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Counting finite semirings

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Abstract

In this short note we count the finite semirings up to isomorphism and up to anti-isomorphism for some small values of n; for which we utilise the existing library of small semigroups in the GAP [20] package SMALLSEMI [18].

1 Introduction

Enumeration of algebra and combinatorial structures of finite order up to isomorphism is a classical topic. Among the algebraic structures considered are groups [**A000001**, 10], rings [8, 11, 19, 21], near-rings [**Chow2024**, 7, 9], semigroups and monoids [**Distler2010**, **Distler2010**, **Distler2013**, **Forsythe1955**, **Grillet1996**, **Grillet2014**, **Jrgensen1977**, **Motzkin1955**, **Plemmons1967**, **Satoh1994**, **A001426**, **A001423**, 4, 5], inverse semigroups and monoids [**Malandro2019**, **A234843**, **A234844**, **A234845**, 2], and many more too numerous to mention[1]. In this short note we count the number of finite semirings up to isomorphism and up to isomorphism and anti-isomorphism for $n \leq 6$. We also count several special classes of semirings for (slightly) larger values of n.

This short note was initiated by an email from M. Volkov to the first author in February of 2025 asking if it was possible to verify with GAP [20] that the number of ai-semirings up to isomorphism with 4 elements is 866 in the paper [**Ren2025**]; see also [**Zhao2020**] where it is shown that there are 61 ai-semirings of order 3. After some initial missteps it was relatively straightforward to verify that this number is correct, by using the library of small semigroups in the GAP [20] package SMALLSEMI [18]. This short note arose out from these first steps. In contrast to groups or rings, where the numbers of non-isomorphic objects of order n is known for relatively large values of n, the number of semigroups of order 11 (up to isomorphism) is not known exactly. Given that 99.4% of the semigroups of order 8 are 3-nilpotent, that the number of 3-nilpotent semigroups of order 11 is approximately 10^{26} [**Distler2012ab**], this number is likely close to the exact value; see also [**Kleitman1976**]. Perhaps unsurprisingly, from the perspective of counting up to isomorphism, it seems that semirings have more in common with semigroups than with rings or groups. Roughly speaking, rings and groups are highly structured, providing strong constraints that enable their enumeration. On the other hand, semigroups, and seemingly semirings also, are less structured, more numerous, and consequently harder to enumerate.

We begin with the definition of a semiring; which is a natural generalisation of the notion of a ring.

Definition 1.1 (Semiring)**.** A *semiring* is a set S equipped with two binary operations $+$ and $\times$ such that:

(a) $(S, +)$ is a commutative semigroup ($(x + y) + z = x + (y + z)$ and $x + y = y + x$ for all $x, y, z \in S$);

(b) $(S, \times)$ is a semigroup ($x \times (y \times z) = (x \times y) \times z$ for all $x, y, z \in S$); and

(c) multiplication $\times$ distributes over addition $+$ ($x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$ for all $x, y, z \in S$).

We note that some authors require that $(S, +)$ is a commutative monoid with (additive) identity denoted 0 where $x \times 0 = 0 \times x = 0$ for all $x \in S$; see, for example, [**Lothaire2005**, **Sakarovitch2009**]. We do not add this requirement, and refer to such objects as *semirings with zero*. The numbers of semirings with zero are discussed in [**stackexchange**] and in [**jipsen**]; see also [**baueralg**] which we will discuss a little further below.

Recall that if $(S, \times)$ and $(T, \otimes)$ are semigroups, then $\phi : S \to T$ is a *(semigroup) homomorphism* if $(x \times y)\phi = (x)\phi \otimes (y)\phi$ for all $x, y \in S$. Note that we write mappings to the right of their arguments and compose them from left to right. If $\phi : S \to T$ is a semigroup homomorphism and ϕ is bijective, then ϕ is an *(semigroup) isomorphism.* If a semigroup isomorphism from a semigroup $(S, \times)$ to itself is called an *automorphism* and the group of all such automorphisms is denoted $\operatorname{Aut}(S, \times)$.

In this short note we are concerned with counting semirings up to isomorphism, and so our next definition is that of an isomorphism.

