Scalar Fields

(*) Suppose a scalar is defined associated with each point in some region of space, then it is called a scalar field.

Scalar fields are denoted by $\varphi, \psi, \xi, \ldots$

Level surfaces

For the scalar field $\varphi = \varphi(x, y, z, t)$ the surface given by $\varphi = c$, are called **level** surfaces.

Directional Derivative

$$\boxed{\nabla = \left(\underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} + \underline{k} \frac{\partial}{\partial z} \right)}$$

Dell Operator

$D_{\underline{\ell}}(P) \Leftarrow$ Directional Derivative of the scalar field at P in the direction PQ

$$\boxed{D_{\underline{\ell}}(P) = \nabla \phi(P) . \underline{\ell}}$$

Unit Vector

Scalar field

~~$D_{\underline{\ell}}(P)$ increasi~~

If $D_{\underline{\ell}}(P)$ (+) : Scalar field is increasing from P to Q

If $D_{\underline{\ell}}(P)$ (−) : Scalar field is decreasing from P to Q

$$\begin{aligned} |D_{\underline{\ell}}(P)| &= |\nabla \phi(P) . \underline{\ell}| \\ &= |\nabla \phi(P)| \times \underbrace{|\underline{\ell}|}_{=1} . \cos\theta \\ &= |\nabla \phi(P)| . \cos\theta \end{aligned}$$

Angle between $\nabla \phi(P)$ and $\underline{\ell}$

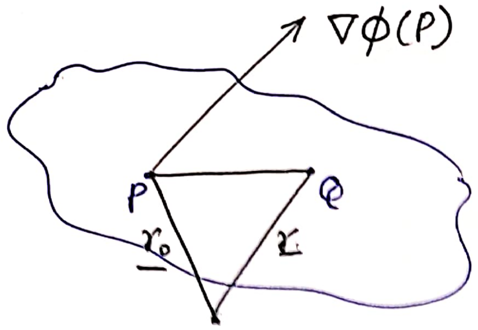
(*) $\theta = \pi/2 \Rightarrow D_{\underline{\ell}}(P) = 0$

ϕ ~~Value~~ does not change in the perpendicular direction to $\nabla \phi(P)$

(*) $\theta = 0$ or $\theta = \pi$

- Max value along the direction of the gradient
- Min Value along the opposite direction.

Equation of the Tangent Plane



$\vec{PQ} = \underline{r} - \underline{r}_0$

equation,

$$\boxed{(\underline{r} - \underline{r}_0) \cdot \nabla\phi(P) = 0}$$

Cartesian Equation,

Let $\underline{r} = (x, y, z)$ and $\underline{r}_0 = (x_0, y_0, z_0)$, $\nabla\phi(P) = a\underline{i} + b\underline{j} + c\underline{k}$

$\cancel{(x - x_0)\underline{i}}$ $(\underline{r} - \underline{r}_0) \cdot \nabla\phi(P) = 0$

$$\left[(x - x_0)\underline{i} + (y - y_0)\underline{j} + (z - z_0)\underline{k}\right] \cdot (a\underline{i} + b\underline{j} + c\underline{k}) = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Cartesian Equation of the Normal,

$\underline{r} - \underline{r}_0 = \lambda \nabla\phi(P)$ becomes,

$$\left[(x - x_0)\underline{i} + (y - y_0)\underline{j} + (z - z_0)\underline{k}\right] = \lambda(a\underline{i} + b\underline{j} + c\underline{k})$$

$$\Rightarrow \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = \lambda}$$

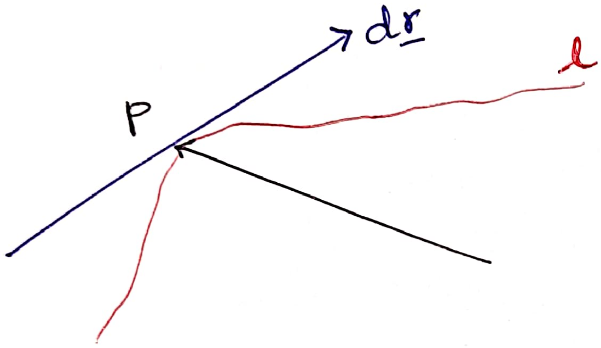
Vector Fields

Suppose a vector is defined associated with each point in some region of space, then it is called a vector field.

Field line

Suppose $\underline{A}$ is a vector field, and l is a curve in the domain of the vector field. If at any point P on l, A(P) is parallel to the tangent to l at P, then l is called a field line.

Equation of a Field line



$$\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$$

then $$d\underline{r} = \lambda(a\underline{i} + b\underline{j} + c\underline{k})$$

$$dx\underline{i} + dy\underline{j} + dz\underline{k} = \lambda(a\underline{i} + b\underline{j} + c\underline{k})$$

so $dx = \lambda a$, $dy = \lambda b$, $dz = \lambda c$

$$\Rightarrow \boxed{\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = \lambda}$$

differential Equation of field Lines.

Gradient (අනුක්‍රමණය)

Gradient $\Rightarrow$ $\nabla\underline{\phi}$ or grad $\underline{\phi}$

$$\boxed{\nabla\underline{\phi} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right)\underline{\phi}}$$

$$\boxed{\nabla\phi = \left[\sum \underline{i}\frac{\partial}{\partial x}\right]\phi}$$

$$\boxed{\nabla \equiv \sum \underline{i}\frac{\partial}{\partial x}}$$

Divergence of a vector field.

divergence of a vector field A,

$$\boxed{\text{div } \underline{A} = \nabla . \underline{A}}$$

$$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right).(a\underline{i} + b\underline{j} + c\underline{k})$$

$$= \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \quad ; \underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$$

Curl of $\underline{A}$.

Curl of $\underline{A}$ is defined by,

$$\boxed{\text{curl } \underline{A} = \nabla \wedge \underline{A}}$$

$$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \wedge (a\underline{i} + b\underline{j} + c\underline{k})$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix}$$

$$\boxed{\text{div } \underline{A} = \nabla . \underline{A} = \left(\sum \underline{i} . \frac{\partial}{\partial x}\right) \underline{A}}$$

$$\boxed{\text{curl } \underline{A} = \nabla \wedge \underline{A} = \left[\sum \underline{i} \wedge \frac{\partial}{\partial x}\right] \underline{A}}$$

Identities

φ and ψ are scalar fields, $\underline{A}$ and $\underline{B}$ are scalar fields

1. $\nabla.(\underline{A}+\underline{B}) = \nabla.\underline{A} + \nabla.\underline{B}$
2. $\nabla\wedge(\underline{A}+\underline{B}) = \nabla\wedge\underline{A} + \nabla\wedge\underline{B}$
3. $\nabla.(\varphi\underline{A}) = \varphi\,\nabla.\underline{A} + \underline{A}.\nabla\varphi$
4. $\nabla\times(\varphi\underline{A}) = \varphi(\nabla\times\underline{A}) + (\nabla\varphi)\times\underline{A}$

$\cancel{5.\ \nabla(\underline{A}.\underline{B}) = \underline{B}.(\nabla\times\underline{A}) + \underline{A}.(\nabla\times\underline{B})}$

5. $\nabla(\underline{A}.\underline{B}) = (\underline{A}.\nabla)\underline{B} + (\underline{B}.\nabla)\underline{A} + \underline{A}\times(\nabla\times\underline{B}) + \underline{B}\times(\nabla\times\underline{A})$
6. $\nabla.(\underline{A}\times\underline{B}) = \underline{B}.(\nabla\times\underline{A}) - \underline{A}.(\nabla\times\underline{B})$
7. $\nabla\times(\underline{A}\times\underline{B}) = \underline{A}(\nabla.\underline{B}) - \underline{B}(\nabla.\underline{A}) + (\underline{B}.\nabla)\underline{A} - (\underline{A}.\nabla)\underline{B}$
8. $\nabla\times(\nabla\varphi) = 0$ curl grad φ is always zero.
9. $\nabla.(\nabla\times\underline{A}) = 0$ div curl A is always zero.
10. $\nabla\times(\nabla\times\underline{A}) = \nabla(\nabla.\underline{A}) - \nabla^2 A$

$$\text{grad}\,(\varphi\psi) = \varphi\ \text{grad}\,\psi + \psi\ \text{grad}\,\varphi$$

$$\text{div}\,(\varphi\underline{A}) = \varphi\ \text{div}\,\underline{A} + (\text{grad}\,\varphi).\underline{A}$$

