Theorem 1. *Let $(A, +)$ be an abelian group and $(M, \cdot)$ a group such that (A, M) forms a semiring with the usual operations. For any permutation $\sigma \in \mathrm{Sym}(M)$, let M^σ denote the semigroup obtained by permuting M via σ. Then the following statements are equivalent:*

1. *The semirings (A, M^σ) and (A, M^τ) are isomorphic.*
2. *σ and τ lie in the same double coset of $\mathrm{Aut}(A) \backslash \mathrm{Sym}(M) / \mathrm{Aut}(M)$.*

Proof. We will denote the product $i \cdot j$ in M by $M(i, j)$.

For the forward direction, suppose $(A, M^\sigma) \cong (A, M^\tau)$. Equivalently, there exists an isomorphism $\phi : (A, M^\sigma) \to (A, M^\tau)$ such that:

$$\phi \in \mathrm{Aut}(A) \tag{1}$$
$$\phi(M^\sigma(i, j)) = M^\tau(\phi(i), \phi(j)) \tag{2}$$

Writing $\gamma = \tau^{-1}\phi\sigma$, $x = \sigma^{-1}(i)$ and $y = \sigma^{-1}(j)$ where necessary, we have

$$\begin{aligned}
(2) &\implies \phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = \tau(M(\tau^{-1}\phi(i), \tau^{-1}\phi(j))) \\
&\implies \tau^{-1}\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = M(\tau^{-1}\phi(i), \tau^{-1}\phi(j)) \\
&\implies \gamma(M(\sigma^{-1}(i), \sigma^{-1}(j)) = M(\gamma\sigma^{-1}(i), \gamma\sigma^{-1}(j)) \\
&\implies \gamma M(x, y) = M(\gamma(x), \gamma(y)) \\
&\implies \gamma \in \mathrm{Aut}(M) \\
&\implies \gamma^{-1} \in \mathrm{Aut}(M)
\end{aligned}$$

Now, by rearranging to $\tau = \phi\sigma\gamma^{-1}$, and noting that from (1) we have that $\phi \in \mathrm{Aut}(A)$, we have shown that τ and σ are in the same double coset of $\mathrm{Aut}(A) \backslash \mathrm{Sym(M)} / \mathrm{Aut}(M)$

For the reverse direction, we begin by supposing that σ and τ are in the same double coset.

Choose $\alpha \in \mathrm{Aut}(M)$ and $\beta \in \mathrm{Aut}(A)$ such that

$$\tau = \beta\sigma\alpha$$

Note that for a function ϕ to be an isomorphism from $(A, M^\sigma) \to (A, M^\tau)$, it must satisfy properties (1) and (2) detailed in the forward direction.

We claim that β satisfies these properties. The first property follows trivially from the definition of β. For the second property,

$$
\begin{aligned}
\beta(M^\sigma(i,j) &= \beta\sigma(M(\sigma^{-1}(i),\sigma^{-1}(j))) \\
&= \tau\alpha^{-1}M(\sigma^{-1}(i),\sigma^{-1}(j)) \\
&\stackrel{*}{=} \tau M(\alpha^{-1}\sigma^{-1}(i),\alpha^{-1}\sigma^{-1}(j)) \\
&= \tau M(\tau^{-1}\beta(i),\tau^{-1}\beta(j)) \\
&= M^\tau(\beta(i),\beta(j))
\end{aligned}
$$

where we have used the fact that $\alpha \in \mathrm{Aut}(M)$ and therefore $\alpha^{-1} \in \mathrm{Aut}(M)$ to justify the equality labelled '*'.

Hence, we have shown that β satisfies (1) and (2) and is therefore the necessary isomorphism. $\square$

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	?
4	866	?
5	15,751	?
6	354,409	?

Table 1: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism.