COM1009
Introduction to Algorithms and Data Structures

Topic 02: Runtime and Asymptotic Notation

Essential reading:

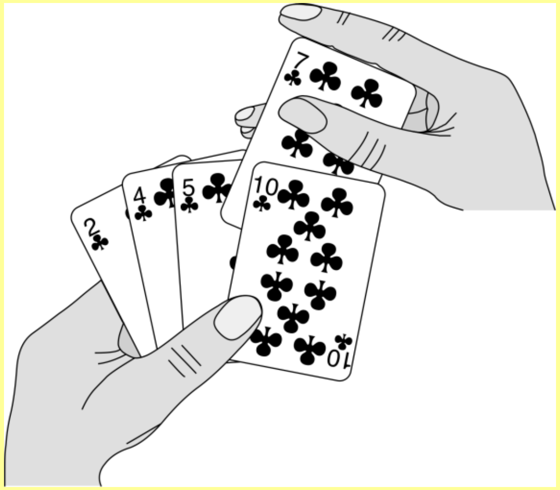
Section 3.1

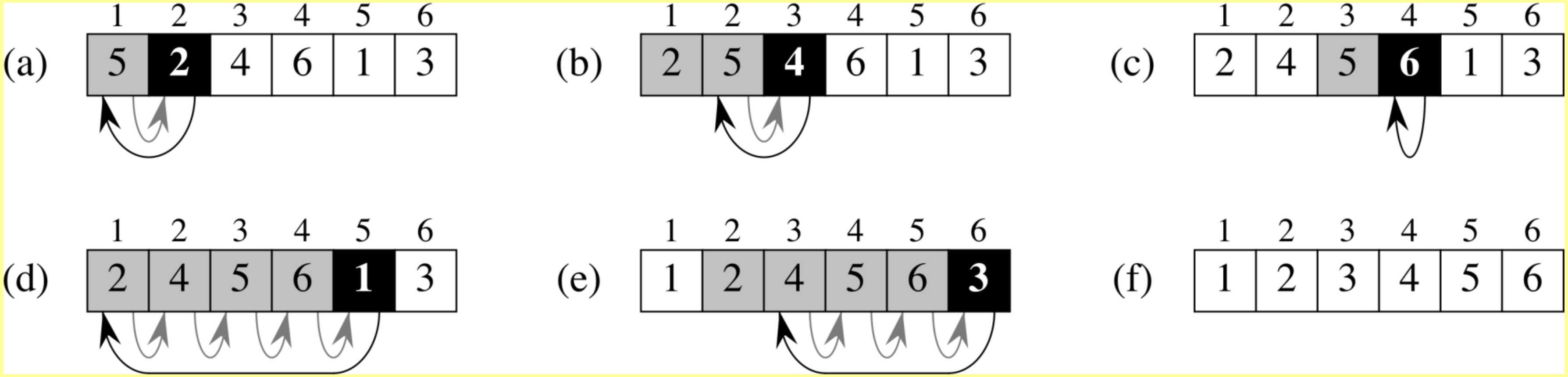
(and have a quick look at the formulas in Section 3.2, they may come in handy)

Aims of this lecture

- To recap and simplify the runtime analysis of InsertionSort.
- To talk about growth of runtime with problem size.
- To introduce asymptotic notation
- To show how to apply asymptotic notation

▶Recap: Runtime of InsertionSort (1)





▶Recap: Runtime of InsertionSort (2)

- General formula:

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j +$$

$$c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- **Best case** simplifies to $T(n) = an + b$ for constants $a > 0, b$ composed of c_1, c_2, etc.
  - A **linear** function in *n*.
- **Worst case** simplifies to $T(n) = an^2 + bn + c$ for constants $a > 0, b, c$ composed of c_1, c_2, etc.
  - A **quadratic** function in *n*.

On best case and worst case

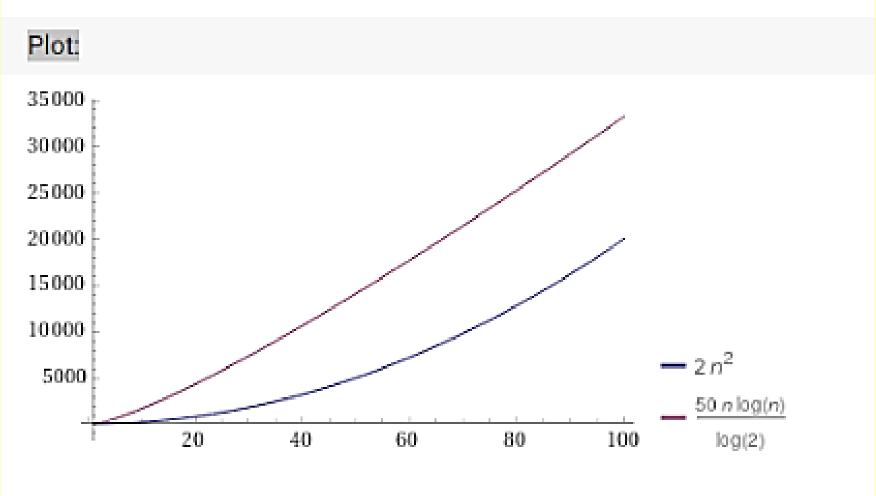
- The running time of every instance is sandwiched between the best case and the worst case running time.
- Average case: performance on “average” input.
  - For sorting: assume each permutation is equally likely
  - For other problems it’s not always clear what an average input is
- Why worst case is important:
  - Guarantee that the algorithm will never take longer
  - For some algorithms, the worst case is quite frequent
  - Often (not always) the average case is as bad as the worst case

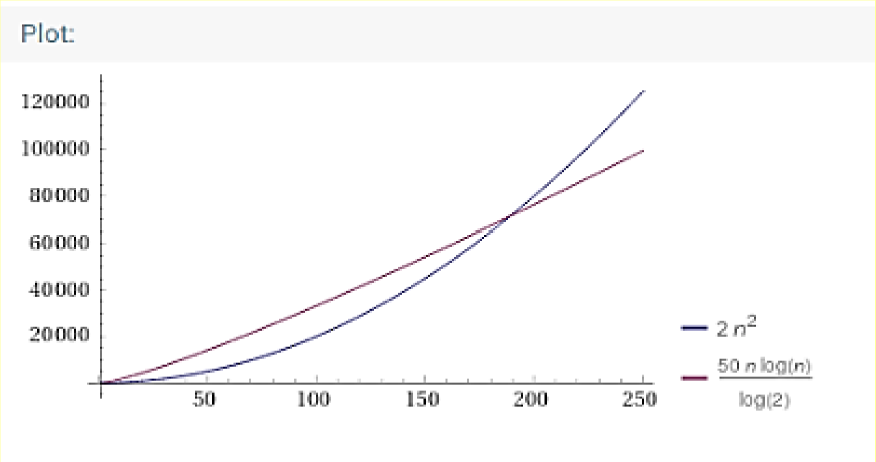
Comparison of two runtimes

Let's compare two algorithms:

- A has runtime $2n^2$
- B has runtime $50n \log n$

Which is faster?





