Calculus

* Calculus is the study of differentiation and integration. Calculus explains the Changes in values on a small & large scale, related to any functions.

* Differential calculus is the rate of change of a variable or a quantity with respect to another variable/quantity. It is represented by:

$$f'(x) = dy/dx$$

Differentiation formulas:

Derivatives of Basic functions.

1) $\frac{dk}{dx} = 0$; k is a constant

2) $\frac{d(x)}{dx} = 1$

3) $\frac{d(kx)}{dx} = k$; k is a constant.

4) $\frac{d(x^n)}{dx} = nx^{n-1}$

Derivatives of Logarithmic and Exponential Functions

1) $\frac{d(e^x)}{dx} = e^x$

2) $\frac{d(\ln(x))}{dx} = \frac{1}{x}$

3) $\frac{d(a^x)}{dx} = a^x \log a$

4) $\frac{d(x^x)}{dx} = x^x(1 + \ln x)$

5) $$\frac{d(\log_e a^x)}{dx} = \frac{1}{x} \times \frac{1}{\ln a}$$

Derivatives of Trignometric Functions

1) $$\frac{d(\sin x)}{dx} = \cos x$$

2) $$\frac{d(\cos x)}{dx} = -\sin x$$

3) $$\frac{d(\tan x)}{dx} = \sec^2 x$$

4) $$\frac{d(\cot x)}{dx} = \operatorname{cosec}^2 x$$

5) $$\frac{d(\sec x)}{dx} = \sec x \tan x$$

6) $$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$$

Derivatives of Inverse Trignometric Functions.

1) $$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

2) $$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

3) $$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

4) $$\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$$

Derivatives of logarithmic and Exponential Functions

1) $\frac{d(e^x)}{dx} = e^x$

2) $\frac{d(\ln(x))}{dx} = \frac{1}{x}$

3) $\frac{d(a^x)}{dx} = a^x \log a$

4) $\frac{d(x^x)}{dx} = x^x(1+\ln x)$

5) $\frac{d(\log_a x)}{dx} = \frac{1}{x} \times \frac{1}{\ln a}$

Differentiation Rules

1) Sum or Difference Rule:
When the function is the sum or difference of two functions, the derivative is the sum or difference of derivative of each function, i.e.
If $f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$

2) Product Rule:
When $f(x)$ is the sum of two $u(x)$ & $v(x)$ functions, it is the function derivative

If $f(x) = u(x) \times v(x)$,
Then $f'(x) = u'(x) \times v(x) + u(x) \times v'(x)$

Quotient Rule.
If the function $f(x)$ is in the form of two functions $\frac{u(x)}{v(x)}$ the derivative of the function can be,

Expressed as:

$$\text{If } f(x) = \frac{u(x)}{v(x)}$$

$$\text{Then } f(x) = \frac{u(x) \times v(x) - u(x) \times v'(x)}{[v(x)]^2}$$

Example 1): Calculate the derivative of

$$f(x) = 2x^3 - 4x^2 + x - 33$$

Recall that $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\frac{d}{dx}(2x^3 - 4x^2 + x - 33) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(x)$$

$$- \cancel{\frac{d}{dx}(33)} \to 0$$

$$= 2\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + \frac{d}{dx}(x)$$

$$= 2(3x^2) - 4(2x) + 1$$

$$= 6x^2 - 8x + 1$$

2) Calculate the derivative of $f(x) = 5x^3 - \tan x$

Recall from the table that $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\frac{d}{dx}(5x^3 - \tan x) = 5\frac{d}{dx}(x^3) - \frac{d}{dx}(\tan x)$$

$$= 5(3x^2) - \sec^2 x$$

$$= 15x^2 - \sec^2 x$$

3) Calculate the derivative of $f(x) = (e^x+1)\tan x$

Product Rule, Recall that $\frac{d}{dx}(e^x+1) = e^x$, and

that $d/_{dx} \tan x = \sec^2 x$

$$\frac{d}{dx}[(e^x+1)\tan x] = \left[\frac{d}{dx}(e^x+1)\right]\tan x + (e^x+1)\left[\frac{d}{dx}\tan x\right]$$
$$= [e^x]\tan x + (e^x+1)[\sec^2 x]$$
$$= e^x \tan x + (e^x+1)\sec^2 x$$

4) Differentiate $f(x) = \frac{\sin x}{x}$

Since the function is the quotient of two seprate functions, $\sin x$ & x, we must use the Quotient Rule. Recall that

$$\frac{d}{dx}\sin x = \cos x, \text{ and } \frac{d}{dx}x = 1$$

$$= \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{\left(\frac{d}{dx}\sin x\right)x - \sin x\left(\frac{d}{dx}x\right)}{x^2}$$

$$= \frac{(\cos x)x - \sin x(1)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Integration

The integration is the process of finding the antiderivative of a function. It is a similar way to add the slices to make it whole. The integration is the inverse process of differentiation.

List of Integral Formulas

1) $\int 1\, dx = x + C$
2) $\int a\, dx = ax + C$
3) $\int x^n\, dx = ((x^{n+1})/(n+1)) + C \; ; \; n \neq 1$
4) $\int \sin x\, dx = -\cos x + C$
5) $\int \cos x\, dx = \sin x + C$
6) $\int \sec^2 x\, dx = \tan x + C$
7) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
8) $\int \sec x\, (\tan x)\, dx = \sec x + C$
9) $\int \operatorname{cosec} x\, (\cot x)\, dx = -\operatorname{cosec} x + C$
10) $\int (1/x)\, dx = \ln|x| + C$
11) $\int e^x\, dx = e^x + C$
12) $\int a^x\, dx = (a^x/\ln a) + C \; : \; a > 0,\ a \neq 1$
13) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
14) $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
15) $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$
16) $\int \sin^n(x)\, dx = \frac{-1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx$
17) $\int \cos^n(x)\, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x)\, dx$

18) $\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$

19) $\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan x + \frac{n-2}{n-1}$

$\int \sec^{n-2}(x)\,dx$

20) $\int \operatorname{cosec}^n(x)\,dx = \frac{-1}{n-1}\operatorname{cosec}^{n-2}(x)\cot(x)$

$+ \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2}(x)\,dx$

Examples 1) Find the Integral of the function $\int_0^3 x^2\,dx$

$\int_0^3 x^2\,dx$

$\left(\frac{x^3}{3}\right)_0^3$

$\left(\frac{3^3}{3}\right) - \left(\frac{0^3}{3}\right) = 9$

2) $\int x^2\,dx$

$\rightarrow = \int x^2\,dx$

$= \left(x^3/3\right) + C$

3) Integrate $\int (x^2-1)(4+3x)\,dx$

$\rightarrow \int (x^2-1)(4+3x)\,dx$

Multiply the terms.

$= \int 4x^2 + 3x^3 - 3x + 4\,dx$

Now Integrate

$= 4\left(x^3/3\right) + 3\left(\frac{x^4}{4}\right) - 3\left(\frac{x^2}{2}\right) - 4x + C$

$= \int (x^2-1)(4+3x)\,dx$ is $4\left(x^3/3\right) + 3\left(\frac{x^4}{4}\right) - 3\left(\frac{x^2}{2}\right) - 4x + C$

4) $\int (4x^2 - 2x)\, dx = \int 4x^2 dx - \int 2x dx$

$= 4\int x dx - 2\int x dx$

$= \frac{4x^3}{3} - 2\frac{x^2}{2} + c$

$= \frac{4x^3}{3} - x^2 + c$

$= x^2\left(\frac{4x}{3} - 1\right) + c$

Integration By Parts Formula.

$\int uv\, dx = u\int v\, dx - \int (u'\int v dx)\, dx$

ILATE Rule :

Identify the function that comes first on the following list and select it as $f(x)$

ILATE Stands for.

I : ~~Logari~~ Inverse trignometric functions.
L : Logarithmic functions : $\ln x$, $\log 5(x)$.
A : Algebraic functions.
T : Trignometric functions. $\cos x$, $\sin x$ etc.
E ; Exponential functions.

Example : $\int x e^x dx$

From ILATE Theorem $f(x) = x$ & $g(x) = e^2$

Thus using the formula for integration by parts, we have

$= \int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\cdot\left(\int g(x)\,dx\right)dx$

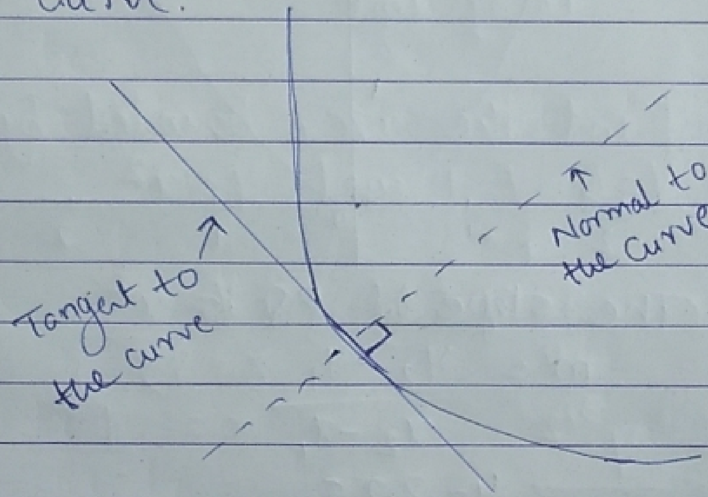
$= \int x\cdot e^{x}\,dx = x\cdot\int e^{x}\,dx - \int 1.\left(\int e^{x}\,dx\right)dx$

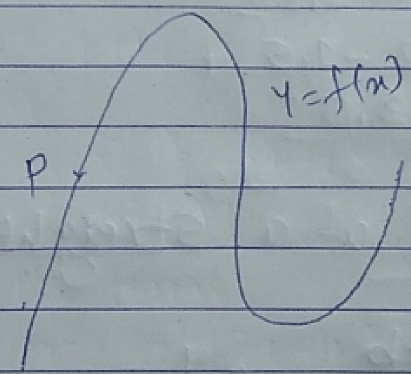
$= x.e^{x} - e^{x} + C$

Tangents & Normals

A tangent to a curve is a line that touches the curve at one point and has the slope as the curve at that point.

A normal to a curve is a line $\perp^r$ to a tangent to the curve.



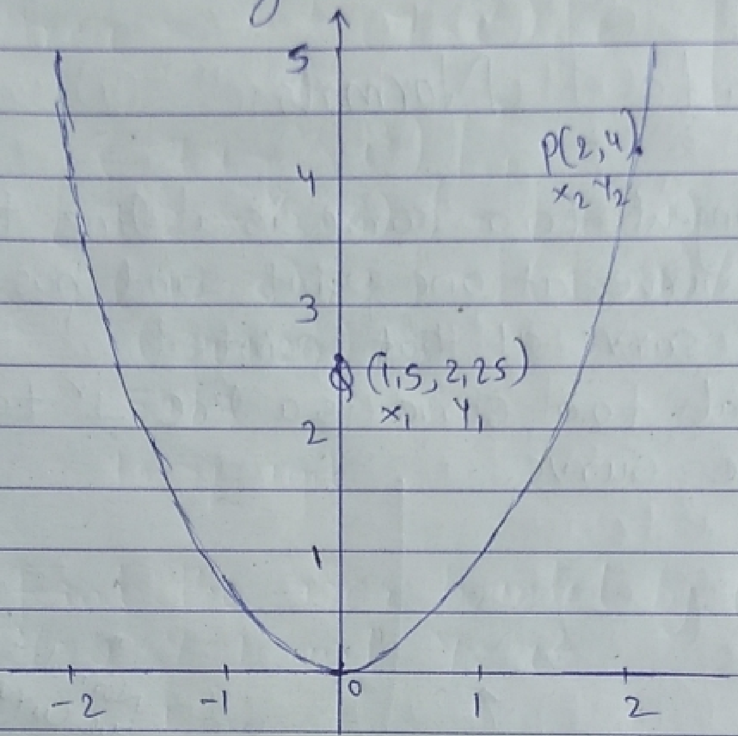


slope of the tangent at P.

The slope of curve $y = f(x)$ at the point P means the slope of the tangent at the point.

By defination the slope is given by

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



The slope of PQ is

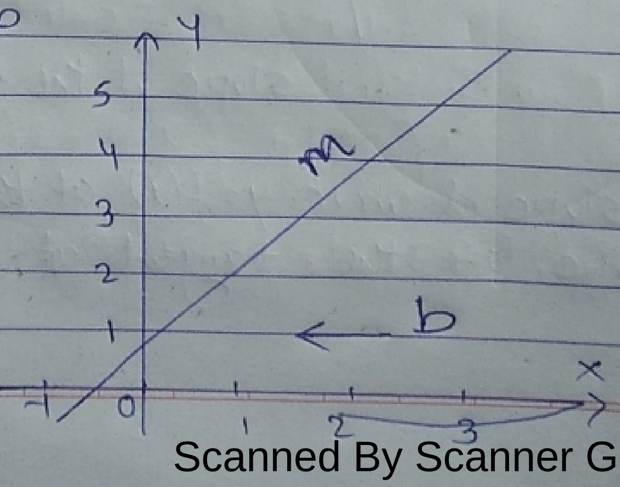
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4 - 2.25}{2 - 1.5}$$

$$= 3.5$$

Equation of a streight Line

* $y = mx + b$



Ex: Find the slope of a line that passes through the points (2,-1) & (-5,3)

$$\text{slope } (m) = \frac{y_2 - y_1}{x_2 - x_1}$$

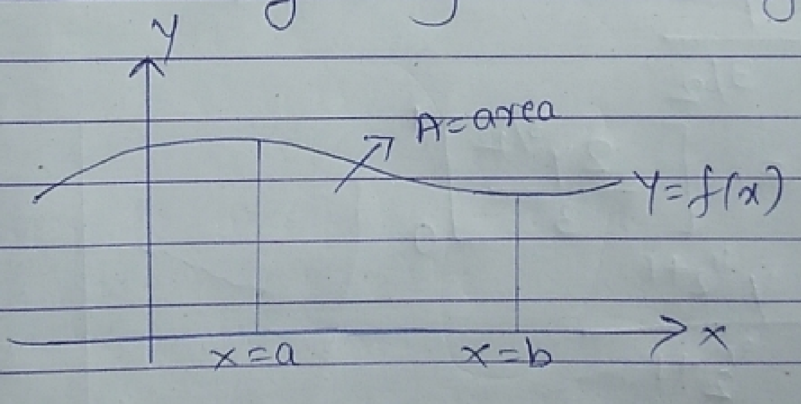
$$= \frac{3-(-1)}{-5-2}$$

$$\text{slope } (m) = -\frac{4}{7}$$

Application of integration

* Area under the Curve:

The area under a curve between 2 points is found out by doing a definite integral between the 2 point. To find the area under curve $y=f(x)$ bet^n $x=a$ & $x=b$, integrate $y=f(x)$ bet^n the limits a & b. This area can be calculated using integration with given limits



$$\int_a^b f(x)\,dx$$

$y=f(x)$

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Examples:

1) Calculate the area under the curve function, $f(x) = 7-x^2$ limit is given as $x = -1$ to 2

$\to$ Area $= \int_{-1}^{2} (7-x^2)\,dx$ $= \boxed{\int_a^b f(x)\,dx}$

$= \left(7x - \frac{1}{3}x^3\right)\Big|_{-1}^{2}$

$= \left[7.2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right]$

$= \left[(42-8)/3\right] - \left[(1-21)/3\right]$

$= (34+20)/3$

$= 54/3$

$= 18$ sq. unit

2) Find the area bounded by the line $y=0$, $y=1$ and $y=x^2$

Area $= \int_0^1 y^{1/2}\,dy$

$y = x^2$

$x = \sqrt{y}$

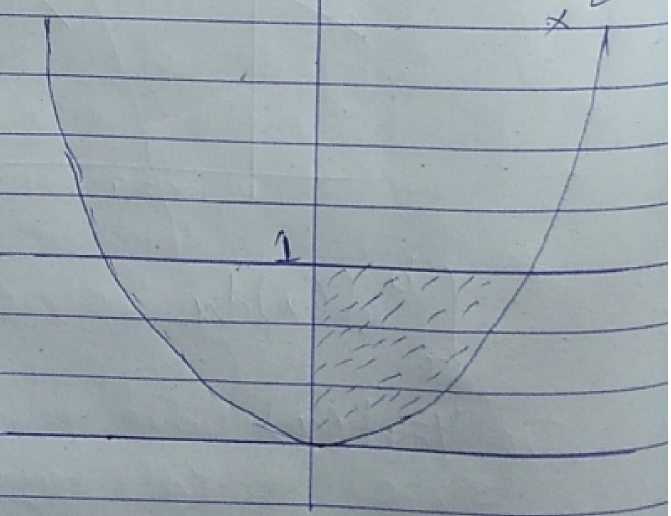
$x = y^{1/2}$

$= \dfrac{y^{3/2}}{3/2}$

$= \left[\dfrac{2y^{3/2}}{3}\right]_0^1$

$= \dfrac{y^{1/2+1}}{1/2+1}$

$= 2/3.$



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