floor equation:

$$k=\left\lfloor \frac{4\left(\frac{2k+1}{3}\right)+2}{3} \right\rfloor = \left\lfloor \frac{\frac{8k+4}{3}+\frac{6}{3}}{3} \right\rfloor = \left\lfloor \frac{8k+10}{9} \right\rfloor$$

We now solve for integers k satisfying $k=\lfloor\frac{8k+10}{9}\rfloor$. By definition of the floor function we get

$$k \le \frac{8k+10}{9} < k+1$$

Solving the first and the second inequalities gives

$$k \le 10 \quad \text{and} \quad k \ge 2$$

respectively. Thus the candidates for k are $\{2,3,\ldots,10\}$. Since each of these k yields a unique x, there are exactly 9 solutions. ■

Illustration 10.4.3 Find all real numbers x such that

$$2x+3\{x\}=4\lfloor x\rfloor-2$$

Solution. Using $\lfloor x\rfloor = x-\{x\}$, the given equation is equivalent to

$$7\{x\}=2x-2 \qquad \Longleftrightarrow \qquad 7\{x-1\}=2(x-1) \tag{*}$$

Let X be the set of all x that satisfy the above equation. Let T be the set of all the t that satisfy

$$7\{t\}=2t$$

From (*) it is clear that$X=\{t+1:\ t\in T\}$. So once we find T we can find X. Fix $t\in T$, and let $t=n+\alpha$ where n is an integer and $0\le\alpha<1$. Then we have

$$7\{t\}=2t \qquad \Rightarrow \qquad 7\alpha=2n+2\alpha \qquad \Rightarrow \qquad 5\alpha=2n \qquad \Rightarrow \qquad 0\le 2n\le 5$$

Thus $n\in\{0,1,2\}$. We consider three cases. From the equation $5\alpha=2n$, we have

$$n=0\Rightarrow\alpha=0, \quad n=1\Rightarrow\alpha=\frac{2}{5}, \quad n=2\Rightarrow\alpha=\frac{4}{5}$$

It is easy to verify that $t=0,\frac{7}{5}$ and $\frac{14}{5}$ are indeed solutions of the equation $7\{t\}=2t$, and thus

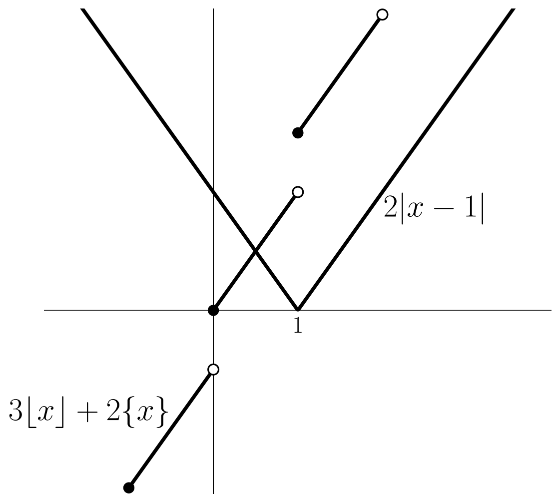
$$T=\{0,\frac{7}{5},\frac{14}{5}\} \qquad \Rightarrow \qquad X=\{1,\frac{12}{5},\frac{19}{5}\}$$

and we are done. ■

Illustration 10.4.4 Find all real numbers x such that

$$2\lfloor x-1\rfloor = 3\lfloor x\rfloor + 2\{x\}$$

Solution. The following graph shows that there is only one solution, and that is 1/2.



We now give an analytical proof. Using $\lfloor x\rfloor = x - \{x\}$, the given equation is equivalent to

$$2|x-1| = 3x - \{x\}$$

Let X be the set of all the solutions of the above. To deal with the modulus, we define

$$G = \{x \in X : \ x \geq 1\} \quad \text{and} \quad L = \{x \in X : \ x < 1\}$$

Clearly $X = G \cup L$. So we just need to find G and L. Say $g \in G$. Then

$$\begin{aligned} 2(g-1) = 3g - \{g\} \qquad &\Rightarrow \qquad \{g\} = g + 2 \\ &\Rightarrow \qquad 0 \leq g+2 < 1 \qquad \Rightarrow \qquad -2 \leq g < -1 \end{aligned}$$

But each element of G is positive, and thus we deduce that G must be empty. Now for L. Say $\ell \in L$. Then

$$\begin{aligned} 2(1-\ell) = 3\ell - \{\ell\} \qquad &\Rightarrow \qquad \{\ell\} = 5\ell - 2 \\ &\Rightarrow \qquad 0 \leq 5\ell - 2 < 1 \qquad \Rightarrow \qquad \frac{2}{5} \leq \ell < \frac{3}{5} \end{aligned}$$

Thus we can write

$$\ell = \frac{2}{5} + \frac{\alpha}{5}$$

for all $0 \leq \alpha < 1$. From $\{\ell\} = 5\ell - 2$ we now deduce that

$$\frac{2}{5} + \frac{\alpha}{5} = 5\left(\frac{2}{5} + \frac{\alpha}{5}\right) - 2 \qquad \Rightarrow \qquad \frac{2}{5} + \frac{\alpha}{5} = \alpha \qquad \Rightarrow \qquad \alpha = \frac{1}{2}$$

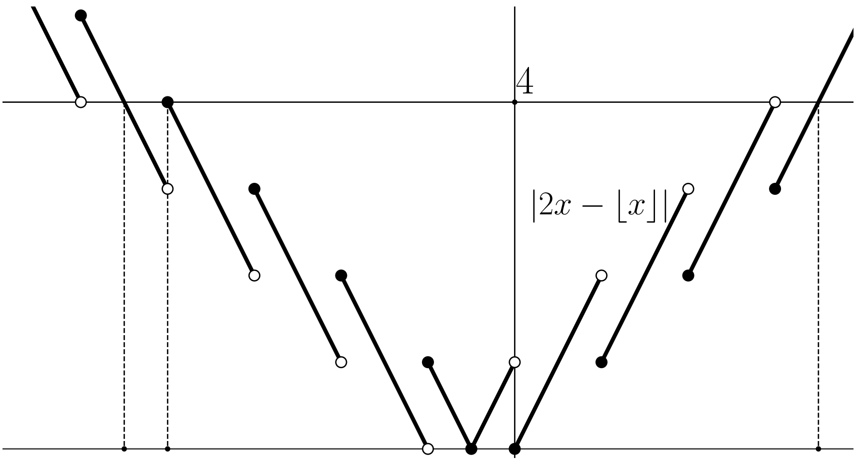
Thus there is only one candidate for ℓ, namely $2/5 + (1/2)/5 = 1/2$. It is easy to check that $1/2$ indeed lies in L. So we conclude that $G = \emptyset$ and $L = \{1/2\}$. Consequently $X = \{1/2\}$. ■

Illustration 10.4.5 The number of real solutions of **[UGA 2020]**

$$|2x - \lfloor x \rfloor| = 4$$

is (a) 4 (b) 3 (c) 2 (d) 1

Solution. The following graph shows that there are exactly 3 solutions.



We invite the reader to justify the above graph and provide an analytical solution. ■

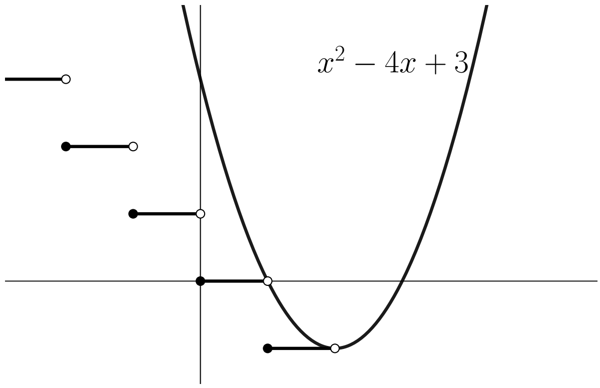
Illustration 10.4.6 Find all real numbers x such that

$$x^2 - 4x + \lfloor x \rfloor + 3 = 0$$

Solution. The given equation is equivalent to

$$x^2 - 4x + 3 = -\lfloor x \rfloor$$

To get an idea about the set of all the solutions, we can draw the graph of the two curves defined by the left hand side and the right hand side and see where all they intersect. The left hand side is a quadratic polynomial, and its graph can be drawn by using Point 4.3.5.



The graphs immediately make it clear that there is no solution to the given equation. Let us outline how one can write an analytical proof. The left hand side is a quadratic in x and hence "grows like x^2." On the other hand, the right hand side is the floor value of x, and hence "grows like x." Thus there can only be a small range of values where the two expressions can possibly be equal. On top of this, the right hand side is an integer. This further narrows down the candidate values for x. The reader is invited to execute this informal idea. ■

Exercise 10.4.1. Find all the real numbers x such that

$$\lceil x \rceil^2 = \lfloor x \rfloor^2 + 2x$$

Exercise 10.4.2. Prove that the equation $\{x\} + x^2 = 0$ has only one solution.

Exercise 10.4.3. Find the number of values of x satisfying

$$\left\{x^2\right\} + \left\lfloor x^4 \right\rfloor = 1$$

Exercise 10.4.4. Find all real numbers x such that

$$3x + 5\{x\} = 4\lfloor 2x \rfloor + 3$$

Exercise 10.4.5. Solve the equations

a) $\left\lfloor \dfrac{3x+1}{2} \right\rfloor = x + 2.$

b) $\left\lfloor \dfrac{x+1}{2} \right\rfloor + 1 = \dfrac{x-1}{3}.$

Exercise 10.4.6. Find all real numbers x such that

$$4\{x\} = x + \lfloor x \rfloor$$

Exercise 10.4.7. Find all the real x for which

$$\left\lfloor 4 - \lfloor x \rfloor^2 \right\rfloor \; + \; (4 - \lfloor x \rfloor^2) = 6$$

Exercise 10.4.8. If

$$y = 2\lfloor x \rfloor + 3 = 3\lfloor x - 2 \rfloor + 5$$

then find the value of $[x + y]$.

Exercise 10.4.9. Find the number of solutions of the equation

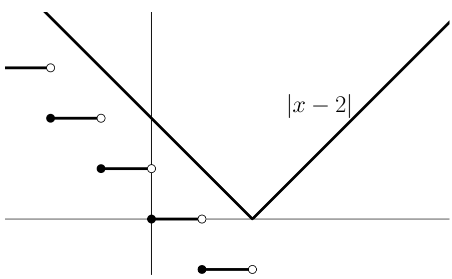
$$2\lfloor x \rfloor - x = \{x\} + 1$$

10.5 INEQUALITIES

Illustration 10.5.1 Find all the real x in $[-1, 3]$ such that

$$\lfloor x \rfloor + |x - 2| \leq 0$$

Solution. The given inequality is equivalent to $|x - 2| \leq -\lfloor x \rfloor$. We thus want to locate those x corresponding to which the graph of the function given by $|x - 2|$ is below the graph of the function given by $-\lfloor x \rfloor$. Plotting the two graphs we see the following.



The graphs make it clear that there is no solution since the graph for $|x - 2|$ is always above the graph for $-\lfloor x \rfloor$. ■

Illustration 10.5.2 Find all real numbers x such that

$$\lfloor x \rfloor^2 - \lfloor x \rfloor - 6 > 0$$

Solution. Let X be the set of all the x such that $\lfloor x \rfloor^2 - \lfloor x \rfloor - 6 > 0$. Let Y be the set of all the real numbers y such that $y^2 - y - 6 > 0$. In symbols

$$X = \{x \in \mathbb{R} : \ \lfloor x \rfloor^2 - \lfloor x \rfloor - 6 > 0\} \quad \text{and} \quad Y = \{y \in \mathbb{R} : \ y^2 - y - 6 > 0\}$$

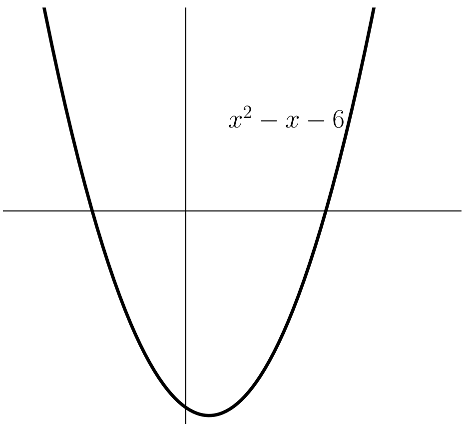
Clearly, x is in X if and only if $\lfloor x \rfloor$ is in Y. So first we find Y, which is visible from the first graph below; Y is the union of two semi-infinite open intervals. Using Point 4.3.5 we can explicitly compute Y. We get

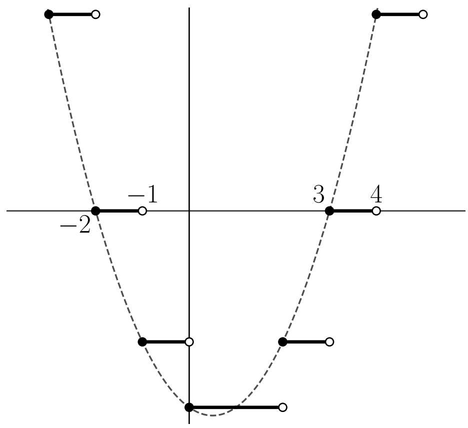
$$Y = (-\infty, -2) \cup (3, \infty)$$

Now, as mentioned above, X is precisely the set of those x such that $\lfloor x \rfloor$ lies in this open interval. It is now easy to see that

$$X = (-\infty, -2) \cup [4, \infty)$$

The figure on the right below shows that the graph of the function corresponding to $\lfloor x \rfloor^2 - \lfloor x \rfloor - 6$, which helps visualize this conclusion.





We invite the reader to convince themselves of this conclusion by means of an analytic justification. ■

Illustration 10.5.3 Find all real numbers x such that

$$x\lfloor x \rfloor - x^2 - 3\lfloor x \rfloor + 3x > 0$$

Solution. We have

$$\begin{aligned} x\lfloor x \rfloor - x^2 - 3\lfloor x \rfloor + 3x &= \lfloor x \rfloor(x-3) - x(x-3) \\ &= (x-3)(\lfloor x \rfloor - x) \\ &= -(x-3)\{x\} \end{aligned}$$

Thus we want to find the set of all the x such that

$$-(x-3)\{x\} > 0 \quad \iff \quad (x-3)\{x\} < 0$$

Since $\{x\}$ is always non-negative, we see that the set of all the solutions is nothing but $(-\infty, 3)$. ■

Illustration 10.5.4 Find the set of all the real numbers x such that

$$-1 \leq \lfloor x \rfloor - x^2 + 4 \leq 2$$

Solution. The given inequality is equivalent to

$$x^2 - 5 \leq \lfloor x \rfloor \leq x^2 - 2$$

The region solving both these inequalities simultaneously is shown below

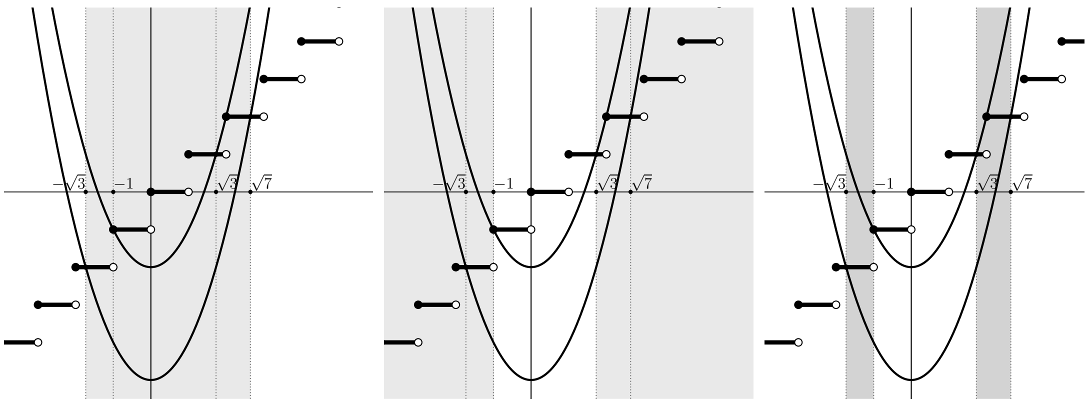


Figure 10.1: The image on the left shows the region for $x^2 - 5 \leq \lfloor x \rfloor$. The center image depicts $\lfloor x \rfloor \leq x^2 - 2$. The one on the right shows the intersection of these two regions.

What we have done is plot the graphs of the quadratic polynomials given by $x^2 - 2$ and $x^2 - 5$, and the graph of the floor function. We are then looking for those x such that the point in the graph of the floor function is above the graph of the quadratic below and below the graph of the quadratic above. This region is

$$[-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$$

The logic is simple. Taking a cue from the plot above, we first see at what points do the graph of the quadratic below meets the graph of the floor function. This boils down to finding those integers n for which $x^2 - 5 = n$ has a solution x such that $\lfloor x \rfloor = n$. Now n cannot be too big since the equation $x^2 - 5 = n$ gives that $|x| \approx \sqrt{n}$ while $\lfloor x \rfloor = n$ gives $|x| \approx n$. And n cannot be too small since then x^2 would be negative. So we have only a small set of candidates for n, and we can try for each of them. Similarly we analyze against the other quadratic. The formal details are left for the reader as an exercise. ■

Exercise 10.5.1. Find the complete solution set of the inequality $[\sqrt{x}] \geq [x^2]$.

Exercise 10.5.2. Find the complete set of value of x satisfying

$$\operatorname{sgn}(\lfloor x \rfloor) > \operatorname{sgn}(x)$$

Exercise 10.5.3. Find the set of all the real numbers x such that $\lfloor x \rfloor - 1 + x^2 \geq 0$.

Exercise 10.5.4. If

$$\lfloor x \rfloor^2 - 7\lfloor x \rfloor + 10 < 0 \quad \text{and} \quad 4\lfloor y \rfloor^2 - 16\lfloor y \rfloor + 7 < 0$$

then $\lfloor x+y \rfloor$ cannot be: (a) 7 (b) 8 (c) 9 (d) both (b) and (c) [(c)]

10.6 LEGENDRE'S FORMULA (OPTIONAL)

10.6.1 Legendre's Formula Let p be a prime. For any integer n, let $\nu_p(n)$ denote the highest power of p that divides n. For example

- $\nu_3(1) = 0$
- $\nu_3(2) = 0$
- $\nu_3(3) = 1$
- $\nu_3(4) = 0$
- $\nu_3(5) = 0$
- $\nu_3(6) = 1$
- $\nu_3(7) = 0$
- $\nu_3(8) = 0$
- $\nu_3(9) = 2$

Note that

$$\nu_p(ab) = \nu_p(a) + \nu_p(b)$$

for any two positive integers a and b.[1] Also, recall that for any natural number n we define $n!$ as the product of all the natural numbers from 1 to n. The following formula gives an expression for $\nu_p(n!)$.

Legendre. For any prime p and any natural number n we have

$$\nu_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

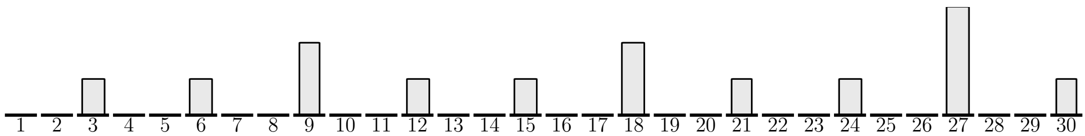
[1]This may be intuitively obvious to the reader, but it should be pointed out that the fact that p is a prime is crucial to make this equation true. A rigorous explanation requires a short excursion into the terrain of elementary number theory.

The sum above is only a finite sum, despite the deceptive notation; all the terms $\lfloor n/p^k \rfloor$ are 0 once k is too large.

Proof. The proof is by 'double-counting.' The quantity $\nu_p(n!)$ is nothing but

$$\nu_p(1) + \nu_p(2) + \nu_p(3) + \cdots + \nu_p(n)$$

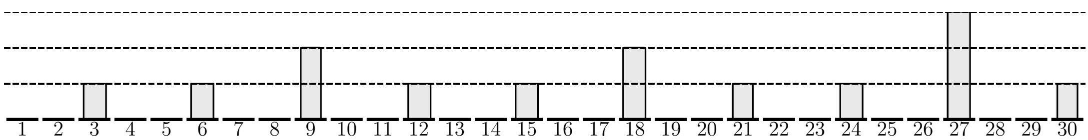
We can think of the above sum as follows. Arrange the n integers $1, 2, 3, \ldots, n$ in a row, and for the integer k erect a rectangle of height $\nu_p(k)$. The sum above is then the sum of the heights of all the rectangles. The following shows the corresponding image for $n = 30$ and $p = 3$.



We can count the sum of heights of these rectangles in a different way. If h_k denotes the number of rectangles of height k, then the sum of the heights of all the rectangles is nothing but

$$h_1 + h_2 + h_3 + \cdots$$

This is illustrated in the following image.



The number h_k is the number of rectangles that the "k-th band" cuts through. But what is h_k? Observing that for any two natural number N and d, the quantity $\lfloor N/d \rfloor$ is the number of positive integers which are divisible by d and which do not exceed N, we see that

$$h_k = \left\lfloor \frac{n}{p^k} \right\rfloor$$

Now using

$$\nu_p(n!) = \nu_p(1) + \nu_p(2) + \nu_p(3) + \cdots + \nu_p(n) = h_1 + h_2 + h_3 + \cdots$$

finishes the proof. ■ ♢

10.7 FLOOR AND SQUARE ROOTS (OPTIONAL)

In this section we discuss miscellaneous results pertaining to the floor functions which also feature square roots. The results are not interconnected and are meant to be treated as one-off puzzles.

10.7.1 An identity due to Ramanujan. For each natural number n we will show that

$$\lfloor\sqrt{n}+\sqrt{n+1}\rfloor = \lfloor\sqrt{4n+2}\rfloor$$

Proof. Let $f(n) = \sqrt{n}+\sqrt{n+1}$. Then we have

$$f(n)^2 = (\sqrt{n}+\sqrt{n+1})^2 = n+(n+1)+2\sqrt{n(n+1)} = 2n+1+\sqrt{4n^2+4n}.$$

Since $4n^2 < 4n^2+4n < (2n+1)^2$, we have $2n < \sqrt{4n^2+4n} < 2n+1$. Thus

$$\begin{aligned} 2n+1+2n < f(n)^2 < 2n+1+2n+1 \quad &\Rightarrow \quad 4n+1 < f(n)^2 < 4n+2 \\ &\Rightarrow \quad \sqrt{4n+1} < f(n) < \sqrt{4n+2} \end{aligned}$$

Now let $g(n) = \sqrt{4n+2}$. We want to show that $\lfloor f(n)\rfloor = \lfloor g(n)\rfloor$. Assume on the contrary that the equality does not hold. Since it clear that $f(n) < g(n)$, by our assumption, there must be an integer k such that $f(n) < k \le g(n)$. But then

$$4n+1 < f(n)^2 = k^2 \le g(n)^2 = 4n+2$$

The above forces that $k^2 = g(n)^2$. But then $4n+2$ would be a perfect square. However, this is not possible.[2] ♢

10.7.2 A useful estimate. An elementary calculation shows that

$$2(\sqrt{n+1}-\sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n}-\sqrt{n-1})$$

for all natural numbers n. This leads to the fact that

$$2\sqrt{n}-2 < \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}$$

by summing telescopically. ♢

Illustration 10.7.3 Let

$$S = \frac{1}{\sqrt{10000}}+\frac{1}{\sqrt{10001}}+\cdots+\frac{1}{\sqrt{160000}}$$

Then the largest positive integer not exceeding S is

(a) 200 (b) 400 (c) 600 (d) 800 **[UGA 2024]**

Solution. Let $a = 10000$ and $b = 160000$. Using Point 10.7.2 we immediately get by a telescopic sum that

$$2(\sqrt{b+1}-\sqrt{a}) < \sum_{k=a}^{b}\frac{1}{\sqrt{k}} < 2(\sqrt{b}-\sqrt{a-1})$$

[2]No number of the form $4n+2$ can be a perfect square. The reader is encouraged to supply a proof of it.

Now

$$2(\sqrt{b+1}-\sqrt{a}) = 2(\sqrt{160001}-\sqrt{10000}) = 600+\varepsilon \quad \text{and}$$
$$2(\sqrt{b}-\sqrt{a-1}) = 2(\sqrt{160000}-\sqrt{9999}) = 600+\delta$$

where $0<\varepsilon<\delta<0.1$. Thus the integer part of the given sum is 600. ■

Exercise 10.7.1. Find all the natural numbers n such that $\sum_{k=1}^{n}\lfloor\sqrt{k(k+1)}\rfloor = 496$.

Exercise 10.7.2. Show that

$$\lfloor 1+\sqrt{2}\rfloor + \left\lfloor \frac{2+\sqrt{3}}{2} \right\rfloor + \left\lfloor \frac{3+\sqrt{4}}{3} \right\rfloor + \cdots + \left\lfloor \frac{n+\sqrt{n+1}}{n} \right\rfloor = n+1$$

Exercise 10.7.3. Find all natural numbers n satisfying the equation:

$$\lfloor\sqrt{1\cdot 2}\rfloor + \lfloor\sqrt{2\cdot 3}\rfloor + \lfloor\sqrt{3\cdot 4}\rfloor + \cdots + \lfloor\sqrt{n\cdot(n+1)}\rfloor = 496$$

Exercise 10.7.4. The integral part of

$$\sum_{n=2}^{9999}\frac{1}{\sqrt{n}}$$

equals (a) 196 (b) 197 (c) 198 (d) 199 **[UGA 2020]**

Exercise 10.7.5. (British Mathematical Olympiad 1996) Define

$$q(n) = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor \quad (n=1,2,\ldots)$$

Determine (with proof) all positive integers n for which $q(n) > q(n+1)$.

10.8 CHALLENGING PROBLEMS

Problem 10.8.1. (IMO 1989 Shortlist.)[3] Let N be a positive integer and α be the positive root of the equation $x^2-Nx-1=0$. Prove that for all natural numbers n we have

$$\lfloor \alpha n + N\alpha\lfloor\alpha n\rfloor\rfloor = 1989n + \left(N^2+1\right)\lfloor\alpha n\rfloor$$

Solution. The number α is the positive root of the quadratic equation $x^2-Nx-1=0$. This implies that

$$\alpha^2 = N\alpha+1 \quad \text{and} \quad \alpha = N+\frac{1}{\alpha}$$

[3]The original formulation was slightly different and made the problem look harder than it is.

We want to prove that

$$\lfloor \alpha n + N\alpha\lfloor \alpha n\rfloor\rfloor = Nn + (N^2+1)\lfloor \alpha n\rfloor$$

Let $k = \lfloor \alpha n\rfloor$, so we can write $\alpha n = k + \{\alpha n\}$. Let us simplify the left hand side

$$\lfloor \alpha n + N\alpha k\rfloor = \lfloor (k + \{\alpha n\}) + N\alpha k\rfloor = k + \lfloor \{\alpha n\} + N\alpha k\rfloor$$

Equating this with the right hand side, we need to prove:

$$k + \lfloor \{\alpha n\} + N\alpha k\rfloor = Nn + N^2k + k \qquad \Longleftrightarrow \qquad \lfloor \{\alpha n\} + N\alpha k\rfloor = Nn + N^2k$$

Let us analyze the term $N\alpha k$ inside the floor function. Using the property $\alpha = N + \frac{1}{\alpha}$:

$$N\alpha k = N\left(N + \frac{1}{\alpha}\right)k = N^2k + \frac{Nk}{\alpha}$$

Substituting this into the floor expression:

$$\left\lfloor \{\alpha n\} + N^2k + \frac{Nk}{\alpha}\right\rfloor = N^2k + \left\lfloor \{\alpha n\} + \frac{Nk}{\alpha}\right\rfloor$$

So the equation to prove reduces to:

$$\left\lfloor \{\alpha n\} + \frac{Nk}{\alpha}\right\rfloor = Nn$$

Now, express k back in terms of n: $k = \alpha n - \{\alpha n\}$. The term inside the floor becomes:

$$\{\alpha n\} + \frac{N}{\alpha}(\alpha n - \{\alpha n\}) = \{\alpha n\} + Nn - \frac{N}{\alpha}\{\alpha n\} = Nn + \{\alpha n\}\left(1 - \frac{N}{\alpha}\right)$$

We need to show that $\lfloor Nn + \{\alpha n\}(1 - \frac{N}{\alpha})\rfloor = Nn$. Since Nn is an integer, this equality holds if and only if the added term is non-negative and strictly less than 1:

$$0 \le \{\alpha n\}\left(1 - \frac{N}{\alpha}\right) < 1$$

Let us simplify the factor $(1 - \frac{N}{\alpha})$. Using $\alpha = N + \frac{1}{\alpha}$, we have

$$\alpha - N = \frac{1}{\alpha}$$

so

$$1 - \frac{N}{\alpha} = \frac{\alpha - N}{\alpha} = \frac{1/\alpha}{\alpha} = \frac{1}{\alpha^2}$$

Thus, our goal reduces to showing that

$$0 \le \frac{\{\alpha n\}}{\alpha^2} < 1$$

We know that $0 \le \{\alpha n\} < 1$. Also, since $N \ge 1$, the root

$$\alpha = \frac{N + \sqrt{N^2+4}}{2}$$

is strictly greater than 1, which implies $\alpha^2 > 1$ and $0 < \frac{1}{\alpha^2} < 1$. Therefore $\frac{\{\alpha n\}}{\alpha^2}$ is strictly bounded between 0 and 1 and we are done. ■

There was one more IMO shortlist

CHAPTER 11

EXPONENTIAL FUNCTIONS

To truly appreciate the function we are about to explore, we must first step back from formal definitions and consider a story from antiquity. It is a legend that serves as a warning about the deceptively quiet nature of the concept we call *exponential growth.*

The story takes us to ancient India, where the creator of the game of chess presented his invention to the local ruler. The ruler, delighted by the strategic depth of the game, offered the inventor any reward he desired. The inventor, a man of mathematical mind, appeared humble in his request. He pointed to the chessboard and said:

> *"My request is simple, O King. Place one grain of wheat on the first square of the chessboard. Place two grains on the second square, four on the third, eight on the fourth, and so on. Simply double the amount of wheat for each subsequent square until the board is full."*

We can imagine the ruler laughing. It sounds like a trifle—a sack of wheat, perhaps two? He ordered his treasurer to fulfill the request.

Let us trace the treasurer's steps.

- Square 1: 1 grain.
- Square 2: 2 grains.
- Square 3: 4 grains.
- Square 10: 512 grains.

So far, the amount is manageable; it fits in a small bowl. But as we proceed past the middle of the board, something terrifying happens. The numbers do not just grow; they explode. By the time we reach the 64th square, the number of grains required is 2^{63}.

$$2^{63} \approx 9,223,372,036,854,775,808 \text{ grains}$$

This is not a sack of wheat. This is not a barn full of wheat. This is roughly 1,000 times the current global annual production of wheat, piled onto a single chessboard. The ruler fell short.

This story illustrates the fundamental nature of the functions we are about to study. While the linear functions we discussed in previous chapters grow by adding a fixed amount at each step (additive growth), the exponential functions grow by multiplying by a fixed amount (multiplicative growth). At first, the difference seems small. Eventually, it becomes insurmountable.

11.1 THE DEFINITION AND KEY PROPERTIES

Let a be a positive real. Our goal is to define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = a^x$ where the input x can be *any* real number. This is not as straightforward as it might seem. The meaning of a^x is clear when x is a positive integer, and can be easily extended to accommodate all integers. However, the meaning of a^x is not so clear when x is a fraction, and is entirely mysterious if x is an irrational number. A fully rigorous treatment can only be given in a course on calculus. However, below we attempt to convey the idea as best as we can at the moment, starting with the integral case.

11.1.1 Integral exponent. We all share a common understanding of what an exponent means when the power is a positive integer. If we write 2^5, we are giving a shorthand instruction: "multiply 2 by itself 5 times." In general, for any real a and for any positive integer n we define a^n as

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$$

If n is negative, we define a^n as $1/a^{|n|}$. For instance, 2^{-5} simply means $1/2^5$. ♢

11.1.2 Rational exponent. What happens when we encounter a rational number, like $x = \frac{3}{2}$? We cannot multiply a number by itself "one and a half" times. That sentence has no semantic meaning.

However, we can rely on the rules of algebra we want to preserve. We want the rule $(a^m)^n = a^{mn}$ to hold true. Therefore, if we have $x = 1/2$, we know that:

$$(a^{1/2})^2 = a^{1/2 \cdot 2} = a^1 = a$$

This implies that $a^{1/2}$ must be the number which, when squared, gives a. In other words, the square root. This logic extends to any rational number p/q:

$$a^{p/q} = \sqrt[q]{a^p}$$

This allows us to exponentiate numbers like 0.5, 1.25, or 3.333.... ♢

11.1.3 Real exponent. Now we arrive at the difficult part—the part that challenged mathematicians for centuries. What does it mean to raise a number to an *irrational* power?[1] Consider 2^x. What is the value of $2^{\sqrt{2}}$? We have

$$\sqrt{2} \approx 1.41421356...$$

[1]The reader is not expected to fully grasp the ideas presented below but hopefully gains at least some understanding.

We cannot use the "repeated multiplication" definition (clearly). And we cannot use the "root of a power" definition, because $\sqrt{2}$ cannot be written as a clean fraction p/q. To define $2^{\sqrt{2}}$, we use a process of **approximation**. We can squeeze the irrational number $\sqrt{2}$ between two rational numbers that are very close to each other.

$$1.4 < \sqrt{2} < 1.5$$
$$1.41 < \sqrt{2} < 1.42$$
$$1.414 < \sqrt{2} < 1.415$$

Since we know how to calculate $2^{1.4}$, $2^{1.41}$, etc., we can form a sequence of calculations that get closer and closer to a specific target.

$$\text{Approximating } 2^{\sqrt{2}}: \quad 2^{1.4}, \quad 2^{1.41}, \quad 2^{1.414}, \quad \ldots$$

We define $2^{\sqrt{2}}$ to be the unique real number that this sequence approaches (or "converges to"). By doing this for every irrational number, we effectively "fill in the holes" of our graph, turning a dotted line into a beautiful, continuous smooth curve. ♢

11.1.4 The "Natural" Base. We have talked about 2^x and 10^x, but in advanced mathematics, there is one base that is more important than all others. It seems strange at first—it is an irrational number, messy and unending, much like π. We call it e. Why would we choose a messy number as our favorite base? Let us perform a thought experiment with money. Imagine a generous bank offers you 100% interest on rupee 1 for one year.

- *Scenario A:* They pay once at the end of the year. We get your original rupee 1 and 1 rupee interest, which is a total of 2 rupees.
- *Scenario B:* We ask them to pay 50% every 6 months. After 6 months, we have a total of 1.50. We then redeposit this amount in the bank and wait for another 6 months. In the second 6 months, we earn 50% on this \$1.50. The total is

$$(1 + \frac{1}{2})^2 = 2.25$$

  We gained 25 cents by compounding!
- *Scenario C:* We get greedy. We ask for interest every month ($\frac{1}{12}$th of 100%). The total after one year would be

$$(1 + \frac{1}{12})^{12} \approx 2.61$$

What if we compounded every day? Every second? Every nanosecond? Does the amount of money grow to infinity? Surprisingly, it does not. It hits a ceiling. No matter how frequently we compound, the money will never exceed a specific number. It gets closer and closer to:

$$e \approx 2.718281828...$$

This number e is called Euler's number, and this is only one of infinitely many scenarios where it spontaneously shows up. The function $x \mapsto e^x$ is called the **natural exponential**

function. In calculus, we learn that this is the *only* function that is its own derivative, which is the way the author likes to think of the number e. $\diamondsuit$

11.1.5 A defining property. Let a be a positive real number. It should be clear to the reader that

$$a^{m+n} = a^m \cdot a^n$$

for all integers m and n. It turns out that this property holds even if x and y are real numbers.

$$a^{x+y} = a^x \cdot a^y$$

for all real numbers x and y. In fact, in a certain sense this is a defining property of the exponential function with a given base, a discussion which again requires some initiation into calculus.

$$\boxed{f(x+y) = f(x) \cdot f(y)}$$

This is a profound statement. It tells us that *addition in the domain corresponds to multiplication in the target.* If we take a step forward in x (adding), we scale the value of y (multiplying). This property is unique to exponential functions. If you ever find a natural phenomenon where adding a little bit to the input multiplies the output by a factor—whether it is bacteria growth, radioactive decay, or compound interest—you are looking at an exponential function. $\diamondsuit$

11.1.6 Law of indices. We list the main properties of exponentiation below. The proofs of these are easy when the exponents are integral. However, the case of general exponents can only be discussed in a course on real analysis.

Lemma. Let a be a positive real number and x and y be arbitrary reals. Then

a) $a^0 = 1$.

b) *Positivity.* $a^t > 0$ for all $t \in \mathbb{R}$.

c) *Monotonocity.* Suppose $x < y$. Then we have

$$a^x < a^y \text{ if } a > 1 \quad \text{and} \quad a^x > a^y \text{ if } a < 1$$

d) *Sum to Product.* $a^{x+y} = a^x \cdot a^y$.

e) *Power of power.* $a^{xy} = (a^x)^y$.[a]

[a] This property in particular says that $a^{p/q} = \sqrt[q]{a^p}$ if p and q are integers.

The monotonicity property implies that the map $x \mapsto a^x : \mathbb{R} \to \mathbb{R}$ is injective whenever $a \neq 1$.

11.1.7 Graph of exponential functions. We have discussed the algebraic nature of exponential functions but to truly befriend a function, one must see its face. Let us examine the geometry of the exponential curve. We assume that the base a is positive and $a \neq 1$. There is a stark dichotomy in the behavior of these graphs, dictated entirely by whether the base a is greater than or less than 1.

a) *Case 1: Growth* ($a > 1$). Think of $a = 2$ or $a = e$. As we move to the right, the function explodes upwards, growing faster than any polynomial. As we move to the left, the function decays towards zero, hugging the x-axis but never quite touching it. The line $y = 0$ is a horizontal asymptote.

b) *Case 2: Decay* ($0 < a < 1$). Think of $a = 1/2$. This is simply the reflection of the growth case across the y-axis (since $(1/a)^x = a^{-x}$). Here, the function explodes as we go left and vanishes towards zero as we go right.

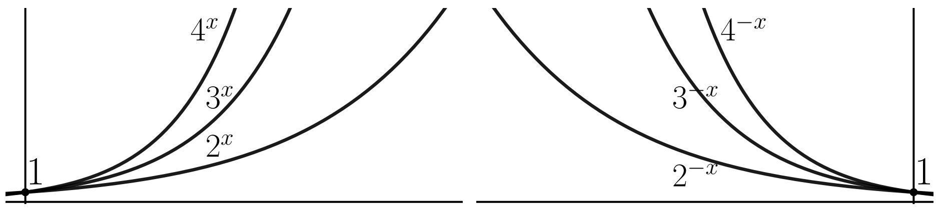


Figure 11.1: On the left we have graphs of exponential functions with base exceeding 1. On the right the bases are between 0 and 1.

The graphs are shown in Figure 11.1. ♢

11.2 ALGEBRA

To navigate the landscape of exponential functions with confidence, one must first achieve a certain fluency in the laws of indices. Think of the properties listed in Point 11.1.6 not merely as rules to be memorized, but as the grammar of a language. Just as one cannot compose poetry without understanding syntax, one cannot solve intricate exponential equations without an intuitive grasp of how to manipulate bases and exponents.

Often, the difficulty in these problems lies not in the exponential concept itself, but in recognizing a familiar algebraic structure—such as a quadratic equation or a system of linear equations—hiding beneath the exponential notation. The following exercises are designed to build this recognition. We begin with standard simplifications to warm up the algebraic machinery, and then progress to equations that require more strategic substitutions and structural insight.

Illustration 11.2.1 Simplify

$$\left(\frac{81}{16}\right)^{-3/4} \times \left[\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}\right]$$

Solution. We simplify the expression term by term, applying the laws of indices carefully.

a) *First term:* $\left(\frac{81}{16}\right)^{-3/4}$. We recognize that $81 = 3^4$ and $16 = 2^4$. Thus, $\frac{81}{16} = \left(\frac{3}{2}\right)^4$. Therefore

$$\left(\frac{81}{16}\right)^{-3/4} = \left[\left(\frac{3}{2}\right)^4\right]^{-3/4} = \left(\frac{3}{2}\right)^{4\times(-3/4)} = \left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

b) *Second term (inside the brackets):* $\left(\frac{25}{9}\right)^{-3/2}$. We know $25 = 5^2$ and $9 = 3^2$, so $\frac{25}{9} = \left(\frac{5}{3}\right)^2$. Thus

$$\left(\frac{25}{9}\right)^{-3/2} = \left(\left(\frac{5}{3}\right)^2\right)^{-3/2} = \left(\frac{5}{3}\right)^{-3} = \left(\frac{3}{5}\right)^3 = \frac{27}{125}.$$

c) *Third term (inside the brackets):* $\left(\frac{5}{2}\right)^{-3}$. This simplifies directly to $\left(\frac{2}{5}\right)^3 = \frac{8}{125}$.

d) *Combining the terms inside the brackets:* The expression inside is $\left(\frac{25}{9}\right)^{-3/2} \div \left(\frac{5}{2}\right)^{-3}$.

$$\frac{27}{125} \div \frac{8}{125} = \frac{27}{125} \times \frac{125}{8} = \frac{27}{8}.$$

e) *Final calculation:* We multiply the first term by the result of the bracketed expression:

$$\frac{8}{27} \times \frac{27}{8} = 1.$$

Thus, the value of the expression is 1. ■

Illustration 11.2.2 If $x = 3^{1/3} + 3^{-1/3}$, determine the value of the expression $3x^3 - 9x$.

Solution. Let $a = 3^{1/3}$ and $b = 3^{-1/3}$. Notice that a and b are reciprocals, meaning $a = 1/b$. We are given $x = a + b$. We want to compute $3x^3 - 9x$. First, let us cube x:

$$x^3 = (a+b)^3 = a^3 + b^3 + 3ab(a+b).$$

Substituting the known values:

- $a^3 = (3^{1/3})^3 = 3$.
- $b^3 = (3^{-1/3})^3 = 3^{-1} = \frac{1}{3}$.
- $ab = 1$.
- $a + b = x$.

So the equation becomes:

$$x^3 = 3 + \frac{1}{3} + 3(1)(x) = \frac{10}{3} + 3x.$$

Rearranging terms to match the target expression:

$$x^3 - 3x = \frac{10}{3}.$$

Multiplying the entire equation by 3:

$$3x^3 - 9x = 10.$$

Thus, the value of the expression is 10. ■

Exercise 11.2.1. Simplify

a)

$$\frac{2^{n+4} - 2 \cdot 2^n}{2 \cdot 2^{n+3}} + 2^{-3}$$

b)

$$\sqrt[4]{x^8 y^{-12} z^4} \times (xy^{-2})^{-1}$$

Exercise 11.2.2. If $a^x = b$, $b^y = c$, and $c^z = a$, show that $xyz = 1$.

11.3 DOMAIN AND RANGE

Illustration 11.3.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined via the equation

$$e^x + e^{f(x)} = e$$

Show that the image of f is $(-\infty, 1)$.

Solution. We are asked to find the set of all possible values that $f(x)$ can take as x ranges over all real numbers. Rearranging the given equation to isolate the term involving $f(x)$, we get

$$e^{f(x)} = e - e^x$$

We know a fundamental property of the exponential function: for any real input, the output is strictly positive. That is, $e^{f(x)} > 0$. Consequently, the right-hand side must also be strictly positive:

$$e - e^x > 0 \qquad \iff \qquad e^1 > e^x \qquad \iff \qquad 1 > x$$

where the last inequality arises because the base $e \approx 2.718 > 1$). This tells us that the domain of f is contained in $(-\infty, 1)$. However, as long as $e - e^x$ is positive, there is some (exactly one) suitable value of $f(x)$ that satisfies $e^{f(x)} = e - e^x$, since the image of the exponential function with base e is $(0, \infty)$. So we conclude that the domain of f is exactly $(-\infty, 1)$.

Now for the range. As x varies over the interval $(-\infty, 1)$, the term e^x takes all values in the interval $(0, e)$. Therefore, the term $e - e^x$ also takes all values in the interval $(0, e)$. Thus the range of f consists of precisely those y for which e^y is in $(0, e)$. This set is $(-\infty, 1)$. ∎

Illustration 11.3.2 Determine the values of a and b for which

$$\left|e^{|x-b|} - a\right| = 2$$

has four distinct solutions.

Solution. Let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to R$ be functions defined as

$$g(x) = |x - b| \quad \text{and} \quad f(t) = |e^t - a|$$

for all $x, t \in \mathbb{R}$. Since $f \circ g(x) = |e^{|x-b|} - a|$, the problem asks to find those values of a and b for which $(f \circ g)^{-1}(2)$ has size 4. But

$$(f \circ g)^{-1}(2) = g^{-1}(f^{-1}(2))$$

So let us understand what is $f^{-1}(2)$ in terms of a. We have

$$\begin{aligned} t \in f^{-1}(2) \quad &\iff \quad f(t) = 2 \quad \iff \quad |e^t - a| = 2 \\ &\iff \quad e^t = 2 + a \quad \text{or} \quad e^t = -2 + a \end{aligned}$$

Thus $f^{-1}(2)$ has exactly two elements in it, let us call them t_a^+ and t_a^-. Now

$$g^{-1}(f^{-1}(2)) = g^{-1}(\{t_a^+, t_a^-\}) = g^{-1}(t_a^+) \cup g^{-1}(t_a^-)$$

Observing that every fiber of g has size at most 2, we deduce that $g^{-1}(f^{-1}(2))$ has size 4 if and only if both $g^{-1}(t_a^+)$ and $g^{-1}(t_a^-)$ have size 2. This happens precisely when both t_a^+ and t_a^- are positive, which, in turn, happens precisely when $2 + a$ and $-2 + a$ are both greater than 1. Summarizing, the necessary and sufficient condition for the given equation to have exactly 4 distinct solutions is that $a > 3$, and no constraints on b. ∎

Illustration 11.3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$$

Is f injective? Is f surjective? Is f monotonic?

Solution. To understand the behavior of this function, we must dismantle the modulus operator.

Case 1: $x < 0$. Here, $|x| = -x$. Substituting this into the numerator, we get:

$$f(x) = \frac{e^{-x} - e^{-x}}{e^x + e^{-x}} = \frac{0}{e^x + e^{-x}} = 0$$

For every negative input, the function returns zero. Thus f is not injective.

Case 2: $x \geq 0$. Here, $|x| = x$. The expression transforms into a familiar hyperbolic function:

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

As x ranges over all non-negative reals, so does $2x$, and hence e^{2x} ranges from 1 to ∞. Therefore, the image of $[0, \infty)$ under f is same as the set

$$\left\{\frac{y-1}{y+1} : y \geq 1\right\} = \left\{1 - \frac{2}{y+1} : y \geq 1\right\} = \left\{1 - \frac{2}{z} : z \geq 2\right\} = [0, 1)$$

Further, since e^{2x} rises monotonically from 1 to ∞ as x increasing from 0 to ∞, it follows by the expression of f that f also grows monotonically on $[0, \infty)$.

Conclusion. We conclude that f is not injective, that the image of f is $[0, 1)$ and hence it is not surjective either. However, f is monotonic. ■

Exercise 11.3.1. Let $f : \mathbb{R} \to \mathbb{R}$ is a function whose image is $[-1, 1]$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \frac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)} + e^{|f(x)|}}$$

Then the image of g is ___________.

Exercise 11.3.2. Consider the following function defined for all real numbers x

$$f(x) = \frac{2018}{100 + e^x}$$

How many integers are there in the range of f? **[CMI 2018 Part A]**

Exercise 11.3.3. Comment if the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{|x|} - e^{-x}$$

is injective, surjective, or bijective.

Exercise 11.3.4. The set of all real numbers x for which **[UGA 2024]**

$$3^{2^{1-x^2}}$$

is an integer has (a) 3 elements (b) 15 elements (c) 24 elements (d) infinitely many elements

Exercise 11.3.5. Find the range of the function given by the expression

$$\left(\frac{1}{3}\right)^{-|x+2|}$$

11.4 EQUATIONS

Before we begin, we remind the reader of the message in Section 4.5.

Illustration 11.4.1 Find the values of x and y if

$$3^x - 4^y = 77 \quad \text{and} \quad 3^{x/2} - 2^y = 7$$

Solution. This system of equations calls for a substitution. We notice that the first equation involves powers that are squares of the terms in the second equation. Specifically,

$$3^x = (3^{x/2})^2 \quad \text{and} \quad 4^y = (2^2)^y = (2^y)^2$$

Let us set $u = 3^{x/2}$ and $v = 2^y$. Substituting these into our system transforms the exponential problem into a simpler algebraic one:

$$u^2 - v^2 = 77 \quad \text{and} \quad u - v = 7$$

The first equation factors as the difference of squares: $(u - v)(u + v) = 77$. Since we know $u - v = 7$, we can substitute this value in:

$$7(u + v) = 77 \implies u + v = 11$$

Now we have a trivial linear system: $u - v = 7$ and $u + v = 11$. Adding the two gives $2u = 18$, so $u = 9$. Subtracting them gives $2v = 4$, so $v = 2$. Finally, we return to our original variables to find x and y:

$$u = 3^{x/2} = 9 = 3^2 \qquad \Rightarrow \qquad \frac{x}{2} = 2 \qquad \Rightarrow \qquad x = 4$$

$$v = 2^y = 2 = 2^1 \qquad \Rightarrow \qquad y = 1$$

Thus $x = 4$ and $y = 1$. ■

Illustration 11.4.2 Find all real numbers x that satisfy following equation

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

[CMI 2019 Part B]

Solution. To unravel this equation, we should look at the prime factorization of the bases. We have

$$8 = 2^3, \quad 27 = 3^3, \quad 12 = 2^2 \cdot 3 \qquad \text{and} \quad 18 = 2 \cdot 3^2$$

This suggests that the fundamental building blocks here are 2^x and 3^x. Let $a = 2^x$ and $b = 3^x$. We can rewrite the terms in the equation as follows:

- $8^x = (2^3)^x = a^3$
- $27^x = (3^3)^x = b^3$
- $12^x = (2^2 \cdot 3)^x = a^2b$
- $18^x = (2 \cdot 3^2)^x = ab^2$

Substituting these into the original equation gives us a surprisingly neat algebraic form:

$$\frac{a^3+b^3}{a^2b+ab^2} = \frac{7}{6} \qquad \Rightarrow \qquad \frac{(a+b)(a^2-ab+b^2)}{ab(a+b)} = \frac{7}{6}$$

Since a and b are exponentials (2^x and 3^x), they are strictly positive, so $a+b \neq 0$. We can safely cancel the $(a+b)$ term:

$$\frac{a^2-ab+b^2}{ab} = \frac{7}{6}$$

Dividing each term in the numerator by the denominator separates the variables nicely:

$$\frac{a}{b} - 1 + \frac{b}{a} = \frac{7}{6}$$

Let $t = \frac{a}{b} = \left(\frac{2}{3}\right)^x$. Then $\frac{b}{a} = \frac{1}{t}$. The equation becomes a quadratic in disguise:

$$t + \frac{1}{t} = 1 + \frac{7}{6} = \frac{13}{6}$$

By inspection (or by solving $6t^2 - 13t + 6 = 0$), the roots are $t = \frac{2}{3}$ and $t = \frac{3}{2}$. We now solve for x in each case:

$$\left(\frac{2}{3}\right)^x = \frac{2}{3} \qquad \Rightarrow \qquad x = 1 \qquad\qquad \text{and} \qquad\qquad \left(\frac{2}{3}\right)^x = \frac{3}{2} \qquad \Rightarrow \qquad x = -1$$

Thus, the solutions are $x = 1$ and $x = -1$. ■

Illustration 11.4.3 The number of value(s) of x satisfying the equation

$$(2011)^x + (2012)^x + (2013)^x - (2014)^x = 0$$

holds is

(a) exactly 2 (b) exactly 1 (c) more than one (d) 0

Solution. This equation pits a sum of exponentials against a single, larger exponential. To make the comparison fair (and the analysis simpler), let us divide the entire equation by the largest term, $(2014)^x$.

$$\left(\frac{2011}{2014}\right)^x + \left(\frac{2012}{2014}\right)^x + \left(\frac{2013}{2014}\right)^x - 1 = 0$$

which rearranges to:

$$\left(\frac{2011}{2014}\right)^x + \left(\frac{2012}{2014}\right)^x + \left(\frac{2013}{2014}\right)^x = 1$$

Let us define a function $f : \mathbb{R} \to \mathbb{R}$ which at x takes the left-hand side of this last equation above. Notice that each base—$\frac{2011}{2014}$, $\frac{2012}{2014}$, and $\frac{2013}{2014}$—is a positive number strictly less than 1. Since exponential functions with bases less than 1 are strictly decreasing, and since the sum of strictly decreasing functions is itself strictly decreasing, f is a monotonic decreasing function.

Now let us look at the range of f. As x becomes smaller and smaller and approaches $-\infty$, each term approaches ∞ (since the bases are smaller than 1). Also, as x becomes larger and larger and approaches ∞, each term approaches 0, and thus so does f. Thus f continuously descends from infinity down to 0. Therefore it must cross the horizontal line $y = 1$ at exactly one point, and hence there is exactly one value of x that satisfies the equation. Option (b) is correct. ■

Illustration 11.4.4 Solve for x:

$$\left(\sqrt{5+\sqrt{24}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$$

Solution. At first glance, the bases appear to be complicated surds. However, a closer inspection reveals a "conjugate" relationship. Let us verify their product:

$$(5+\sqrt{24})(5-\sqrt{24}) = 25 - 24 = 1$$

This implies that the two terms inside the parentheses are reciprocals of each other. Let $a = \sqrt{5+\sqrt{24}}$. Then $\sqrt{5-\sqrt{24}} = 1/a$. The equation simplifies beautifully to:

$$a^x + \left(\frac{1}{a}\right)^x = 10 \qquad \Rightarrow \qquad a^x + \frac{1}{a^x} = 10$$

Writing y in place of a^x, the equation becomes

$$y + 1/y = 10 \qquad \Longleftrightarrow \qquad y^2 - 10y + 1 = 0$$

Solving this quadratic for y:

$$y = \frac{10 \pm \sqrt{100-4}}{2} = \frac{10 \pm \sqrt{96}}{2} = \frac{10 \pm 4\sqrt{6}}{2} = 5 \pm 2\sqrt{6}$$

Recall that $\sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$. So our original base was $a = \sqrt{5+2\sqrt{6}}$. Notice that

$$5 + 2\sqrt{6} = a^2 \quad \text{and} \quad 5 - 2\sqrt{6} = a^{-2}$$

Thus the two values of y are a^2 and a^{-2}. Therefore, the corresponding values of x are 2 and -2. ■

Illustration 11.4.5 Find the number of values of x which satisfies the equation

$$|x|^{x^2-2x} = 1$$

Solution. An expression of the form $a^b = 1$ can be satisfied under three specific conditions. We investigate each case carefully.

Case 1: The exponent is zero ($b = 0$) and the base is non-zero ($a \neq 0$). In this case we would have

$$x^2 - 2x = 0 \quad \Rightarrow \quad x(x-2) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 2$$

However, if $x = 0$, the base $|x|$ becomes 0, leading to the indeterminate form 0^0. Thus, we must reject $x = 0$. If $x = 2$, the base is $|2| = 2 \neq 0$. So, $x = 2$ is a valid solution.

Case 2: The base is one ($a = 1$). In this case

$$|x| = 1 \quad \Rightarrow \quad x = 1 \text{ or } x = -1$$

Since $1^{\text{anything}} = 1$, these are always valid solutions. So, $x = 1$ and $x = -1$ are solutions.

Case 3: The base is negative one ($A = -1$) and the exponent is an even integer. Here, the base is $|x|$, which is always non-negative. Thus, $|x|$ can never be -1. This case yields no solutions.

Summing up, the valid values for x are $\{2, 1, -1\}$. There are exactly 3 solutions. ∎

Exercise 11.4.1. Solve for x:

$$4^{x+1.5} + 9^x = 6^{x+1}$$

Exercise 11.4.2. Solve for x:

$$2^{2x^2} + 2^{x^2+2x+2} = 2^{5+4x}$$

Exercise 11.4.3. Solve for x:

$$4^{\sqrt{3x^2-2x}+1} + 2 = 9.2^{\sqrt{3x^2-2x}}$$

Exercise 11.4.4. Solve for x:

$$9^{x^2-1} - 36 \times 3^{x^2-3} + 3 = 0$$

Exercise 11.4.5. Solve for x:

$$3^{3x+1} - 4 \times 27^{x-1} + 9^{1.5x-1} - 80 = 0$$

Exercise 11.4.6. Solve for x:

$$(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$$

Exercise 11.4.7. Solve for x:

$$(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{4}{2-\sqrt{3}}$$

Exercise 11.4.8. Solve for x:

$$5^{x-2} \times 2^{\frac{3x}{x+1}} = 4$$

Exercise 11.4.9. Solve for x:

$$\left(x^2 - x - 1\right)^{x^2-1} = 1$$

Exercise 11.4.10. Solve for x:

$$(x-2)^{x^2-x} = (x-2)^{12}$$

Exercise 11.4.11. Find the number of solution of the equation

$$\left(\frac{3}{5}\right)^x - 2x + \frac{7}{5} = 0$$

11.5 INEQUALITIES

Illustration 11.5.1 Find the set of all the values of x which satisfy:

$$8 - x \cdot 2^x + 2^{3-x} - x > 0$$

Solution. We seek the set of all real x satisfying the inequality

$$8 - x \cdot 2^x + 2^{3-x} - x > 0.$$

To reveal the underlying structure of this expression, it is helpful to group terms that share common functional forms. We rewrite the inequality as:

$$(8 - x \cdot 2^x) + (2^3 \cdot 2^{-x} - x) > 0$$

Factorizing 2^x out of the first pair and recognizing $2^3 = 8$, we have:

$$2^x(2^{3-x} - x) + (2^{3-x} - x) > 0$$

Now, we observe a common factor of $(2^{3-x} - x)$, allowing us to factorize the entire expression:

$$(2^x + 1)(2^{3-x} - x) > 0.$$

Since exponential functions take only positive values, the factor (2^x+1) is strictly positive for all $x \in \mathbb{R}$. We can therefore divide both sides by this term without reversing the inequality, reducing the problem to:

$$2^{3-x} - x > 0 \qquad \iff \qquad 2^{3-x} > x$$

To solve this, let $f(x) = 2^{3-x} = 82^{-x}$ and $g(x) = x$. The function f is strictly decreasing while g is strictly increasing. The graphs of such functions can intersect at most once. By inspection, at $x = 2$, we have $f(2) = 2^{3-2} = 2$ and $g(2) = 2$. Thus, $x = 2$ is the unique point of intersection.

Because f is decreasing and g is increasing, for any $x < 2$, we must have $f(x) > f(2) = 2$ and $g(x) < g(2) = 2$, which implies $f(x) > g(x)$. Conversely, for $x > 2$, $f(x) < g(x)$. Summarizing, the set of all the solutions is $x \in (-\infty, 2)$. ■

Illustration 11.5.2 Find the set of all the values of x which satisfy:

$$|x|^{x^2-x-2} < 1$$

Solution. Inequalities of the form $a^b < 1$ require careful partitioning of the domain based on the value of the base $a = |x|$. We note first that for the expression to be well-defined, we must have $x \neq 0$ to avoid a non-positive exponent on 0.

Case 1: $|x| > 1$. In this region the base is greater than unity. For a number greater than 1 raised to a power to remain less than 1, the exponent must be strictly negative:

$$x^2 - x - 2 < 0 \qquad \iff \qquad (x-2)(x+1) < 0 \qquad \iff \qquad x \in (-1, 2)$$

Intersecting this with our condition $|x| > 1$, we find the valid values are $x \in (1, 2)$.

Case 2: $0 < |x| < 1$. In this region, the base is a fraction between 0 and 1. For such a base, the inequality $a^b < 1$ is satisfied if and only if the exponent b is strictly positive:

$$x^2 - x - 2 > 0 \qquad \iff \qquad (x-2)(x+1) > 0 \qquad \iff \qquad x \in (-\infty, -1) \cup (2, \infty)$$

Taking the intersection of $(-\infty, -1) \cup (2, \infty)$ and $(-1, 1) \setminus \{0\}$, we see there are no overlapping points. Thus, this case yields no solutions.

Case 3: $|x| = 1$. If $x = 1$ or $x = -1$, the expression becomes 1^{power}, which is exactly 1. Since our inequality is strict (< 1), these boundary points are excluded.

Summarizing: Combining the results, the complete solution set is $x \in (1, 2)$. ■

Illustration 11.5.3 Find the set of all the values of x which satisfy:

$$\frac{2^{1-x} - 2^x + 1}{2^x - 1} \leq 0$$

Solution. To simplify the algebraic handling, we let $t = 2^x$. Note that since the exponential function is strictly positive, we have $t > 0$. The expression becomes:

$$\frac{\frac{2}{t} - t + 1}{t - 1} \leq 0$$

Multiplying the numerator by t (which is valid as $t > 0$) to clear the fraction, we obtain:

$$\frac{-(t^2 - t - 2)}{t(t-1)} \leq 0 \qquad \Longleftrightarrow \qquad \frac{(t-2)(t+1)}{t(t-1)} \geq 0$$

We apply the wavy curve method to this rational expression in t. The critical points are $t = 2, t = 1, t = 0$, and $t = -1$. However, we must respect the constraint $t > 0$. On the interval $t > 0$, the factor $(t + 1)$ is always positive and thus does not affect the sign of the inequality. We are reduced to:

$$\frac{t-2}{t-1} \geq 0 \quad \text{for } t > 0$$

The critical points for this simplified expression are $t = 1$ (from the denominator) and $t = 2$ (from the numerator). The wavy curve tells us the expression is positive for $t \in (0, 1) \cup [2, \infty)$. We exclude $t = 1$ because it makes the denominator zero. Translating back to x:

$$t \in (0,1) \qquad \Longleftrightarrow \qquad 0 < 2^x < 1 \qquad \Longleftrightarrow \qquad x \in (-\infty, 0)$$

and

$$t \geq 2 \qquad \Longleftrightarrow \qquad 2^x \geq 2 \qquad \Longleftrightarrow \qquad x \in [1, \infty)$$

Summarizing the solution set is $x \in (-\infty, 0) \cup [1, \infty)$. ■

Illustration 11.5.4 For what values of x does

$$5^x + (2\sqrt{3})^{2x} - 169 \leq 0$$

hold?

Solution. At first glance, the bases seem unrelated. However, let us simplify the second term. Observe that $(2\sqrt{3})^2 = 4 \times 3 = 12$. Thus, using the power of a power rule, we can rewrite the term $(2\sqrt{3})^{2x}$ as $((2\sqrt{3})^2)^x$, which simplifies to 12^x. The inequality now takes a more familiar form:

$$5^x + 12^x \leq 169$$

The number 169 should ring a bell; it is 13^2. Furthermore, the numbers 5 and 12 form a Pythagorean triple with 13, that is, $5^2 + 12^2 = 13^2$. This observation immediately tells us that $x = 2$ is a solution to the equation $5^x + 12^x = 169$.

To solve the inequality, we must consider the behavior of the function $f(x) = 5^x + 12^x$. Since the bases 5 and 12 are both greater than 1, their respective exponential functions are strictly increasing. Consequently, their sum $f(x)$ is also a strictly increasing function. Since $f(2) = 169$ and f is strictly increasing, for any $x < 2$ we must have $f(x) < 169$, and for any $x > 2$ we must have $f(x) > 169$. Therefore, the inequality holds precisely when $x \in (-\infty, 2]$. ■

Exercise 11.5.1. Find the set of all the values of x which satisfy:

$$0.1^{4x^2-2x-2} \leq 0.1^{2x-3}$$

Exercise 11.5.2. Find the set of all the values of x which satisfy:

$$x^2 \cdot 5^x - 5^{2+x} < 0$$

Exercise 11.5.3. Find the set of all the values of x which satisfy:

$$\left(\frac{1}{3}\right)^{\frac{|x|+1}{x-2}} > 9$$

Exercise 11.5.4. Find the set of all the values of x which satisfy:

$$4^x + 2^{x+1} - 6 \leq 0$$

Exercise 11.5.5. Find the set of all the values of x which satisfy:

$$\left(\frac{1}{3}\right)^{\sqrt{x+4}} > \left(\frac{1}{3}\right)^{\sqrt{x^2++3x+4}}$$

Exercise 11.5.6. Find the set of all the values of x which satisfy:

$$9^{\sqrt{x^2-3}} + 3 < 3^{\sqrt{x^2-3}-1} \cdot 28$$

Exercise 11.5.7. Find the set of all the values of x which satisfy:

$$\sqrt{9^x + 3^x - 2} \geq 9 - 3^x$$

Exercise 11.5.8. Find the set of all the values of x which satisfy:

$$3^{\sqrt{x}} > 2^a$$

CHAPTER 12

LOGARITHMIC FUNCTIONS

To truly appreciate the logarithm, one must first descend into the "computational hell" of the 16th and 17th centuries. This was an era of exploding scientific ambition—Kepler was mapping the laws of planetary motion, and navigators were charting the oceans—yet the only tools available for calculation were the abacus and the pen.

Astronomers, in particular, faced a paralyzing bottleneck. To calculate a single position of Mars, Kepler might need to multiply two seven-digit numbers, a task that is not only tedious but prone to catastrophic error. A single slip of the quill could invalidate weeks of work. The situation was so dire that the Danish astronomer Tycho Brahe employed a staff of "human computers" just to crunch numbers. The scientific revolution was effectively stalled by the sheer difficulty of multiplication and division.

The breakthrough appeared in 1614 with the publication of *Mirifici Logarithmorum Canonis Descriptio* ("A Description of the Wonderful Table of Logarithms") by John Napier, the Baron of Merchiston. Napier had spent twenty years brooding over this problem. His insight was not algebraic, but kinetic. He imagined two particles: one moving with constant velocity (arithmetic progression) and another moving with velocity proportional to its distance from a fixed point (geometric progression). By synchronizing these two motions, he created a mapping where the difficult task of multiplying the geometric values could be replaced by the simple task of adding the arithmetic ones.

Independently, the Swiss clockmaker Joost Bürgi had stumbled upon a similar concept around 1588, constructing tables based on powers of the number 1.0001. However, Bürgi—historically reticent—failed to publish his work until 1620, by which time Napier's tables were already revolutionizing European mathematics.

The English mathematician Henry Briggs immediately recognized the genius of Napier's work but suggested a crucial modification: shifting the base to 10 to align with our decimal number system. This birth of the "common logarithm" turned Napier's theoretical wonder into a practical necessity. For the next 350 years, until the advent of the pocket calculator in the 1970s, the slide rule and the log table were the primary weapons of every engineer, physicist, and architect. As Pierre-Simon Laplace famously eulogized, the invention of logarithms, "by shortening the labors, doubled the life of the astronomer."

12.1 DEFINITION AND KEY PROPERTIES

Just as subtraction undoes addition and division undoes multiplication, the logarithm is the operation that undoes exponentiation.

Definition. Let a be a positive real number such that $a \neq 1$. For any positive real number x, the **logarithm** of x to the base a, denoted $\log_a x$, is the unique real number y such that:

$$a^y = x$$

In the language of functions, if $f : \mathbb{R} \to (0, \infty)$ is the exponential function defined by $f(y) = a^y$, then the logarithmic function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \log_a x$ is its inverse. It immediately follows that

$$x^{\log_x y} = y$$

whenever x and y are positive.

12.1.1 The laws of logarithms. The power of logarithms stems from their ability to downgrade operations: they turn products into sums, quotients into differences, and powers into multipliers.

Lemma. Let a, x, y be positive reals with $a \neq 1$, and let p be any real number.

a) (Product Rule.) $\log_a(xy) = \log_a x + \log_a y$

b) (Quotient Rule.) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

c) (Power Rule.) $\log_a(x^p) = p \log_a x$ and $\log_{a^p}(x) = \frac{1}{p} \log_a p$.

d) (Base Change Formula.) $\log_a x = \frac{\log_b x}{\log_b a}$

Proof. Let $u = \log_a x$ and $v = \log_a y$, which implies $x = a^u$ and $y = a^v$ by definition. We provide the formal derivations for the fundamental laws below:

(a) We have

$$xy = a^u \cdot a^v = a^{u+v}$$

Now, converting this back to logarithmic form:

$$\log_a(xy) = u + v = \log_a x + \log_a y$$

proving the desired result.

(b) We have

$$\frac{x}{y} = \frac{a^u}{a^v} = a^{u-v}$$

By the definition of the logarithm, the exponent required to produce the ratio x/y from base a is $u - v$. Thus:

$$\log_a\left(\frac{x}{y}\right) = u - v = \log_a x - \log_a y$$

proving the quotient rule.

(c) Consider the quantity x raised to an arbitrary real power p. Using our substitution $x = a^u$ we have $x^p = (a^u)^p$. By the "power of a power" rule of indices, we multiply the exponents:

$$x^p = a^{up}$$

Converting this back into logarithmic form, we find the power to which base a must be raised to yield x^p is up. Therefore:

$$\log_a(x^p) = up = p(u) = p\log_a x$$

showing the power rule.

(d) Let $y = \log_a x$. In exponential form, this is:

$$a^y = x$$

We now take the logarithm of both sides with respect to a *new* base b:

$$\log_b(a^y) = \log_b x$$

Applying the power rule derived above to the left-hand side:

$$y\log_b a = \log_b x \qquad \Rightarrow \qquad y = \frac{\log_b x}{\log_b a}$$

Substituting $y = \log_a x$ back in, we arrive at the identity:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

and we are done. $\diamondsuit$

Example 12.1.2. Let us try to get a hang of these laws by means of a few exercises. Let us simplify the expressions:

a)

$$\log_3 2 \cdot \log_8 7 \cdot \log_4 3 \cdot \log_7 6 \cdot \log_5 4$$

b)

$$\log_3 \log_2 \log_{\sqrt{3}} 81$$

Solution. *a):* Convert all to base b using the base change formula to get:

$$\log_3 2 \cdot \log_8 7 \cdot \log_4 3 \cdot \log_7 6 \cdot \log_5 4 = \frac{\log 2}{\log 3} \cdot \frac{\log 7}{\log 8} \cdot \frac{\log 3}{\log 4} \cdot \frac{\log 6}{\log 7} \cdot \frac{\log 4}{\log 5}$$

Canceling common terms in the numerators and denominators:

$$\frac{\log 2 \cdot \log 6}{\log 8 \cdot \log 5} = \frac{\log 2 \cdot \log 6}{3 \log 2 \cdot \log 5} = \frac{\log_5 6}{3}$$

b) Work from the innermost logarithm outwards Using the power rule we have

$$\log_3(\log_2(\log_{\sqrt{3}}(81))) = \log_3(\log_2(2 \log_3(81))) = \log_3(\log_2 8) = \log_3(3) = 1$$

Thus the given expression reduces to 1. ■

Example 12.1.3. Let us show that $\log_2(3)$ is irrational. Assume on the contrary that $\log_2 3 = p/q$ for some positive integers p and q, where we may assume that p and q have no common factors other than 1. Thus we get

$$3 = 2^{p/q} \qquad \Rightarrow \qquad 3^q = 2^p$$

But the last equation is impossible since the right hand side is divisible by 2 but the left hand side is not.

12.2 ELEMENTARY ALGEBRA

Illustration 12.2.1 If **[UGA 2016]**

$$\log_{10} x = 10^{\log_{100} 4}$$

then find the value of x.

Solution. We simplify the right-hand side using the base change properties of logarithms. Recall that $\log_{a^n}(b^m) = \frac{m}{n} \log_a b$. Applying this to $\log_{100} 4$:

$$\log_{100} 4 = \log_{10^2}(2^2) = \frac{2}{2} \log_{10} 2 = \log_{10} 2.$$

Alternatively, we can write

$$\log_{100} 4 = \frac{\ln 4}{\ln 100} = \frac{\ln 4}{2 \ln 10} = \frac{1}{2} \log_{10} 4 = \log_{10}(4^{1/2}) = \log_{10} 2$$

Now substitute this back into the original equation:

$$\log_{10} x = 10(\log_{10} 2) = \log_{10}(2^{10}).$$

Exponentiating both sides (with base 10), we get:

$$x = 2^{10} = 1024.$$

and we are done. ■

Illustration 12.2.2 Prove that

$$\log_{a+b} m + \log_{a-b} m = 2\log_{a+b} m \log_{a-b} m$$

if it is given that $m^2 = a^2 - b^2$.

Solution. We use the change of base formula to transform the left hand side to the right hand side:

$$\begin{aligned}
\log_{a+b} m + \log_{a-b} m &= \frac{\log m}{\log(a+b)} + \frac{\log m}{\log(a-b)} \\
&= \log m \cdot \frac{\log(a+b) + \log(a-b)}{\log(a+b)\log(a-b)} \\
&= \log m \cdot \frac{\log((a+b)(a-b))}{\log(a+b)\log(a-b)} \\
&= \frac{\log m}{\log(a+b)} \cdot \frac{\log(a^2-b^2)}{\log(a-b)} = \log_{a+b} m \log_{a-b}(m^2)
\end{aligned}$$

which gives the right hand side by an application of the power rule. ■

Illustration 12.2.3 If

$$\frac{\log_2\left(b^3/8\right)}{\log_3\left(27/a^2\right)} = 1 \quad \text{and} \quad \log_3\left(\frac{9}{a}\right) = \log_2\left(\frac{b}{4}\right)$$

then find the value of $\frac{a}{b}$.

Solution. We are presented with a system of logarithmic equations. To resolve it, we first simplify each expression to find a relationship between a and b. Consider the first equation:

$$\frac{\log_2\left(b^3/8\right)}{\log_3\left(27/a^2\right)} = 1 \qquad \Rightarrow \qquad \log_2\left(b^3/8\right) = \log_3\left(27/a^2\right)$$

Using the properties $\log(x/y) = \log x - \log y$ and $\log x^k = k\log x$, we obtain:

$$3\log_2 b - 3 = 3 - 2\log_3 a \qquad \Rightarrow \qquad 3\log_2 b + 2\log_3 a = 6 \quad \cdots (1)$$

Now consider the second equation:

$$\log_3\left(\frac{9}{a}\right) = \log_2\left(\frac{b}{4}\right) \qquad \Rightarrow \qquad \log_3 9 - \log_3 a = \log_2 b - \log_2 4$$

$$2 - \log_3 a = \log_2 b - 2 \qquad \Rightarrow \qquad \log_2 b + \log_3 a = 4 \quad \cdots (2)$$

We now have a linear system in terms of $u = \log_2 b$ and $v = \log_3 a$:

$$\begin{cases} 3u + 2v = 6 \\ u + v = 4 \end{cases}$$

Multiplying (2) by 2 gives $2u + 2v = 8$. Subtracting this from (1) yields:

$$u = -2 \qquad \Rightarrow \qquad \log_2 b = -2 \qquad \Rightarrow \qquad b = 2^{-2} = \frac{1}{4}$$

Substituting $u = -2$ into (2) gives:

$$-2 + v = 4 \qquad \Rightarrow \qquad v = 6 \qquad \Rightarrow \qquad \log_3 a = 6 \qquad \Rightarrow \qquad a = 3^6 = 729$$

The problem asks for the ratio $\frac{a}{b}$:

$$\frac{a}{b} = \frac{729}{1/4} = 2916$$

■

Illustration 12.2.4 Simplify

$$b \cdot a^{\frac{2}{\log_b a}+1} - 2a^{\log_a b+1} \cdot b^{\log_b a+1} + a \cdot b^{\frac{2}{\log_a b}+1}$$

Solution. We leverage the identity $\frac{1}{\log_b a} = \log_a b$ and the fundamental property $x^{\log_x y} = y$. First, we expand the exponents using $x^{m+n} = x^m \cdot x^n$:

- $a^{\frac{2}{\log_b a}+1} = a^{2\log_a b+1} = (a^{\log_a b})^2 \cdot a^1 = b^2 \cdot a$
- $a^{\log_a b+1} = a^{\log_a b} \cdot a = b \cdot a$
- $b^{\log_b a+1} = b^{\log_b a} \cdot b = a \cdot b$
- $b^{\frac{2}{\log_a b}+1} = b^{2\log_b a+1} = (b^{\log_b a})^2 \cdot b^1 = a^2 \cdot b$

Thus the left hand side is

$$b(b^2a) - 2(ba)(ab) + a(a^2b) = ab^3 - 2a^2b^2 + a^3b$$

Factorizing the common term ab reveals a perfect square trinomial:

$$ab(b^2 - 2ab + a^2) = ab(a-b)^2$$

and we end up with a neat expression. ■

Illustration 12.2.5 Simplify

a)

$$\frac{81^{1/\log_5 9} + 3^{3/\log_{\sqrt{6}} 3}}{409}\left((\sqrt{7})^{2/\log_{25} 7} - 125^{\log_{25} 6}\right)$$

b)

$$\left(2^{\log_{\sqrt[4]{2}} a} - 3^{\log_{27}\left(a^2+1\right)^3} - 2a\right) / \left(7^{4\log_{49} a} - a - 1\right)$$

Solution. (a): Let

$$X = \frac{81^{1/\log_5 9} + 3^{3/\log_{\sqrt{6}} 3}}{409} \quad \text{and} \quad Y = (\sqrt{7})^{2/\log_{25} 7} - 125^{\log_{25} 6}$$

To simplify X we use $\frac{1}{\log_a b} = \log_b a$. The terms in the numerator of X are

$$81^{\log_9 5} = (9^2)^{\log_9 5} = 9^{\log_9 25} = 25 \quad \text{and} \quad 3^{3\log_3 \sqrt{6}} = 3^{\log_3 (\sqrt{6})^3} = (\sqrt{6})^3 = 6\sqrt{6}$$

Thus

$$X = \frac{25 + 6\sqrt{6}}{409}$$

Similarly the terms in Y are

$$(\sqrt{7})^{2\log_7 25} = 7^{\log_7 25} = 25 \quad \text{and} \quad 125^{\log_{25} 6} = (5^3)^{\log_{5^2} 6} = 5^{\frac{3}{2}\log_5 6} = 6^{3/2} = 6\sqrt{6}$$

So

$$Y = 25 - 6\sqrt{6}$$

The product is

$$\frac{(25 + 6\sqrt{6})(25 - 6\sqrt{6})}{409} = \frac{625 - 216}{409} = \frac{409}{409} = 1$$

(b) We simplify the numerator first.

$$2^{\log_{\sqrt[4]{2}} a} = 2^{4\log_2 a} = a^4 \quad \text{and} \quad 3^{\log_{27}(a^2+1)^3} = 3^{\log_{3^3}(a^2+1)^3} = 3^{\frac{3}{3}\log_3(a^2+1)} = a^2 + 1$$

So the numerator becomes

$$a^4 - (a^2+1) - 2a = a^4 - a^2 - 2a - 1 = a^4 - (a+1)^2 = (a^2 - a - 1)(a^2 + a + 1)$$

The denominator is

$$7^{4\log_{49} a} - a - 1 = 7^{4(\frac{1}{2})\log_7 a} - a - 1 = 7^{\log_7 a^2} - a - 1 = a^2 - a - 1$$

The result is

$$\frac{(a^2 - a - 1)(a^2 + a + 1)}{a^2 - a - 1} = a^2 + a + 1$$

and we are done. ∎

Illustration 12.2.6 Given a and b are positive numbers satisfying

$$4\left(\log_{10} a\right)^2 + \left(\log_2 b\right)^2 = 1$$

then find the possible range of values of a and b.

Solution. Writing u and v for $\log_{10} a$ and $\log_{10} b$ respectively, we can write the given equation as

$$(2u)^2 + v^2 = 1$$

Thus $(2u, v)$ lies on the unit circle in the Cartesian plane. Conversely, any point on the unit circle furnishes a valid pair $(2u, v)$. This implies the bounds

$$-1 \le 2u \le 1 \qquad \Rightarrow \qquad -\frac{1}{2} \le \log_{10} a \le \frac{1}{2} \qquad \Rightarrow \qquad 10^{-1/2} \le a \le 10^{1/2}$$

We deduce that $a \in [\frac{1}{\sqrt{10}}, \sqrt{10}]$. Similarly, for v:

$$-1 \le v \le 1 \qquad \Rightarrow \qquad -1 \le \log_2 b \le 1 \qquad \Rightarrow \qquad 2^{-1} \le b \le 2^1$$

It follows that $b \in [\frac{1}{2}, 2]$. ■

Illustration 12.2.7 If

$$\sum_{k=0}^{n-1} \log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99} \log_r(r+1)$$

then find the value of n.

Solution. We simplify the product on the right hand side using the base change formula:

$$\prod_{r=10}^{99} \frac{\log(r+1)}{\log r} = \frac{\log 11}{\log 10} \cdot \frac{\log 12}{\log 11} \cdots \frac{\log 100}{\log 99} = \frac{\log 100}{\log 10} = \log_{10} 100 = 2$$

The left hand side is a telescoping sum:

$$\sum_{k=0}^{n-1} (\log_2(r+2) - \log_2(r+1)) = \log_2(n+1) - \log_2(1) = \log_2(n+1)$$

Equating both sides:

$$\log_2(n+1) = 2 \qquad \Rightarrow \qquad n+1 = 2^2 \qquad \Rightarrow \qquad n = 3$$

and we are done. ■

Illustration 12.2.8 Find the value of

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 66 \rfloor$$

Solution. We are computing the sum $S = \sum_{k=1}^{66} \lfloor \log_2 k \rfloor$. The term $\lfloor \log_2 k \rfloor$ tells us the integer part of the binary logarithm of k. This value remains constant for blocks of integers between consecutive powers of 2. Specifically, $\lfloor \log_2 k \rfloor = m$ if and only if $m \le \log_2 k < m+1$, which is equivalent to $2^m \le k < 2^{m+1}$. We invite the reader to do the necessary book keeping to convince themselves that the sum boils down to

$$0 + 2 + 8 + 24 + 64 + 160 + 18$$

which is 276. ■

Exercise 12.2.1. If

$$\frac{\log_a \sqrt{a^2-1} \cdot \log^2_{1/a} \sqrt{a^2-1}}{\log_{a^2}\left(a^2-1\right) \cdot \log_{\sqrt[3]{a}} \sqrt[6]{a^2-1}} = \frac{1}{2}$$

then find the value of a.

Exercise 12.2.2. If

$$\log_{18} 36 = a \quad \text{and} \quad \log_{24} 72 = b$$

then find the value of $4(a+b) - 5ab$.

Exercise 12.2.3. Simplify

1.
$$\frac{\log_a b + \log_a \left(b^{\frac{1}{2}\log_b a^2}\right)}{\log_a b - \log_{ab} b} \frac{\log_{ab} b \log_a b}{b^{2\log_b \log_a b} - 1}$$

2.
$$5^{\log_{1/5}\left(\frac{1}{2}\right)} + \log_{\sqrt{2}} \frac{4}{\sqrt{7}+\sqrt{3}} + \log_{1/2} \frac{1}{10+2\sqrt{21}}$$

Exercise 12.2.4. If

$$x = \log_d(abc), \quad y = \log_b(acd), \quad z = \log_c(abd) \quad \text{and} \quad t = \log_a(bcd)$$

then find the value of

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{t+1}$$

Exercise 12.2.5. Let

$$\log_{12} 27 = \alpha$$

What is the value of $\log_6(16)$ in terms of α.

Exercise 12.2.6. Suppose a and b are positive reals such that

$$\log_{27}(a) + \log_9(b) = \frac{7}{2} \quad \text{and} \quad \log_{27}(b) + \log_9(a) = \frac{2}{3}$$

Find the values of a and b.

Exercise 12.2.7. Suppose

$$\log_{a^2}(a^2+1) = 16$$

What is the value of

$$\log_{a^{32}}\left(a + \frac{1}{a}\right)$$

Exercise 12.2.8. If

$$\sum_{r=1}^{n} \lfloor \log_2 r \rfloor = 2010$$

then find the value of n.

Exercise 12.2.9. Let a, b and c be real numbers, each greater than 1 , such that **[UGA 2017]**

$$\frac{2}{3}\log_b a + \frac{3}{5}\log_c b + \frac{5}{2}\log_a c = 3.$$

If the value of b is 9, then the value of a must be (a) $\sqrt[3]{81}$ (b) $\frac{27}{2}$ (c) 18 (d)

27

Exercise 12.2.10. If a, b, c are real numbers > 1, then show that

$$\frac{1}{1+\log_{a^2 b}\frac{c}{a}} + \frac{1}{1+\log_{b^2 c}\frac{a}{b}} + \frac{1}{1+\log_{c^2 a}\frac{b}{c}} = 3$$

12.3 DOMAIN AND RANGE

Illustration 12.3.1 Find the domain of

$$\log_{2x}\left(2x - x^2 + 3\right)$$

Solution. To find the domain of the given expression, we must satisfy three fundamental conditions inherent to the definition of a logarithmic function: the argument must be strictly positive, the base must be strictly positive, and the base cannot be equal to one.

Argument Condition: $2x - x^2 + 3 > 0$. Rearranging gives $x^2 - 2x - 3 < 0$. Factorizing the quadratic expression, we have $(x-3)(x+1) < 0$. This inequality holds for $x \in (-1, 3)$.

Base Positivity: $2x > 0 \implies x > 0$.

Base Non-Unity: $2x \neq 1 \implies x \neq \frac{1}{2}$.

The domain is the intersection of these three sets: $x \in (-1,3) \cap (0, \infty) \cap \{x \neq 1/2\}$. Thus, the domain is:

$$\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 3\right)$$

and we are done. ∎

Illustration 12.3.2 Find the domain of the function f given by

$$f(x) = \sqrt{\log_{x+2\{x\}}\left(\lfloor x \rfloor^2 - 5\lfloor x \rfloor + 7\right)}$$

Solution. For the square root to yield a valid real number, the radicand must be non-negative. There are two ways this can happen:

a) The base of the logarithm exceeds 1 and the argument is at least 1.
b) The base of the logarithm is strictly between 0 and 1 and the argument of the log is in $(0, 1]$.

In case (a) we want to find the set of all the x such that

$$\begin{aligned} & x + 2\{x\} > 1 \quad \text{and} \quad \lfloor x\rfloor^2 - 5\lfloor x\rfloor + 7 \geq 1 \\ \iff & x + 2\{x\} > 1 \quad \text{and} \quad (\lfloor x\rfloor - 2)(\lfloor x\rfloor - 3) \geq 0 \\ \iff & x + 2\{x\} > 1 \quad \text{and} \quad \lfloor x\rfloor \in (-\infty, 2] \cup [3, \infty) \end{aligned}$$

This is same as asking for those x which satisfy $x + 2\{x\} \geq 1$ as the second condition is satisfied for all x. From Figure 12.1 it is clear that this region is $(-1/3, 0) \cup (1/3, 1) \cup (1, \infty)$

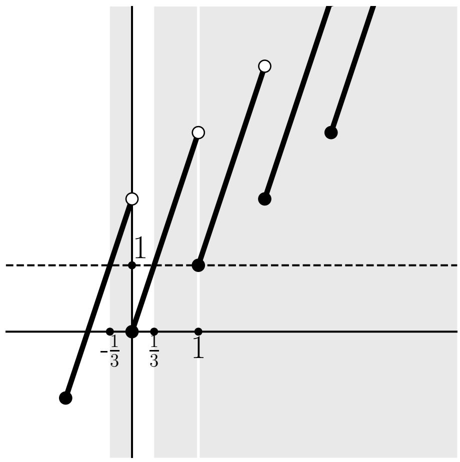


Figure 12.1

In case (b) we want to find the set of all x such that

$$\begin{aligned} & 0 < x + 2\{x\} < 1 \quad \text{and} \quad \lfloor x\rfloor^2 - 5\lfloor x\rfloor + 7 \leq 1 \\ \iff & 0 < x + 2\{x\} < 1 \quad \text{and} \quad (\lfloor x\rfloor - 2)(\lfloor x\rfloor - 3) \leq 0 \\ \iff & 0 < x + 2\{x\} < 1 \quad \text{and} \quad \lfloor x\rfloor \in [2, 3] \end{aligned}$$

From the graph it is clear that this region is empty. Thus the domain of the given function is $(-1/3, 0) \cup (1/3, 1) \cup (1, \infty)$. ■

Illustration 12.3.3 Find the range of the function h given by

$$h(x) = \left\lfloor \ln \frac{x}{e} \right\rfloor + \left\lfloor \ln \frac{e}{x} \right\rfloor$$

Solution. We start by simplifying the arguments inside the floor functions using the quotient rule for logarithms:

$$\ln(x/e) = \ln x - \ln e = \ln x - 1 \quad \text{and} \quad \ln(e/x) = \ln e - \ln x = 1 - \ln x$$

Substituting these back into the function definition:

$$h(x) = \lfloor \ln x - 1 \rfloor + \lfloor 1 - \ln x \rfloor$$

Using the property that integers can be pulled out of the floor function, we see that

$$h(x) = \lfloor \ln x \rfloor + \lfloor -\ln x \rfloor$$

Since $\ln x$ ranges over all real numbers as x ranges over positive reals, we are thus asking the image of the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(t) = \lfloor t \rfloor + \lfloor -t \rfloor, \text{ for all } t \in \mathbb{R}$$

But the expression $\lfloor t \rfloor + \lfloor -t \rfloor$ takes only two values:

$$\lfloor t \rfloor + \lfloor -t \rfloor = \begin{cases} 0 & \text{if } t \text{ is an integer} \\ -1 & \text{if } t \text{ is not an integer} \end{cases}$$

Thus, the range of the function is simply $\{-1, 0\}$. ■

Illustration 12.3.4 Range of the function f given by

$$f(x) = \log_2\left(\frac{4}{\sqrt{x+2} + \sqrt{2-x}}\right)$$

is ____________.

Solution. Let $g(x) = \sqrt{x+2} + \sqrt{2-x}$ so that $f(x) = \log_2(4/g(x))$. The domain of g is $[-2, 2]$. To find the range of $f(x) = \log_2(4/g(x))$, we first find the range of $g(x)$. Using the properties of square functions:

$$g(x)^2 = (x+2) + (2-x) + 2\sqrt{(x+2)(2-x)} = 4 + 2\sqrt{4-x^2}$$

As x varies in $[-2, 2]$, $4 - x^2$ varies from 0 to 4. Thus, $g(x)^2$ varies from $4 + 2(0) = 4$ to $4 + 2\sqrt{4} = 8$. Keeping in mind that g takes only non-negative values, we deduce that the range of g is $[2, \sqrt{8}] = [2, 2\sqrt{2}]$. Thus the range of $4/(g(x))$ is this $[\sqrt{2}, 2]$. Finally, the range of $f(x)$ is $[\log_2 \sqrt{2}, \log_2 2] = [1/2, 1]$. ■

Illustration 12.3.5 Consider the two equations **[CMI 2021 Part A]**

$$\log_{2021} a + a = 2022 \quad \cdots (1) \qquad \text{and} \qquad 2021^b + b = 2022 \quad \cdots (2)$$

Then

(a) Equation (1) has a unique solution.

(b) Equation (2) has a unique solution.

(c) There exists a solution a for (1) and a solution b for (2) such that $a = b$.

(d) There exists a solution a for (1) and a solution b for (2) such that $a + b$ is an integer.

Solution. A rigorous argument would need the machinery of calculus. Here we draw the graph (Figure 12.2) and leave the reader to convince themselves based on the geometry of the sketches.

The first equation is $\log_{2021} a = 2022 - a$. Thus we are looking for the intersection of the graphs of the functions

$$x \mapsto \log_{2021} x : (0, \infty) \to \mathbb{R} \quad \text{and} \quad x \mapsto 2022 - x : (0, \infty) \to \mathbb{R}$$

The second equation is $2021^b = 2022 - b$. So we are looking for the intersections of graphs of

$$x \mapsto 2021^x : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad x \mapsto 2022 - x : \mathbb{R} \to \mathbb{R}$$

These graphs show that (a) and (b) are true. Graph cannot be used to address statement (d), and hence we resort to algebra. Let a_0 be the solution to (1):

$$\log_{2021} a_0 = 2022 - a_0 \qquad \Rightarrow \qquad a_0 = 2021^{2022 - a_0}$$

Using this, it is a simple check that $b = 2022 - a_0$ satisfies the second equation and hence statement (d) is also correct. ∎

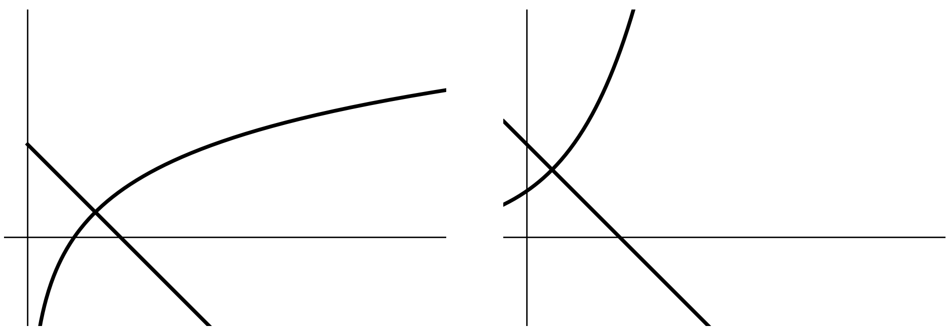


Figure 12.2

Illustration 12.3.6 Find the greatest value of the expression

$$\log_2^4 x + 12 \log_2^2 x \log_2 \frac{8}{x}$$

when x varies between 1 and 64.

Solution. We wish to maximize Let $t = \log_2 x$. Since $1 \le x \le 64$, we have $0 \le t \le 6$. Noting that

$$\log(8/x) = \log 8 - \log x = 3 - t$$

we rewrite the original expression as a function of t:

$$E(t) = t^4 + 12t^2(3-t) = t^4 - 12t^3 + 36t^2$$

This expression is a perfect square:

$$E(t) = (t^2 - 6t)^2$$

Let $g(t) = t^2 - 6t$. For $t \in [0,6]$, the vertex of this parabola is at $t = -(-6)/2 = 3$. At the endpoints $t = 0$ and $t = 6$, $g(t) = 0$. At the vertex $t = 3$, $g(3) = 9 - 18 = -9$. Thus, the range of $g(t)$ for $t \in [0,6]$ is $[-9,0]$. Squaring this range to find the values of $E(t)$, we get $[0,81]$. The greatest value is therefore 81. ■

Exercise 12.3.1. Find the number of integral values of x in the domain of function f defined as

$$f(x) = \sqrt{\ln|\ln|x||} + \sqrt{7|x| - |x|^2 - 10}$$

Exercise 12.3.2. The domain of function f given by

$$f(x) = \log_{\lfloor x+\frac{1}{2} \rfloor}\left(2x^2 + x - 1\right)$$

is: (a) $\left[\frac{3}{2}, \infty\right)$ (b) $(2,\infty)$ (c) $\left(-\frac{1}{2}, \infty\right) - \left\{\frac{1}{2}\right\}$ (d) $\left(\frac{1}{2}, 1\right) \cup (1,\infty)$

Exercise 12.3.3. Find the domain and range of the function f given by

$$f(x) = \frac{(\ln x)\left(\ln x^2\right) + \ln x^3 + 3}{\ln^2 x + \ln x^2 + 2}$$

Exercise 12.3.4. Consider the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : (0,\infty) \to \mathbb{R}$ defined as

$$f(x) = x \quad \text{and} \quad g(x) = 2 + \log_e x$$

The graphs of the functions intersect:

(a) once in $(0,1)$ and never in $(1,\infty)$ (b) once in $(0,1)$ and once in $\left(e^2, \infty\right)$
(c) once in (0,1) and once in (e, e^2) (d) more than twice in $(0,\infty)$

Exercise 12.3.5. The number of solutions of the equation $e^x - \log|x| = 0$ is :

(a) 0 (b) 1 (c) 2 (d) 3 [(b)]

12.4 EQUATIONS

Before we begin, we remind the reader of the message in Section 4.5.

Illustration 12.4.1 Suppose

$$\log_2(\log_8 x) = \log_8(\log_2 x)$$

Find the value(s) of x.

Solution. Using the base change property $\log_8 z = \frac{\log_2 z}{\log_2 8} = \frac{1}{3}\log_2 z$:

$$\begin{aligned}
\log_2(\log_8 x) = \log_8(\log_2 x) \quad &\iff \quad \log_2\left(\frac{1}{3}\log_2 x\right) = \frac{1}{3}\log_2(\log_2 x) \\
&\iff \quad \log_2(\log_2 x) - \log_2 3 = \frac{1}{3}\log_2(\log_2 x) \\
&\iff \quad \log_2(\log_2 x) = \frac{3}{2}\log_2 3 = \log_2(3\sqrt{3}) \\
&\iff \quad \log_2 x = 3\sqrt{3}
\end{aligned}$$

Thus the solution to the given equation is $x = 2^{3\sqrt{3}}$. ■

Illustration 12.4.2 Solve for x:

$$20\log_{4x}\sqrt{x} + 7\log_{16x} x^3 - 3\log_{x/2} x^2 = 0$$

Solution. We first simplify each term by converting them to base x using the identity $\log_a b = \frac{1}{\log_b a}$. For the equation to be defined, we must have $x > 0$, $x \neq 1$, $x \neq 1/4$, $x \neq 1/16$, and $x \neq 2$. Applying base change and power rules:

- $20\log_{4x}\sqrt{x} = 20 \cdot \dfrac{1/2}{\log_x(4x)} = \dfrac{10}{\log_x 4 + 1}$
- $7\log_{16x} x^3 = 7 \cdot \dfrac{3}{\log_x(16x)} = \dfrac{21}{\log_x 16 + 1} = \dfrac{21}{2\log_x 4 + 1}$ (Note: $16 = 4^2$)
- $3\log_{x/2} x^2 = 3 \cdot \dfrac{2}{\log_x(x/2)} = \dfrac{6}{1 - \log_x 2} = \dfrac{6}{1 - \frac{1}{2}\log_x 4} = \dfrac{12}{2 - \log_x 4}$

Let $t = \log_x 4$. The equation becomes:

$$\frac{10}{t+1} + \frac{21}{2t+1} - \frac{12}{2-t} = 0$$

Multiplying by the common denominator $(t+1)(2t+1)(2-t)$ and simplifying:

$$13t^2 - 3t - 10 = 0 \qquad \Rightarrow \qquad (13t+10)(t-1) = 0$$

Thus we have two possible values of t:

$$t = 1 \qquad \Rightarrow \qquad \log_x 4 = 1 \qquad \Rightarrow \qquad x = 4$$

and

$$\begin{aligned}
t = -10/13 \qquad \Rightarrow \qquad \log_x 4 = -10/13 \qquad &\Rightarrow \qquad x^{-10/13} = 4 \\
&\Rightarrow \qquad x = 4^{-13/10} = 2^{-13/5}
\end{aligned}$$

Both solutions satisfy the initial constraints and hence these are the two solutions of the given equation. ■

Illustration 12.4.3 Solve for x:

$$(x+1)\log_3^2 x + 4x\log_3 x - 16 = 0$$

Solution. The given equation is a quadratic in terms of $u = \log_3 x$ with coefficients that are functions of x:

$$(x+1)u^2 + (4x)u - 16 = 0$$

Using the quadratic formula for u:

$$u = \frac{-4x \pm \sqrt{(4x)^2 - 4(x+1)(-16)}}{2(x+1)} = \frac{-4x \pm \sqrt{16x^2+64x+64}}{2(x+1)}$$

which gives

$$u = \frac{-4x \pm \sqrt{16(x+2)^2}}{2(x+1)} = \frac{-4x \pm 4(x+2)}{2(x+1)}$$

Using the solution with the positive sign, we get

$$u = \frac{-4x+4x+8}{2x+2} = \frac{4}{x+1} \qquad \Rightarrow \qquad \log_3 x = \frac{4}{x+1}$$

The function $\log_3 x$ is an increasing function and the function $\frac{4}{x+1}$ is a decreasing function as x ranges over the positive reals. By inspection, $x = 3$ gives $f(3) = 1$ and $g(3) = 1$. Thus $x = 3$ is the unique solution for this case.

Now for the solution with negative sign:

$$\begin{aligned} u = \frac{-4x-4x-8}{2x+2} = \frac{-8x-8}{2x+2} = -4 \qquad &\Rightarrow \qquad \log_3 x = -4 \\ &\Rightarrow \qquad x = 3^{-4} = 1/81 \end{aligned}$$

The solutions are $x = 3$ and $x = 1/81$. ■

Illustration 12.4.4 Show that the equation

$$\log_2(2x^2) + \log_2 x \cdot x^{\log_x(\log_2 x+1)} + \frac{1}{2}\log_4^2(x^4) + 2^{-3\log_{1/2}(\log_2 x)} = 1$$

has no solutions.

Solution. Let $u = \log_2 x$. The expression requires $x > 0$ and $u > 0$ for the $\log_2(\log_2 x)$ term to be defined. We simplify each term:

- $\log_2(2x^2) = 1 + 2u$.
- $x^{\log_x(u+1)} = u + 1$. Thus $\log_2 x \cdot (u+1) = u(u+1) = u^2 + u$.
- $\frac{1}{2}\log_4^2(x^4) = \frac{1}{2}(\log_4 x^4)^2 = \frac{1}{2}(2u)^2 = 2u^2$.

- $2^{-3\log_{1/2}(u)} = 2^{3\log_2(u)} = u^3$.

The equation becomes:

$$1 + 2u + u^2 + u + 2u^2 + u^3 = 1 \quad \Rightarrow \quad u^3 + 3u^2 + 3u = 0$$
$$\Rightarrow \quad u(u^2 + 3u + 3) = 0$$

Since $u > 0$ is required for the expression to be defined, $u = 0$ is excluded. The quadratic $u^2 + 3u + 3$ has discriminant $9 - 12 = -3$, so it has no real roots. Thus, the equation has no solutions. ■

Illustration 12.4.5 Let α and β be the roots of

$$\log^2\left(1 + \frac{4}{x}\right) + \log^2\left(1 - \frac{4}{x+4}\right) = 2\log^2\left(\frac{2}{x-1} - 1\right)$$

then find the value of $\alpha^2 + \beta^2$.

Solution. After elementary algebraic manipulations, the given equation becomes

$$\log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

which is same as

$$2\log^2\left(\frac{x+4}{x}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

where we used $\log(a/b) = -\log(b/a)$ to modify the left hand side. The equation reduces to:

$$\log^2\left(\frac{x+4}{x}\right) = \log^2\left(\frac{3-x}{x-1}\right) \quad \Rightarrow \quad \log\left(\frac{x+4}{x}\right) = \pm\log\left(\frac{3-x}{x-1}\right)$$

Taking the positive sign, we get

$$\frac{x+4}{x} = \frac{3-x}{x-1} \quad \Rightarrow \quad (x+4)(x-1) = x(3-x) \quad \Rightarrow \quad x^2 = 2$$

Using the negative sign we get

$$\frac{x+4}{x} = \frac{x-1}{3-x} \quad \Rightarrow \quad (x+4)(3-x) = x(x-1) \quad \Rightarrow \quad x^2 = 6$$

Checking domain constraints, we find the roots α, β. From $x^2 = 2$ and $x^2 = 6$, the sum $\alpha^2 + \beta^2 = 2 + 6 = 8$. ■

Illustration 12.4.6 Prove that solution of the equation

$$2\log_9\left(2\left(\frac{1}{2}\right)^x - 1\right) = \log_{27}\left(\left(\frac{1}{4}\right)^x - 4\right)^3$$

is an irrational number.

Solution. Using base change to base 3, the given equation becomes

$$2\log_{3^2}(2\cdot 2^{-x}-1)=\log_{3^3}(2^{-2x}-4)^3 \quad\Rightarrow\quad \frac{2}{2}\log_3(2\cdot 2^{-x}-1)=\frac{3}{3}\log_3(2^{-2x}-4)$$
$$\Rightarrow\quad 2\cdot 2^{-x}-1=2^{-2x}-4$$

Writing t in place of 2^{-x}, we get

$$t^2-2t-3=0 \quad\Rightarrow\quad (t-3)(t+1)=0$$

Since $t=2^{-x}>0$, we have $t=3$. Thus

$$2^{-x}=3 \quad\Rightarrow\quad -x=\log_2 3 \quad\Rightarrow\quad x=-\log_2 3$$

So $-\log_2 3$ is the only solution to the equation, which is an irrational number. ■

Illustration 12.4.7 Solve for x:

$$3^x=10-\log_2 x$$

Solution. We solve $3^x=10-\log_2 x$. Let $f(x)=3^x$ and $g(x)=10-\log_2 x$. For $x>0$, $f(x)$ is a strictly increasing function, while $3-\log_2 x$ is strictly decreasing. A strictly increasing function and a strictly decreasing function can intersect at most once. By inspection, let's test small integers: If $x=2$: $3^2=9$ and $10-\log_2 2=10-1=9$. Since $f(2)=g(2)$, $x=2$ is the unique solution. ■

Exercise 12.4.1. Solve for x:

$$\left(\log_{1/\sqrt{1+x}} 10\right)\log_{10}\left(x^2-3x+2\right)=(\log_{10}(x-3))\log_{1/\sqrt{1+x}} 10-2$$

Exercise 12.4.2. Solve for x:

$$\log_x\left(x^2+1\right)=\sqrt{\log_{\sqrt{x}}\left(x^2\left(1+x^2\right)\right)+4}$$

;

Exercise 12.4.3. Solve for x:

$$\log_4(6+\sqrt{x}-|\sqrt{x}-2|)=\frac{1}{2}+\log_2|\sqrt{x}-|\sqrt{x}-2||$$

Exercise 12.4.4. Solve the system of equations:

$$\log_{12} x\left(\frac{1}{\log_x 2}+\log_2 y\right)=\log_2 x \quad\text{and}\quad \log_2 x\cdot(\log_3(x+y))=3\log_3 x$$

Exercise 12.4.5. Find the sum of the roots of the equation

$$(x+1) = 2\log_2\left(2^x+3\right) - 2\log_4\left(1980 - 2^{-x}\right)$$

Exercise 12.4.6. Find the sum of all integral solution of the equation

$$4\log_{x/2}(\sqrt{x}) + 2\log_{4x}\left(x^2\right) = 3\log_{2x}\left(x^3\right)$$

Exercise 12.4.7. Solve for x:

$$|3-x|^{\log^2 x - \log x^2} = |3-x|^3$$

Exercise 12.4.8. Solve for x:

$$|\log_2(3x-1) - \log_2 3| = |\log_2(5-2x) - 1|$$

Exercise 12.4.9. Solve for x:

$$\log_2\log_3\left(x^2-16\right) - \log_{1/2}\log_{1/3}\frac{1}{x^2-16} = 2$$

Exercise 12.4.10. Solve for x:

$$20\log_{4x}\sqrt{x} + 7\log_{16x}x^3 - 3\log_{x/2}x^2 = 0$$

Exercise 12.4.11. Solve for x:

$$\log_{\frac{1}{2+|x|}}\left(5+x^2\right) = \log_{3+x^2}(15+\sqrt{x})$$

Exercise 12.4.12. Solve for x:

$$(1+x/2)\log_2 3 - \log_2\left(3^x-13\right) = 3\log_{\sqrt{5}/25}5 + 4$$

Exercise 12.4.13. Solve for x:

$$\log_{10}2x + \log_{10}x^2 = \log_{10}^2 2 - 1$$

Exercise 12.4.14. Solve for x:

$$9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$$

Exercise 12.4.15. Solve for x:

$$\frac{x^{\frac{1}{2}\log_2 x}}{4} = 2^{\frac{\log_2 \frac{2}{2}x}{4}}$$

Exercise 12.4.16. Solve for x:

$$x^{\log x+5} = 10^{15+3\log x}$$

Exercise 12.4.17. Solve for x:

$$\left|1 - \log_{1/5} x\right| + 2 = \left|3 - \log_{1/5} x\right|$$

Exercise 12.4.18. The sum of all the solutions of **[UGA 2021]**

$$2 + \log_2(x-2) = \log_{(x-2)} 8$$

in the interval $(2, \infty)$ is (a) $\frac{35}{8}$ (b) 5 (c) $\frac{49}{8}$ (d) $\frac{55}{8}$

12.5 INEQUALITIES

Illustration 12.5.1 Find the set of all the values of x such that

$$\frac{\log_2\left(4x^2 - x - 1\right)}{\log_2\left(x^2+1\right)} > 1$$

Solution. First, we establish the domain constraints:

- $4x^2 - x - 1 > 0 \quad \Rightarrow \quad x \in \left(-\infty, \frac{1-\sqrt{17}}{8}\right) \cup \left(\frac{1+\sqrt{17}}{8}, \infty\right)$.
- $x^2 + 1 > 0$ (always true for $x \in \mathbb{R}$).
- $\log_2(x^2+1) \neq 0 \quad \Rightarrow \quad x^2 + 1 \neq 1 \quad \Rightarrow \quad x \neq 0$.

Now since $x^2 + 1 \geq 1$ for all x, we know that $\log_2(x^2+1)$ is non-negative, and hence we can multiply by $\log_2(x^2+1)$ on both sides without changing the sign of the inequality. So the given inequality is equivalent to

$$\log_2(4x^2 - x - 1) > \log_2(x^2+1) \quad \Rightarrow \quad 4x^2 - x - 1 > x^2 + 1$$

which gives

$$4x^2 - x - 1 > x^2 + 1 \quad \Rightarrow \quad 3x^2 - x - 2 > 0 \quad \Rightarrow \quad (3x+2)(x-1) > 0$$

This yields $x \in (-\infty, -2/3) \cup (1, \infty)$. Intersecting this with the domain constraints $4x^2 - x - 1 > 0$, we find that the set

$$(-\infty, -2/3) \cup (1, \infty)$$

is the set of all the solutions of the given inequality. ■

Illustration 12.5.2 Find the set of all the values of x such that

$$\log_{\frac{1}{x}}\left(\frac{2(x-2)}{(x+1)(x-5)}\right) \geq 1$$

Solution. The domain constraints are that the argument of the logarithm and the base should both be positive, and the base should be different from 1. Thus we have $x > 0, x \neq 1$ and

$$\frac{2(x-2)}{(x+1)(x-5)} > 0$$

The wavy curve on the argument shows that the solution to the above inequality is precisely the set

$$(-1, 2) \cup (5, \infty)$$

Thus the domain of the given expression is

$$(0, 1) \cup (1, 2) \cup (5, \infty)$$

To solve the given inequality, we consider cases. First suppose that the base is between 0 and 1. In this case we have $x > 1$. The inequality reverses to give

$$\begin{aligned}
\frac{2(x-2)}{(x+1)(x-5)} \leq \frac{1}{x} \qquad &\Longleftrightarrow \qquad \frac{2x^2-4x-(x^2-4x-5)}{x(x+1)(x-5)} \leq 0 \\
&\Longleftrightarrow \qquad \frac{x^2+5}{x(x+1)(x-5)} \leq 0
\end{aligned}$$

Keeping in mind that $x > 1$, this gives $(1, 2)$ as the solution. Now suppose that the base exceeds 1. In this case $0 < x < 1$. The given inequality yields

$$\frac{x^2+5}{x(x+1)(x-5)} \geq 0$$

which has no solutions in the range $(0, 1)$. Final Solution: $(1, 2)$. ■

Illustration 12.5.3 Find the set of all the values of x such that

$$\sqrt{\log_3(9x-3)} \leq \log_3\left(x - \frac{1}{3}\right)$$

Solution. Let $\log_3(x - 1/3) = t$. Then

$$\log_3(9x-3) = \log_3(9(x-1/3)) = \log_3 9 + \log_3(x-1/3) = 2 + t$$

The inequality in terms of t is therefore $\sqrt{2+t} \leq t$, whose solution set is $(2, \infty)$. Thus

$$\log_3(x-1/3) \geq 2 \qquad \Rightarrow \qquad x - 1/3 \geq 9 \qquad \Rightarrow \qquad x \geq 28/3$$

So the solution set is $(28/3, \infty)$. ■

Illustration 12.5.4 Find the set of all the values of x such that

$$\log_{1/2}(\sqrt{5-x}-x+1) > -3$$

Solution. The domain constraints are that $5-x \geq 0$ and, the argument $\sqrt{5-x}-x+1$ of the logarithm is positive. Thus the domain is the set of all those x such that $x \leq 5$ and $\sqrt{5-x} > x-1$. We leave the reader to verify that the domain is $(-\infty, \frac{1+\sqrt{17}}{2})$. Now we have

$$\log_{1/2}(\sqrt{5-x}-x+1) > -3 \qquad \Longleftrightarrow \qquad \sqrt{5-x}-x+1 < (1/2)^{-3} = 8$$

The inequality becomes

$$\sqrt{5-x} < x+7 \tag{*}$$

To solve this, we square both the sides: Since $x+7$ needs to be positive, the above can be solved by squaring both sides:

$$5-x < x^2+14x+49 \qquad \Rightarrow \qquad x^2+15x+44 > 0 \qquad \Rightarrow \qquad (x+11)(x+4) > 0$$

The solution to the last inequality is $x \in (-\infty, -11) \cup (-4, \infty)$. Keeping in mind that $x+7$ should be positive, we get that the solution to (*) is $(-4, \infty)$. Intersection with $x \leq 5$ and the domain constraint, we conclude that the set of all the solutions of the given inequality is $(-4, \frac{1+\sqrt{17}}{2})$. ■

Illustration 12.5.5 Find the set of all the values of x such that

$$\frac{3\log_a x+6}{\log_a^2 x+2} > 1$$

Solution. Write $\log_a x = t$. The inequality becomes

$$\begin{aligned} \frac{3t+6}{t^2+2} > 1 \qquad &\Longleftrightarrow \qquad 3t+6 > t^2+2 \\ &\Longleftrightarrow \qquad t^2-3t-4 < 0 \\ &\Longleftrightarrow \qquad (t-4)(t+1) < 0 \qquad \Longleftrightarrow \qquad t \in (-1, 4) \end{aligned}$$

Thus, $-1 < \log_a x < 4$. The solution depends on a: If $a > 1$ then the solution set is $(1/a, a^4)$ and if $0 < a < 1$ then the solution set is $(a^4, 1/a)$. ■

Illustration 12.5.6 Find the set of all the values of x such that

$$\log_{(x-3)}\left(2\left(x^2-10x+24\right)\right) \geq \log_{(x-3)}\left(x^2-9\right)$$

Solution. The domain constraint is that the base of the logarithm should be positive and not equal to one, and that the arguments of the logarithms should also be positive. These give

$$x-3 > 0, \neq 1 \qquad \Longleftrightarrow \qquad x > 3, x \neq 4$$

and

$$2(x-4)(x-6)>0 \iff x \in (3,4)\cup(6,\infty) \qquad \text{and} \qquad x^2-9>0 \iff x>3$$

Combining these, the domain is $(3,4)\cup(6,\infty)$. To solve the given inequality, we will consider two cases, one where the base exceeds 1 and the second when the base is between 0 and 1.

Case 1: Base exceeds 1. We have $x-3>1$. The given inequality is then equivalent to

$$2x^2-20x+48 \geq x^2-9 \quad \Rightarrow \quad x^2-20x+57 \geq 0 \quad \Rightarrow \quad (x-3)(x-17)\geq 0$$

Intersecting with the domain, this leads to $x \in [17,\infty)$.

Case 2: Base is between 0 and 1. Here we have $0<x-3<1$, that is $3<x<4$. The given inequality, in this region, is equivalent to

$$x^2-20x+57 \leq 0 \qquad \iff \qquad x \in [3,17]$$

Overlapping with the domain, we get $x \in (3,4)$.

The set of all the solutions is $(3,4)\cup[17,\infty)$. ■

Exercise 12.5.1. Find the set of all the values of x such that

$$\frac{(x-1)^2(x-2)\log(1+x)}{x^3(x-3)(x-4)} \leq 0$$

Exercise 12.5.2. Find the set of all the values of x such that

$$\sqrt{\log^2_{1/2} x + 4\log_2\sqrt{x}} < \sqrt{2}\left(4-\log_{16} x^4\right)$$

Exercise 12.5.3. Find the set of all the values of x such that

$$\log_a(x-1)+\log_a x > 2$$

Exercise 12.5.4. Find the set of all the values of x such that

$$\log_x 2x \leq \sqrt{\log_x\left(2x^3\right)}$$

Exercise 12.5.5. Find the set of all the values of x such that

$$\log_x 2.\log_{2x} 2.\log_2 4x > 1$$

Exercise 12.5.6. Find the set of all the values of x such that

$$\log_3 \frac{|x^2-4x|+3}{x^2+|x-5|} \geq 0$$

Exercise 12.5.7. Find the set of all the values of x such that

$$\frac{1}{\log_3(x+1)} < \frac{1}{2\log_9 \sqrt{x^2+6x+9}}$$

Exercise 12.5.8. Find the set of all the values of x such that

$$\log_a \left(1-8a^{-x}\right) \geq 2(1-x)$$

Exercise 12.5.9. Find the set of all the values of x such that

$$\log_{\frac{2}{3}|x-2|} 2^{1-x^2} \geq 0$$

Exercise 12.5.10. Find the set of all the values of x such that

$$\log_{x^2} \frac{4x-5}{|x-2|} \geq \frac{1}{2}$$

Exercise 12.5.11. Find the set of all the values of x such that

$$\frac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$$

Exercise 12.5.12. Find the set of all the values of x such that

$$\frac{x-1}{\log_3 (9-3^x)-3} \leq 1$$

Exercise 12.5.13. Find the set of all the values of x such that

$$\frac{(x-0.5)(3-x)}{\log_2 |x-1|} > 0$$

Exercise 12.5.14. The set of all the x such that

$$2-\log_2 \left(x^2+3x\right) \geq 0$$

is ____________.

Exercise 12.5.15. Describe the set of all the x such that

$$\log_{1/3}\left(2^{x+2} - 4^x\right) \geq -2$$

Exercise 12.5.16. Find all the x such that

$$\log_{0.6}\left(\log_6\left(\frac{x^2+x}{x+4}\right)\right) < 0$$

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