Chris Kelliher

Quantitative Finance with Python

A Practical Guide to
Investment Management,
Trading, and Financial Engineering

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To my amazing daughter, Sloane, my light and purpose.
To my wonderful, loving wife, Andrea, without whom none of my achievements would be possible.
To my incredible, supportive parents and sister and brother, Jen and Lucas.

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Foreword

In March 2018, the Federal Reserve ("Fed") was in the midst of its first hiking cycle in over a decade, and the European Central Bank ("ECB"), still reeling from the Eurozone debt crisis, continued to charge investors for the privilege of borrowing money. US sovereign bonds ("Treasuries") were yielding 3% over their German counterparts ("Bunds"), an all-time high, and unconventional monetary policy from the two central banks had pushed the cost of protection to an all-time low.

Meanwhile, across the pond, a sophisticated Canadian pension flipped a rather esoteric coin: A so-called digital put-on Euro/Dollar, a currency pair that trades over a trillion dollars a day. On this crisp winter morning, the EURUSD exchange rate ("spot") was 1.2500. If the flip resulted in heads and spot ended below 1.2500 in 2 years, the pension would receive $10 million. If the flip were tails and spot ended above 1.2500, the pension would have to pay $2.5 million. Naturally, the 4 to 1 asymmetry in the payout suggests that the odds of heads were only 25%. Interestingly, the flip yielded heads, and in 2 years, spot was below 1.2500.

After the trade, I called Chris, reiterated the pitch, and explained that since January 1999, when EURUSD first started trading, the market implied odds of heads had never been lower. As macroeconomic analysis and empirical realizations suggest that the coin is fair, and there is about 50% chance of getting heads, should the client perhaps consider trading the digital put in 10x the size? In his quintessentially measured manner, Chris noted, "We must have a repeatable experiment to isolate a statistical edge". Ten separate flips, for instance, could reduce the risk by over 2/3. Moreover, negative bund yields, which guarantee that investors will lose money, incentivize capital flows to Treasuries, and the anomalous rates "carry is a well-rewarded risk premium". Furthermore, as "investors value $1 in risk-off more than $1 in risk-on", does the limited upside in the payout of the digital put also harness a well-rewarded tail risk premium?

I wish I were surprised by Chris's nuance, or objectivity, or spontaneity. Having known him for 8 years, though, I have come to realize that he is the most gifted quant I have had the privilege to work with, and this book is a testament to his ability to break complex ideas down to first principles, even in the treatment of the most complex financial theory. The balance between rigor and intuition is masterful, and the textbook is essential reading for graduate students who aspire to work in investment management. Further, the depth of the material in each chapter makes this book indispensable for derivative traders and financial engineers at investment banks, and for quantitative portfolio managers at pensions, insurers, hedge funds and mutual funds. Lastly, the "investment perspectives" and case studies make this an invaluable guide for practitioners structuring overlays, hedges and absolute return strategies in fixed income, credit, equities, currencies and commodities.

In writing this book, Chris has also made a concerted effort to acknowledge that markets are not about what is true, but rather what can be true, and when: With negative yields, will Bunds decay to near zero in many years? If so, will $1 invested in Treasuries, compound and buy all Bunds in the distant future? Or will the inflation differential between the Eurozone and the US lead to a secular decline in the purchasing power of $1? One may conjecture that no intelligent investor will buy perpetual Bunds with a negative yield. However, even if the Bund yield in the distant future is positive, but less than the Treasury yield, market implied odds of heads, for a perpetual flip, must be zero. As the price of the perpetual digital put is zero, *must* the intelligent investor add this option to her portfolio?

Since the global financial crisis, the search for yield has increasingly pushed investors down the risk spectrum, and negative interest rates, and unconventional monetary policy, are likely just the tip of the iceberg. This book recognizes that unlike physics, finance has no universal laws, and an asset manager must develop an investment philosophy to navigate the known knowns, known unknowns and unknown unknowns. To allow a portfolio manager to see the world as it was, and as it can be, this book balances the traditional investment finance topics with the more innovative quant techniques, such as machine learning. Our hope is that the principles in this book transcend the outcome of the perpetual flip.

– *Tushar Arora*

Author

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I

Foundations of Quant Modeling

CHAPTER 1

Setting the Stage: Quant Landscape

1.1 INTRODUCTION

QUANTITATIVE finance and investment management are broad fields that are still evolving rapidly. Technological and modeling innovations over the last decades have led to a series of fundamental shifts in the industry that are likely just beginning. This creates challenges and opportunities for aspiring quants who are, for example, able to harness the new data sources, apply new machine learning techniques or leverage cutting edge derivative models. The landscape of these fields is also fairly broad, ranging from solving investment problems related to optimal retirement outcomes for investors to providing value to firms by helping them hedge certain undesirable exposures.

In spite of the innovations, however, quants must remember that models are by definition approximations of the world. As George Box famously said "All Models are Wrong, Some are Useful", and this is something all quants should take to heart. When applying a model, the knowledge of its strengths, weaknesses and limitations must be at the forefront of our mind. Exacerbating this phenomenon is the fact that finance, unlike hard sciences like Physics or Chemistry, does not allow for repeatable experiments. In fact, in finance we don't even know for sure that consecutive experiments come from the same underlying distribution because of the possibility of a regime change. Making this even more challenging is the potential feedback loop created by the presence of human behavior and psychology in the market. This is a key differentiating factor relative to other, harder, sciences. As a result, quants should aim to make models that are parsimonious and only as complex as the situation requires, and to be aware and transparent about the limitations and assumptions[1].

The goal of this book is to bridge the gap between theory and practice in this world. This book is designed to help the reader understand the underlying financial theory, but also to become more fluent in applying these concepts in Python. To achieve this, there will be a supplementary coding repository with a bevy of practical

[1]Along these lines, Wilmott and Derman have created a so-called modeling manifesto designed to emphasize the differences between models and actual, traded markets. [151]

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applications and Python code designed to build the reader's intuition and provide a coding baseline.

In this chapter we aim to orient the reader to the landscape of quant finance. In doing so, we provide an overview of what types of firms make up the financial world, what some common quant careers tend to look like, and what the instruments are that quants are asked to model. In later chapters, we discuss the specifics of modeling each instrument mentioned here, and many of the modeling techniques that are central to the quant careers discussed. Before diving into the modeling techniques, however, our goal is to provide more context around why the techniques and instruments matter and how they fit into the larger picture of quant finance and investment management.

We then discuss what a typical quant project looks like, with an emphasis on what is common across all roles and seats at different organizations. Of critical importance is not only being able to understand the models and the mathematical theory, but also being fluent with the corresponding data and having the required tools to validate the models being created. Lastly, we try to provide the reader with some perspective on trends in the industry. Unlike other less dynamic and more mature fields, finance is still rapidly changing as new data and tools become available. Against this backdrop, the author provides some perspective on what skills might be most valuable going forward, and what the industry might look like in the future.

1.2 QUANT FINANCE INSTITUTIONS

The fields of finance and investment management contain many interconnected players that serve investors in different ways. For some, such as buy-side hedge funds and asset managers, their main function is to generate investments with positive returns and create products that are attractive to investors. For others, such as dealers, market makers and other sell-side institutions, their main function is to match buyers and sellers in different markets and create customized derivative structures that can help their clients hedge their underlying risks. Lastly, the field includes a lot of additional service providers. These providers may provide access to data, including innovative data sources, or an analytics platform for helping other institutions with common quant calculations. This space includes financial technology companies, whose primary role is to leverage technology and new data sources to create applications or signals that can be leveraged by buy or sell-side institutions to make more efficient decisions. In the remainder of this section we provide the reader some context on what functions these organizations provide. We then proceed to discuss where quants might fit in at these various entities in the following section.

1.2.1 Sell-Side: Dealers & Market Makers

Sell-side institutions facilitate markets by providing liquidity through making markets and by structuring deals that are customized to meet client demands. On the market making side, sell-side institutions provide liquidity by stepping in as buyers or sellers when needed and subsequently looking to offload the risk that they take on quickly as the market returns to an equilibrium. Market makers are compensated for their liquidity provision by collecting the bid-offer spread on these transactions. In many markets, market makers use automated execution algorithms to make markets. In other cases, traders may fill this role by manually surveying the available order stack and stepping in when the order book becomes skewed toward buys or sells respectively.

Dealers also facilitate markets by creating structures for clients that help them to hedge their risks, or instrument a macroeconomic view, efficiently. This often involves creating derivative structures and exotic options that fit a clients needs. When doing this, a dealer may seek another client willing to take on the offsetting trade that eliminates the risk of the structure completely. In other cases, the dealer/market maker may warehouse the risk internally, or may choose to hedge certain sensitivities of the underlying structure, but allow certain other sensitivities to remain on their books. We will discuss this hedging process for various derivative structures in more detail in chapter 11 .

1.2.2 Buy-Side: Asset Managers & Hedge Funds

Buy-side institutions, such as hedge funds and asset managers, are responsible for managing assets on behalf of their clients. The main goal of a buy-side firm is to deliver strong investment returns. To this end, hedge funds and asset managers may employ many different types of strategies and investment philosophies, ranging from purely discretionary to purely systematic. For example, hedge funds and asset managers may pursue the following types of strategies:

Global Macro: Global Macro strategies may be discretionary in nature, where views are generated based on a portfolio manager's assessment of economic conditions, or systematic, where positions are based on quantitative signals that are linked to macroeconomic variables.

Relative Value: Relative value strategies try to identify inconsistencies in the pricing of related instruments and profit from their expected convergence. These relative value strategies may be within a certain asset class, or may try to capture relative value between two asset classes, and may be pursued via a discretionary approach, or systematically.

Event Driven: Event driven strategies try to profit from upcoming corporate events, such are mergers or acquisitions. These strategies tend to bet on whether these transactions will be completed, benefiting from subsequently adjusted valuations.

Risk Premia: Risk premia strategies try to identify risks that are well-rewarded and harvest them through consistent, isolated exposure to the premia. Common risk premia strategies include carry, value, volatility, quality and momentum. Some of these risk premia strategies are discussed in chapter 20.

Statistical Arbitrage: Statistical arbitrage strategies are a quantitative form of relative value strategies where a quantitative model is used to identify anomalies between assets, for example, through a factor model. In other cases, a pairs trading approach may be used, where we bet on convergence of highly correlated stocks that have recently diverged. These types of statistical arbitrage models are discussed further in detail in chapter 20.

A key differentiating factor between hedge funds and asset managers is the level and structure of their fees. Hedge funds generally charge significantly higher fees, and have sizable fees linked to their funds performance. Additionally, most hedge funds are so-called absolute return funds, meaning that their performance is judged in absolute terms. Asset managers, by contrast, often do not collect performance fees and measure their performance relative to benchmark indices with comparable market exposure. Hedge funds also tend to have considerably more freedom in the instruments and structures that they can trade, and require less transparency[2].

1.2.3 Financial Technology Firms

Generally speaking, financial technology firms leverage data and technology to create products that they can market to buy-side and sell-side institutions. The proliferation of available data over the last decade has led to a large increase in Fin-Tech companies. Many FinTech companies at their core solve big data problems, where they take non-structured data and transform it into a usable format for their clients. As an example, a FinTech company might track traffic on different companies websites and create a summary signal for investment managers. In another context, FinTech companies might leverage technological innovations and cloud computing to provide faster and more accurate methods for pricing complex derivatives. Buy and sell-side institutions, would then purchase these firms services and then incorporate them into their processes, either directly or indirectly.

1.3 MOST COMMON QUANT CAREER PATHS

Aspiring quants may find themselves situated in any of these organizations and following many disparate career paths along the way. In many ways, the organization that a quant chooses will determine what type of modeling skills will be most emphasized. In **buy-side** institutions, such as hedge funds and asset managers, a heavy focus will be placed on econometric and portfolio construction techniques. Conversely, while working on the **sell-side**, at dealers, or investment banks, understanding of stochastic processes may play a larger role. Even within these institutions, a quant's role may vary greatly depending on their group/department. To provide additional context, in the following sections, we briefly describe what the most common quant functions are at buy-side, sell-side and fin-tech companies.

1.3.1 Buy Side

At buy-side institutions, building investment products and delivering investment outcomes are at the core of the business. As such, many quants join these shops, such as asset management firms, pension funds, and hedge funds, and focus on building models that lead to optimal portfolios, alpha signals or proper risk management.

Examples of roles within buy-side institutions include:

Desk Quants: A desk quant sits on a trading floor at a hedge fund or other

[2]Because of this, there are tighter restrictions on who can invest in hedge funds.

buy-side firm and supports portfolio managers through quantitative analysis. The function of this support can vary greatly from institution to institution and may in some cases involve a great deal of forecasting models and have a heavy emphasis on regression methods, machine learning techniques and time series/econometric modeling, such as those described in chapter 3. In other cases, this support may involve more analysis of derivative valuations, and identifying hedging strategies, as discussed in section II of this text. Desk quants may also helps provide portfolio managers with quantitative analysis that supports discretionary trading process.

Asset Management Quants: A quant at an asset manager will have a large emphasis placed on portfolio construction and portfolio optimization techniques. As such, understanding of the theory of optimization, and the various ways to apply it to investment portfolios is a critical skill-set. Asset management quants may also be responsible for building alpha models and other signals, and in doing so will leverage econometric modeling tools. Asset management quants will rely heavily on the material covered in section IV of this text.

Research Quants: Research quants tend to focus more on longer term research projects and try to build new innovative models. These may be relative value models, proprietary alpha signals or innovative portfolio construction techniques. Research quants utilize many of the same skills as other quants, but are more focused on designing proprietary, groundbreaking models rather than providing ad-hoc analysis for portfolio managers.

Quant Developers: Quant developers are responsible for building production models and applications for buy-side shops. In this role, quant developers must be experts in programming, but also have a mastery of the underlying financial theory. Quant Developers will rely on the programming skills described in chapters 4 and 5 and also must be able to leverage the financial theory and models described in the rest of the book.

Quant Portfolio Managers: Generally these quants are given a set risk-budget that they use to deploy quantitative strategies. These roles have a heavy market facing component, but also require an ability to leverage quant tools such as regression and machine learning to build systematic models. As such, these roles also require a strong background in finance in order to understand the dynamics of the market and uncover attractive strategies to run quantitatively. Simplified versions of some of the strategies that might be employed by quantitative portfolio managers are described in more detail in chapter 20.

1.3.2 Sell Side

At sell-side institutions, making markets and structuring products for clients are crucial drivers of success, and quants at these institutions can play a large role in both of these pursuits. Many quants join sell-side shops and are responsible for creating automated execution algorithms that help the firm make markets. Other quants may be responsible for helping build customized derivative products catered to clients hedging needs. This process is commonly referred to on the sell-side as structuring. Examples of quant roles on the sell-side include:

Desk Quants: Like a buy-side desk quant, a desk quant on the sell-side sits on a trading desk and supports traders and market makers. Sell-side desk quants will often help create structured products that are customized for clients. This may involve creating exotic option payoffs that provide a precise set of desired exposures. It may also involve creating pricing models for exotic options, modeling sensitivities (Greeks) for complex derivatives and building different types of hedging portfolios. Sell-side desk quants will leverage the concepts discussed in section II with a particular emphasis on the exotic option pricing topics discussed in chapter 10 and the hedging topics discussed in chapter 11.

Risk Quants: Risk quants help sell-side institutions measure various forms of risk, such as market risk, counterparty risk, model risk and operational risk. These are often significant roles at banks as they determine their capital ratios and subsequently the cash that banks must hold. Market risk quants are responsible for determining risk limits and designing stress tests. Many of the topics relevant to risk quants are discussed in chapter 19. Risk quants also often need to work with the modeling concepts presented in the rest of the book, such as the time series analysis and derivatives modeling concepts discussed in chapter 3 and section II, respectively.

Model Validation Quants: Many sell-side institutions have separate teams designed to validate newly created production models and production model changes. The quants on these teams, model validation quants, are responsible for understanding the financial theory behind the models, analyzing the assumptions, and independently verifying the results. The principles of model validation are discussed in detail in chapter 7. Additionally, model validation quants gain exposure to the underlying models that they are validating through implementing them independently and the verification process.

Quant Trader/Automated Market Maker: Quant traders and quant market makers are responsible for building automated market making algorithms. These algorithms are used to match buyers and sellers in a highly efficient manner, while collecting the bid-offer spread. Unlike a Quantitative Portfolio manager, a market making quant tends to build higher frequency trading models and will hold risk for very short time periods. This leads to an increased emphasis on coding efficiency and algorithms. The foundations for a Quant Trader's coding background are discussed in more detail in chapter 5, and some examples of the types of trading algorithms that they rely on are presented in chapter 20.

Quant Developer: Much like quant developers on the buy-side, sell-side quant developers are responsible for building scalable, production code leveraged by sell-side institutions. This requires mastery of coding languages like Python, and also strong knowledge of financial theory. In contrast to buy-side quant developers, sell-side quant developers will have a larger emphasis on the options modeling techniques discussed in section II.

1.3.3 Financial Technology

Data Scientist: Financial technology is a burgeoning area of growth within the finance industry and is a natural place for quants. Technological advances have led to a proliferation of data over the last decade, leading to new opportunities for quants to analyze and try to extract signals from the data. At these firms, quants generally serve in data scientist type roles, apply machine learning techniques and solve big data problems. For example, a quant may be responsible for building or applying Natural Language Processing algorithms to try to company press releases and trying to extract a meaningful signal to market to buy-side institutions.

1.3.4 What's Common between Roles?

Regardless of the type of institution a quant ends up in, there are certain common themes in their modeling work and required expertise. In particular, at the heart of the majority of most quant problem is trying to understand the underlying distribution of assets. In chapters 2 and 3 we discuss the core mathematical tools used by buy- and sell-side institutions for understanding these underlying distributions. Additionally, solving quant problems in practice generally involves implementing a numerical approximation to a chosen model, and working with market data. As a result, a strong mastery of coding languages is central to success as a quant.

1.4 TYPES OF FINANCIAL INSTRUMENTS

There are two main types of instruments in financial markets, cash instruments, such as stocks and bonds, and derivatives, whose value is contingent on an underlying asset, such as a stock or a bond.

Analysis of different types of financial instruments requires a potentially different set of quantitative tools. As we will see, in some cases, such as forward contracts, simple replication and no arbitrage arguments will help us to model the instrument. Other times we will need more complex replication arguments, based on dynamically replicated portfolios, such as when valuing options. Lastly, in some circumstances these replication arguments may fail and, we may need to instead invoke the principles of market efficiency and behavioral finance to calculate an expectation.

The following sections briefly describe the main financial instruments that are of interest to quants and financial firms:

1.4.1 Equity Instruments

Equity instruments enable investors to purchase a stake in the future profits of a company. Some equities may pay periodic payments in the form of dividends, whereas others might forego dividends and rely on price appreciation in order to generate returns. Equity instruments may include both public and private companies and arise when companies issue securities (i.e. stocks) to investors. Companies may issue these securities in order to help finance new projects, and in doing so are sharing the future profits of these projects directly with investors. Equity investments embed a significant amount of risk as they are at the bottom of the capital structure, meaning that they are last to be re-paid in the event of a bankruptcy or default event. Because of this, it is natural to think that equity investors would expect to be paid a premium for taking on this risk. Market participants commonly refer to this as the equity risk premium. The equity market also has other equity like products such as exchange traded products and exchange traded notes. Common models for equities are discussed in section IV and the underlying techniques that tend to belie these models are discussed in chapter 3.

1.4.2 Debt Instruments

Governments and private companies often gain access to capital by borrowing money to fund certain projects or expenses. When doing so, they often create a debt security, such as a government or corporate bond. These bonds are the most common type of fixed income or debt instrument, and are structured such that the money is repaid at a certain maturity date. The value that is re-paid at the maturity date is referred to as a bond's principal. In addition, a periodic stream of coupons may be paid between the initiation and maturity dates. In other cases, a bond might not pay coupons but instead have a higher principal relative to its current price.[3]

A bond that does not pay any coupons is referred to as a zero-coupon bond. Pricing debt instruments, such as zero-coupon bonds, relies on present value and time value of money concepts, which are further explored in chapter 3.[4] As an example, the pricing equation for a zero-coupon bond can be written as:

$$V_0 = \exp\left(-yT\right) P \tag{1.1}$$

where P is the bond's principal and V_0 is the current value or price of the bond. Further, y is the yield that is required in order for investors to bear the risks associated with the bond. In the case of a government bond, the primary risk would be that this yield would change, which would change the market value of the bond. This is referred to as interest rate risk. In the case of corporate bonds, another critical risk would be that the underlying corporation might go bankrupt and fail to repay their debt. Investors are likely to require a higher yield, or a lower initial price, in order to withstand this risk of default. It should be noted, however, that debt holders are above equities in the capital structure, meaning they will be re-paid, or partially re-paid, prior to any payment to equity holders. In practice this means that in the event of a default bond holders often receive some payment less than the bond's principal, which is known as the recovery value.

Astute readers may notice that this implies that both equity and corporate bond holders share the same default risk for a particular firm. That is, they are both linked to the same firm, and as a result the same underlying earnings and future cashflows, however, are characterized by different payoffs and are at different places in the firms capitalization structure. This creates a natural relationship between equities and corporate bonds, which we explore further in chapter 14.

There are many different types and variations of bonds within fixed income markets. Some bonds are linked to nominal rates whereas others are adjusted for changes

[3] Assuming positive interest rates.

[4] More detail on these concepts can also be found in [191].

in inflation. Additionally, bonds may contain many other features, such as gradual repayment of principal[5] or have coupons that vary with a certain reference rate. More information is provided on modeling debt instruments in chapter 13.

1.4.3 Forwards & Futures

Forwards and futures are, generally speaking, the simplest derivatives instrument. A forward or futures contract is an agreement to buy or sell a specific asset or security at a predetermined date. Importantly, in a forward or futures contract it is *required* that the security be bought or sold at the predetermined price, regardless of whether it is economically advantageous to the investor. Forwards and futures contracts themselves are quite similar, with the main differentiating factor being that forwards are over-the-counter contracts and futures are exchange traded.

Forward and futures contracts can be used to hedge against price changes in a given asset by locking in the price today. As an example, an investor with foreign currency exposure may choose to hedge that currency risk via a forward contract rather than bear the risk that the currency will move in an adverse way.

The payoff for a long position in a forward contract can be written as:

$$V_0 = S_T - F \tag{1.2}$$

where F is the agreed upon delivery price and S_T is the asset price on the delivery date. Similarly, a short position in a forward contract can be written as:

$$V_0 = F - S_T \tag{1.3}$$

F is typically chosen such that the value of the forward contract when initiating the, V_0, is equal to zero [179].

Replication arguments can be used to find the relationship between spot (current asset) prices and forward prices. To see this, consider the following two portfolios:

- Long Position in Forward Contract
- Borrow S_0 dollars at Risk Free Rate & Buy Asset for S_0 dollars.

The following table summarizes the payoffs of these respective portfolios both at trade initiation ($t = 0$) and expiry ($t = T$):

Time	**Portfolio 1: Long Forward**	**Portfolio 2: Borrow and Long Stock**
0	0	0
T	$S_T - F$	$S_T - S_0 \exp(rT)$

Note that we are assuming a constant interest rate, no transactions costs and that the underlying asset does not pay dividends. As you can see, the value of both of these portfolios at $t = 0$ is zero. The value at expiry in both cases depends on S_T, and the other terms, such as F, are known at trade initiation.

[5] Which is referred to as an Amortizing Bond

These portfolios have the same economics, in that they both provide exposure the underlying asset on the expiry/delivery date. To see this, consider a portfolio that is long portfolio 1 and short portfolio 2. In that case, the payoff for this investor becomes:

$$V_0 = 0 \tag{1.4}$$
$$V_T = F - S_0 \exp(rT) \tag{1.5}$$

This portfolio has zero cost, therefore, in the absence of arbitrage, its payoff must also be equal to zero. This leads to a forward pricing equation of:

$$F = S_0 \exp(rT) \tag{1.6}$$

If the forward prices diverges from this, it can easily be shown that this results in an arbitrage opportunity. For example, if $F - S_0 \exp(rT) > 0$, then we can go long a forward contract, sell the stock at the current price and lend the proceeds. Conversely, if $F - S_0 \exp(rT) > 0$, then we can enter a short position in the forward contract and borrow money to buy the stock at the current price.

It should be emphasized that this replication is **static**. This means that we built a single replicating portfolio and were able to wait until expiry without adjusting our hedge. Later in the book we will use replication arguments to value option payoffs, and in this case **dynamic replication** will be required.

These forward pricing replication arguments can easily extended to include dividends [100]. Incorporation of dividends[6] leads to the following formula for forward pricing:

$$F = S_0 \exp((r - q)T) \tag{1.7}$$

where q is an asset's dividend yield. Under certain assumptions, such as constant or deterministic interest rates, it can also be shown that futures and forward prices will be the same.

Forward contracts are particularly common in foreign exchange markets. Equity and commodity markets, in contrast, have liquid markets for many index futures. In interest rate markets, both forwards and futures contracts are traded, and the role of a stochastic discount factor creates another level of complexity in valuing and differentiating between the two[7]. Futures and forwards contracts in these different markets are discussed in more detail in section III. Additionally, more detailed treatment of forwards and futures contracts can be found in [100].

1.4.4 Options

An option provides an investor with the right, rather than the obligation to buy or sell an asset at a predetermined price on a specified date. This contrasts with a forward contract where that exchange is required. Thus, as the name implies, an option

[6] In the form of a continuous dividend yield.

[7] More details on futures modeling in an interest rate setting, and the convexity correction that arises, can be found in chapter 13

provides the investor with a choice of whether to engage in the transaction depending on whether it is economically favorable[8]. It turns out that this right to choose whether to exercise leads to some interesting and subtle mathematical properties.

The most common options are **call** and **put** options which are often referred to as vanilla options. A call option provides an investor the right to buy a security or asset for a given price at the option's expiry date. Clearly, situations when the asset price at expiry are highest are best for holders of call options.

A put option, conversely, gives an investor the right to sell a security for a prespecified price at expiry. Put options will be most valuable when the asset price at expiry is lowest. The agreed upon price specified in an option is referred to as the **strike price**.

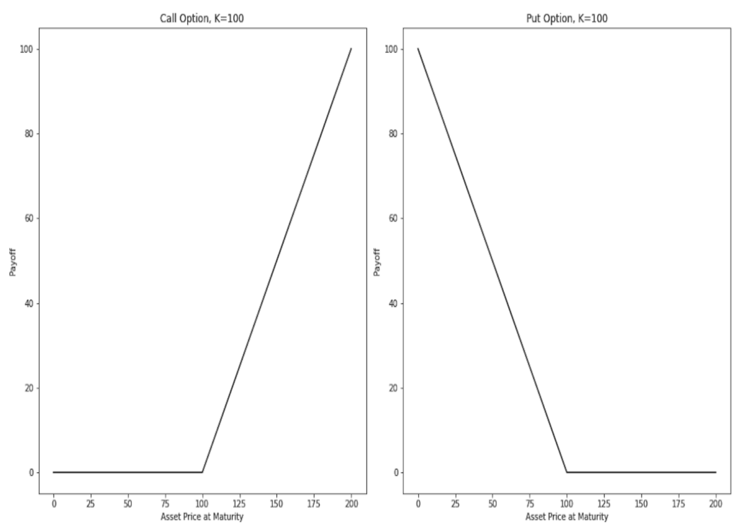
The payoff for a call option, C and put option, P, respectively, can be written as:

$$C = \max(S_T - K, 0) \tag{1.8}$$

$$P = \max(K - S_T, 0) \tag{1.9}$$

where the max in the payoff functions reflects that investors will only exercise their option if it makes sense economically[9]. We can see that a call option payoff looks like a long position in a forward contract when the option is exercised, and will only be exercised if $S_T > K$. Similarly, a put option payoffs are similar to a short position in a forward contract conditional on the option being exercised, which will only happen if $S_T < K$.

In the following chart, we can see what a payoff diagram for a call and put option look like:



[8]This is referred to as an exercise decision

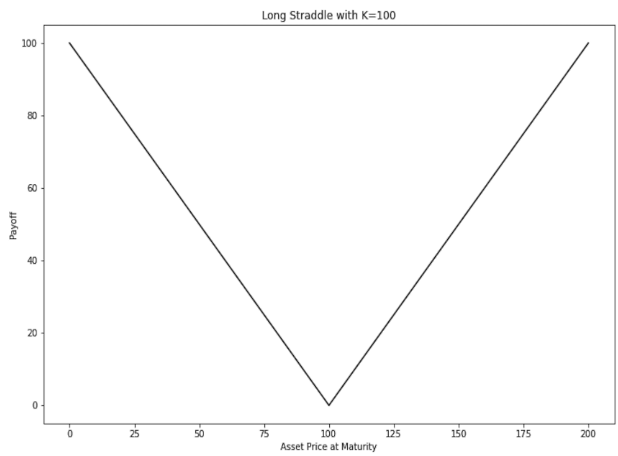
[9]Because the payoff is greater than zero.

As we can see, the payoff of a long position in a call or a put option is always greater than or equal to zero. As a result, in contrast to forwards, where the upfront cost is zero, options will require an upfront payment to receive this potential positive payment in the future. Additionally, we can see that the inclusion of the max function, created by an option holder's right to choose, leads to non-linearity in the payoff function.

A significant portion of this book is dedicated to understanding, modeling and trading these types of options structures. In particular, chapter 2 and section II provide the foundational tools for modeling options.

1.4.5 Option Straddles in Practice

A commonly traded option strategy is a so-called straddle, where we combine a long position in a put option with a long position in a call option at the same strike. In the following chart, we can see the payoff diagram for this options portfolio:



From the payoff diagram, we can see an interesting feature of this strategy is that the payoff is high when the underlying asset moves in either direction. It is also defined by a strictly non-negative payoff, meaning that we should expect to pay a premium for it. At first glance, this appears to be an appealing proposition for investors. We are indifferent in how the asset moves, we just need to wait for it to move in order for us to profit. The truth, however, turns out to be far more nuanced. Nonetheless, straddles are fundamental options structures that we will continue to explore throughout the book, most notably in chapter 11.

1.4.6 Put-Call Parity

A relationship between call and put options can also be established via a static replication argument akin to the one we saw earlier for forward contracts. This relationship is known as put-call parity, and is based on the fact that a long position in a call, combined with a short position in a put, replicates a forward contract in the underlying asset.

To see this, let's consider an investor with the following options portfolio:
Portfolio 1

- Buy a call with strike K and expiry T
- Short a put with strike K and expiry T

The payoff at expiry for an investor in this portfolio can be expressed via the following formula:

$$V_T = \max\left(S_T - K, 0\right) - \max\left(K - S_T, 0\right) \tag{1.10}$$

If S_T is above K at expiry, then the put we sold will expiry worthless, and the payoff will simply be $S_T - K$. Similarly, if S_T expires below K, then the call expires worthless and the payoff is $-\left(K - S_T\right) = S_T - K$. Therefore, the payoff of this portfolio is $V_T = S_T - K$ regardless of the value at expiry.

Now let's consider a second portfolio:
Portfolio 2

- Buy a single unit of stock at the current price, S_0
- Borrows K dollars in the risk-free bond

At time T, this second portfolio will also have a value of $S_T - K$ regardless of the value of S_T. Therefore, these two portfolios have the same terminal value at all times, and consequently must have the same upfront cost in the absence of arbitrage. This means we must have the following relationship:

$$C - P = \left(S_0 - Ke^{-rT}\right) \tag{1.11}$$

where C is the price of a call option with strike K and time to expiry T, P is the price of a put option with the same strike and expiry, S_0 is the current stock price, and r is the risk-free rate. It should be emphasized that the left-hand side of this equation is the cost of portfolio 1 above, a long position in a call and a short position in a put. Similarly, the right hand side is the cost of the second portfolio, a long position in the underlying asset and a short position in the risk-free bond. This formula can be quite useful as it enables us to establish the price of a put option given the price of a call, or vice versa.

1.4.7 Swaps

A swap is an agreement to a periodic exchange of cashflows between two parties. The swap contract specifies the dates of the exchanges of cash flows, and defines the reference variable or variables that determine the cashflow. In contrast to forwards, swaps are characterized by multiple cash flows on different dates.

Swaps define at least one so-called **floating leg**, whose cashflow is dependent on the level, return or change in some underlying reference variable. It may also include a **fixed leg** where the coupon is set at contract initiation. The most common types of swaps are fixed for floating swaps, where one leg is linked to an interest rate, equity return or other market variable, and the fixed leg coupon, is set when the contract is entered. Other swaps may be floating for floating swaps, with each leg referencing a different market variable.

The cashflows for the fixed leg of a swap can be calculated using the present value concepts detailed in chapter 3[10]. In particular, the present value for each leg of a swap is calculated by discounting each cashflow to today. The present value for the cashflows of the fixed leg of a swap can then be written as:

$$\mathrm{PV}(\text{fixed}) = \sum_{i=1}^{N} \delta_{t_i} C D(0, t_i) \tag{1.12}$$

where C is the set fixed coupon and $D(0, t_i)$ is the discount factor from time t_i to today, and δ_{t_i} is the time between payments. The reader may also notice that, because C is a constant set when the trade is entered, it can be moved outside the summation. We can see that the present value of the fixed leg of a swap is the sum of the discounted cashflows weighted by the time interval. While this part may seem trivial, it turns out there is actually some ambiguity in how we calculate the time between payments, δ_i. For example, do we include all days or only business days? Do we assume each month has 30 days or count the actual number of days in each month? In practice a **daycount convention** is specified to help us measure time intervals precisely. The most common daycount conventions are discussed in more detail in chapter 13.

To calculate the present value of the cashflows of a floating leg in a swap we can use the following equation:

$$\mathrm{PV}(\text{float}) = \sum_{i=1}^{N} \delta_{t_i} F_{t_i} D(0, t_i) \tag{1.13}$$

where F_{t_i} is the value of the reference rate, index or return at time t_i. Unlike the fixed leg, this value is not known at trade initiation and requires knowledge of the expected future value of the reference variable[11]. However, in many cases, a forward or futures contract may directly tell us the expected value of the reference variable, $\mathbb{E}\left[F_{t_i}\right]$.

The present value of the swap then becomes:[12]

$$\mathrm{PV} = \mathrm{PV}(\text{float}) - \mathrm{PV}(\text{fixed}) \tag{1.14}$$

[10] For more information see [191]

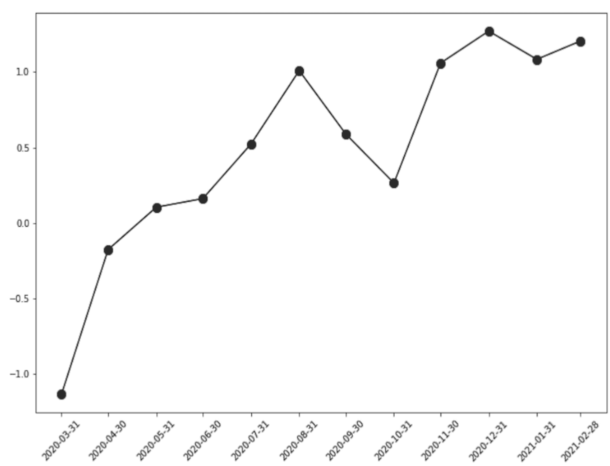
[11] It should be noted that only the expected value of the reference variable is needed to value a swap contract, in contrast to options structures where the entire distribution will be required

[12] Note that this is from the perspective of the buyer of the floating leg.

The most common types of swaps are interest rate swaps and credit default swaps.[13] In interest rate swaps, the most common products are fixed for floating swaps where Libor is the reference rate for the floating leg. In credit default swaps, a fixed coupon is paid regularly in exchange for a payment conditional on default of an underlying entity.[14] In the next sections, we look at an example of finding the fair swap rate for a swap contract and then look at a market driven example: equity index total return swaps.

1.4.8 Equity Index Total Return Swaps in Practice

In this section we leverage the coding example found in the supplementary materials to value a total return swap on the S&P 500 where the return of the index is defined as the floating leg and the fixed leg is a constant, financing leg. The rate for this financing leg is set by market participants via hedging and replication arguments similar to those introduced in this chapter for forwards. In particular, in the following chart we show the cumulative P&L to an investor who receives the floating S&P return and pays the fixed financing cost:



We can see that, in this example the investor who receives the S&P return makes a substantial profit, leading to a loss on the other side of the trade. This should be unsurprising, however, as the period of observation was defined by particularly strong equity returns as the drawdown from the Covid crisis eased, resulting in initially negative returns but a steep uptrend in the above chart.

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[13] Interest Rate swaps are discussed in more detail in chapter 13. Credit Default Swaps are discussed in more detail in chapter 14. More information on swaps can also be found in [100] and [96].

[14] Given their structure, credit default swaps can be viewed as analogous to life insurance contracts on corporations or governments.

1.4.9 Over-the-Counter vs. Exchange Traded Products

In many markets, instruments trade on an exchange. These instruments tend to be liquid and are characterized by standardized terms, such as contract size and expiration date. Exchange traded markets naturally lend themselves to automated trading and low-touch execution[15]. The majority of the products in equity and equity derivatives markets are exchange traded.

In other cases, over-the-counter (OTC) contracts are the market standard. In these cases, an intermediary such as a dealer will customize a structure to cater to the needs of a client. These markets are far less standardized with customizable terms/features, and are characterized by high-touch execution. Execution in OTC markets often happens via phone or Bloomberg chat requiring interaction and potential negotiation with a sell-side counterpart. In these markets, clients, such as buy-side institutions often reach out to dealers with a desired structure, and the dealer responds with pricing. This process may then include several iterations working with the dealer to finalize the trades terms and negotiate the price. Many derivatives, including the vast majority of exotic options, are traded OTC.

The primary benefit of an OTC contract is that an investor may customize the terms and exposure to meet their exact needs. This is not possible in exchange traded products which consist of only standard instruments with preset features. As an example, a client looking to hedge currency risk that is contingent on future sales of a product might want to enter a customized contract that hedges this currency risk conditional on positive equity returns, when sales are likely to be strongest. This type of OTC product is commonly referred to as a **hybrid** option, as it is contingent on returns in both the foreign exchange and equity markets. The primary drawbacks of OTC contracts are the less automated, higher touch execution, and the corresponding lower levels of liquidity that can lead to higher bid-offer spreads. This leads to a situation where investors must solicit pricing for a dealer to start the execution process, rather than observing a set of market data and choosing the pockets where pricing looks most competitive.

In section III, we will highlight which markets trade which instruments on exchanges vs. OTC, and discuss the implications for investors looking to leverage those products, and for quants looking to model them. We will find that the distinction is most important in interest rate markets, where we use non-deterministic interest rates[16].

1.5 STAGES OF A QUANT PROJECT

Quant projects can vary greatly by organization and role, as we saw earlier in this chapter, however, there is a great deal of commonality in how these varying quant tasks are structured. In particular, quant projects, generally speaking consist of the following four main steps: data collection, data cleaning, model implementation and

[15]Low touch execution refers to the ability to execute trades in an automated manner with minimal dealer contact.

[16]See chapter 13 for more details

model validation. In the following sections, we briefly describe what each of theses steps entails. Throughout the book, we will highlight approaches to tackling these steps efficiently, both from a technical and quantitative perspective.

1.5.1 Data Collection

Data collection is the process of identifying the proper source for our model and gathering it in a desired format for use in the model. This part of a quant project requires being able to interact with different types of data sources, such as databases, flat/CSV files, ftp sites and websites. As such, we need to be familiar with the libraries in Python that support these tasks. We discuss this stage, and some of the more commonly used financial data sources, in more detail in chapter 6.

1.5.2 Data Cleaning

Once we have obtained the relevant data for our model the next step is to make sure that it is in proper order for our model to use. This process is traditionally referred to as data cleaning, where we analyze the set of data that we are given to make sure there are no invalid values, missing data or data that is not in line with the assumptions of the model. Depending on the type of data we are working with, this process can range from very simple to extremely complex. For example, with equity data we may need to verify that corporate actions are handled in an appropriate manner. In contrast, when working with derivatives, we need to check the data for arbitrage that would violate our model assumptions. Completing this stage of a project will require a level of mastery of Python data structures and an ability to use them to transform and manipulate data. These data structures are discussed in more detail in chapter 5, and more details on the typical cleaning procedure for different types of financial data is discussed in chapter 6.

1.5.3 Model Implementation

Model implementation is the core task we face as quants. It is the process of writing a piece of code that efficiently implements our model of choice. In practical quant finance applications, closed form solutions are rarely available to solve realistic problems. As a result, quant projects require implementation of an approximation of a given model in code. The underlying models and techniques vary by application, ranging from using simulation of an Stochastic Differential Equation to estimate an option price to using econometric techniques to forecast stock returns. As models become more complex, this step becomes increasingly challenging.

