Colorado CSCI 5454: Algorithms
Homework 11

Instructor: Bo Waggoner
Due: Saturday, December 14, 2019 at 11:59pm
Turn in electronically via Gradescope.

Remember to list the people you worked with and any outside sources used (if none, write "none").

Problem 1 (4 points)

Recall from HW3 the max flow problem with vertex capacities $w(v)$, where the total flow into a vertex can be at most $w(v)$.

We solved that problem by modifying the graph to create a different instance of max flow. Here, instead, modify the linear programming formulation of max flow to give a linear program for this problem.

Make sure to explain what the variables are, the objective function, and the constraints. The constraints should include bounds on the variables (e.g. nonnegative) if applicable.

Solution. We assume there are no edges into s or out of t. We also assume the flows and capacities are all between 0 and 1.

We are given the capacities c_e on each edge $e \in E$. The variables we get to choose are the amount of flow f_e on each edge $e \in E$.

The objective is to maximize total flow into t, which is the sum of the flows on all edges into e.

The "capacity constraints" were $f_e \leq c_e$. The "flow constraints" said flow out of u equals flow into u, as long as $u \notin \{s, t\}$.

The final constraints we need to add are for vertex capacities. This says the total flow into v is at most $w(v)$.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{u \in V:(u,t)\in E} f_e \\
\text{subject to} \quad & \\
& f_e \le c_e & \text{(for all } e \in E) \\
\sum_{u \in V:(u,v)\in E} & f_e = \sum_{u \in V:(v,u)\in E} f_e & \text{(for all } v \in V) \\
\sum_{u \in V:(u,v)\in E} & f_e \le w(v) & \text{(for all } v \in V) \\
& f_e \le 1 & \text{(for all } e \in E) \\
& f_e \ge 0 & \text{*for all } e \in E)
\end{aligned}
$$

Problem 2 (8 points)

Consider a finite two-player, zero-sum game with utility function $u(a_1, a_2)$. Our goal is to find an equilibrium strategy p_1 for player one, i.e.

$$\max_{p_1 \in \Delta_{A_1}} \min_{a_2 \in A_2} \sum_{a_1 \in A_1} p_1(a_1) u(a_1, a_2) \tag{1}$$

In this problem, we will cast this as a linear programming problem.

Part a (2 points) The original variables are $p_1(1), \ldots, p_1(n)$ where there are n actions of player 1. What linear constraints must these variables satisfy to form a valid probability distribution?

Solution.

$$\sum_{a_1 \in A_1} p_1(a_1) = 1$$

and

$$p_1(a_1) \ge 0 \quad \text{(for all } a_1 \in A_1)$$

Part b (2 points) Let us create a variable v that represents the worst-case utility of player 1, i.e. think of v as representing $\min_{a_2} \sum_{a_1 \in A_1} p_1(a_1) u(a_1, a_2)$.

Let us pick a particular action for player 2, call it a_2. What inequality below must be true? Fill in the blank with one of $<, \le, =, \ge, >$:

$$v \ \_\_\_\_\_ \ \sum_{a_1 \in A_1} p_1(a_1) u(a_1, a_2).$$

Justify your answer briefly.

Hint: In rock-paper-scissors, if a_2 is "rock", then the right side represents "utility of player 1 for playing p_1 if player 2 plays rock."

Solution. $\leq$

The worst-case utility over all the actions is at most the utility for any one action. For example, my utility if my opponent plays rock (right side) can only be larger than my worst-case utility over all her actions.

Part c (4 points) Building on the previous parts, create a linear program for an equilibrium strategy of player 1. Briefly justify your answer: why does this solve Expression 1?

Hint: try "maximize v subject to..."

The variables are v and $p_1(a_1)$ for all $a_1 \in A_1$.

$$
\begin{aligned}
\text{maximize} \quad & v \\
\text{subject to} \quad & \\
v \leq \sum_{a_1 \in A_1} p_1(a_1) u(a_1, a_2) & \qquad \text{(for all } a_2 \in A_2\text{)} \\
\sum_{a_1 \in A_1} p_1(a_1) = 1 & \\
p_1(a_1) \geq 0 & \qquad \text{(for all } a_1 \in A_1\text{)}
\end{aligned}
$$