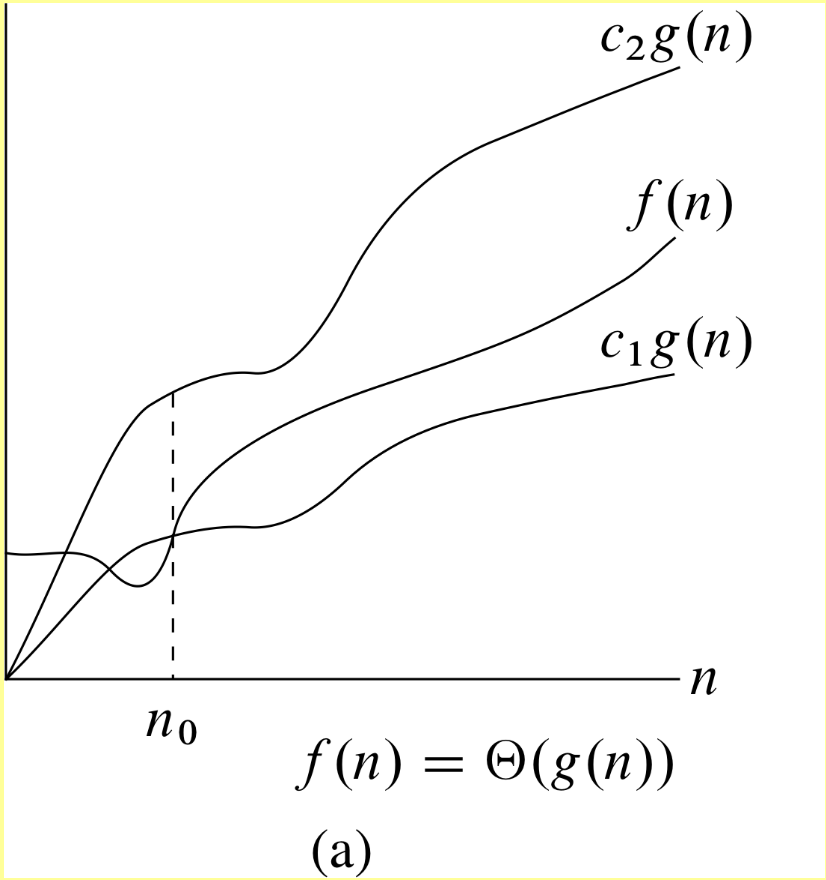
Using Wolfram Alpha

Observations

- The biggest-order term (n^2 vs. $n \log n$) dominates the runtime as n grows.
- How the runtime scales with n is more important than constant factors (for large n).
- Additive smaller order terms (e.g. "$+10n$" in "$2n^2 + 10n$") become **irrelevant** for large n.
- Care about large n, small problems (small n) are easy anyway.
- Recommendations:
  - If your problem is always very small, use the simplest algorithm.
  - Otherwise, use most **efficient** algorithm (**best growth** in n)

▶ Asymptotic Notation: Θ ("Theta")

- Idea: capture **asymptotic growth**
- Ignore constant factors
- Ignore small-order terms
- Ignore "blips" for tiny *n*
- Intuition: "Θ" **captures fastest growing term** e.g. $2n^2 + 3n = \Theta(n^2)$.
- More details in the book, Section 3.1.



Definition of $\Theta(g(n))$

For a given (non-negative) function $g(n)$ we denote by $\Theta(g(n))$ the set of functions

$$\Theta(g(n)) = \{f(n) \colon \text{there exist constants } 0 < c_1 \leq c_2 \text{ and } n_0 \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$$

A function $f(n)$ belongs to the set $\Theta(g(n))$ if it can be "sandwiched" between $c_1 g(n)$ and $c_2 g(n)$, for sufficiently large n.

We ought to write: $f(n) \in \Theta(g(n))$.

However, the common notation is: $f(n) = \Theta(g(n))$, the "equality" being read from left to right!

We say that $g(n)$ is an asymptotically tight bound for $f(n)$.

Example for Θ notation

Example: $\frac{3}{2}n^2 + \frac{7}{2}n - 4 = \Theta(n^2)$.

- To show this, we need to find constants c_1, c_2, n_0 such that for all $n \geq n_0$

$$0 \leq c_1 n^2 \leq \frac{3}{2}n^2 + \frac{7}{2}n - 4 \leq c_2 n^2$$

- To do this, let's divide by n^2: $0 \leq c_1 \leq \frac{3}{2} + \frac{7}{2n} - \frac{4}{n^2} \leq c_2$

- This is true, e.g., for $c_1 = \frac{3}{2}, c_2 = 2, n_0 = 7$.
(Other choices are possible so long as the inequalities hold.)

