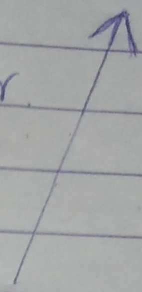
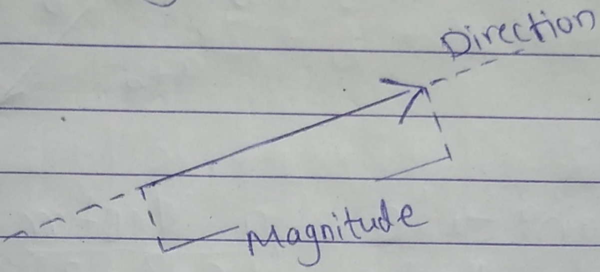
Linear Algebra

Vectors in Geometry



The image to the left is a Vector.
The length shows the Magnitude.
The Arrow shows the Direction.

Vector: A vector is a quantity or phenomenon that has two independent properties: magnitude and direction.

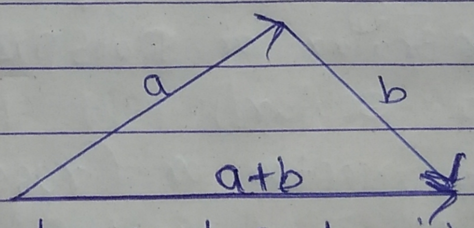


Vector in Geometry

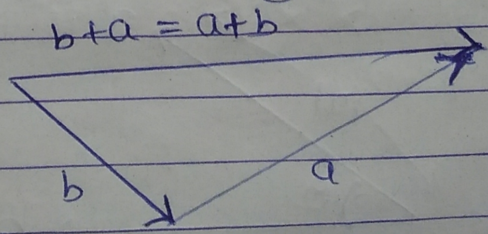
Representation: $a = \overrightarrow{AB}$ (Tail → Head)

Vector Addition:

the line from the tail of a to the head of b is the vector a+b.

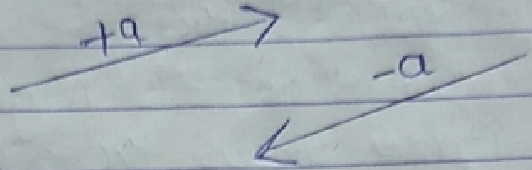


We can add two vectors by joining them head-to-tail

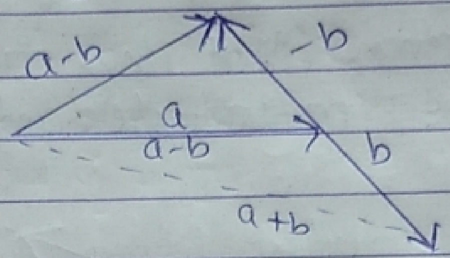


And it doesn't matter which order we add them, we get the same result.

Vector Substracting :
-a opposite of +a



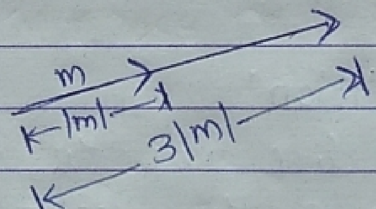
Substracting :



* Magnitude of a Vector

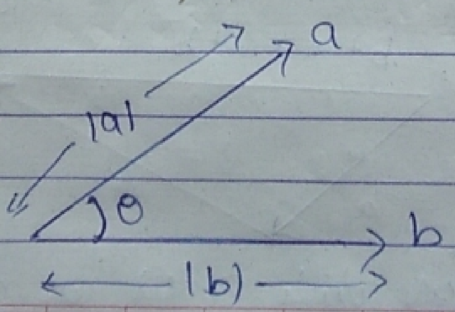
Formula $|a| = \sqrt{(x^2+y^2)}$

* Multiply a vector by a Scalar form



* Vector by Vector form

- The Scalar or Dot product (the result is a scalar).
- The vector or Cross product (the result is a vector).

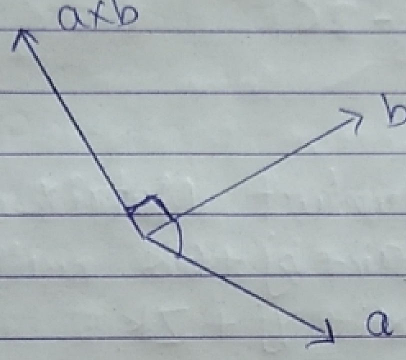


Dot product $(a \cdot b)$

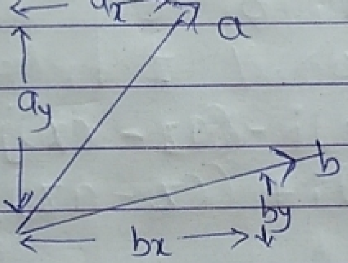
$$a \cdot b = |a| \times |b| \times \cos(\theta)$$

getting scalar value

Cross product :

