Preliminaries

Definition 1. A *semiring* is a set S equipped with two binary operations $(+, \cdot)$ such that:

- The *additive reduct* $(S, +)$, which we denote by A, is a commutative semigroup.
- The *multiplicative reduct* $(S, \cdot)$, which we denote by M, is a semigroup.
- Multiplication distributes over addition, i.e. for all $a, b, c \in S$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

Thus, we may identify the semiring S with the pair (A, M).

Note that since both A and M have the same underlying set S, the symmetric group $\mathrm{Sym}(S)$ acts naturally on both structures.

Definition 2 (Semiring isomorphism)**.** We say that two semirings $S = (A, M)$ and $S' = (A', M')$ are *isomorphic* if there exists a bijection $\phi : S \to S'$ such that $A \overset{\phi}{\cong} A'$ and $M \overset{\phi}{\cong} M'$, where $\overset{\phi}{\cong}$ denotes a semigroup isomorphism under ϕ.

Theorem 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S. For any permutation $\sigma \in \mathrm{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ. Suppose that for two permutations $\sigma, \tau \in \mathrm{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are isomorphic.
2. σ and τ lie in the same double coset of $\mathrm{Aut}(A) \backslash \mathrm{Sym}(S) / \mathrm{Aut}(M)$.

Proof. We will denote the product $i \cdot j$ in M by $M(i, j)$.

For the forward direction, suppose $(A, M^\sigma) \cong (A, M^\tau)$. Equivalently, there exists a bijection $\phi : (A, M^\sigma) \to (A, M^\tau)$ such that for all $i, j \in S$:

$$\phi \in \mathrm{Aut}(A) \tag{1}$$
$$\phi(M^\sigma(i, j)) = M^\tau(\phi(i), \phi(j)) \tag{2}$$

From (2), we have

$$\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = \tau(M(\tau^{-1}\phi(i), \tau^{-1}\phi(j)))$$
$$\tau^{-1}\phi\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) = M(\tau^{-1}\phi(i), \tau^{-1}\phi(j))$$

Now, let $\gamma = \tau^{-1}\phi\sigma$, $x = \sigma^{-1}(i)$ and $y = \sigma^{-1}(j)$. Then,

$$\gamma(M(x, y)) = M(\gamma(x), \gamma(y))$$

implying that $\gamma \in \mathrm{Aut}(M)$ and in particular, $\gamma^{-1} \in \mathrm{Aut}(M)$.

Rearranging the definition of γ, we obtain $\tau = \phi\sigma\gamma^{-1}$. Since $\phi \in \mathrm{Aut}(A)$, we conclude that τ and σ lie in the same double coset of $\mathrm{Aut}(A)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}(M)$.

For the reverse direction, we begin by supposing that σ and τ are in the same double coset.

Choose $\alpha \in \mathrm{Aut}(M)$ and $\beta \in \mathrm{Aut}(A)$ such that

$$\tau = \beta\sigma\alpha$$

Note that for a function ϕ to be an isomorphism from $(A, M^\sigma) \to (A, M^\tau)$, it must satisfy properties (1) and (2) detailed in the forward direction. In particular, if we have property (1), then as an automorphism of A, ϕ would automatically be bijective on the underlying set S.

We claim that β satisfies these properties. The first property follows trivially from the definition of β. For the second property,

$$\begin{aligned}
\beta(M^\sigma(i,j)) &= \beta\sigma(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\
&= \tau\alpha^{-1}(M(\sigma^{-1}(i), \sigma^{-1}(j))) \\
&\overset{*}{=} \tau(M(\alpha^{-1}\sigma^{-1}(i), \alpha^{-1}\sigma^{-1}(j))) \\
&= \tau(M(\tau^{-1}\beta(i), \tau^{-1}\beta(j))) \\
&= M^\tau(\beta(i), \beta(j))
\end{aligned}$$

where we have used $\alpha \in \mathrm{Aut}(M)$ and therefore $\alpha^{-1} \in \mathrm{Aut}(M)$ to justify the equality labelled '*'.

Hence, we have shown that β satisfies (1) and (2) and is therefore the necessary isomorphism. $\square$

Note that the above theorem can be easily adapted to yield the following result for equivalence of semirings.

Corollary 1. Let A be a commutative semigroup and M be a semigroup, such that A and M are both defined on the same underlying set S. For any permutation $\sigma \in \mathrm{Sym}(S)$, let M^σ denote the semigroup obtained by permuting M via σ. Suppose that for two permutations $\sigma, \tau \in \mathrm{Sym}(S)$, the pairs (A, M^σ) and (A, M^τ) both form semirings. Then, the following statements are equivalent:

1. (A, M^σ) and (A, M^τ) are equivalent, i.e. isomorphic or anti-isomorphic.
2. σ and τ lie in the same double coset of $\mathrm{Aut}(A)\backslash \mathrm{Sym}(S)/ \mathrm{Aut}^*(M)$.

where $\mathrm{Aut}^*(M)$ denotes the group of automorphisms and anti-automorphisms of M.

Remark 1. $\mathrm{Aut}(M)$ and $\mathrm{Aut}^*(M)$ are equal if M is non-self-dual.

Remark 2. Suppose M is a self-dual semigroup and τ is an anti-automorphism on M. Then,

$$\mathrm{Aut}^*(M) = \langle \mathrm{Generators}(\mathrm{Aut}(M)) \cup \{\tau\} \rangle.$$

Results

The following results were obtained using the `semirings` package in GAP, which (for small n) provides functions for counting and enumerating various semiring-related objects. First, we introduce a few definitions for the less known structures.

Note that we take the convention that a ring has a multiplicative identity, although a semiring might not.

Definition 3. The prefix 'ai' refers to *additive idempotence*. For instance, an ai-semiring S is a semiring such that the additive reduct is idempotent, i.e. it satisfies $a + a = a$ for all $a \in S$.

Definition 4. A *rng* is a ring without the requirement for multiplicative identity. Maintaining the language from Definition 1, this may be thought of as the pair (A, M) where A is an abelian group and M is a semigroup.

Definition 5. A *rig* S is a ring without the requirement for negatives (additive inverses) such that

$$0 \cdot a = 0 \qquad \forall a \in S, \tag{3}$$

where 0 denotes the additive identity in S.

Note that although generally in a ring, Property (3) follows directly from the axioms, this might not hold if we do not have negatives, and so is instead specified explicitly as an axiom.

As above, this may be thought of as the pair (A, M) where A is a commutative monoid and M is a monoid. In fact, many authors use the term 'rig' to refer to what we have defined as a semiring.

Definition 6. A *rg* S is a ring without the requirement for negatives or multiplicative identity, such that

$$0 \cdot a = 0 \qquad \forall a \in S,$$

where 0 denotes the additive identity in S.

The reasoning for this additional axiom is as in Definition 5. As above, this may be thought of as the pair (A, M) where A is a commutative monoid and M is a semigroup. The term 'rg' is very non-standard in the literature.

The semiring-like structures that can be counted using the `semirings` package are:

- Semirings
- Semirings with one
- Ai-semirings
- Ai-semirings with one
- Rigs
- Ai-rigs
- Rgs
- Ai-rgs
- Rngs
- Rings

For instance, one could count the number of semirings with n elements up to isomorphism using `NrSemirings(n)` or up to equivalence using `NrSemirings(n, true)`. `AllSemirings` could be used to enumerate these objects. Functions for the other objects mentioned above are constructed similarly. Using the helper function `SETUPFINDER`, the package could also easily be used to count/enumerate any object which is a semiring with additional constraints, as long as the sets of valid additive and multiplicative reducts are expressible as families of semigroups. With a little more effort, it would be possible to count semirings for arbitrary sets of valid additive and multiplicative reducts.

The algorithm used to count these objects up to isomorphism is fairly rudimentary and is based on Theorem 1. As the condition given by this theorem is precise, we can make use of the `smallsemi` package in GAP to loop over possible semirings (A, M) in a minimal way, such that no two semirings yielded by this process can be isomorphic.

Similarly, we can count semirings up to equivalence by using the condition given in Corollary 1, again in a minimal way.

Below are some tables of results for the aforementioned structures. As far as we know, no results are published the number of any of these structures up to equivalence. For results up to isomorphism, those that have not been previously published (as far as we know) are marked '†'. Results that we are in the process of computing are marked '?'. As a sanity check, various results that are already published are available at Peter Jipsen's Mathematical Structures Library, though he makes use of different naming conventions[1].

These tables are merely a sample of the results that can be obtained using the `semirings` package.

[1] Note that Jipsen's page for "semirings with one", seems to be mistitled and actually provides results for ai-semirings with one (which can be counted using the `ai-semirings` package!). This is not a difference in naming convention, but seems to just be a mistake. As far as we know, all results in Table 5 are unpublished.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	10	9
3	132	106
4	2,341	1,713
5	57,427†	38,247
6	7,571,579†	4,102,358

Table 1: Numbers of semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 4$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	6	5
3	61	45
4	866	581
5	15,751†	9,750
6	354,409†	205,744
7	9,908,909†	5,470,437

Table 2: Numbers of ai-semirings with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 4$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	2	2
3	6	6
4	40	38
5	295	262
6	3,246	2,681
7	59,314†	43,331

Table 3: Numbers of rigs with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 6$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	1	1
3	3	3
4	20	18
5	149	125
6	1,488	1,150
7	18,554	13,171
8	295,292†	116,274

Table 4: Numbers of ai-rigs with n elements up to isomorphism and up to isomorphism or anti-isomorphism. See Jipsen's library for $n \leq 7$ up to isomorphism.

n	up to isomorphism	up to isomorphism + anti-isomorphism
1	1	1
2	4	4
3	22	21
4	169	155
5	1,819	1,561
6	41,104	30,112
7	?	?

Table 5: Numbers of semirings with one (unital semirings) with n elements up to isomorphism and up to isomorphism or anti-isomorphism.