Examples (1)

Task: find constants $c_1, c_2, n_0 > 0$ from definition of Θ.

- $2n^2 = \Theta(n^2)$ since for all $n \geq n_0$
  $0 \leq c_1 n^2 \leq 2n^2 \leq c_2 n^2$
  when choosing, say, $c_1 = 1, c_2 = 2, n_0 = 1$

- $2n^2 - 10n = \Theta(n^2)$ since for all $n \geq n_0$
  $0 \leq c_1 n^2 \leq 2n^2 - 10n \leq c_2 n^2$
  when choosing, say, $c_1 = 1, c_2 = 2, n_0 = 10$
  (as after division by n^2 we have $1 \leq 2 - 10/n \leq 2$ for $n \geq 10$)

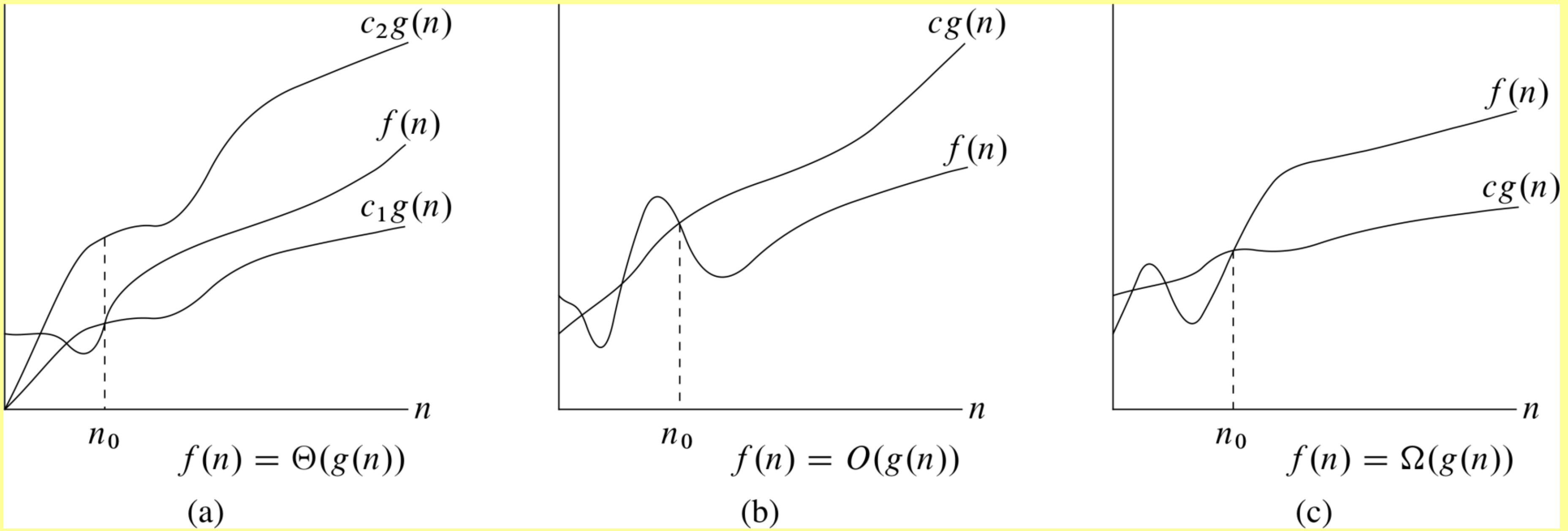
- $50n \log n = \Theta(n \log n)$ since for all $n \geq n_0$
  $0 \leq c_1 n \log n \leq 50n \log n \leq c_2 n \log n$
  when choosing, say, $c_1 = 50, c_2 = 50, n_0 = 1$

Examples (2)

- but: $2n^2 \neq \Theta(n)$ since there is no constant c_2 such that $2n^2 \leq c_2 n$ **for all** $n \geq n_0$.
- and: $2n^2 \neq \Theta(n^3)$ since there is no constant c_1 such that $2n^2 \geq c_1 n^3$ **for all** $n \geq n_0$.

Asymptotic Notation: Θ, O, Ω

- Θ expresses tight upper and lower bounds on $f(n)$.
- Use O ("big-Oh") if we only want to express an upper bound.
- Use Ω if we only want to express a lower bound.



Definition of $O\big(g(n)\big)$, $\Omega\big(g(n)\big)$

For a given (non-negative) function $g(n)$ we denote by $O\big(g(n)\big)$ and $\Omega\big(g(n)\big)$ the following sets of functions:

$$O(g(n)) = \{f(n) \colon \text{there exist constants } 0 < c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}$$

$$\Omega(g(n)) = \{f(n) \colon \text{there exist constants } 0 < c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}$$

O and Ω are weaker than Θ. Together, they give Θ:

For any $f(n)$ and $g(n)$ we have $f(n) = \Theta\big(g(n)\big)$ if and only if $f(n) = O\big(g(n)\big)$ and $f(n) = \Omega\big(g(n)\big)$.

Faster and slower growth

- Little-Oh "o" and little omega "ω" indicate strictly slower and faster growth, respectively:

$$f(n) = o(g(n)) \text{ if } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = \omega(g(n)) \text{ if } g(n) = o(f(n))$$

Asymptotic Notation: Overview

Notation	Meaning	Analogy
$f(n) = O(g(n))$	f grows at most as fast as g	"$f \leq g$"
$f(n) = \Omega(g(n))$	f grows at least as fast as g	"$f \geq g$"
$f(n) = \Theta(g(n))$	f grows as fast as g	"$f = g$"
$f(n) = o(g(n))$	f grows slower than g	"$f < g$"
$f(n) = \omega(g(n))$	f grows faster than g	"$f > g$"

- "Equalities" are to be **read from left to right** – think of $f(n) = O(g(n))$ as actually meaning $f(n) \in O(g(n))$
- So $n = O(n^2)$ is **true** but $O(n^2) = n$ is **false**!
- We can chain equalities, e. g. $n = O(n) = O(n^2)$

Common runtimes

$$
\begin{array}{cl}
\Theta(1) & \text{constant time} \\
\Theta(\log n) & \text{logarithmic time} \\
\Theta(n) & \text{linear time} \\
\Theta(n^2) & \text{quadratic time} \\
\Theta(n^3) & \text{cubic time} \\
n^k \text{ for } k = \Theta(1) & \text{polynomial time} \\
2^n & \text{exponential time}
\end{array}
$$

- Every polynomial of *log n* grows strictly slower than every polynomial of n, e. g. $(\log n)^{100} = o(n^{0.01})$
- Every polynomial of *n* grows strictly slower than every exponential function $2^{n^{\varepsilon}}$, e. g. $n^{100} = o(2^{n^{0.01}})$

Examples

Examples of using the various symbols:

- $2n + 1 = O(n)$
- $42 = O(n)$ (but not $\Theta(n)$)
- $n - 9 = \Omega(n)$
- $n^2 + n = \Omega(n)$ (but neither $O(n)$, nor $\Theta(n)$)
- $n^3 = o(n^4) = o(2^n)$
- $\sqrt{n} = \omega(\log n)$ (note that $\sqrt{n} = n^{1/2}$)

▶How to read asymptotic notation

How to read *The runtime of Algorithm XYZ is* $O(n^2)$?

The runtime of Algorithm XYZ is some (anonymous) function that grows at most as fast as n^2.

Or, more briefly,
"The runtime of Algorithm XYZ grows at most as fast as n^2."

Think of asymptotic notation as a **placeholder** for some anonymous function from the specified class.

- runtime is $\Theta(n^2)$ → runtime grows as fast as n^2
- runtime is $\Omega(n^2)$ → runtime grows at least as fast as n^2
- runtime is $o(n^2)$ → runtime grows slower than n^2
- runtime is $\omega(n^2)$ →runtime grows faster than n^2

▶Asymptotic runtime of InsertionSort

- The runtime of InsertionSort is …

$$\Omega(n) \qquad\qquad O(n^2)$$

(it grows at least as fast as *n* and at most as fast as n^2)

- This is because:
  - The best-case runtime is $\Theta(n)$
  - The worst-case runtime is $\Theta(n^2)$
  - So for every input, the runtime is
    - at least $\Omega(n)$
    - and at most $O(n^2)$

How to find c_1, c_2, n_0

- It is often helpful (though not compulsory) to divide by g(n), e.g.

$$c_1 n \leq 10n + 5 \leq c_2 n \quad \Leftrightarrow \quad c_1 \leq 10 + \frac{5}{n} \leq c_2$$

  Then try c_1, c_2 **sandwiching the constant term**, e.g. $c_1 = 10, c_2 = 15$.

- Remember that $\boldsymbol{c_1 > 0}$: to show that $1 - \frac{3}{n} = \Omega(1)$ we cannot use $n_0 = 3$ as then there is no suitable $c_1 > 0$!
  However, say, $n_0 = 6$ and $c_1 = \frac{1}{2}$ works as $1/2 \leq 1 - \frac{3}{n}$ for all $n \geq 6$.

- Also remember that inequalities need to hold **for all $\boldsymbol{n \geq n_0}$**.
  For instance, to show $1 - \frac{3}{n} = O(1)$ we cannot use $c_2 = \frac{1}{2}$ as $1 - \frac{3}{n} \leq \frac{1}{2}$ is false for $n > 6$! Need to choose $c_2 \geq 1$ (e.g. $c_2 = 1$).

- No need to invest time to find the best possible constants.

Rules to make runtime analysis simple

- For two non-negative functions *f(n)*, *g(n)*:
  1. Slower functions can be ignored:

$$f(n) + g(n) = \Theta(\max(f(n), g(n)))$$

  2. Asymptotic times can be multiplied:

$$\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$$

```
Foo
foo
foo
for i = 1 to n do
    foo
    foo
    foo
```

Example of how to use this:

- First two lines take time $\Theta(1)$
- One iteration of the for loop takes time $\Theta(1)$
- The for loop is executed $\Theta(n)$ times
- Total time is:

$$\Theta(1) + \Theta(n) \cdot \Theta(1) = \Theta(n).$$

Asymptotic Notation: Comparing Sets

- Is $2n^2 + \Theta(n) = \Theta(n^2)$ true or false?
  (Think of $\Theta(n)$ as a placeholder for an anonymous function from the set $\Theta(n)$ of all functions that grow linearly in *n*.)
- Such a statement is true if **no matter how the anonymous functions are chosen on the left of the equal sign, there is a way to choose the anonymous functions on the right of the equal sign** to make the equation valid.
- Example: is $O(n) = O(n^2)$?

  True, because $O(n) \subseteq O(n^2)$
- Example: is $O(n^2) = O(n)$?

  False, for example n^2 is in $O(n^2)$ but not in $O(n)$!

Summary

- We may consider best-case, average-case, and worst-case runtime. Often the focus is on **worst-case runtime**.
- The most important aspect of efficiency is **scalability**: how the runtime grows with the input size, n.
  - Asymptotic perspective: $n \geq n_0$ (smaller problems are easy)
  - Scalability is more important than constant factors
  - Small-order terms become more insignificant as n grows.
- **Asymptotic notation** $(O, \Omega, \Theta, o, \omega)$ hides constant factors and small-order terms, revealing asymptotic runtimes.
- Asymptotic notation refers to **sets of functions**, but for convenience is written with equalities read from **left to right**.