Counting finite semirings

J. Edwards, J. D. Mitchell, and P. Ragavan

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Abstract

In this short note we count the finite semirings up to isomorphism, and up to isomorphism and anti-isomorphism for some small values of n; for which we utilise the existing library of small semigroups in the GAP [18] package Smallsemi [11].

1   Introduction

Enumeration of algebraic and combinatorial structures of finite order up to isomorphism is a classical topic. Among the algebraic structures considered are groups [4, 40], rings [5, 16, 26, 42], near-rings [1, 6, 41], semigroups and monoids [12, 13, 15, 17, 19, 20, 23, 31, 36, 37, 38, 39, 43, 46], inverse semigroups and monoids [28, 32, 33, 34, 35], and many more too numerous to mention[1]. In this short note we count the number of finite semirings up to isomorphism and up to isomorphism and anti-isomorphism for $n \leq 6$. We also count several special classes of semirings for (slightly) larger values of n.

This short note was initiated by an email from M. Volkov to the second author in February of 2025 asking if it was possible to verify the claim in [44] that the number of ai-semirings up to isomorphism with 4 elements is 866. After some initial missteps it was relatively straightforward to verify that this number is correct, by using the library of small semigroups in the GAP [18] package Smallsemi [11]. This short note arose out from these first steps. In contrast to groups or rings, where the numbers of non-isomorphic objects of order n is known for relatively large values of n, the number of semigroups of order 10 (up to isomorphism) is apparently not known exactly (although the number up to isomorphism and anti-isomorphism is known [15]). The paper [25] purports to show that almost all semigroups of order n (as n tends to infinity) up to isomorphism and anti-isomorphism are 3-nilpotent (and indeed 99.4% of the semigroups of order 8 are 3-nilpotent), and it is shown in [14] that the number of 3-nilpotent semigroups of order 10 is approximately 10^{19}, this number is likely close to the exact value. Perhaps unsurprisingly, from the perspective of counting up to isomorphism, it seems that semirings have more in common with semigroups than with rings or groups. Very roughly speaking, rings and groups are highly structured, providing strong constraints that facilitates their enumeration. On the other hand, semigroups, and seemingly semirings also, are less structured, more numerous, and harder to enumerate.

We begin with the definition of a semiring; which is a natural generalisation of the notion of a ring.

Definition 1.1 (Semiring). A *semiring* is a set S equipped with two binary operations $+$ and $\times$ such that:
(a) $(S, +)$ is a commutative semigroup $((x + y) + z = x + (y + z)$ and $x + y = y + x$ for all $x, y, z \in S)$;
(b) $(S, \times)$ is a semigroup $(x \times (y \times z) = (x \times y) \times z$ for all $x, y, z \in S)$; and
(c) multiplication $\times$ distributes over addition $+$ $(x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$ for all $x, y, z \in S)$.

We note that some authors require that $(S, +)$ is a commutative monoid with (additive) identity denoted 0 where $x \times 0 = 0 \times x = 0$ for all $x \in S$; see, for example, [27, 45]. We do not add this requirement, and refer to such objects as *semirings with zero*. The numbers of semirings with zero are discussed in [47] and in [7]; see also ALG [2] which we will discuss a little further below. In [48] it was shown that there are 61 ai-semirings of order 3.

Recall that if $(S, \times)$ and $(T, \otimes)$ are semigroups, then $\phi : S \to T$ is a *(semigroup) homomorphism* if $(x \times y)\phi = (x)\phi \otimes (y)\phi$ for all $x, y \in S$. Note that we write mappings to the right of their arguments and compose them from left to right. If $\phi : S \to T$ is a semigroup homomorphism and ϕ is bijective, then ϕ is an *(semigroup) isomorphism*. A semigroup isomorphism from a semigroup $(S, \times)$ to itself is called an *automorphism* and the group of all such automorphisms is denoted $\mathrm{Aut}(S, \times)$.

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[1]The disparity in the number of references for semigroups and monoids and the other algebraic structures is a consequence of the author's familiarity with the literature for semigroups and monoids, and there are likely many other references that could have been included were it not for us not knowing about them.

In this short note we are concerned with counting semirings up to isomorphism, and so our next definition is that of an isomorphism.

Definition 1.2 (Semiring isomorphism)**.** We say that two semirings $(S, +, \times)$ and $(S, \oplus, \otimes)$ are *isomorphic* if there exists a bijection $\phi : S \to S$ which is simultaneously a semigroup isomorphism from $(S, +)$ to $(S, \oplus)$ and from $(S, \times)$ to $(S, \otimes)$. We refer to ϕ as a *(semiring) isomorphism*.

Since a semiring is comprised of two semigroups, enumerating semirings is equivalent to enumerating those pairs consisting of an additive semigroup $(S, +)$ and a multiplicative semigroup $(S, \times)$ such that $\times$ distributes over $+$. The next theorem indicates which $(S, \times)$ we should consider for each of the additive semigroups $(S, +)$.

We denote the symmetric group on the set S by $\mathrm{Sym}(S)$. If $\sigma \in \mathrm{Sym}(S)$ and $\cdot : S \times S \to S$ is a binary operation, then we define the binary operation $\cdot^{\sigma} : S \times S \to S$ by

$$x \cdot^{\sigma} y = ((x)\sigma^{-1} \cdot (y)\sigma^{-1})\sigma. \tag{1}$$

It is straightforward to verify that (1) is a (right, group) action of $\mathrm{Sym}(S)$ on the set of all binary operations on S. Clearly if $\cdot$ is associative, then so too is $\cdot^{\sigma}$ for every $\sigma \in \mathrm{Sym}(S)$. The group of automorphisms $\mathrm{Aut}(S, \cdot)$ of a semigroup $(S, \cdot)$ coincides with the stabiliser of the operation $\cdot$ under the action of $\mathrm{Sym}(S)$ defined in (1).

Recall that if H and K are subgroups of a group G, then the *double cosets* $H \backslash G / K$ are the sets of the form $\{hgk \mid h \in H, k \in K\}$ for $g \in G$. The next theorem is key to our approach for counting semirings.

Theorem 1.3. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \mathrm{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic if and only if σ and τ belong to the same double coset of $\mathrm{Aut}(S, \times) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, +)$.*

Proof. ($\Rightarrow$) Suppose that ϕ is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Then $\phi \in \mathrm{Aut}(S, +)$ and $(x \times^{\sigma} y)\phi = (x)\phi \times^{\tau} (y)\phi$ for all $x, y \in S$. It follows that

$$((x)\sigma^{-1} \times (y)\sigma^{-1})\sigma\phi\tau^{-1} = (x \times^{\sigma} y)\phi\tau^{-1} = ((x)\phi \times^{\tau} (y)\phi)\tau^{-1} = (x)\phi\tau^{-1} \times (y)\phi\tau^{-1}.$$

If we set $\gamma = \sigma\phi\tau^{-1}$, $p = (x)\sigma^{-1}$, and $q = (y)\sigma^{-1}$, then $(p \times q)\gamma = (p)\gamma \times (q)\gamma$ and so $\gamma, \gamma^{-1} \in \mathrm{Aut}(S, \times)$. Rearranging we obtain $\tau = \gamma^{-1}\sigma\phi$. Since $\phi \in \mathrm{Aut}(S, +)$, we conclude that τ and σ lie in the same double coset of $\mathrm{Aut}(S, \times) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, +)$.

($\Leftarrow$) Suppose that σ and τ are in the same double coset of $\mathrm{Aut}(S, +) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, \times)$. Then there exists $\alpha \in \mathrm{Aut}(S, \times)$ and $\beta \in \mathrm{Aut}(S, +)$ such that $\tau = \alpha\sigma\beta$.

We will show that β is a semiring isomorphism from $(S, +, \times^{\sigma})$ to $(S, +, \times^{\tau})$. Since $\beta \in \mathrm{Aut}(S, +)$, it suffices to show that β is an isomorphism from $(S, \times^{\sigma})$ and $(S, \times^{\tau})$:

$$(x \times^{\sigma} y)\beta = (x\sigma^{-1} \times y\sigma^{-1})\sigma\beta = (x\sigma^{-1} \times y\sigma^{-1})\alpha^{-1}\tau = (x\sigma^{-1}\alpha^{-1} \times y\sigma^{-1}\alpha^{-1})\tau \qquad \alpha^{-1} \in \mathrm{Aut}(S, \times)$$

$$= (x\beta\tau^{-1} \times y\beta\tau^{-1})\tau = (x\beta \times^{\tau} y\beta). \qquad \square$$

If $(S, \times)$ and $(S, \otimes)$ are semigroups, then $(S, \times)$ is said to be *anti-isomorphic* to $(S, \otimes)$ if there exists a bijection $\phi : S \to S$ such that $(x \times y)\phi = y \otimes x$ for all $x, y \in S$. The bijection ϕ is referred to as an *anti-isomorphism*. If the operations $\times$ and $\otimes$ coincide, then ϕ is an *anti-automorphism*. Clearly the composition of two anti-automorphisms is an automorphism, and the composition of an anti-automorphism and an automorphism is an anti-automorphism. As such the set of all automorphisms or anti-automorphisms forms a group under composition of functions; we denote this group by $\mathrm{Aut}^{*}(S, \times)$.

Similarly, using the obvious analogue of Definition 1.2, we can define anti-isomorphic semirings. It is routine to adapt the proof of Definition 1.3 to prove the following.

Corollary 1.4. *Let $(S, +)$ be a commutative semigroup, let $(S, \times)$ be a semigroup, and let $\sigma, \tau \in \mathrm{Sym}(S)$ be such that $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are semirings. Then $(S, +, \times^{\sigma})$ and $(S, +, \times^{\tau})$ are isomorphic or anti-isomorphic if and only if σ and τ belong to the same double coset of $\mathrm{Aut}^{*}(S, \times) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, +)$.*

2   Tables of results

It follows from Definition 1.3 and Definition 1.4 that to enumerate the semirings on a set S up to isomorphism (or up to isomorphism and anti-isomorphism) it suffices to consider every commutative semigroup $(S, +)$ and semigroup $(S, \times)$ and to compute representatives of the double cosets $\mathrm{Aut}(S, \times) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, +)$ (or $\mathrm{Aut}^{*}(S, \times) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(S, +)$). The

semigroups up to isomorphism and anti-isomorphism are available in SMALLSEMI [11]. Since S is small, it is relatively straightforward to can compute $\mathrm{Aut}(S, \times)$ and $\mathrm{Aut}(S, +)$ (using the SEMIGROUPS [29] package for GAP [18], which reduces the problem to that of computing the automorphism group, using BLISS [21, 22], of a graph associated to $\mathrm{Aut}(S, \times)$). GAP [18] contains functionality for computing double coset representatives based. This is the approach implemented by the authors of the current paper in the GAP [18] package SEMIRINGS [30] to compute the numbers in this section.

Below are some tables listing the numbers of semirings (Definition 1.1) up to isomorphism, and up to isomorphism or anti-isomorphism, with various properties for some small values of $n \in \mathbb{N}$. As far as we know, many of the numbers in these tables were not previously known. In particular, we are not aware of any results in the literature about the number of semirings up to isomorphism and anti-isomorphism. Some of the numbers in the tables below can be found using ALG [2], although this approach is considerably slower than the approach described here, largely because the precomputed data for small semigroups available in SMALLSEMI [11] does a lot of the heavy lifting.

As a sanity check, where possible, we have checked the numbers produced by SEMIRINGS [30] against those in the literature, and those we could produce using ALG [2]. With one exception, these numbers agreed with each other. ALG [2] computes that there are $57,443$ semirings (using Definition 1.1) of order 5 up to isomorphism, whereas SEMIRINGS [30] computes $57,427$. Using GAP [18] and SEMIRINGS [30] we attempted to verify whether or not the output of ALG [2] was correct. Unfortunately, it appears that 16 of the pairs of multiplication tables output by ALG [2] do not satisfy one or the other of the distributivity conditions from Definition 1.1(iii); see [3] for more details. This precisely accounts for the difference between the number output by SEMIRINGS [30] and the number output by ALG [2].

These tables are only a sample of the results that can be obtained using SEMIRINGS [30].

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1	1	1	1	1
2	10	4	4	2	9	4	4	2
3	132	22	22	6	106	20	21	6
4	2,341	283	169	40	1,713	226	155	38
5	57,427	4,717	1,819	295	38,247	3,365	1,561	262
6	7,571,579	108,992	41,104	3,246	4,102,358	71,138	30,112	2,681
7	-	-	?	59,314	-	-	6,268,858	43,331

Table 1: Numbers of semirings (Definition 1.1) with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See [8, 9, 10] for some of these numbers up to isomorphism.

n	up to isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1
2	8	4	4	2
3	80	18	20	6
4	1,067	169	141	36
5	18,188	1,990	1,276	228
6	543,458	32,212	17,621	2,075
7	162,744,745	799,354	690,924	25,640

Table 2: Numbers of commutative semirings (i.e. those satisfying $x \times y = y \times x$ for all $x, y \in S$) with n elements.