[1]The disparity in the number of references for semigroups and monoids and the other algebraic structures is a consequence of the authors familiarity with the literature for semigroups and monoids, and there are likely many other references that could have been included were it not for us not knowing about them.

Definition 1.2 (Semiring isomorphism)**.** We say that two semirings $(S, +, \times)$ and $(S, \oplus, \otimes)$ are *isomorphic* if there exists a bijection $\phi : S \to S$ which is simultaneously a semigroup isomorphism from $(S, +)$ to $(S, \oplus)$ and from $(S, \times)$ to $(S, \otimes)$. We refer to ϕ as a *(semiring) isomorphism*.

Since a semiring is comprised of two semigroups, enumerating semirings is equivalent to enumerating those pairs consisting of an additive semigroup $(S, +)$ and a multiplicative semigroup $(S, \times)$ such that $\times$ distributes over $+$. The next theorem indicates which $(S, \times)$ we should consider for each of the additive semigroups $(S, +)$.

We denote the symmetric group on the set S by $\mathrm{Sym}(S)$. If $\sigma \in \mathrm{Sym}(S)$ and $\cdot : S \times S \to S$ is a binary operation, then we define the binary operation $\cdot^{\sigma} : S \times S \to S$ by

$$x \cdot^{\sigma} y = ((x)\sigma^{-1} \cdot (y)\sigma^{-1})\sigma. \tag{1}$$

It is straightforward to verify that (**??**) is a (right, group) action of $\mathrm{Sym}(S)$ on the set of all binary operations on S. Clearly if $\cdot$ is associative, then so too is $\cdot^{\sigma}$ for every $\sigma \in \mathrm{Sym}(S)$. The group of automorphisms $\mathrm{Aut}(S, \cdot)$ of a semigroup $(S, \cdot)$ coincides with the stabiliser of the operation $\cdot$ under the action of $\mathrm{Sym}(S)$ defined in (**??**).

Recall that if H and K are subgroups of a group G, then the *double cosets* $H\backslash G/K$ are the sets of the form $\{hgk \mid h \in H, k \in K\}$ for $g \in G$. The next theorem is key to our approach for counting semirings.

Theorem 1.3. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \mathrm{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic if and only if σ and τ belong to the same double coset of $\mathrm{Aut}(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$.*

Proof. ($\Rightarrow$) Suppose that ϕ is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Then $\phi \in \mathrm{Aut}(S, +)$ and $(x \times^{\sigma} y)\phi = (x)\phi \times^{\tau} (y)\phi$ for all $x, y \in S$. It follows that

$$((x)\sigma^{-1} \times (y)\sigma^{-1})\sigma\phi\tau^{-1} = (x \times^{\sigma} y)\phi\tau^{-1} = ((x)\phi \times^{\tau} (y)\phi)\tau^{-1} = (x)\phi\tau^{-1} \times (y)\phi\tau^{-1}.$$

If we set $\gamma = \sigma\phi\tau^{-1}$, $p = (x)\sigma^{-1}$, and $q = (y)\sigma^{-1}$, then $(p \times q)\gamma = (p)\gamma \times (q)\gamma$ and so $\gamma, \gamma^{-1} \in \mathrm{Aut}(S, \times)$. Rearranging we obtain $\tau = \gamma^{-1}\sigma\phi$. Since $\phi \in \mathrm{Aut}(S, +)$, we conclude that τ and σ lie in the same double coset of $\mathrm{Aut}(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$.

($\Leftarrow$) Suppose that σ and τ are in the same double coset of $\mathrm{Aut}(S, +)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, \times)$. Then there exists $\alpha \in \mathrm{Aut}(S, \times)$ and $\beta \in \mathrm{Aut}(S, +)$ such that $\tau = \alpha\sigma\beta$.

We will show that β is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Since $\beta \in \mathrm{Aut}(S, +)$, it suffices to show that β is an isomorphism from $(S, \times^{\sigma})$ and $(S, \times^{\tau})$:

$$\begin{aligned} (x \times^{\sigma} y)\beta &= (x\sigma^{-1} \times y\sigma^{-1})\sigma\beta = (x\sigma^{-1} \times y\sigma^{-1})\alpha^{-1}\tau = (x\sigma^{-1}\alpha^{-1} \times y\sigma^{-1}\alpha^{-1})\tau \qquad \alpha^{-1} \in \mathrm{Aut}(S, \times) \\ &= (x\beta\tau^{-1} \times y\beta\tau^{-1})\tau = (x\beta \times^{\tau} y\beta). \end{aligned}$$ □

If $(S, \times)$ and $(S, \otimes)$ are semigroups, then $(S, \times)$ is said to be *anti-isomorphic* to $(S, \otimes)$ if there exists a bijection $\phi : S \to S$ such that $(x \times y)\phi = y \otimes x$ for all $x, y \in S$. The bijection ϕ is referred to as an *anti-isomorphism.* If the operations $\times$ and $\otimes$ coincide, then ϕ is an anti-isomorphism. Clearly the composition of two anti-isomorphisms is an automorphism, and the composition of an anti-isomorphism and an isomorphism is an anti-isomorphism. As such the set of all automorphisms or anti-automorphisms forms a group under composition of functions; we denote this group by $\mathrm{Aut}^*(S, \times)$. Note that $\mathrm{Aut}(S, \times)$ is an index two subgroup of $\mathrm{Aut}^*(S, \times)$.

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Corollary 1.4. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \mathrm{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic or anti-isomorphic if and only if σ and τ belong to the same double coset of $\mathrm{Aut}^*(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$. where $\mathrm{Aut}^*(S, \times)$ denotes the group of automorphisms and anti-automorphisms of $(S, \times)$. ======= Similarly, using the obvious analogue of* **??***, we can define anti-isomorphic semirings. It is routine to adapt the proof of* **??** *to prove the following.*

Corollary 1.5. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \mathrm{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic or anti-isomorphic if and only if σ and τ belong to the same double coset of $\mathrm{Aut}^*(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$. ¿¿¿¿¿¿¿ 4d0cf1396620bd46a3bff7547b7be0fab31ea333*

2 Tables of results

It follows from **??** *and* **??** *that to enumerate the semirings on a set S up to isomorphism (or up to isomorphism and anti-isomorphism) it suffices to consider every commutative semigroup $(S, +)$ and semigroup $(S, \times)$ and to compute representatives of the double cosets $\mathrm{Aut}(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$ (or $\mathrm{Aut}^*(S, \times)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(S, +)$). The semigroups up* *to isomorphism and anti-isomorphism are available in* Smallsemi *[18]. Since S is small, it is relatively straightforward to can compute* $\mathrm{Aut}(S, \times)$ *and* $\mathrm{Aut}(S, +)$ *(using the* Semigroups *[22] package for* GAP *[20], which reduces the problem to that of computing the automorphisms of a graph associated to* $\mathrm{Aut}(S, \times)$ *using* bliss *[**bliss**, **junttila2007**]).* GAP *[20] contains functionality for computing double coset representatives based. This is the approach implemented by the authors of the current paper in the* GAP *[20] package* Semirings *[**Semirings**] to compute the numbers in this section.*

*Below are some tables listing the numbers of semirings (**??**), up to isomorphism, and up to isomorphism or anti-isomorphism, with various properties for some small values of $n \in \mathbb{N}$. As far as we know, many of the numbers in these tables were not previously known. In particular, we are not aware of any results in the literature about the number of semirings up to isomorphism and anti-isomorphism. Some of the numbers in the tables below can be found using [**baueralg**], although this approach is considerably slower than the approach described here, largely because the precomputed data for small semigroups available in* Smallsemi *[18] does a lot of heavy lifting.*

¡¡¡¡¡¡¡ HEAD Some structures that can be counted using the `semirings` *package are semirings, semirings with one, ai-semirings, ai-semirings with one and zero, semirings with one and zero, ai-semirings with one and zero, semirings with zero, ai-semirings with zero, rings, and rings with one. By the prefix 'ai-' we mean additively idempotent, i.e. $a + a = a$ for all $a \in S$.*

Below are some tables of results for the aforementioned structures. As far as we know, no results are published the number of any of these structures up to equivalence. For results up to isomorphism, those that have not been previously published (as far as we know) are marked '†'. As a sanity check, various results that are already published are available at Jipsen's Mathematical Structures Library [17], though he may make use of different naming conventions[2]. ======= These tables are only a sample of the results that can be obtained using Semirings *[**Semirings**].*

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n	up to isomorphism	up to isomorphism or anti-isomorphism
1	1	1
2	10	9
3	132	106
4	2,341	1,713
5	57,427[3]	38,247
6	7,571,579	4,102,358

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Table 1: Numbers of semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See [14] for $n \leq 4$ up to isomorphism.

