Homework-9

Resources - Notes, MIT open course.

Collaboration - Nischal and Abhilash

**Problem 1:**

**Part a**

$$U = \begin{bmatrix} u_{11}\ u_{12}\ \ldots\ u_{1k} \\ u_{21}\ u_{22}\ \ldots\ u_{2k} \\ \ldots \\ u_{n1}\ u_{n2}\ \ldots u_{nk} \end{bmatrix}_{n*k}$$

$$\sum = \begin{bmatrix} \sigma_1\ 0\ldots\ 0 \\ 0\ \sigma_2\ldots\ 0 \\ \ldots \\ 0\ \ldots\ 0\ \sigma_k \end{bmatrix}_{k*k}$$

$$V^T = \begin{bmatrix} v_{11}\ v_{12}\ \ldots\ v_{1d} \\ v_{21}\ v_{22}\ \ldots\ v_{2d} \\ \ldots. \\ v_{k1}\ v_{k2}\ \ldots\ v_{kd} \end{bmatrix}_{k*d}$$

$$A_k = \begin{bmatrix} u_{11}\ u_{12}\ \ldots\ u_{1k} \\ u_{21}\ u_{22}\ \ldots\ u_{2k} \\ \ldots \\ u_{n1}\ u_{n2}\ \ldots u_{nk} \end{bmatrix}_{n*k} \begin{bmatrix} \sigma_1\ 0\ldots\ 0 \\ 0\ \sigma_2\ldots\ 0 \\ \ldots \\ 0\ \ldots\ 0\ \sigma_k \end{bmatrix}_{k*k} \begin{bmatrix} v_{11}\ v_{12}\ \ldots\ v_{1d} \\ v_{21}\ v_{22}\ \ldots\ v_{2d} \\ \ldots. \\ v_{k1}\ v_{k2}\ \ldots\ v_{kd} \end{bmatrix}_{k*d}$$

$$A_k = \begin{bmatrix} u_{11}\sigma_1 & u_{12}\sigma_2 & \ldots & u_{1k}\sigma_k \\ u_{21}\sigma_1 & u_{22}\sigma_2 & \ldots & u_{2k}\sigma_k \\ \ldots.. & & & \\ u_{n1}\sigma_1 & u_{n2}\sigma_2 & \ldots & u_{nk}\sigma_k \end{bmatrix}_{n*k} \begin{bmatrix} v_{11}\ v_{12}\ \ldots\ v_{1d} \\ v_{21}\ v_{22}\ \ldots\ v_{2d} \\ \ldots. \\ v_{k1}\ v_{k2}\ \ldots\ v_{kd} \end{bmatrix}_{k*d}$$

$$A_k(1,1) = u_{11} * \sigma_1 * v_{11} + u_{12} * \sigma_2 * v_{21} + \ldots. + u_{1k} * \sigma_k * v_{k1}$$

Generalizing for all rows(j) and columns(l) we can rewrite the above equation as follows:

$$A_k(j,l) = \sum_{i=1}^{k} \sigma_i * u_i(j) * v_i(l)$$

## Part b

$$A_k = \begin{bmatrix} \sum_{i=1}^k \sigma_i * u_i(1) * v_i(l) & \sum_{i=1}^k sigma_i * u_i(1) * v_i(2) & ... & \sum_{i=1}^k sigma_i * u_i(1) * v_i(l) \\ \sum_{i=1}^k \sigma_i * u_i(2) * v_i(1) & \sum_{i=1}^k sigma_i * u_i(2) * v_i(2) & ... & \sum_{i=1}^k sigma_i * u_i(2) * v_i(l) \\ \sum_{i=1}^k \sigma_i * u_i(j) * v_i(l) & \sum_{i=1}^k sigma_i * u_i(j) * v_i(2) & ... & \sum_{i=1}^k sigma_i * u_i(j) * v_i(l) \end{bmatrix}$$

$$= \sigma_1 * \begin{bmatrix} u_1(1) * v_1(1) & u_1(1) * v_1(2) & ... & u_1(1) * v_1(l) \\ u_1(2) * v_1(1) & u_1(2) * v_1(2) & ... & u_1(2) * v_1(l) \\ u_1(j) * v_1(1) & u_1(j) * v_1(2) & ... & u_1(j) * v_1(l) \end{bmatrix} + \sigma_2 * \begin{bmatrix} u_2(1) * v_2(1) & u_2(1) * v_2(2) & ... & u_2(1) * v_2(1) \\ u_2(2) * v_2(1) & u_2(2) * v_2(2) & ... & u_2(2) * v_2(l) \\ u_2(j) * v_2(1) & u_2(j) * v_2(2) & ... & u_2(j) * v_2(l) \end{bmatrix} +$$

$$... + \sigma_i * \begin{bmatrix} u_i(1) * v_i(1) & u_i(1) * v_i(2) & ... & u_i(l) * v_i(l) \\ u_i(2) * v_i(1) & u_i(2) * v_i(2) & ... & u_i(2) * v_i(l) \\ u_i(j) * v_i(1) & u_i(j) * v_i(2) & ... & u_i(j) * v_i(l) \end{bmatrix}$$

$$= \sigma_1 * u_1 * v_1^T + \sigma_2 * u_2 * v_2^T + .... + \sigma_k * u_k * v_k^T$$

$$A_k = \sum_{i=1}^{k} \sigma_i * u_i * v_i^T$$

Hence proved.

## Part c

$$A_{k+1} = A_k + \sigma_{k+1} * \begin{bmatrix} u_{k+1}(1) * v_{k+1}(1) & u_{k+1}(1) * v_{k+1}(2) & ... & u_{k+1}(1) * v_{k+1}(l) \\ u_{k+1}(2) * v_{k+1}(1) & u_{k+1}(2) * v_{k+1}(2) & ... & u_{k+1}(2) * v_{k+1}(l) \\ u_{k+1}(j) * v_{k+1}(1) & u_{k+1}(j) * v_{k+1}(2) & ... & u_{k+1}(j) * v_{k+1}(l) \end{bmatrix}$$

$$A_{k+1} = A_k + \sigma_{k+1} * u_{k+1} * v_{k+1}^T$$

**Part d**

$$A = \sum_{i=1}^{r} \sigma_i * u_i(j) * v_i(l)$$

$$A_k = \sum_{i=1}^{k} \sigma_i * u_i(j) * v_i(l)$$

$$A - A_k = \sum_{i=1}^{r} \sigma_i * u_i(j) * v_i(l) - \sum_{i=1}^{k} \sigma_i * u_i(j) * v_i(l) = \sum_{i=k+1}^{r} \sigma_i * u_i(j) * v_i(l)$$

$$R_k(j,l) = \sum_{i=k+1}^{r} \sigma_i * u_i(j) * v_i(l)$$

**Part e**

$$R_k(j,l) = \sum_{i=k+1}^{r} \sigma_i * u_i(j) * v_i(l)$$

$$R_k(j,l)^2 = \sum_{i=k+1}^{r} (\sigma_i * u_i(j) * v_i(l))^2$$

When we expand the above and multiply the terms we can write it

$$||R||_F^2 = \sum_{j,l} \sum_{i=k+1}^{r} \sigma_i^2 * u_i^2(j) * v_i^2(l)$$

The rest of terms in cases where a!=b

$$\sum_{j,l} \sigma_a * \sigma_b * u_a * u_b * v_a * v_b = 0$$

$$||R||_F^2 = \sum_{j,l} \sum_{i=k+1}^{r} \sigma_i^2 * u_i^2 * v_i^2$$

And as U and V are unit vectors and as individual components summation is 1 we can write

$$||A - A_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

**Problem 2**

- We can use JL lemma to reduce the dimensions. We've pairwise distances between points as 1 $||x_i - x_j|| = 1$
- We are placing m points such that pairwise distance between points is in range 0.9 to 1.1 i.e $\varepsilon = 0.1$
- Let $||y_i - y_j||$ be distance between points after Gaussian projection of original points in d dimensions.
- From JL lemma-1 let $d \geq \frac{8}{\varepsilon^2} ln(\frac{m}{\delta})$

  Then for any $\varepsilon \in (0, 1]$ with a probability $1 - \delta$ that for all (i,j) pairs we can claim $(1 - \varepsilon)||x_i - x_j|| \leq ||y_i - y_j|| \leq (1 + \varepsilon)||x_i - x_j||$
- We now need a minimum value of d and we know $\varepsilon = 0.1$ and if we substitute $\delta = 0.1$
- $d = \frac{8}{0.1^2} ln(\frac{m}{0.1}) = 800(ln(m) - ln(0.1)) = 800 * ln(m) - 800 * ln(0.1)$
- $O(d) = O(ln(m))$

Hence,"I can place m points with all pairwise distances between 0:9 and 1:1, and I only need O (ln(m) ) dimensions!"