References

[1] E. Aichinger et al. *SONATA, System of nearrings and their applications, Version 2.9.6*. Refereed GAP package. Dec. 2022. URL: https://gap-packages.github.io/sonata/.

[2] A. Bauer. *alg: Algebraic library for constructive mathematics*. https://github.com/andrejbauer/alg. Accessed: 2025-07-03. 2019.

n	up to isomorphism				up to isomorphism or anti-isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1	1	1	1	1
2	6	2	2	1	5	2	2	1
3	61	12	11	3	45	10	10	3
4	866	129	73	20	581	93	64	18
5	15,751	1,852	703	149	9,750	1,207	574	125
6	354,409	33,391	9,195	1,488	205,744	20,142	6,835	1,150
7	9,908,909	729,629	164,163	18,554	5,470,437	415,527	109,880	13,171
8	-	?	?	295,292	?	?	?	116,274

Table 3: Numbers of ai-semirings (i.e. those satisfying $x + x = x$ for all $x \in S$, where "ai" stands for "additively idempotent") with n elements.

n	up to isomorphism			
	no additional constraints	with 0	with 1	with 0 + 1
1	1	1	1	1
2	4	2	2	1
3	29	8	9	3
4	289	57	55	16
5	3,589	580	437	100
6	53,661	6,639	4,296	794
7	949,843	96,264	52,043	7,493
8	?	?	?	?

Table 4: Numbers of commutative ai-semirings (i.e. those satisfying $x \times y = y \times x$ and $x + x = x$ for all $x, y \in S$) with n elements.

[3]   A. Bauer. *Issue #16: Incorrect results.* https://github.com/andrejbauer/alg/issues/16. GitHub issue, opened Jul 4, 2025. 2025.

[4]   H. U. BESCHE, B. EICK, and E. A. O'BRIEN. "A millennium project: constructing small groups". In: *International Journal of Algebra and Computation* 12.05 (Oct. 2002), pp. 623–644. ISSN: 1793-6500. DOI: 10.1142/s0218196702001115. URL: http://dx.doi.org/10.1142/S0218196702001115.

[5]   S. R. Blackburn and K. R. McLean. "The enumeration of finite rings". In: *Journal of the London Mathematical Society* 106.4 (July 2022), pp. 3240–3262. ISSN: 1469-7750. DOI: 10.1112/jlms.12661. URL: http://dx.doi.org/10.1112/jlms.12661.

[6]   C. Chow, M. Janota, and J. Araújo. "Cube-Based Isomorph-Free Finite Model Finding". In: *ECAI 2024*. IOS Press, Oct. 2024. ISBN: 9781643685489. DOI: 10.3233/faia240980. URL: http://dx.doi.org/10.3233/FAIA240980.

[7]   A. Contributors. *MathStructures.* [Online; accessed 3-July-2025]. 2025. URL: %5Curl%7Bhttps://math.chapman.edu/~jipsen/structures/doku.php?id=start&rev=1728801324%7D.

[8]   A. Contributors. *semirings — MathStructures.* [Online; accessed 4-July-2025]. 2021. URL: https://math.chapman.edu/~jipsen/structures/doku.php?id=semirings&rev=1614028296.

[9]   A. Contributors. *semirings_with_identity — MathStructures.* [Online; accessed 4-July-2025]. 2021. URL: https://math.chapman.edu/~jipsen/structures/doku.php?id=semirings_with_identity&rev=1614028296.

[10] A. Contributors. *semirings_with_identity_and_zero — MathStructures.* [Online; accessed 4-July-2025]. 2021. URL: https://math.chapman.edu/~jipsen/structures/doku.php?id=semirings_with_identity_and_zero&rev=1614028296.

[11] A. Distler and J. Mitchell. *Smallsemi, A library of small semigroups, Version 0.7.2.* GAP package. Feb. 2025. URL: https://gap-packages.github.io/smallsemi/.

[12] A. Distler. "Classification and enumeration of finite semigroups". Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported. PhD thesis. University of St Andrews, May 2010. URL: https://hdl.handle.net/10023/945.

[13] A. Distler and T. Kelsey. "The semigroups of order 9 and their automorphism groups". In: *Semigroup Forum* 88.1 (June 2013), pp. 93–112. ISSN: 1432-2137. DOI: 10.1007/s00233-013-9504-9. URL: http://dx.doi.org/10.1007/s00233-013-9504-9.

[14] A. Distler and J. D. Mitchell. "The Number of Nilpotent Semigroups of Degree 3". In: *The Electronic Journal of Combinatorics* 19.2 (June 2012). ISSN: 1077-8926. DOI: 10.37236/2441. URL: http://dx.doi.org/10.37236/2441.

[15] A. Distler et al. "The Semigroups of Order 10". In: *Principles and Practice of Constraint Programming*. Springer Berlin Heidelberg, 2012, pp. 883–899. ISBN: 9783642335587. DOI: 10.1007/978-3-642-33558-7_63. URL: http://dx.doi.org/10.1007/978-3-642-33558-7_63.

[16] B. Fine. "Classification of Finite Rings of Order p^2". In: *Mathematics Magazine* 66.4 (Oct. 1993), p. 248. ISSN: 0025-570X. DOI: 10.2307/2690742. URL: http://dx.doi.org/10.2307/2690742.

[17] G. E. Forsythe. "SWAC Computes 126 Distinct Semigroups of Order 4". In: *Proceedings of the American Mathematical Society* 6.3 (June 1955), p. 443. ISSN: 0002-9939. DOI: 10.2307/2032786. URL: http://dx.doi.org/10.2307/2032786.

[18] *GAP – Groups, Algorithms, and Programming, Version 4.14.0*. The GAP Group. 2024. URL: https://www.gap-system.org.

[19] P. A. Grillet. "Computing finite commutative semigroups". In: *Semigroup Forum* 53.1 (Dec. 1996), pp. 140–154. ISSN: 1432-2137. DOI: 10.1007/bf02574129. URL: http://dx.doi.org/10.1007/BF02574129.

[20] P. A. Grillet. "Counting Semigroups". In: *Communications in Algebra* 43.2 (Aug. 2014), pp. 574–596. ISSN: 1532-4125. DOI: 10.1080/00927872.2013.790036. URL: http://dx.doi.org/10.1080/00927872.2013.790036.

[21] T. Junttila and P. Kaski. *bliss: A Tool for Computing Automorphism Groups and Canonical Labelings of Graphs*. Version as of July 2025. 2007. URL: https://users.aalto.fi/~tjunttil/bliss/.

[22] T. Junttila and P. Kaski. "Engineering an efficient canonical labeling tool for large and sparse graphs". In: *Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments (ALENEX)* (2007), pp. 135–149. DOI: 10.1137/1.9781611972870.13. URL: https://epubs.siam.org/doi/10.1137/1.9781611972870.13.

[23] H. Jürgensen, P. Wick, and V. von Št. Schwarz. "Die halbgruppen der ordnungen ≤ 7". In: *Semigroup Forum* 14.1 (Dec. 1977), pp. 69–79. ISSN: 1432-2137. DOI: 10.1007/bf02194655. URL: http://dx.doi.org/10.1007/BF02194655.

[24] V. L. Klee Jr. "The November meeting in Los Angeles". In: *Bull. Amer. Math. Soc.* 64.2 (1958), p. 56. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1958-10159-7. URL: https://doi.org/10.1090/S0002-9904-1958-10159-7.

[25] D. J. Kleitman, B. R. Rothschild, and J. H. Spencer. "The Number of Semigroups of Order n". In: *Proceedings of the American Mathematical Society* 55.1 (Feb. 1976), p. 227. ISSN: 0002-9939. DOI: 10.2307/2041879. URL: http://dx.doi.org/10.2307/2041879.

[26] R. L. Kruse and D. T. Price. "Enumerating Finite Rings". In: *Journal of the London Mathematical Society* s2-2.1 (Jan. 1970), pp. 149–159. ISSN: 0024-6107. DOI: 10.1112/jlms/s2-2.1.149. URL: http://dx.doi.org/10.1112/jlms/s2-2.1.149.