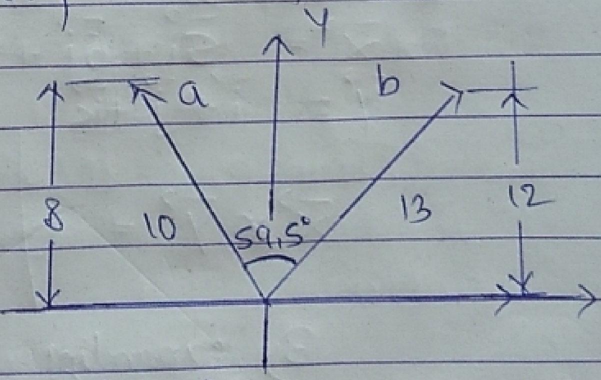


$$a \cdot b = a_x \times b_x + a_y \times b_y$$



Both Method works

Ex: 1) Calculate the dot product of vectors a & b.



$$a \cdot b = |a| \times |b| \times \cos(\theta)$$

$$a \cdot b = 10 \times 13 \times \cos(59.5^\circ)$$

$$a \cdot b = 130 \times 0.5075$$

$$a \cdot b = 65.98\ldots = 66$$

Ex 2) Two vectors A & B are given by
$A = 2i - 3j + 7k$ & $B = -4i + 2j - 4k$
Find the product of 2 vectors

$\rightarrow$ $A \cdot B = (2i - 3j + 7k) \cdot (-4i + 2j - 4k)$
$= 2(-4) + (-3)2 + 7(-4)$
$= -8 - 6 - 28$
$= -42$

Cross Product Matrix

$A = ai + bj + ck$
$B = xi + yj + zk$

$$A \times B = \begin{vmatrix} i & j & k \\ a & b & c \\ x & y & z \end{vmatrix}$$

$A \times B = (bz - cy)i - (az - cx)j + (ay - bx)k$
$= (bz - cy)i + (cx - az)j + (ay - bx)k$

Ex 1) Find the cross product of the given 2 vectors.

$\vec{x} = \vec{5i} + \vec{6j} + \vec{2k}$
$\vec{y} = \vec{i} + \vec{j} + \vec{k}$

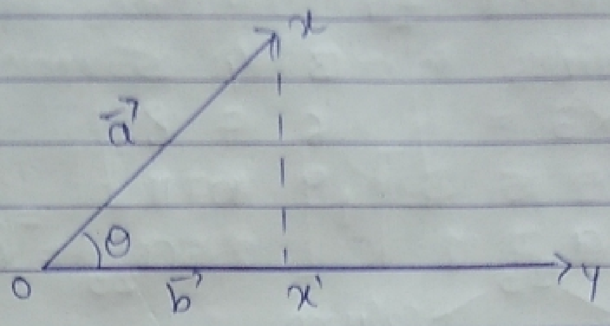
$$\vec{x} \times \vec{y} = \begin{vmatrix} i & j & k \\ 5 & 6 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

By expanding

$\vec{x} \times \vec{y} = (6-2)\vec{i} - (5-2)\vec{j} + (5-6)\vec{k}$

$\therefore$ $\vec{x} \times \vec{y} = 4\vec{i} - 3\vec{j} - \vec{k}$

⊛ Vector Projection :-



* Projection $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

* Projection $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$

* $P_b(a) = \left(\dfrac{a\cdot b}{|b|}\right)\dfrac{b}{|b|}$

Formula.

Ex 1) Find the scalar projection of $b = \langle -4, 1\rangle$ onto $a = \langle 1, 2\rangle$

→ Projection b onto a is the number

$$P_a(b) = |b|\cos(\theta) = \frac{b\cdot a}{|a|}$$

$$= \frac{(-4)(1) + (1)(2)}{\sqrt{1^2+2^2}}$$

$$P_a(b) = -\frac{2}{\sqrt{5}}$$

Ex 2) Find a vector projection $a = \langle 1, 2\rangle$ onto $b = \langle -4, 1\rangle$

→ The vector projection of a onto b is vector

$$P_b(a) = \left(\frac{a\cdot b}{|b|}\right)\frac{b}{|b|}$$

$$= \left(\frac{-2}{\sqrt{17}}\right)\frac{1}{\sqrt{17}}(-4, 1)$$

$$P_a(b) = \left\langle \frac{8}{17}, \frac{-2}{17}\right\rangle$$

Matrix & Determinants

Matrix introduction

Matrix 3×3 = $\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix}$

a_{23} — row, column $\qquad a_{23} = 8$

Order of a matrix

Number of row multiply by number of column.

Trace of Matrix

Ex:

$A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$ $\quad tr(A)$

$= 2+10+18$

$= 30$

*) Square Matrix : Number of row & number of column both are same.

Ex:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 6 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$2\times2 \qquad 3\times3$

*) Diagonal Matrix : In a matrix which the entries outside the main diagonal are all zero.

