Definition 1. A *semiring* is a set S equipped with two binary operations $(+, \cdot)$ such that:

- The *additive reduct* $(S, +)$, which we denote by A, is a commutative semigroup.
- The *multiplicative reduct* $(S, \cdot)$, which we denote by M, is a semigroup.
- Multiplication distributes over addition, i.e. for all $a, b, c \in S$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

Thus, we may identify the semiring S with the pair (A, M).

Note that since both A and M have the same underlying set S, the symmetric group $\mathrm{Sym}(S)$ acts naturally on both structures.

Definition 2 (Semiring isomorphism)**.** We say that two semirings $S = (A, M)$ and $S' = (A', M')$ are *isomorphic* if there exists a bijection $\phi : S \to S'$ such that $A \overset{\phi}{\cong} A'$ and $M \overset{\phi}{\cong} M'$, where $\overset{\phi}{\cong}$ denotes a semigroup isomorphism under ϕ.

Theorem 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S. For any permutation $\sigma \in \mathrm{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ. Suppose that for two permutations $\sigma, \tau \in \mathrm{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are isomorphic.
2. σ and τ lie in the same double coset of $\mathrm{Aut}(A) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(M)$.

Proof. We will denote the product $i \cdot j$ in M by $M(i, j)$.

For the forward direction, suppose $(A, M^\sigma) \cong (A, M^\tau)$. Equivalently, there exists a bijection $\phi : (A, M^\sigma) \to (A, M^\tau)$ such that for all $i, j \in S$,:

$$\phi \in \mathrm{Aut}(A) \tag{1}$$

$$\phi(M^\sigma(i, j)) = M^\tau(\phi(i), \phi(j)) \tag{2}$$

From (2), we have

$$\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = \tau(M(\tau^{-1}\phi(i), \tau^{-1}\phi(j)))$$
$$\tau^{-1}\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = M(\tau^{-1}\phi(i), \tau^{-1}\phi(j))$$

Now, let $\gamma = \tau^{-1}\phi\sigma$, $x = \sigma^{-1}(i)$ and $y = \sigma^{-1}(j)$. Then,

$$\gamma(M(x, y)) = M(\gamma(x), \gamma(y))$$

implying that $\gamma \in \operatorname{Aut}(M)$ and in particular, $\gamma^{-1} \in \operatorname{Aut}(M)$.

Rearranging the definition of γ, we obtain $\tau = \phi\sigma\gamma^{-1}$. Since $\phi \in \operatorname{Aut}(A)$, we conclude that τ and σ lie in the same double coset of $\operatorname{Aut}(A)\backslash \operatorname{Sym}(S)/\operatorname{Aut}(M)$.

For the reverse direction, we begin by supposing that σ and τ are in the same double coset.

Choose $\alpha \in \operatorname{Aut}(M)$ and $\beta \in \operatorname{Aut}(A)$ such that

$$\tau = \beta\sigma\alpha$$

Note that for a function ϕ to be an isomorphism from $(A, M^\sigma) \to (A, M^\tau)$, it must satisfy properties (1) and (2) detailed in the forward direction. In particular, if we have property (1), then as an automorphism of A, ϕ would automatically be bijective on the underlying set S.

We claim that β satisfies these properties. The first property follows trivially from the definition of β. For the second property,

$$\begin{aligned}
\beta(M^\sigma(i,j)) &= \beta\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\
&= \tau\alpha^{-1}(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\
&\overset{*}{=} \tau(M(\alpha^{-1}\sigma^{-1}(i), \alpha^{-1}\sigma^{-1}(j))) \\
&= \tau(M(\tau^{-1}\beta(i), \tau^{-1}\beta(j))) \\
&= M^\tau(\beta(i), \beta(j))
\end{aligned}$$

where we have used $\alpha \in \operatorname{Aut}(M)$ and therefore $\alpha^{-1} \in \operatorname{Aut}(M)$ to justify the equality labelled '*'.

Hence, we have shown that β satisfies (1) and (2) and is therefore the necessary isomorphism. □

Note that the above theorem can be easily adapted to yield the following result for equivalence of semirings.

Corollary 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S. For any permutation $\sigma \in \operatorname{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ. Suppose that for two permutations $\sigma, \tau \in \operatorname{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are equivalent, i.e. isomorphic or anti-isomorphic.
2. σ and τ lie in the same double coset of $\operatorname{Aut}(A)\backslash \operatorname{Sym}(S)/\operatorname{Aut}^*(M)$.

where $\operatorname{Aut}^*(M)$ denotes the group of automorphisms and anti-automorphisms of M.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	?
4	866	?
5	15,751	?
6	354,409	?
7	9,908,909	?

Table 1: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism.