Counting finite semirings

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Abstract

In this short note we count the finite semirings up to isomorphism, and up to isomorphism and anti-isomorphism for some small values of n; for which we utilise the existing library of small semigroups in the GAP [**GAP4**] package SMALLSEMI [**Smallsemi**].

1 Introduction

Enumeration of algebraic and combinatorial structures of finite order up to isomorphism is a classical topic. Among the algebraic structures considered are groups [**BESCHE2002**, **A000001**], rings [**Blackburn2022**, **Fine1993**, **Kruse1970**, **A027623**], near-rings [**Chow2024**, **SONATA**, **A305858**], semigroups and monoids [**Distler2010**, **Distler2010**, **Distler2012**, **Distler2013**, **Forsythe1955**, **Grillet1996**, **Grillet2014**, **Jrgensen1977**, **Motzkin1955**, **Plemmons1967**, **A027851**, **A058129**, **Satoh1994**, **A001426**, **A001423**], inverse semigroups and monoids [**Malandro2019**, **A001428**, **A234843**, **A234844**, **A234845**], and many more too numerous to mention[1]. In this short note we count the number of finite semirings up to isomorphism and up to isomorphism and anti-isomorphism for $n \leq 6$; see Tables 1 to 4. We also count several special classes of semirings for (slightly) larger values of n.

This short note was initiated by an email from M. Volkov to the second author in February of 2025 asking if it was possible to verify the claim in [**Ren2025**] that the number of ai-semirings up to isomorphism with 4 elements is 866 (see the caption of Table 3 for the definition). After some initial missteps it was relatively straightforward to verify that this number is correct, by using the library of small semigroups in the GAP [**GAP4**] package SMALLSEMI [**Smallsemi**]. This short note arose out from these first steps. In contrast to groups or rings, where the numbers of non-isomorphic objects of order n is known for relatively large values of n, the number of semigroups of order 10 (up to isomorphism) is apparently not known exactly (although the number up to isomorphism and anti-isomorphism is known [**Distler2012**]). The paper [**Kleitman1976**] purports to show that almost all semigroups of order n (as n tends to infinity) up to isomorphism and anti-isomorphism are 3-nilpotent (and indeed 99.4% of the semigroups of order 8 are 3-nilpotent), and it is shown in [**Distler2012ab**] that the number of 3-nilpotent semigroups of order 10 is approximately 10^{19}, this number is likely close to the exact value. Perhaps unsurprisingly, from the perspective of counting up to isomorphism, it seems that semirings have more in common with semigroups than with rings or groups. Very roughly speaking, rings and groups are highly structured, providing strong constraints that facilitates their enumeration. On the other hand, semigroups, and seemingly semirings also, are less structured, more numerous, and harder to enumerate.

We begin with the definition of a semiring; which is a natural generalisation of the notion of a ring.

Definition 1.1 (Semiring)**.** A *semiring* is a set S equipped with two binary operations $+$ and $\times$ such that:

(a) $(S, +)$ is a commutative semigroup ($(x + y) + z = x + (y + z)$ and $x + y = y + x$ for all $x, y, z \in S$);

(b) $(S, \times)$ is a semigroup ($x \times (y \times z) = (x \times y) \times z$ for all $x, y, z \in S$); and

(c) multiplication $\times$ distributes over addition $+$ ($x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$ for all $x, y, z \in S$).

We note that some authors require that $(S, +)$ is a commutative monoid with (additive) identity; see, for example, [**Lothaire2005**, **Sakarovitch2009**]. We do not add this requirement, and refer to such objects as *semirings with zero*. The numbers of semirings with zero are discussed in [**stackexchange**] and in [**jipsen**]; see also ALG [**baueralg**] which we will discuss a little further below. An *ai-semiring* is a semiring S where $x + x = x$ for all $x \in S$. In [**Zhao2020**] it was shown that there are 61 ai-semirings of order 3.

[1]The disparity in the number of references for semigroups and monoids and the other algebraic structures is a consequence of the authors' familiarity with the literature for semigroups and monoids, and there are likely many other references that could have been included were it not for us not knowing about them.

Recall that if $(S, \times)$ and $(T, \otimes)$ are semigroups, then $\phi : S \to T$ is a *(semigroup) homomorphism* if $(x \times y)\phi = (x)\phi \otimes (y)\phi$ for all $x, y \in S$. Note that we write mappings to the right of their arguments and compose them from left to right. If $\phi : S \to T$ is a semigroup homomorphism and ϕ is bijective, then ϕ is an *(semigroup) isomorphism.* A semigroup isomorphism from a semigroup $(S, \times)$ to itself is called an *automorphism* and the group of all such automorphisms is denoted $\operatorname{Aut}(S, \times)$.

In this short note we are concerned with counting semirings up to isomorphism, and so our next definition is that of an isomorphism.

Definition 1.2 (Semiring isomorphism)**.** We say that two semirings $(S, +, \times)$ and $(S, \oplus, \otimes)$ are *isomorphic* if there exists a bijection $\phi : S \to S$ which is simultaneously a semigroup isomorphism from $(S, +)$ to $(S, \oplus)$ and from $(S, \times)$ to $(S, \otimes)$. We refer to ϕ as a *(semiring) isomorphism.*

Since a semiring is comprised of two semigroups, enumerating semirings is equivalent to enumerating those pairs consisting of an additive semigroup $(S, +)$ and a multiplicative semigroup $(S, \times)$ such that $\times$ distributes over $+$. The next theorem indicates which $(S, \times)$ we should consider for each of the additive semigroups $(S, +)$.

We denote the symmetric group on the set S by $\operatorname{Sym}(S)$. If $\sigma \in \operatorname{Sym}(S)$ and $\cdot : S \times S \to S$ is a binary operation, then we define the binary operation $\cdot^{\sigma} : S \times S \to S$ by

$$x \cdot^{\sigma} y = ((x)\sigma^{-1} \cdot (y)\sigma^{-1})\sigma. \tag{1}$$

It is straightforward to verify that (1) is a (right, group) action of $\operatorname{Sym}(S)$ on the set of all binary operations on S. Clearly if $\cdot$ is associative, then so too is $\cdot^{\sigma}$ for every $\sigma \in \operatorname{Sym}(S)$. The group of automorphisms $\operatorname{Aut}(S, \cdot)$ of a semigroup $(S, \cdot)$ coincides with the stabiliser of the operation $\cdot$ under the action of $\operatorname{Sym}(S)$ defined in (1).

Recall that if H and K are subgroups of a group G, then the *double cosets* $H\backslash G/K$ are the sets of the form $\{hgk \mid h \in H, k \in K\}$ for $g \in G$. The next theorem is key to our approach for counting semirings.

Theorem 1.3. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \operatorname{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic if and only if σ and τ belong to the same double coset of $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$.*

Proof. $(\Rightarrow)$ Suppose that ϕ is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Then $\phi \in \operatorname{Aut}(S, +)$ and $(x \times^{\sigma} y)\phi = (x)\phi \times^{\tau} (y)\phi$ for all $x, y \in S$. It follows that

$$((x)\sigma^{-1} \times (y)\sigma^{-1})\sigma\phi\tau^{-1} = (x \times^{\sigma} y)\phi\tau^{-1} = ((x)\phi \times^{\tau} (y)\phi)\tau^{-1} = (x)\phi\tau^{-1} \times (y)\phi\tau^{-1}.$$

If we set $\gamma = \sigma\phi\tau^{-1}$, $p = (x)\sigma^{-1}$, and $q = (y)\sigma^{-1}$, then $(p \times q)\gamma = (p)\gamma \times (q)\gamma$ and so $\gamma, \gamma^{-1} \in \operatorname{Aut}(S, \times)$. Rearranging we obtain $\tau = \gamma^{-1}\sigma\phi$. Since $\phi \in \operatorname{Aut}(S, +)$, we conclude that τ and σ lie in the same double coset of $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$.

