

Definition 1. A *semiring* is a set S equipped with two binary operations $(+, \cdot)$ such that:

- The *additive reduct* $(S, +)$, which we denote by A , is a commutative semigroup.
- The *multiplicative reduct* (S, \cdot) , which we denote by M , is a semigroup.
- Multiplication distributes over addition, i.e. for all $a, b, c \in S$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

Thus, we may identify the semiring S with the pair (A, M) .

Note that since both A and M have the same underlying set S , the symmetric group $\text{Sym}(S)$ acts naturally on both structures.

Definition 2 (Semiring isomorphism). We say that two semirings $S = (A, M)$ and $S' = (A', M')$ are *isomorphic* if there exists a bijection $\phi : S \rightarrow S'$ such that $A \xrightarrow{\phi} A'$ and $M \xrightarrow{\phi} M'$, where $\xrightarrow{\phi}$ denotes a semigroup isomorphism under ϕ .

Theorem 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S . For any permutation $\sigma \in \text{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ . Suppose that for two permutations $\sigma, \tau \in \text{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are isomorphic.
2. σ and τ lie in the same double coset of $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}(M)$.

Proof. We will denote the product $i \cdot j$ in M by $M(i, j)$.

For the forward direction, suppose $(A, M^\sigma) \cong (A, M^\tau)$. Equivalently, there exists a bijection $\phi : (A, M^\sigma) \rightarrow (A, M^\tau)$ such that for all $i, j \in S$:

$$\phi \in \text{Aut}(A) \tag{1}$$

$$\phi(M^\sigma(i, j)) = M^\tau(\phi(i), \phi(j)) \tag{2}$$

From (2), we have

$$\begin{aligned} \phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) &= \tau(M(\tau^{-1}\phi(i), \tau^{-1}\phi(j))) \\ \tau^{-1}\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) &= M(\tau^{-1}\phi(i), \tau^{-1}\phi(j)) \end{aligned}$$

Now, let $\gamma = \tau^{-1}\phi\sigma$, $x = \sigma^{-1}(i)$ and $y = \sigma^{-1}(j)$. Then,

$$\gamma(M(x, y)) = M(\gamma(x), \gamma(y))$$

implying that $\gamma \in \text{Aut}(M)$ and in particular, $\gamma^{-1} \in \text{Aut}(M)$.

Rearranging the definition of γ , we obtain $\tau = \phi\sigma\gamma^{-1}$. Since $\phi \in \text{Aut}(A)$, we conclude that τ and σ lie in the same double coset of $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}(M)$.

For the reverse direction, we begin by supposing that σ and τ are in the same double coset.

Choose $\alpha \in \text{Aut}(M)$ and $\beta \in \text{Aut}(A)$ such that

$$\tau = \beta\sigma\alpha$$

Note that for a function ϕ to be an isomorphism from $(A, M^\sigma) \rightarrow (A, M^\tau)$, it must satisfy properties (1) and (2) detailed in the forward direction. In particular, if we have property (1), then as an automorphism of A , ϕ would automatically be bijective on the underlying set S .

We claim that β satisfies these properties. The first property follows trivially from the definition of β . For the second property,

$$\begin{aligned} \beta(M^\sigma(i, j)) &= \beta\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\ &= \tau\alpha^{-1}(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\ &\stackrel{*}{=} \tau(M(\alpha^{-1}\sigma^{-1}(i), \alpha^{-1}\sigma^{-1}(j))) \\ &= \tau(M(\tau^{-1}\beta(i), \tau^{-1}\beta(j))) \\ &= M^\tau(\beta(i), \beta(j)) \end{aligned}$$

where we have used $\alpha \in \text{Aut}(M)$ and therefore $\alpha^{-1} \in \text{Aut}(M)$ to justify the equality labelled ‘*’.

Hence, we have shown that β satisfies (1) and (2) and is therefore the necessary isomorphism. \square

Note that the above theorem can be easily adapted to yield the following result for equivalence of semirings.

Corollary 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S . For any permutation $\sigma \in \text{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ . Suppose that for two permutations $\sigma, \tau \in \text{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are equivalent, i.e. isomorphic or anti-isomorphic.
2. σ and τ lie in the same double coset of $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}^*(M)$.

where $\text{Aut}^*(M)$ denotes the group of automorphisms and anti-automorphisms of M .

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	?
2	10	?
3	132	?
4	2,341	?
5	57,427	?
6	7,571,579	?

Table 1: Numbers of semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	?
4	866	?
5	15,751	?
6	354,409	?
7	9,908,909	?

Table 2: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	?
2	2	?
3	6	?
4	40	?
5	295	?
6	3,246	?
7	59,314	?

Table 3: Numbers of rigs with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	?
2	1	?
3	3	?
4	20	?
5	149	?
6	1,488	?
7	18,554	?
8	?	?

Table 4: Numbers of ai-rigs with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	?
2	4	?
3	22	?
4	169	?
5	1,819	?
6	41,104	?

Table 5: Numbers of unital semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism.