

**Definition 1.** A *semiring* is a set  $S$  equipped with two binary operations  $(+, \cdot)$  such that:

- The *additive reduct*  $(S, +)$ , which we denote by  $A$ , is a commutative semigroup.
- The *multiplicative reduct*  $(S, \cdot)$ , which we denote by  $M$ , is a semigroup.
- Multiplication distributes over addition, i.e. for all  $a, b, c \in S$ ,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

Thus, we may identify the semiring  $S$  with the pair  $(A, M)$ .

Note that since both  $A$  and  $M$  have the same underlying set  $S$ , the symmetric group  $\text{Sym}(S)$  acts naturally on both structures.

**Definition 2** (Semiring isomorphism). We say that two semirings  $S = (A, M)$  and  $S' = (A', M')$  are *isomorphic* if there exists a bijection  $\phi : S \rightarrow S'$  such that  $A \xrightarrow{\phi} A'$  and  $M \xrightarrow{\phi} M'$ , where  $\xrightarrow{\phi}$  denotes a semigroup isomorphism under  $\phi$ .

**Theorem 1.** Let  $A$  be a commutative semigroup and  $M$  be a semigroup, such that  $A$  and  $M$  are both defined on the same underlying set  $S$ . For any permutation  $\sigma \in \text{Sym}(S)$ , let  $M^\sigma$  denote the semigroup obtained by permuting  $M$  via  $\sigma$ . Suppose that for two permutations  $\sigma, \tau \in \text{Sym}(S)$ , the pairs  $(A, M^\sigma)$  and  $(A, M^\tau)$  both form semirings. Then, the following statements are equivalent:

1.  $(A, M^\sigma)$  and  $(A, M^\tau)$  are isomorphic.
2.  $\sigma$  and  $\tau$  lie in the same double coset of  $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}(M)$ .

*Proof.* We will denote the product  $i \cdot j$  in  $M$  by  $M(i, j)$ .

For the forward direction, suppose  $(A, M^\sigma) \cong (A, M^\tau)$ . Equivalently, there exists a bijection  $\phi : (A, M^\sigma) \rightarrow (A, M^\tau)$  such that for all  $i, j \in S$ :

$$\phi \in \text{Aut}(A) \tag{1}$$

$$\phi(M^\sigma(i, j)) = M^\tau(\phi(i), \phi(j)) \tag{2}$$

From (2), we have

$$\begin{aligned} \phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) &= \tau(M(\tau^{-1}\phi(i), \tau^{-1}\phi(j))) \\ \tau^{-1}\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) &= M(\tau^{-1}\phi(i), \tau^{-1}\phi(j)) \end{aligned}$$

Now, let  $\gamma = \tau^{-1}\phi\sigma$ ,  $x = \sigma^{-1}(i)$  and  $y = \sigma^{-1}(j)$ . Then,

$$\gamma(M(x, y)) = M(\gamma(x), \gamma(y))$$

implying that  $\gamma \in \text{Aut}(M)$  and in particular,  $\gamma^{-1} \in \text{Aut}(M)$ .

Rearranging the definition of  $\gamma$ , we obtain  $\tau = \phi\sigma\gamma^{-1}$ . Since  $\phi \in \text{Aut}(A)$ , we conclude that  $\tau$  and  $\sigma$  lie in the same double coset of  $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}(M)$ .

For the reverse direction, we begin by supposing that  $\sigma$  and  $\tau$  are in the same double coset.

Choose  $\alpha \in \text{Aut}(M)$  and  $\beta \in \text{Aut}(A)$  such that

$$\tau = \beta\sigma\alpha$$

Note that for a function  $\phi$  to be an isomorphism from  $(A, M^\sigma) \rightarrow (A, M^\tau)$ , it must satisfy properties (1) and (2) detailed in the forward direction. In particular, if we have property (1), then as an automorphism of  $A$ ,  $\phi$  would automatically be bijective on the underlying set  $S$ .

We claim that  $\beta$  satisfies these properties. The first property follows trivially from the definition of  $\beta$ . For the second property,

$$\begin{aligned} \beta(M^\sigma(i, j)) &= \beta\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\ &= \tau\alpha^{-1}(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\ &\stackrel{*}{=} \tau(M(\alpha^{-1}\sigma^{-1}(i), \alpha^{-1}\sigma^{-1}(j))) \\ &= \tau(M(\tau^{-1}\beta(i), \tau^{-1}\beta(j))) \\ &= M^\tau(\beta(i), \beta(j)) \end{aligned}$$

where we have used  $\alpha \in \text{Aut}(M)$  and therefore  $\alpha^{-1} \in \text{Aut}(M)$  to justify the equality labelled ‘\*’.

Hence, we have shown that  $\beta$  satisfies (1) and (2) and is therefore the necessary isomorphism.  $\square$

Note that the above theorem can be easily adapted to yield the following result for equivalence of semirings.

**Corollary 1.** Let  $A$  be a commutative semigroup and  $M$  be a semigroup, such that  $A$  and  $M$  are both defined on the same underlying set  $S$ . For any permutation  $\sigma \in \text{Sym}(S)$ , let  $M^\sigma$  denote the semigroup obtained by permuting  $M$  via  $\sigma$ . Suppose that for two permutations  $\sigma, \tau \in \text{Sym}(S)$ , the pairs  $(A, M^\sigma)$  and  $(A, M^\tau)$  both form semirings. Then, the following statements are equivalent:

1.  $(A, M^\sigma)$  and  $(A, M^\tau)$  are equivalent, i.e. isomorphic or anti-isomorphic.
2.  $\sigma$  and  $\tau$  lie in the same double coset of  $\text{Aut}(A) \backslash \text{Sym}(S) / \text{Aut}^*(M)$ .

where  $\text{Aut}^*(M)$  denotes the group of automorphisms and anti-automorphisms of  $M$ .

**Remark 1.**  $\text{Aut}(M)$  and  $\text{Aut}^*(M)$  are equal if  $M$  is non-self-dual.

**Remark 2.** Suppose  $M$  is a self-dual semigroup and  $\tau$  is an anti-automorphism on  $M$ . Then,

$$\text{Aut}^*(M) = \langle \text{Generators}(\text{Aut}(M)) \cup \{\tau\} \rangle.$$

$n$	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	10	9
3	132	106
4	2,341	1,713
5	57,427	38,247
6	7,571,579	4,102,358

Table 1: Numbers of semirings with  $n$  elements up to isomorphism and up to isomorphism or anti-isomorphism.

$n$	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	45
4	866	581
5	15,751	9,750
6	354,409	205,744
7	9,908,909	?

Table 2: Numbers of ai-semirings with  $n$  elements up to isomorphism and up to isomorphism or anti-isomorphism.

$n$	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	2	2
3	6	6
4	40	38
5	295	262
6	3,246	2,681
7	59,314	43,331

Table 3: Numbers of rigs with  $n$  elements up to isomorphism and up to isomorphism or anti-isomorphism.

$n$	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	1	1
3	3	3
4	20	18
5	149	125
6	1,488	1,150
7	18,554	13,171
8	?	?

Table 4: Numbers of ai-rigs with  $n$  elements up to isomorphism and up to isomorphism or anti-isomorphism.

$n$	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	4	4
3	22	21
4	169	155
5	1,819	1,561
6	41,104	30,112
7	?	?

Table 5: Numbers of unital semirings with  $n$  elements up to isomorphism and up to isomorphism or anti-isomorphism.