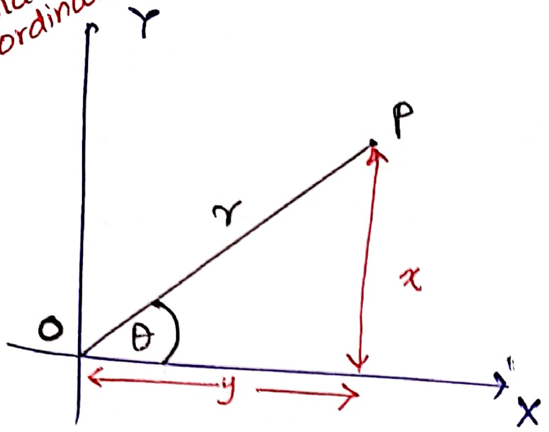
$$\text{curl}\,(\varphi\underline{A}) = \varphi\ \text{curl}\,\underline{A} + (\text{grad}\,\varphi)\times\underline{A}$$

$$\text{div}\,(\underline{A}\times\underline{B}) = (\underline{B}.\nabla)\underline{A} + (\text{div}\,\underline{B})\underline{A} + (\underline{A}.\nabla)\underline{B} + (\text{div}\,\underline{A})\underline{B}$$

$$\text{curl}\,(\underline{A}\wedge\underline{B}) = (\text{div}\,\underline{B})\underline{A} - (\text{div}\,\underline{A})\underline{B} + (\underline{B}.\nabla)\underline{A} - (\underline{A}.\nabla)\underline{B}$$

System of coordinates in 2-D

Polar coordinates



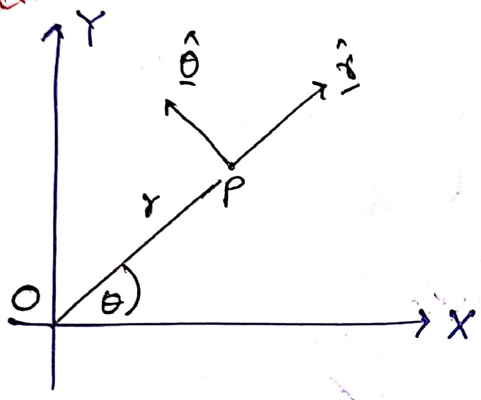
Here $OP = r$

$x = r\cos\theta$, $y = r\sin\theta$

Metric. $ds^2 = dx^2 + dy^2$

$= dr^2 + r^2 d\theta^2$

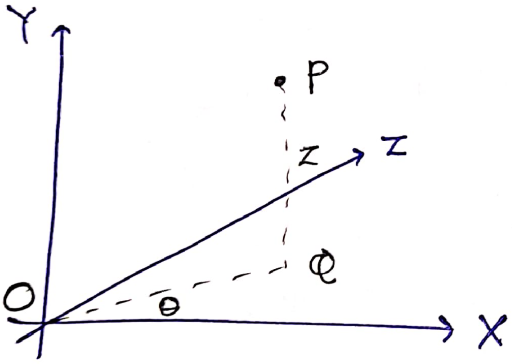
Unit Vectors



$\hat{\underline{r}} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$

$\hat{\underline{\theta}} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$

Different types of coordinates in 3-D



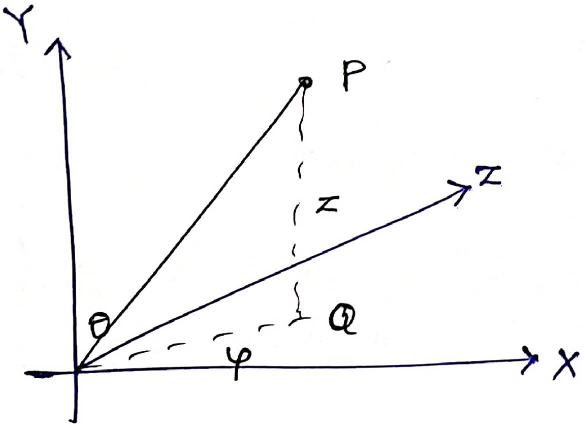
Here $OQ = r$, So $P \equiv (r, \theta, z)$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Metric $ds^2 = PP'^2$

$= dr^2 + r^2 d\theta^2 + dz^2$

Cylindrical Polar Coordinates



Here $OP = r$, So $P \equiv (r, \theta, \varphi)$

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

Metric $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$

Curvilinear Coordinates

Ouvw system of coordinates,
metric,

$$ds^2 = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2$$

The unit vectors in the increasing directions,

$$u : \frac{\partial r}{\partial u} \qquad v : \frac{\partial r}{\partial v} \qquad w : \frac{\partial r}{\partial w}$$

Unit Vectors,

$$\hat{\underline{u}} = \frac{\frac{\partial r}{\partial u}}{\left|\frac{\partial r}{\partial u}\right|}, \quad \hat{\underline{v}} = \frac{\frac{\partial r}{\partial v}}{\left|\frac{\partial r}{\partial v}\right|}, \quad \hat{\underline{w}} = \frac{\frac{\partial r}{\partial w}}{\left|\frac{\partial r}{\partial w}\right|}$$

For Scalar Field φ,

Gradient : $$\boxed{\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u}\hat{\underline{u}} + \frac{1}{h_2}\frac{\partial\varphi}{\partial v}\hat{\underline{v}} + \frac{1}{h_3}\frac{\partial\varphi}{\partial w}\hat{\underline{w}}}$$

For the vector field $\underline{a} = a_1\hat{\underline{u}} + a_2\hat{\underline{v}} + a_3\hat{\underline{w}}$

$$\boxed{\text{div } \underline{a} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 a_1)\frac{\partial}{\partial v}(h_1 h_3 a_2)\frac{\partial}{\partial w}(h_1 h_2 a_3)\right]}$$

$$\boxed{\text{curl } \underline{a} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\underline{u}} & h_2\hat{\underline{v}} & h_3\hat{\underline{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}}$$

$$\boxed{\nabla^2\varphi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial}\left[\frac{h_2 h_3}{h_1}\frac{\partial\varphi}{\partial u}\right] + \frac{\partial}{\partial}\left[\frac{h_1 h_3}{h_2}\frac{\partial\varphi}{\partial v}\right] + \frac{\partial}{\partial}\left[\frac{h_1 h_2}{h_3}\frac{\partial\varphi}{\partial w}\right]\right]}$$

Values for different system of coordinates

	h_1	h_2	h_3
~~Cylindrical~~ Polar	1	r	1
Spherical Polar	1	r	$r\sin\theta$

Cylindrical polar cdts

$$\text{div } \underline{a} = \frac{1}{r}\left[\frac{\partial}{\partial r}(ra_1) + \frac{\partial}{\partial \theta}(a_2) + r\frac{\partial}{\partial z}(a_3)\right]$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\underline{\hat{\theta}} + \frac{\partial\varphi}{\partial z}\underline{k}$$

$$\text{curl } \underline{a} = \frac{1}{r}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & \underline{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ a_1 & ra_2 & a_3 \end{vmatrix}$$

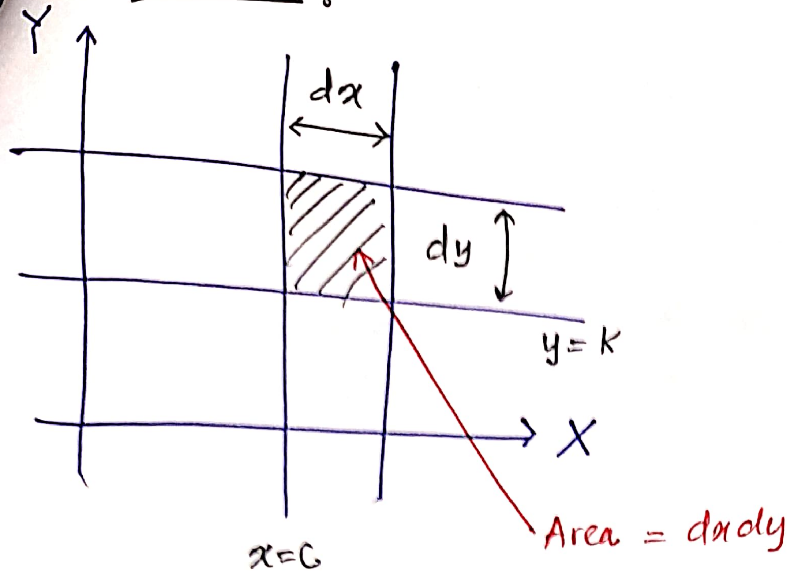
Spherical polar cdts

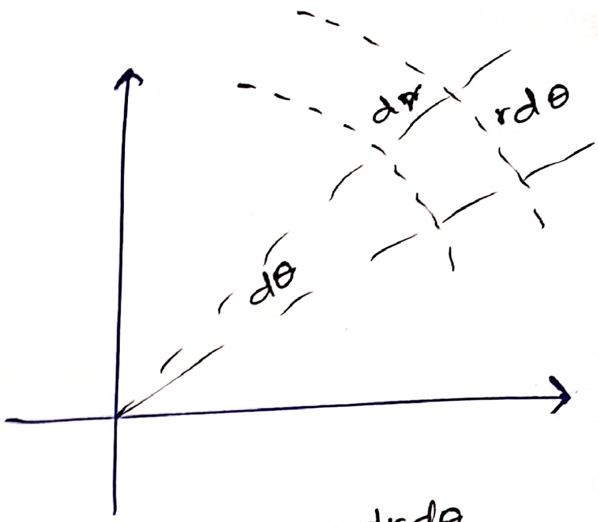
$$\text{div } \underline{a} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, a_1) + \frac{\partial}{\partial \theta}(r\sin\theta\, a_2) + r\frac{\partial}{\partial \varphi}(a_3)\right]$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\psi}\underline{\hat{\psi}}$$

$$\text{curl } \underline{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ a_1 & ra_2 & r^2\sin\theta\, a_3 \end{vmatrix}$$

Small Area in 2-D

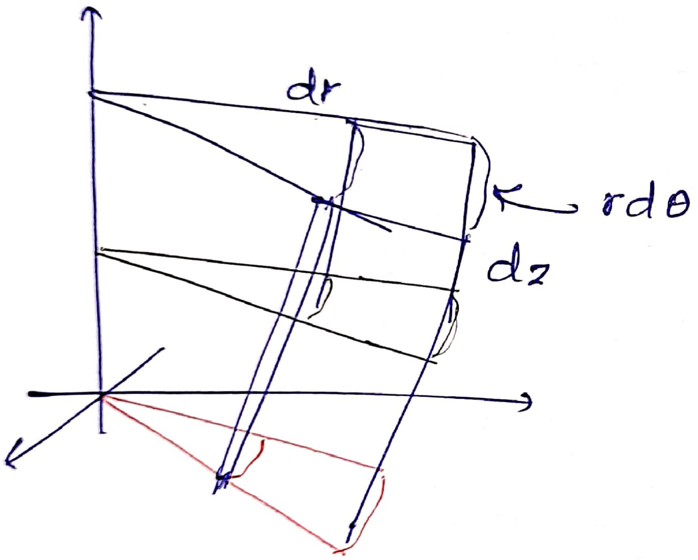




Small Area rdrdθ

Small volume in 3-D

with cylindrical polar cdts.



Volume element is given by,

$\boxed{dv = r\,dr\,d\theta\,d\varphi}$

Coordinate Transformations

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

Note :-

We consider the (+) sign of the determinant.

i.e if determinant is (−) we drop (−) sign

Line integral of a vector field

$$\text{Integral of the Vector field } \underline{A} \text{ over the curve } c = \int_c \underline{A} \cdot d\underline{r}$$

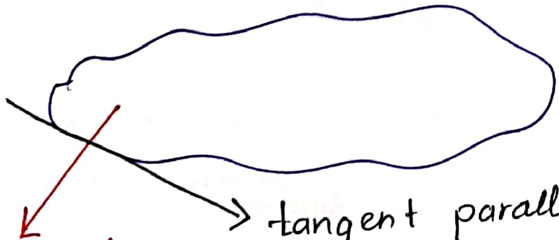
$$\int_c \underline{A} \cdot d\underline{r} = \int_c \left[A_1 \frac{dx}{dt} + A_2 \frac{dy}{dt} + A_3 \frac{dz}{dt} \right] dt$$

Flux

$$\text{Outward Flux of the vector field } \underline{A} \text{ over } c = \oint_c \underline{A} \cdot d\underline{s}$$

It is defined as,

$$\oint_c \underline{A} \cdot \underline{n}\, ds \qquad s - \text{arc length}$$



Let $A = M(x,y)\,\underline{i} + N(x,y)\,\underline{j}$

$$\therefore \oint_c \underline{A} \cdot \underline{n}\, ds = \oint_c \left[M(x,y)\,\underline{i} + N(x,y)\,\underline{j} \right] \cdot \frac{dy\underline{i} - dx\underline{i}}{ds}\, ds$$

$$= \oint_c M\,dy - N\,dx$$

Outward flux of $\underline{A}$ over the simple closed curve C,

$$\boxed{\oint_c \underline{A} \cdot d\underline{s} = \oint_c M\,dy - N\,dx}$$

Circulation

defined by,

$$\boxed{\oint_c \underline{A}\cdot d\underline{r} = \oint_c \underline{A}\cdot\underline{t}\, ds}$$

$$\oint_c \underline{A}\cdot d\underline{r} = \oint_c [M(x,y)\underline{i} + N(x,y)\underline{j}]\cdot\frac{dx\underline{i} + dy\underline{j}}{ds}\, ds$$

$$= \oint_c M dx + N dy$$

~~Outward flux~~ Circulation of $\underline{A}$ over the simple closed curve c,

$$\boxed{\oint_c \underline{A}\cdot d\underline{r} = \oint_c M dx + N dy}$$

Green's Theorem in 2-D

Suppose c is a simple closed curve in a plane, and S is the surface enclosed by c. The outward flux of a vector field $\underline{A}$ over c is same as the surface integral of div A over S

$$\underline{A} = M(x,y)\underline{i} + N(x,y)\underline{j}$$

$$\text{div}\,\underline{A} = \frac{\partial}{\partial x}M(x,y) + \frac{\partial}{\partial y}N(x,y)$$

So Green's Theorem,

$$\boxed{\oint_c M dy - N dx = \iint_S \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right] dx\, dy}$$

Flux Form of Green's Theorem

$$\text{Circulation} = \oint_c M dx + N dy$$

$$= \oint_c N dy - (-M) dx$$

$$= \iint \left[\frac{\partial N}{\partial x} + \frac{\partial(-M)}{\partial y}\right] dx\, dy$$

$$= \iint_S \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy$$

$$\boxed{\oint_c M dx + N dy = \iint_S \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy}$$

Circulation form of Green's Theorem

Stokes Theorem

Let $\underline{F}$ be a vector field with continuous partial derivatives defined on an open surface S and on the perimeter curve c a simple closed curve, of S.

$$\boxed{\oint_c \underline{F}\cdot d\underline{r} = \iint_S (\text{curl}\,\underline{F})\underline{n}\cdot d\underline{A}}$$

Divergence Theorem

Let S be a closed surface and V be the Volume enclosed by S. Then outward flux of continuously differentiable vector field $\underline{F}$ over S is same as the Volume integral of div $\underline{F}$.

$$\boxed{\iint_S \underline{F}\cdot\underline{n}\, d\underline{A} = \iiint_V \text{div}\,\underline{F}\, dv}$$

divergence Theorem connects a volume integral and an integral over a closed surface.