Ex:

$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Scalar Matrix : Scalar matrix is the special kind of diagonal matrix in which diagonal element contain same element. outside values are rest.

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

*) Identity Matrix : The identity matrix of size n is the n×n square matrix with ones on the main diagonal and zeros elsewhere.

$$\underset{2\times 2}{I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3\times 3$$

*) Zero Matrix : In matrix all the elements are zero.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*) Equal Matrix : They have the same order or dimension and the corrosponding elements are equal.

$$a = \begin{array}{c} \begin{matrix} a_{11} & a_{12} & a_{13} \end{matrix} \\ \begin{bmatrix} 2 & 5 & 3 \\ 4 & 7 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} b_{11} & b_{12} & b_{13} \end{matrix} \\ \begin{bmatrix} 2 & 5 & 3 \\ 4 & 7 & 1 \end{bmatrix} \end{array} = b$$

*) Negative Matrix : It is a real or integer matrix for which each matrix element is a -ve number.

$$\begin{array}{ccc} & A & & B \end{array}$$

$$= -\times \begin{bmatrix} -2 & 5 & 3 \\ -4 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 & -3 \\ 4 & -7 & -1 \end{bmatrix}$$

*) Adding Matrix : A matrix can only be added to another matrix if two matrices have the same dimensions

$$\begin{bmatrix} 2 & 5 & 3 \\ 4 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 7 & 1 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 4 \\ 6 & 12 & 4 \end{bmatrix}$$

*) Substraction Matrix : A matrix can only be substract another matrix

$$\begin{bmatrix} 4 & 5 & 7 \\ 3 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 9 \\ 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -1 \end{bmatrix}$$

*) Scalar Multiplication : The term scalar multiplication refers to the product of a real number and a matrix. In scalar multiplication, each entry in the matrix is multiplied by the given scalar.

$$2 \cdot \overset{A}{\begin{bmatrix} 10 & 6 \\ 4 & 7 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 20 & 12 \\ 8 & 14 \end{bmatrix}}$$

$A \times 2 = B$

* Transpose Matrix : It is a interchanging it's rows into columns or columns into rows.

$$A = \begin{bmatrix} 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 2 \\ 6 & 3 \\ 7 & 4 \end{bmatrix}$$

* Matrix Multiplication : It is a binary operation whose output is also a matrix when two matrices are multiplied. In linear algebra, the multiplication of matrices is possible only when the matrices are compatible.

1×3 by 3×1 get 1×1

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\times4 + 2\times5 + 3\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 4+10+18 \end{bmatrix}$$

$$= \begin{bmatrix} 32 \end{bmatrix}$$

3×1 by 1×3 get 3×3

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4\times1 & 4\times2 & 4\times3 \\ 5\times1 & 5\times2 & 5\times3 \\ 6\times1 & 6\times2 & 6\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

$$(m\times n) \times (n\times p) \longrightarrow m\times p$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = B$$

$$= \begin{bmatrix} 2+(-6)+1 & 1+(-4)+1 \\ 4+3+3 & 2+2+3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ 10 & 7 \end{bmatrix}$$

① Deteurminant :

It is a scalar value
Calculate elements of a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ c_{11} & c_{12} & c_{13} \end{bmatrix}$$

$a_{11} c b_{12}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$4 - 6 = -2$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Formula $= \boxed{a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)}$

*) Adjoint of a Matrix :

The Adjoint of a Matrix A is the transpose of the cofactor matrix of A.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

$A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4-3 = 1$

$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9 - 16 = -7$

$A_{21} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6-12$ $+6$

$A_{12} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = +1$

$A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3-3 = 0$

$A_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4-2 = -2$ $A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8-9 = -1$

$A_{32} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4-3 = -1$ $A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3-2 = 1$

$$\text{Adj } A = \begin{vmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{vmatrix} = A^T = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

* Inverse of a Matrix :

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

Where $|A| \neq 0$

$$A \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \text{Adj } A$$

$$\text{Co factor} = \begin{bmatrix} 6 & -2 \\ -7 & 4 \end{bmatrix}$$

$A_{11} = 6$ $A_{12} = 2$ $A_{21} = 7$ $A_{22} = 4$

Ex: $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ What is inverse of A ?

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$A_{11} = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} = -28-(-30) = 2$ $A_{12} = \begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix} = 21$

$A_{13} = -18$, $A_{21} = 6$, $A_{22} = -7$, $A_{23} = 6$,

$A_{31} = 4$, $A_{32} = -8$, $A_{33} = 4$.

Cofactor matrix of $A = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

adj A = transpose of cofactor matrix

$$\text{adj } A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4\times(-7) - (-6)\times 5 - 3\times(-6)\}$$

$$= 28 + 30 + 18 \quad -28 + 30 + 18$$

$$= \quad\quad = 20$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{20} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$