Counting finite semirings

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Abstract

In this short note we count the finite semirings up to isomorphism and up to anti-isomorphism for some small values of n; for which we utilise the existing library of small semigroups in the GAP [**GAP4**] package SMALLSEMI [**Smallsemi**].

1 Introduction

Enumeration of algebra and combinatorial structures of finite order up to isomorphism is a classical topic. Among the algebraic structures considered are groups [**BESCHE2002**, **A000001**], rings [**Blackburn2022**, **Fine1993**, **Kruse1970**, **A027623**], near-rings [**Chow2024**, **SONATA**, **A305858**], semigroups and monoids [**Distler2010**, **Distler2010**, **Distler2013**, **Forsythe1955**, **Grillet1996**, **Grillet2014**, **Jrgensen1977**, **Motzkin1955**, **Plemmons1967**, **A027851**, **A058129**, **Satoh1994**, **A001426**, **A001423**], inverse semigroups and monoids [**Malandro2019**, **A001428**, **A234843**, **A234844**, **A234845**], and many more too numerous to mention[1]. In this short note we count the number of finite semirings up to isomorphism and up to isomorphism and anti-isomorphism for $n \leq 6$. We also count several special classes of semirings for (slightly) larger values of n.

This short note was initiated by an email from M. Volkov to the first author in February of 2025 asking if it was possible to verify with GAP [**GAP4**] that the number of ai-semirings up to isomorphism with 4 elements is 866 in the paper [**Ren2025**]; see also [**Zhao2020**] where it is shown that there are 61 ai-semirings of order 3. After some initial missteps it was relatively straightforward to verify that this number is correct, by using the library of small semigroups in the GAP [**GAP4**] package SMALLSEMI [**Smallsemi**]. This short note arose out from these first steps. In contrast to groups or rings, where the numbers of non-isomorphic objects of order n is known for relatively large values of n, the number of semigroups of order 11 (up to isomorphism) is not known exactly. Given that 99.4% of the semigroups of order 8 are 3-nilpotent, that the number of 3-nilpotent semigroups of order 11 is approximately 10^{26} [**Distler2012ab**], this number is likely close to the exact value; see also [**Kleitman1976**]. Perhaps unsurprisingly, from the perspective of counting up to isomorphism, it seems that semirings have more in common with semigroups than with rings or groups. Roughly speaking, rings and groups are highly structured, providing strong constraints that enable their enumeration. On the other hand, semigroups, and seemingly semirings also, are less structured, more numerous, and consequently harder to enumerate.

We begin with the definition of a semiring; which is a natural generalisation of the notion of a ring.

Definition 1.1 (Semiring)**.** A *semiring* is a set S equipped with two binary operations $+$ and $\times$ such that:

(a) $(S, +)$ is a commutative semigroup ($(x + y) + z = x + (y + z)$ and $x + y = y + x$ for all $x, y, z \in S$);
(b) $(S, \times)$ is a semigroup ($x \times (y \times z) = (x \times y) \times z$ for all $x, y, z \in S$); and
(c) multiplication $\times$ distributes over addition $+$ ($x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$ for all $x, y, z \in S$).

We note that some authors require that $(S, +)$ is a commutative monoid with additive identity denoted 0 where $x \times 0 = 0 \times x = 0$ for all $x \in S$; see, for example, [**Lothaire2005**, **Sakarovitch2009**]. We do not add this requirement, and refer to such objects as *semirings with zero*. The numbers of semirings with zero are discussed in [**stackexchange**]; see also [**baueralg**], which we will discuss further below.

In this short note we are concerned with counting semirings up to isomorphism, and so our next definition is that of an isomorphism.

[1]The disparity in the number of references for semigroups and monoids and the other algebraic structures is a consequence of the authors familiarity with the literature for semigroups and monoids, and there are likely many other references that could have been included were it not for us not knowing about them.

Definition 1.2 (Semiring isomorphism)**.** We say that two semirings $(S, +, \times)$ and $(S, \oplus, \otimes)$ are *isomorphic* if there exists a bijection $\phi : S \to S$ which is simultaneously a semigroup isomorphism from $(S, +)$ to $(S, \oplus)$ and from $(S, \times)$ to $(S, \otimes)$. We refer to ϕ as a *(semiring) isomorphism*.

If $(+, \times) = (\oplus, \otimes)$ in Definition 1.2, then the semiring isomorphism ϕ is called an *automorphism*. The group of all automorphisms of a semiring S is denoted by $\operatorname{Aut}(S)$. Since a semiring is comprised of two semigroups, enumerating semirings is equivalent to enumerating those pairs consisting of an additive semigroup $(S, +)$ and a multiplicative semigroup $(S, \times)$ such that $\times$ distributes over $+$. The next theorem indicates which $(S, \times)$ we should consider for each of the additive semigroups $(S, +)$.

We denote the symmetric group on the set S by $\operatorname{Sym}(S)$. If $\sigma \in \operatorname{Sym}(S)$ and $\cdot : S \times S \to S$ is a binary operation, then we define the binary operation $\cdot^{\sigma} : S \times S \to S$ by

$$x \cdot^{\sigma} y = ((x)\sigma^{-1} \cdot (y)\sigma^{-1})\sigma. \tag{1}$$

It is straightforward to verify that (1) is a (right group) action of $\operatorname{Sym}(S)$ on the set of all binary operations on S. Clearly if $\cdot$ is associative, then so too is $\cdot^{\sigma}$ for every $\sigma \in \operatorname{Sym}(S)$. The group of automorphisms $\operatorname{Aut}(S, \cdot)$ of a semigroup $(S, \cdot)$ coincides with the stabiliser of the operation $\cdot$ under the action of $\operatorname{Sym}(S)$ defined in (1).

Recall that if H and K are subgroups of a group G, then the *double cosets* $H\backslash G/K$ are the sets of the form $\{hgk \mid h \in H, k \in K\}$ for $g \in G$. The next theorem is key to our approach for counting semirings.

Theorem 1.3. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \operatorname{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic if and only if σ and τ belong to the same double coset of $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$.*

Proof. ($\Rightarrow$) Suppose that ϕ is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Then $\phi \in \operatorname{Aut}(S, +)$ and $(x \times^{\sigma} y)\phi = (x)\phi \times^{\tau} (y)\phi$ for all $x, y \in S$. It follows that

$$((x)\sigma^{-1} \times (y)\sigma^{-1})\sigma\phi\tau^{-1} = (x \times^{\sigma} y)\phi\tau^{-1} = ((x)\phi \times^{\tau} (y)\phi)\tau^{-1} = (x)\phi\tau^{-1} \times (y)\phi\tau^{-1}.$$

If we set $\gamma = \sigma\phi\tau^{-1}$, $p = (x)\sigma^{-1}$, and $q = (y)\sigma^{-1}$, then $(p \times q)\gamma = (p)\gamma \times (q)\gamma$ and so $\gamma, \gamma^{-1} \in \operatorname{Aut}(S, \times)$. Rearranging we obtain $\tau = \gamma^{-1}\sigma\phi$. Since $\phi \in \operatorname{Aut}(S, +)$, we conclude that τ and σ lie in the same double coset of $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$.

($\Leftarrow$) Suppose that σ and τ are in the same double coset of $\operatorname{Aut}(S, +)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, \times)$. Then there exists $\alpha \in \operatorname{Aut}(S, \times)$ and $\beta \in \operatorname{Aut}(S, +)$ such that $\tau = \alpha\sigma\beta$.

We will show that β is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Since $\beta \in \operatorname{Aut}(S, +)$, it suffices to show that β is an isomorphism from $(S, \times^{\sigma})$ and $(S, \times^{\tau})$:

$$\begin{aligned}(x \times^{\sigma} y)\beta = (x\sigma^{-1} \times y\sigma^{-1})\sigma\beta = (x\sigma^{-1} \times y\sigma^{-1})\alpha^{-1}\tau &= (x\sigma^{-1}\alpha^{-1} \times y\sigma^{-1}\alpha^{-1})\tau \qquad \alpha^{-1} \in \operatorname{Aut}(S, \times)\\ &= (x\beta\tau^{-1} \times y\beta\tau^{-1})\tau = (x\beta \times^{\tau} y\beta).\end{aligned}$$ ☐

If $(S, \times)$ and $(S, \otimes)$ are semigroups, then $(S, \times)$ is said to be *anti-isomorphic* to $(S, \otimes)$ if there exists a bijection $\phi : S \to S$ such that $(x \times y)\phi = y \otimes x$ for all $x, y \in S$. The bijection ϕ is referred to as an *anti-isomorphism*. Similarly, using the obvious analogue of Definition 1.2, we can define anti-isomorphic semigroups. It is routine to adapt the proof of Definition 1.3 to prove the following.

Corollary 1.4. *Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S. For any permutation $\sigma \in \operatorname{Sym}(S)$, let M^{σ} denote the semigroup obtained by permuting M via σ. Suppose that for two permutations $\sigma, \tau \in \operatorname{Sym}(S)$, the pairs (A, M^{σ}) and (A, M^{τ}) both form semirings. Then, the following statements are equivalent:*

(a) *(A, M^{σ}) and (A, M^{τ}) are equivalent, i.e. isomorphic or anti-isomorphic.*

(b) *σ and τ lie in the same double coset of $\operatorname{Aut}(A)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}^*(M)$.*

where $\operatorname{Aut}^*(M)$ denotes the group of automorphisms and anti-automorphisms of M.

2 Tables of results

It follows from Definition 1.3 and Definition 1.4 that to enumerate the semirings on a set S up to isomorphism (or up to isomorphism and anti-isomorphism) it suffices to consider every commutative semigroup $(S, +)$ and semigroup $(S, \times)$ and to compute representatives of the double cosets $\operatorname{Aut}(S, \times)\backslash \operatorname{Sym}(S)/ \operatorname{Aut}(S, +)$. The semigroups up to isomorphism are available in SMALLSEMI [**Smallsemi**]. Since S is small, it is relatively straightforward to can compute $\mathrm{Aut}(S, \times)$ and $\mathrm{Aut}(S, +)$ (using the SEMIGROUPS [**Semigroups**] package for GAP [**GAP4**]). GAP [**GAP4**] contains functionality for computing double coset representatives based on [**<empty citation>**]. This is the approach implemented by the authors of the current paper in the GAP [**GAP4**] package `semirings` to compute the numbers in this section.

Note that in a structure that does not have additive inverses but does have a zero element, we typically require the additional axiom

$$0 \cdot a = a \cdot 0 = 0 \qquad \forall a \in S, \tag{2}$$

Definition 2.1. The prefix 'ai' refers to *additive idempotence*. For instance, an ai-semiring S is a semiring such that the additive reduct $(S, +)$ is idempotent, i.e. it satisfies $a + a = a$ for all $a \in S$.

The semiring-like structures that can be counted using the `semirings` package are:

- Semirings
- Semirings with one
- Ai-semirings
- Ai-semirings with one
- Semirings with one and zero
- Ai-semirings with one and zero
- Semirings with zero
- Ai-semirings with zero
- Rings
- Rings with one

For instance, one could count the number of semirings with n elements up to isomorphism using `NrSemirings(n)` or up to equivalence using `NrSemirings(n, true)`. `AllSemirings` could be used to enumerate these objects. Functions for the other objects mentioned above are constructed similarly. Using the helper function `SETUPFINDER`, the package could also easily be used to count/enumerate any object which is a semiring with additional constraints, as long as the sets of valid additive and multiplicative reducts are expressible as families of semigroups.

The algorithm used to count these objects up to isomorphism is fairly rudimentary and is based on Theorem 1.3. As the condition given by this theorem is precise, we can make use of the `smallsemi` package in GAP to loop over possible semirings (A, M) in a minimal way, such that no two semirings yielded by this process can be isomorphic.

Similarly, we can count semirings up to equivalence by using the condition given in Corollary 1.4, again in a minimal way.

Below are some tables of results for the aforementioned structures. As far as we know, no results are published the number of any of these structures up to equivalence. For results up to isomorphism, those that have not been previously published (as far as we know) are marked '†'. Results that we are in the process of computing are marked '?'. As a sanity check, various results that are already published are available at Peter Jipsen's Mathematical Structures Library, though he may make use of different naming conventions[2].

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	10	9
3	132	106
4	2,341	1,713
5	57,427†	38,247
6	7,571,579†	4,102,358

Table 1: Numbers of semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 4$ up to isomorphism.

These tables are merely a sample of the results that can be obtained using the `semirings` package.

[2] Note that Jipsen's page for "semirings with one", seems to be mistitled and actually provides results for ai-semirings with one (which can be counted using the `semirings` package). This is not a difference in naming convention, but seems to just be a mistake. As far as we know, all results in Table 5 are unpublished.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	45
4	866	581
5	15,751†	9,750
6	354,409†	205,744
7	9,908,909†	5,470,437

Table 2: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 4$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	2	2
3	6	6
4	40	38
5	295	262
6	3,246	2,681
7	59,314†	43,331

Table 3: Numbers of semirings with one and zero with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 6$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	1	1
3	3	3
4	20	18
5	149	125
6	1,488	1,150
7	18,554	13,171
8	295,292†	116,274

Table 4: Numbers of ai-semirings with one and zero with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 7$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	4	4
3	22	21
4	169	155
5	1,819	1,561
6	41,104	30,112
7	?	?

Table 5: Numbers of semirings with one (unital semirings) with n elements up to isomorphism and up to isomorphism or anti-isomorphism.