$(\Leftarrow)$ Suppose that σ and τ are in the same double coset of $\operatorname{Aut}(S, +)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, \times)$. Then there exists $\alpha \in \operatorname{Aut}(S, \times)$ and $\beta \in \operatorname{Aut}(S, +)$ such that $\tau = \alpha\sigma\beta$.

We will show that β is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Since $\beta \in \operatorname{Aut}(S, +)$, it suffices to show that β is an isomorphism from $(S, \times^{\sigma})$ and $(S, \times^{\tau})$:

$$\begin{aligned}(x \times^{\sigma} y)\beta &= (x\sigma^{-1} \times y\sigma^{-1})\sigma\beta = (x\sigma^{-1} \times y\sigma^{-1})\alpha^{-1}\tau = (x\sigma^{-1}\alpha^{-1} \times y\sigma^{-1}\alpha^{-1})\tau \qquad && \alpha^{-1} \in \operatorname{Aut}(S, \times)\\ &= (x\beta\tau^{-1} \times y\beta\tau^{-1})\tau = (x\beta \times^{\tau} y\beta). && \square\end{aligned}$$

If $(S, \times)$ and $(S, \otimes)$ are semigroups, then $(S, \times)$ is said to be *anti-isomorphic* to $(S, \otimes)$ if there exists a bijection $\phi : S \to S$ such that $(x \times y)\phi = y \otimes x$ for all $x, y \in S$. The bijection ϕ is referred to as an *anti-isomorphism.* If the operations $\times$ and $\otimes$ coincide, then ϕ is an *anti-automorphism.* Clearly the composition of two anti-automorphisms is an automorphism, and the composition of an anti-automorphism and an automorphism is an anti-automorphism. As such the set of all automorphisms or anti-automorphisms forms a group under composition of functions; we denote this group by $\operatorname{Aut}^*(S, \times)$.

Similarly, using the obvious analogue of Definition 1.2, we can define anti-isomorphic semirings. It is routine to adapt the proof of Theorem 1.3 to prove the following.

Corollary 1.4. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \operatorname{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic or anti-isomorphic if and only if σ and τ belong to the same double coset of $\operatorname{Aut}^*(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$.*

2 Tables of results

It follows from Theorem 1.3 and Corollary 1.4 that to enumerate the semirings on a set S up to isomorphism (or up to isomorphism and anti-isomorphism) it suffices to consider every commutative semigroup $(S, +)$ and semigroup $(S, \times)$ and to compute representatives of the double cosets $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$ (or $\operatorname{Aut}^*(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$). The semigroups up to isomorphism and anti-isomorphism are available in SMALLSEMI [**Smallsemi**]. Since S is small, it is relatively straightforward to can compute $\operatorname{Aut}(S, \times)$ and $\operatorname{Aut}(S, +)$ (using the SEMIGROUPS [**Semigroups**] package for GAP [**GAP4**], which reduces the problem to that of computing the automorphism group, using BLISS [**bliss, junttila2007**], of a graph associated to $\operatorname{Aut}(S, \times)$). GAP [**GAP4**] contains functionality for computing double coset representatives. This is the approach implemented by the authors of the current paper in the GAP [**GAP4**] package SEMIRINGS [**Semirings**] to compute the numbers in this section.

Below are some tables listing the numbers of semirings (Definition 1.1) up to isomorphism, and up to isomorphism or anti-isomorphism, with various properties for some small values of $n \in \mathbb{N}$. As far as we know, many of the numbers in these tables were not previously known. In particular, we are not aware of any results in the literature about the number of semirings up to isomorphism and anti-isomorphism. Some of the numbers in the tables below can be found using ALG [**baueralg**], although this approach is considerably slower than the approach described here, largely because the precomputed data for small semigroups available in SMALLSEMI [**Smallsemi**] does a lot of the heavy lifting.

As a sanity check, where possible, we have checked the numbers produced by SEMIRINGS [**Semirings**] against those in the literature, and those we could produce using ALG [**baueralg**]. With one exception, these numbers agreed with each other. ALG [**baueralg**] computes that there are $57,443$ semirings (using Definition 1.1) of order 5 up to isomorphism, whereas SEMIRINGS [**Semirings**] computes $57,427$. Using GAP [**GAP4**] and SEMIRINGS [**Semirings**] we attempted to verify whether or not the output of ALG [**baueralg**] was correct. Unfortunately, it appears that 16 of the pairs of multiplication tables output by ALG [**baueralg**] do not satisfy one or the other of the distributivity conditions from Definition 1.1(iii); see [**bauer˙alg˙issue16**] for more details. This precisely accounts for the difference between the number output by SEMIRINGS [**Semirings**] and the number output by ALG [**baueralg**].

The numbers present in the tables below were computed using a variety of desktop and laptop computers with the following specs: a 2025 MacBook Pro M4 Pro with 14 threads and 48GB of RAM; an Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz processor with 12 threads and 64GB of RAM; an Intel(R) Core(TM) i7-12700KF @ 5GHz with 20 threads and 64GB of RAM. We mostly include this information to indicate that we did not make use of particularly powerful computing resources. The longest running of the computation in the tables below took approximately 2400 CPU hours. This was the time taken to count the semirings with 0 up to isomorphism and anti-isomorphism with 7 elements.

For each column in each table, the number of CPU hours taken to compute the largest value in that column has been included.

These tables are only a sample of the results that can be obtained using SEMIRINGS [**Semirings**].

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1	1	1	1	1
2	10	4	4	2	9	4	4	2
3	132	22	22	6	106	20	21	6
4	2,341	283	169	40	1,713	226	155	38
5	57,427	4,717	1,819	295	38,247	3,365	1,561	262
6	7,571,579	108,992	41,104	3,246	4,102,358	71,138	30,112	2,681
7	-	8,925,672	11,679,328	59,314	-	4,910,824	6,268,858	43,331

Table 1: Numbers of semirings (Definition 1.1) with n elements up to isomorphism and up to isomorphism or anti-isomorphism, and the time t, in hours, taken to calculate the largest value. See [**MSsemirings**, **MSsemiringsWithOneAndZero**, **MSsemiringsWithOne**] for some of these numbers up to isomorphism.

n	up to isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1
2	8	4	4	2
3	80	18	20	6
4	1,067	169	141	36
5	18,188	1,990	1,276	228
6	543,458	32,212	17,621	2,075
7	162,744,745	799,354	690,924	25,640
8	-	-	-	791,061

Table 2: Numbers of commutative semirings (i.e. those satisfying $x \times y = y \times x$ for all $x, y \in S$) with n elements, and the time t, in hours, taken to calculate the largest value.

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1	1	1	1	1
2	6	2	2	1	5	2	2	1
3	61	12	11	3	45	10	10	3
4	866	129	73	20	581	93	64	18
5	15,751	1,852	703	149	9,750	1,207	574	125
6	354,409	33,391	9,195	1,488	205,744	20,142	6,835	1,150
7	9,908,909	729,629	164,163	18,554	5,470,437	415,527	109,880	13,171
8	-	-	-	295,292	-	-	-	116,274

Table 3: Numbers of ai-semirings (i.e. those satisfying $x + x = x$ for all $x \in S$, where "ai" stands for "additively idempotent") with n elements, and the time t, in hours, taken to calculate the largest value.

n	up to isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1
2	4	2	2	1
3	29	8	9	3
4	289	57	55	16
5	3,589	550	437	100
6	53,661	6,639	4,296	794
7	949,843	96,264	52,043	7,493
8	20,054,643	1,639,905	764,329	84,961

Table 4: Numbers of commutative ai-semirings (i.e. those satisfying $x \times y = y \times x$ and $x + x = x$ for all $x, y \in S$) with n elements.