[27] M. Lothaire. *Applied Combinatorics on Words*. Cambridge University Press, July 2005. ISBN: 9781107341005. DOI: 10.1017/cbo9781107341005. URL: http://dx.doi.org/10.1017/CBO9781107341005.

[28] M. E. Malandro. "Enumeration of finite inverse semigroups". In: *Semigroup Forum* 99.3 (Sept. 2019), pp. 679–723. ISSN: 1432-2137. DOI: 10.1007/s00233-019-10054-9. URL: http://dx.doi.org/10.1007/s00233-019-10054-9.

[29] J. D. Mitchell et al. *Semigroups - GAP package, Version 5.5.1*. June 2025. DOI: 10.5281/zenodo.592893. URL: http://dx.doi.org/10.5281/zenodo.592893.

[30] J. D. Mitchell and P. Ragavan. *Semirings, Counting and enumerating semirings Version 0.2.0*. GAP package. July 2025. URL: https://gap-packages.github.io/semirings/.

[31] T. S. Motzkin and J. L. Selfridge. "Semigroups of Order Five". In: Presented in [24].

[32] OEIS Foundation Inc. *Number of commutative inverse monoids of order n. Entry A234845 in The On-Line Encyclopedia of Integer Sequences*. 2025. URL: https://oeis.org/A234845.

[33] OEIS Foundation Inc. *Number of commutative inverse semigroups of order n. Entry A234843 in The On-Line Encyclopedia of Integer Sequences*. 2025. URL: https://oeis.org/A234843.

[34] OEIS Foundation Inc. *Number of inverse monoids of order n. Entry A234844 in The On-Line Encyclopedia of Integer Sequences*. 2025. URL: https://oeis.org/A234844.

[35] OEIS Foundation Inc. *Number of inverse semigroups of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator), Entry A001428 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A001428.

[36] OEIS Foundation Inc. *Number of nonisomorphic monoids (semigroups with identity) of order n, Entry A058129 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A058129.

[37] OEIS Foundation Inc. *Number of nonisomorphic semigroups of order n, Entry A027851 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A027851.

[38] OEIS Foundation Inc. *Number of semigroups of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator), Entry A001423 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A001423.

[39] OEIS Foundation Inc. *The number of commutative semigroups of order n, Entry A001426 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A001426.

[40] OEIS Foundation Inc. *The number of groups of order n, Entry A000001 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A000001.

[41] OEIS Foundation Inc. *The number of near-rings with n elements, Entry A305858 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A305858.

[42] OEIS Foundation Inc. *The number of rings with n elements, Entry A027623 in The On-Line Encyclopedia of Integer Sequences.* 2025. URL: https://oeis.org/A027623.

[43] R. J. Plemmons. "There are 15973 semigroups of order 6". In: *Math. Algorithms* 2 (1967), pp. 2–17. ISSN: 0025-5548.

[44] M. Ren et al. "The finite basis problem for additively idempotent semirings of order four, I". In: *Semigroup Forum* 110.2 (Apr. 2025), pp. 422–457. ISSN: 1432-2137. DOI: 10.1007/s00233-025-10520-7. URL: http://dx.doi.org/10.1007/s00233-025-10520-7.

[45] J. Sakarovitch. *Elements of Automata Theory.* Ed. by R. Thomas. Cambridge University Press, Oct. 2009. ISBN: 9781139195218. DOI: 10.1017/cbo9781139195218. URL: http://dx.doi.org/10.1017/CBO9781139195218.

[46] S. Satoh, K. Yama, and M. Tokizawa. "Semigroups of order 8". In: *Semigroup Forum* 49.1 (Dec. 1994), pp. 7–29. ISSN: 1432-2137. DOI: 10.1007/bf02573467. URL: http://dx.doi.org/10.1007/BF02573467.

[47] *Semirings of small orders.* https://math.stackexchange.com/questions/3961014/semirings-of-small-orders. Viewed Apr 20, 2025; answer by Alex J Best on Dec 25, 2020. 2020.

[48] X. Zhao et al. "The variety generated by an ai-semiring of order three". In: *Ural Mathematical Journal* 6.2 (Dec. 2020), p. 117. ISSN: 2414-3952. DOI: 10.15826/umj.2020.2.012. URL: http://dx.doi.org/10.15826/umj.2020.2.012.