The vast majority of this book is dedicated to model implementation and the most common models and techniques used to solve these problems. We also focus heavily on methods for implementing models in a robust and scalable way by providing the required background in object-oriented programming concepts. Further, in section III we work through different model implementations across asset classes and discuss the key considerations for modeling across different markets.

1.5.4 Model Validation

Once we have implemented our model, a separate process must begin that convinces us that the model has implemented correctly and robustly. This process is referred to as model validation, and it is designed to catch unintended software bugs, identify model assumptions and limitations of the model. For simple models, this process may be relatively straight-forward, as we can verify our model against another independent third party implementation. As models get increasingly realistic, this process becomes much less trivial as the true model values themselves become elusive. In this context, we need to rely on a set of procedures to ensure that we have coded our models correctly. This model validation step is discussed in detail in chapter 7.

1.6 TRENDS: WHERE IS QUANT FINANCE GOING?

In this chapter we have tried to provide somewhat of a roadmap to the quant finance and investment management industry. As a relatively new, younger field, quant finance and investment management is still in a very dynamic phase. This is partly driven by technological innovations and partly driven by fundamental improvements to the underlying models. Along these lines, in the remainder of this section we highlight a few areas of potential evolution in the coming years and decades.

1.6.1 Automation

Automation is a key trend in the finance industry that is likely to continue for the foreseeable future. On the one hand, technological advances have led to the ability to streamline and automate processes that used to require manual intervention or calculations. Automation of these processes generally requires strong programming knowledge and often also requires a solid understanding of the underlying financial concepts. In some cases, automation may involve writing a script to take the place of a manual reconciliation process a portfolio manager used to do. More substantively, automation may also involve replacing human based trading algorithms with automated execution algorithms. This is a trend that we have seen in most exchange traded markets, that is, execution has gotten significantly lower-touch. In the future, there is the potential for this to extend to other segments of the market.

1.6.2 Rapid Increase of Available Data

Over the past few decades, a plethora of new data sources have become available, many of which are relevant for buy-side and sell-side institutions. This has created a dramatic rise in the number of big data problems and data scientists in the quant finance industry. Many of these data sources have substantially different structure than standard financial datasets, such as text data and satellite photographs. The ability to parse text data, such as newspaper articles, could be directly relevant to buy-side institutions who want to process news data using a systematic approach. Similarly, image data of store parking lots may provide insight into the demand for different stores and products that leads balance sheet data.

This trend in the availability of data is likely to continue. It has been said that something like 90% of the financial data available to market participants is from the last decade and further that proliferation of data is likely to make this statement true in the next few decades as well. This is a welcome trend for quants, finTech firms and data scientists looking to apply their skills in finance as it provides a richer opportunity set. This data is not without challenges, however, as the fact that these datasets are new makes it challenging to thoroughly analyze them historically in different regimes. It stands to reason, however, that the reward for being able to process these new data sources robustly should also be quite high before other market participants catch on.

1.6.3 Commoditization of Factor Premias

Another key trend in the quant finance and investment management community has been the evolution of the concepts of **alpha** and **beta**. Traditionally, investment returns have been viewed against a benchmark with comparable *market exposure*. This means that for most hedge funds, if they arguably take minimal market exposure, or beta, over long periods of time, then their performance[17] would be judged in absolute terms. For asset management firms who take large amounts of beta, their performance would be judged against a balanced benchmark with the appropriate beta, such as a 60/40[18]. This ensures that investment managers are compensated for the excess returns, or alpha that they generate but are not compensated for their market exposure, which could easily be replicated elsewhere more cheaply.

More recently, there has been a movement toward identification of additional factors, or risk premia, such as carry, value, momentum, (low) volatility and quality. This has created a headwind for many investment firms as the returns in these premia have become increasingly commoditized, leading to lower fees and cheaper replication. Further, returns from these premia, which used to be classified as **alpha**, have become another type of **beta**. Although in some ways this has been a challenge for the buy-side, it also creates an opportunity for quants who are able to identify and find robust ways to harvest these premia.

1.6.4 Movement toward Incorporating Machine Learning/Artificial Intelligence

In recent years, there has been a push toward using machine learning techniques and artificial intelligence to help solve quant finance and portfolio management problems. This trend is likely not going anywhere, and recently many seminal works in machine learning have discussed the potential applications of these techniques in a financial setting[19]. For example, Machine Learning may help us with many quantitative tasks, such as extracting meaning from unstructured data, building complex optimal hedging schemes and creating higher frequency trading strategies.

[17] And consequently their ability to charge performance fees.

[18] A 60/40 portfolio has a 60% allocation to equities and 40% allocation to fixed income.

[19] Such as Halperin [101] and Lopez de Prado [56] [58]

While this trend toward Machine Learning is likely to continue and potentially accelerate into the future it is important to know the strengths and weaknesses of these different techniques and keep in mind Wilmott's quant manifesto [151]. No model or technique will be a perfect representation of the world or perfect in all circumstances. In the context of machine learning, this may mean that there are certain instances where application of these techniques is natural and leads to significant improvement. Conversely, it is important to keep in mind that in other cases machine learning techniques are likely to struggle to add value. For example, in some cases, such as lower frequency strategies, there might not be sufficient data to warrant use of sophisticated machine learning techniques with large feature sets. In chapter 21 we discuss the potential uses and challenges of leveraged machine learning techniques in finance.

1.6.5 Increasing Prevalence of Required Quant/Technical Skills

Taken together, these trends lead to a larger overarching trend in favor of the importance of quantitative techniques that can help us uncover new data sources, explain the cross-section of market returns via a set of harvestable premia, and help us automate trading and other processes. Over the past few decades, many roles in the financial industry have begun to require more technical skills and a more quantitative inclination. Knowledge of a coding language such as Python and fluency with basic quant techniques has become more widespread, leading to an industry where substantial quantitative analysis is required even for the some of the most fundamentally oriented institutions.

CHAPTER 2

Theoretical Underpinnings of Quant Modeling: Modeling the Risk Neutral Measure

2.1 INTRODUCTION

PERHAPS the canonical mathematical technique associated with quant finance is that of stochastic calculus, where we learn to model the evolution of stochastic processes. As this subject requires a mastery of calculus, probability, differential equations and measure theory, many if not all graduate students find these concepts to be the most daunting. Nonetheless, it is a critical tool in a quants skillset that will permeate a quant's career. In this chapter we provide a broad, high level overview of the techniques required in order to tackle the financial modeling techniques detailed in later chapters. This treatment is meant to augment, and not replace by any means the standard and more rigorous treatment that dedicated stochastic calculus texts and courses provide. Instead, we look to provide an intuitive, accessible treatment that equips the reader with the tools required to handle practical derivatives valuation problems.

As we will see, stochastic processes are the evolution of sequences of random variables indexed by time. Modeling the evolution of the entire sequence then requires a different set of tools than modeling a single probability distribution. In a stochastic process, we need to model a series of interconnected steps, each of which is random, and drawn from a probabilistic distribution. This is in contrast to other applications of probability theory where we look at successive, often independent, single draws from a distribution. Of course, this adds complexity, and means that we need a new set of techniques. For example, it requires a different type of calculus, stochastic calculus, that incorporates the behavior of the entire sequence in a stochastic process instead of working on deterministic functions.

In this chapter, we provide the framework, and mathematical machinery, for dealing with these stochastic processes in the context of options pricing. As we will soon see, this is fundamentally distinct from the pursuit of forecasting or prediction models. In this context, instead of building models based on our estimates of future

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probabilities, we will instead develop arguments that are based on no-arbitrage, hedging and replication. If we can perfectly replicate an option, or contingent claim, for example, with a position in the underlying asset, then we know we have found an asset that mimics its exposure and payoff regardless of the true future probabilities. No arbitrage will then further guide us by telling us that these two claims that are economically equivalent must also have the same price. Thus, to the extent that we are able to do this, which we will soon judge for ourselves, we will then be able to build models that are agnostic to these future probabilities and instead built on our ability to hedge the options risk, or replicate the claim. Instead, we will build a different set of probabilities, under the assumption that investors are risk neutral. Importantly building this set of probabilities will not require forecasting the drift of the asset in the future. Later in the text, in chapter 12, we will generalize this to a market implied distribution, based on the set of risk-neutral probabilities extracted from options prices, using these replication arguments.

In the supplementary materials, we briefly review the foundational tools that stochastic calculus is built on, notably starting with a review of standard calculus concepts and probability theory. This review is intentionally high level and brief. Readers looking for a deeper review of these concepts should consult [180] or [49]. In the following sections, we then extend these concepts to the case of a stochastic process, and highlight many of the challenges that arise. It should again be emphasized that the treatment of stochastic calculus here is meant to be light and intuitive. Those looking for a more rigorous treatment should consult [176], [23] or [145].

2.2 RISK NEUTRAL PRICING & NO ARBITRAGE

2.2.1 Risk Neutral vs. Actual Probabilities

Before delving into the theory of risk-neutral probabilities and no-arbitrage theory, it is worth first understanding it intuitively. Along those lines, Baxter and Rennie [23] present an exceptional example highlighting the difference between risk-neutral and actual probabilities that the reader is strongly encouraged to review. At the core of this example is the idea of replication. If we are able to synthetically replicate a given bet, or contingent claim, then the actual probabilities of the underlying sequence of events are no longer important in how we value the claim. Instead we can rely on the relative pricing of the contingent claim and our replication strategy.

When working with actual probabilities, we look for trades that are significantly skewed in our favor. That is, we want a high probability of favorable outcomes and a relative low probability of unfavorable outcomes. The idea is then, if we can engage in this strategy many times, we should profit over time. On each individual bet, we have no idea what is going to happen, it is random, but over time we should be able to accumulate a profit if the odds are skewed in our favor. In fact this is the basis for most, if not all, systematic investing strategies, and is a major focus on IV. But, importantly, as we will see in the remainder of this chapter, there are times when we are able to be agnostic to the actual probabilities. That is, of course, because we have built a replicating portfolio. If we are able to do this, and the replicating portfolio is priced differently then the contingent claim, then we want to buy (or sell) the contingent claim and sell (or buy) the replicating portfolio. This will be true regardless of the actual probabilities. Even if the contingent claim is in isolation a great investment strategy, that is, even if the probabilities are skewed in our favor, we are better off trading the replicating portfolio against the contingent claim. This is because, in that case, we are agnostic to the outcome, we can simply wait for the claim to expire and collect our profit. The pricing models that we develop in this text will be based on this phenomenon, and the fact that this relationship should keep the pricing of derivatives, or contingent claims, and their replicating portfolios, in line.

Throughout the text, we will emphasize the fundamental difference between risk-neutral probabilities, and the risk-neutral pricing measure and the actual set of probabilities, which we will refer to as the physical measure. In the risk-neutral measure, we work in the context of a replicating portfolio. This makes us indifferent to what happens in the future, as long as our replication strategy holds. In the physical measure, we will not use these replication arguments but instead rely on forecasts of the actual probabilities. This will become necessary in the case where replication arguments do not exist.

2.2.2 Theory of No Arbitrage

The concept of no-arbitrage is a fundamental tenet of quant finance [28]. Arbitrage refers to the ability to engage in a trading strategy that obtains excess profits without taking risk. For example, if we were able to buy and sell the same security at different prices at the same time, then we could simultaneously buy the lower priced and sell the higher priced, leaving us a risk-free profit. Economic theory posits that these types of opportunities are not present in markets, and, to the extent that they appear, market participants are happy to make the appropriate trades that force prices to re-align toward their equilibrium, no-arbitrage state. This means that, if we are able to replicate a claim, whether it is a position in a stock or a complex derivative, then the price of the replicating portfolio should be the same as the price of the original asset. If not, an arbitrage would occur, where we could buy the cheaper of the replicating portfolio and the original asset, and collcet a profit without bearing any risk. This type of no-arbitrage and replication argument is at the heart of so-called risk-neutral pricing theory. Instead of attempting to forecast the asset price and use that to value our derivative, we can instead look to replicate it using simpler instruments. The price of the replicating portfolio, will then help us value the original, more complex derivative.

In the last chapter, we saw an example of this when we derived the pricing formula for a forward or futures contract under certain conditions. In this case we were able to build a replicating portfolio for a forward or a future that involved a position in the underlying asset, and a position in a risk-free bond. Notably, this position was static: we put on our position at trade origination and simply waited for the expiration date. Importantly, we didn't need to re-balance. As the derivatives that we work with get more complex, so too will the replication schemes. In this chapter we will consider options, or contingent claims, which will require a dynamic hedge in order to replicate.

2.2.3 Complete Markets

Another pivotal concept in the pursuit of valuing complex derivative securities is that of a complete vs. incomplete market [157]. In a complete market, any contingent claim, regardless of complexity can be replicated. This ability to replicate these contingent claims will be central to our modeling efforts, and therefore this is an important distinction. An incomplete market, by contrast is one in which not every claim can be replicated. In this case, valuation of claims that we cannot replicate will be more complex, and will not be able to be done using the standard risk-neutral valuation framework that we develop.

2.2.4 Risk Neutral Valuation Equation

The first fundamental theorem of asset pricing [157] states that there is no arbitrage if and only if there exists a risk-neutral measure. In this risk-neutral measure, valuations of all replicable securities can be computed by discounting prices at the risk-free rate. This means that, in this case, asset prices can be written as:

$$S_0 = \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u\,du} S_T\right] \tag{2.1}$$

for a given risk-neutral measure $\tilde{\mathbb{P}}$ and an interest rate process r. Under this measure, $\tilde{\mathbb{P}}$, the current stock price reflects the discounted risk-neutral expectation of all future cashflows.

Similarly, the price today, p_0, of a contingent claim with payoff $V(S_T)$ can be written as:

$$p_0 = \mathbb{E}\left[e^{\int_0^T -r\tau d\tau} V(S_T)\right] \tag{2.2}$$

In the case of a constant interest rate, this equation simplifies to:

$$p_0 = e^{-rT}\mathbb{E}\left[V(S_T)\right] \tag{2.3}$$

This risk-neutral valuation formula, built on replication and hedging arguments, will be one of the key pillars in our derivative modeling efforts, and as such, will be used frequently throughout the text.

2.2.5 Risk Neutral Discounting, Risk Premia & Stochastic Discount Factors

The use of risk-neutral discounting, and the corresponding risk-neutral measure is a noteworthy part of our framework that is worth emphasizing. As we can see from equation (2.2), discounting in a risk-neutral context is done via a deterministic function, which can easily be obtained from the current interest rate curve. Importantly, these discount factors are not state dependent, or stochastic. They are also based on the assumption that investors are risk-neutral. Said differently, they are built independently of investors varying risk preferences. This brings a great deal of convenience and simplicity to the risk-neutral valuation framework. If we were forced to incorporate investor risk preferences into our model, it would result in a different more complex discount or drift term, that would vary not only from state to state, but also potentially from investor to investor. We would also then need to model the correlation structure between our stochastic discount factors, and the underlying asset that our derivative is based on.

In the next chapter, we transition to the physical measure where the assumption of risk-neutrality no longer holds. Instead, we will then have to incorporate investor preferences. Further, these investor preferences may be different from investor to investor, which will manifest itself in different utility functions. A share of Apple stock, for example, may provide one investor with more utility than another, if the payoffs for Apple are high in a particular state that the first investor values greatly. The amount of utility for a given security will be dependent on how risk averse they are which gives rise to the concept of a risk premia. Risk premia is a foundational concept in quantitative finance, and refers to the fact that investors may earn a premia for bearing risks that others are averse from. In this context, the discount factors that we compute will themselves be stochastic, and be functions of state and risk preferences. The fact that we are able to obviate these challenges associated with risk premia and risk preferences in the context of derivatives valuation is a fundamental point, and one that the reader is encouraged to think through.

2.3 BINOMIAL TREES

2.3.1 Discrete vs. Continuous Time Models

The study of stochastic processes is broken into discrete and continuous models. Continuous models rely on the same tools that we rely on in ordinary calculus problem, albeit updated to handle the stochastic nature of the processes we are working with. They assume that asset prices themselves are continuous processes, that can be traded and move instantaneously, rather than at set, discrete increments. In a discrete model, conversely, we might assume that prices move every d days, or every m minutes, but importantly the movements would be finite.