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Table 2: Numbers of semirings (**??**) with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	ai	ai with 0	ai with 1	ai with 0 + 1	ai	ai with 0	ai with 1	ai with 0 + 1
1								
2								
3								
4								
5								
6								
7								
8								

Table 3: Numbers of semirings (**??**) with n elements up to isomorphism and up to isomorphism or anti-isomorphism. ¿¿¿¿¿¿¿ 4d0cf1396620bd46a3bff7547b7be0fab31ea333

[2]Note that Jipsen's page for *semirings with one* [15], seems to be mistitled and actually provides results for ai-semirings with one (which can be counted using the `semirings` package). This is not a difference in naming convention, but seems to just be a mistake. As far as we know, all results in Table **??** are unpublished.

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	ai	ai with 0	ai with 1	ai with 0 + 1	ai	ai with 0	ai with 1	ai with 0 + 1
1								
2								
3								
4								
5								
6								
7								
8								

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Table 4: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See [12] for $n \leq 4$ up to isomorphism.

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Table 5: Numbers of commutative semirings ($x \times y = y \times x$ for all $x, y \in S$) with n elements.

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	ai	ai with 0	ai with 1	ai with 0 + 1	ai	ai with 0	ai with 1	ai with 0 + 1
1	1			1	1			1
2	6			1	5			1
3	61			3	45			3
4	866			20	581			18
5	15,751			149	9,750			125
6	354,409			1,488	205,744			1,150
7	9,908,909			18,554	5,470,437			13,171
8				295,292				116,274

Table 6: Numbers of ai-semirings ($x + x = x$ for all $x \in S$) with n elements.

References

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[4] O. F. I. (2025). *Number of nonisomorphic monoids (semigroups with identity) of order n, Entry A058129 in The On-Line Encyclopedia of Integer Sequences.*

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[6] O. F. I. (2025). *Number of reduced Latin squares of order n; also number of labeled loops (quasigroups with an identity element) with a fixed identity element, Entry A000315 in The On-Line Encyclopedia of Integer Sequences.*

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n	up to isomorphism				up to isomorphism or anti-isomorphism			
	ai	ai with 0	ai with 1	ai with 0 + 1	ai	ai with 0	ai with 1	ai with 0 + 1
1								
2								
3								
4								
5								
6								
7								
8								

Table 7: Numbers of commutative ai-semirings ($x \times y = y \times x$ and $x + x = x$ for all $x, y \in S$) with n elements. ¿¿¿¿¿¿¿ 4d0cf1396620bd46a3bff7547b7be0fab31ea333

n	up to isomorphism	up to isomorphism or anti-isomorphism
1	1	1
2	2	2
3	6	6
4	40	38
5	295	262
6	3,246	2,681
7	59,314	43,331

Table 8: Numbers of semirings with one and zero with n elements ¡¡¡¡¡¡¡ HEAD up to isomorphism and up to isomorphism or anti-isomorphism. See [16] for $n \leq 6$ up to isomorphism.
======= up to isomorphism, and up to isomorphism or anti-isomorphism.
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n	up to isomorphism	up to isomorphism or anti-isomorphism
1 2 3 4 5 6 7 8		

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Table 9: Numbers of ai-semirings with one and zero with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See [13] for $n \leq 7$ up to isomorphism.

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Table 10: Numbers of ai-semirings with one and zero with n elements up to isomorphism and up to isomorphism or anti-isomorphism.

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n	up to isomorphism	up to isomorphism or anti-isomorphism
1	1	1
2	4	4
3	22	21
4	169	155
5	1,819	1,561
6	41,104	30,112

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