Of course, in practice markets actually trade more in alignment with discrete time models. Yet, in spite of this, quants have a tendency to gravitate toward continuous time models, and more work has been done building continuous models that closely resemble market dynamics. A significant reason for this is that, as quants, we have a more robust toolset for solving continuous problems (calculus) relative to discrete time problems (tree methods). Thus, working in a continuous time model is more mathematically convenient. This is analogous to calculus itself, where the assumption of the infinitesimal limit may likewise be unrealistic, however, it still provides a useful framework and accurate approximation for solving the problem of interest even as we relax the relevant assumptions. Nonetheless, in this section we detail the canonical discrete time tool for modeling a stochastic process, a binomial tree. We then move on to continuous stochastic processes, which are the focus of the remainder of the text.

2.3.2 Scaled Random Walk

Let's begin by building our first building block for stochastic calculus, a discrete random walk. A random walk is a stochastic process that is a martingale, meaning its expected future value is equal to the current value of the process. A random walk can be motivated by considering the example of flipping a fair coin with equal probability of heads and tails. Let's assume there is a process that moves up one if a head occurs and back one if a tail occurs. Mathematically, we can write this as:

$$X_t = \begin{cases} 1 & \text{if heads} \\ -1 & \text{if tails} \end{cases} \tag{2.4}$$

where since we are tossing a fair coin, p, the probability of a head is 0.5, as is the probability of tails, $1-p$. We further assume that each increment, X_i is independent. Readers may notice that this sequence of random variables, X_t, each follow a Bernoulli distribution. However, unlike the case considered in the supplementary materials, when the distribution took on 0 or 1 values, we now consider the case of ± 1.

To construct a random walk, we can sum these independent Bernoulli increments, that is:

$$W_n = \sum_{i=1}^{n} X_i \tag{2.5}$$

Using the expected value and variance formulas presented in the supplementary materials, we can compute the mean and variance of the random walk process, W_t that we have created:

$$\begin{aligned} \mathbb{E}\left[W_n\right] &= 0 && (2.6) \\ \operatorname{var}(W_n) &= \sum_{i=1}^{n} \operatorname{var}(X_i) && (2.7) \\ &= n\left(\mathbb{E}\left[X_i^2\right]\right) && (2.8) \\ &= n && (2.9) \end{aligned}$$

where in (2.7) we are using the independence property to eliminate the covariance terms. In the next line, we are using the definition of variance of the random variable X_i and, as we have shown the mean is equal to zero, this term can be eliminated as well. Finally, in the last line, we use the fact that $\operatorname{var}(X_i) = (p)(1)^2 + (1-p)(-1)^2 = 1$.

An unappealing feature of the random walk that we just created was that the variance is a function of the number of steps in our grid. In reality, this isn't a desireable quality, instead we will want it to be proportional to the length of time in our interval. For example, a path that spans one-year should intuitively have more variance than one that spans one-week, however, a path with 100 steps in a year should have the same variance as one with 500 steps that also spans a year.

To overcome this, let's create a grid that incorporates the timeframe that we are looking to model. Consider the grid defined by k, the number of total coin flips, or steps, and n the number of trials per period. Further, the time interval, t, can then be defined in terms of k and n via the following equation:

$$t = \frac{k}{n} \tag{2.10}$$

Using this grid, lets construct a scaled version of the previously constructed random walk which is defined as follows:

$$\hat{W}_t = \frac{1}{\sqrt{n}} W_k \tag{2.11}$$

Notice that our new grid is still based on a random walk that takes k steps, however, these steps are now rescaled to reflect the time period, by normalizing by $\frac{1}{\sqrt{n}}$. This scaled random walk has the following mean and variance:

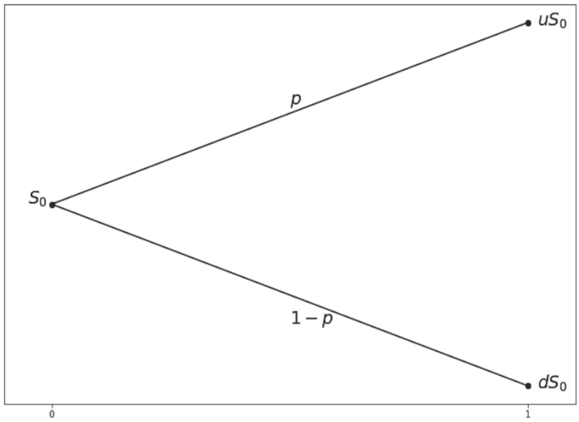
$$\begin{aligned}
\mathbb{E}\left[\hat{W}_t\right] &= 0 && (2.12)\\
\mathrm{var}(\hat{W}_t) &= \mathrm{var}(\frac{1}{\sqrt{n}} W_k) && (2.13)\\
&= \frac{k}{n} = t && (2.14)
\end{aligned}$$

Importantly, notice that the variance is independent of k and n individually and is instead only a function of the time interval t. This rescaled random walk will become a building block for our discrete time models, and, as we transition to continuous time models, we will leverage this process and take the limit as the step size gets infinitesimally small.

2.3.3 Discrete Binomial Tree Model

The first model that we will consider is a discrete time model that relies on similar principles to a random walk in that we assume at each increment the asset moves up or down by the same amount. This model is referred to as a binomial tree [175], and is the most common discrete model applied in practice.

Let's begin with a one-period binomial tree model, with an asset that starts at S_0. As stated, the asset may then either go up and end at uS_0 or go down and finish at $dS_0 = \frac{1}{u} S_0$. Lastly, we need to define the probability of up and down moves, which will be p and $1 - p$, respectively. Visually, our one-period binomial tree model can be expressed in the following chart:

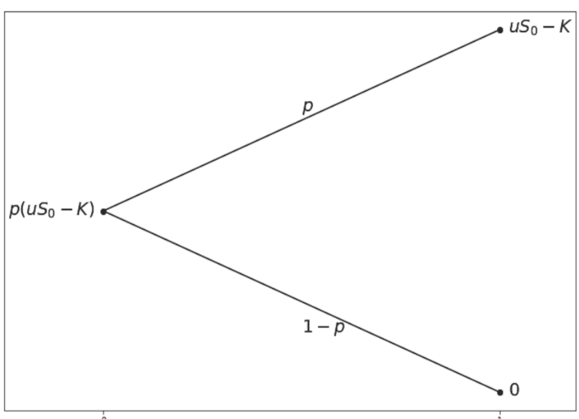


Notice that the up and down movements in the model we have created are symmetric. This ends up being an important feature as it causes the tree to recombine as we add more periods, leaving us with less nodes overall.

In order to value an option on our binomial tree we can iterate backward, beginning at the end of the tree where the payoff of the option should be known. For example, suppose the derivative of interest is a call option that expiries at the end of the first period and whose strike is equal to K. The payoffs of this structure at the end of the first period are known. If the movement of the asset is down, then the payoff is zero, otherwise it is $uS_0 - K$. Further, we know that the up movement occurs with probability p, and the down movement occurs with probability $1 - p$. Therefore, according to risk-neutral pricing theory, the expected value at time zero is this payoff, discounted by the risk-free rate, and can be written as:

$$c_0 = e^{-rt} p \left(uS_0 - K \right) \tag{2.15}$$

where we have omitted the second term for a down movement because the payoff is zero by definition. This can be seen via the following tree of the option valuation:



This simple one-period example highlights how we use a binomial tree in order to value a derivative. In particular, we first choose the parameters in our tree, notably u, d and p. u and d are chosen such that the movements are symmetric, resulting in a recombining tree. Additionally, we will want to choose u and d such that the variance in the underlying asset matches our desired level. p, in contrast is set such that the drift aligns with the risk-neutral valuation framework introduced above. The following equations show how u, d and p should be set if we are targeting an annualized volatility σ, and have a constant interest rate, r:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{2.16}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{2.17}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2.18}$$

Once we have selected the parameters, we can evolve asset prices forward along the tree until we reach the terminal nodes. We can then work backward along the tree of asset prices computing the value for the derivative of interest at each node.

At the end, the value of the derivative is known as it is just the payoff function of that derivative:

$$c_{n,i} = V(S_i) \tag{2.19}$$

We can then find the value one period sooner by leveraging the terminal values, and the probabilities of each terminal node:

$$c_{n-1,i} = e^{-r\tau}\left(pc_{n,i} + (1-p)c_{n,i+1}\right) \tag{2.20}$$

This process can be repeated until we reach the beginning of the tree, where we obtain a price for the derivative.

In the following visualization, we show how a recombining binomial tree works over multiple periods: