CHAPTER 28

Numerical Solution of Equations

1. Introduction. 2. Solution of algebraic and transcendental equations—Bisection method, Method of false position, Newton's method. 3. Useful deductions from the Newton-Raphson Formula. 4. Approximate solution of equations—Horner's method. 5. Solution of linear simultaneous equations. 6. Direct methods of solution—Gauss elimination method, Gauss-Jordan method, Factorization method. 7. Iterative methods of solution—Jacobi's method, Gauss-Seidal method, Relaxation method. 8. Solution of non-linear simultaneous equations—Newton-Raphson method. 9. Determination of eigen values by iteration. 10. Objective Type of Questions.

28.1 INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. As seen in § 1.8, the graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We shall close the chapter by describing an iterative method for the solution of eigen-value problem. For a detailed study of these topics, the reader should refer to author's book *'Numerical Methods in Engineering & Science'*.

28.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

To find the roots of an equation $f(x) = 0$, we start with a known approximate solution and apply any of the following methods :

(1) Bisection method. This method consists in locating the root of the equation $f(x) = 0$ between a and b. If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

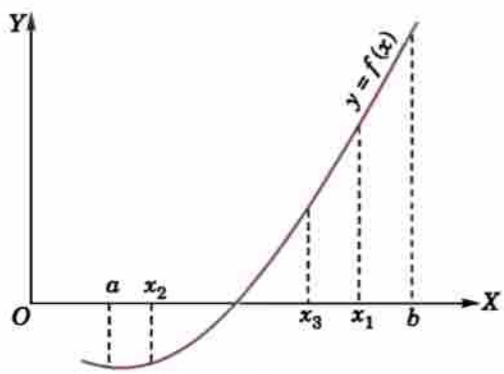

Fig. 28.1

In the Fig. 28.1, $f(x_1)$ is +ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is – ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Example 28.1. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* *(Mumbai, 2003)*
(b) Using bisection method, find the negative root of the equation $x^3 - 4x + 9 = 0$. *(J.N.T.U., 2009)*

Solution. (a) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3

$\therefore$ first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $\quad f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \quad$ *i.e.*, + ve

$\therefore$ the root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $\quad f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$$x_5 = 2.71875, \quad x_6 = 2.70313, \quad x_7 = 2.71094$$
$$x_8 = 2.70703, \quad x_9 = 2.70508, \quad x_{10} = 2.70605$$
$$x_{11} = 2.70654, \quad x_{12} = 2.70642$$

Hence the root is 2.7064

(b) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$

$\therefore$ the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is – 2.7064.

Example 28.2. *By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.* *(V.T.U., 2003 S)*

Solution. Let $f(x) = x \sin x - 1$. We know that $1^r = 57.3°$.

Since $\quad f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $\quad f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

$\therefore$ first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25) \sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$

and $f(x_2) > 0$, *i.e.* now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

$\therefore$ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

$\therefore$ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

(2) Method of false position *or* **Regula-falsi method.** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 28.2). This indicates that a root lies between x_0 and x_1 consequently $f(x_0) f(x_1) < 0$.

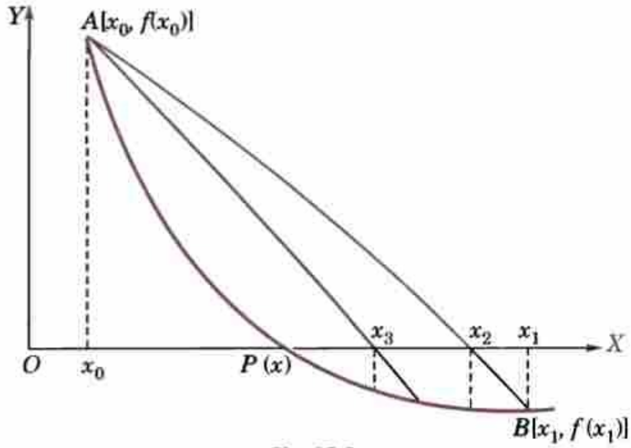

Fig. 28.2

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad ...(1)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad ...(2)$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (2), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (1) is known as the *method of false position*.

Example 28.3. *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.* *(Manipal, 2005)*

Solution. Let $f(x) = x^3 - 2x - 5$

so that $f(2) = -1$ and $f(3) = 16$ *i.e.*, A root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \quad ...(i)$$

Now $f(x_2) = f(2.0588) = -0.3908$ *i.e.*, the root lies between 2.0588 and 3.

$\therefore$ taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3908$, $f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862,\ x_5 = 2.0915,\ x_6 = 2.0934,\ x_7 = 2.0941,\ x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 28.4. *Find the root of the equation $\cos x = xe^x$ using the regula-falsi method correct to four decimal places.* *(Bhopal, 2009)*

Solution. Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1$, $f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ taking $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \quad ...(i)$$

Now $f(0.31467) = 0.51987$

i.e., the root lies between 0.31467 and 1.

$\therefore$ taking $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$, $f(x_1) = -2.17798$ in (i), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$

i.e., the root lies between 0.44673 and 1.

$\therefore$ taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.20356$, $f(x_1) = -2.17798$ in (i), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995,\quad x_6 = 0.51520,\quad x_7 = 0.51692$$
$$x_8 = 0.51748,\quad x_9 = 0.51767,\quad x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 28.5. *Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places.* *(V.T.U., 2010 ; J.N.T.U., 2008 ; Kottayam, 2005)*

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -$ ve, $f(2) = -$ ve and $f(3) = +$ ve.

$\therefore$ a root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3$, $f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \quad ...(i)$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e., the root lies between 2.72102 and 3.

$\therefore$ taking $x_0 = 2.72102$, $x_1 = 3$, $f(x_0) = -0.01709$

and $f(x_1) = 0.23136$ in (i), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024,\ x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 28.6. *Use the method of false position, to find the fourth root of 32 correct to three decimal places.*

Solution. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.*, a root lies between 2 and 3.

∴ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \qquad ...(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

∴ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (*i*), we get
$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

∴ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (*i*), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

(3) Newton-Raphson method*. Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

∴ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + ... = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \qquad ...(1)$$

∴ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ...(2)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

Obs. 1. *Newton's method is useful in cases of large values of* $f'(x)$ *i.e.* when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

Obs. 2. Newton's method has a second order of quadratic convergence. Suppose x_n differs from the root α by a small quantity ε_n so that $x_0 = \alpha + \varepsilon_n$ and $x_{n+1} = \alpha + \varepsilon_{n+1}$.

Then (2) becomes $\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \dfrac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$

i.e.,
$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + ...}{f'(\alpha) + \varepsilon_n f''(\alpha) + ...} \qquad \text{[By Taylor's expansion.]}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + ...}{f'(\alpha) + \varepsilon_n f''(\alpha) + ...} \qquad [\because\ f(\alpha) = 0]$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]} = \frac{\varepsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}. \qquad \left[\begin{array}{l}\text{neglecting third and}\\ \text{higher powers of } \varepsilon_n.\end{array}\right]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. (*P.T.U., 2005*)

Obs. 3. Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 28.3). Then the equation of the tangent at $A_0\,[x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

*See footnote p. 466. Named after the English mathematician *Joseph Raphson* (1648–1715) who suggested a method similar to Newton's method.

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence *the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.*]

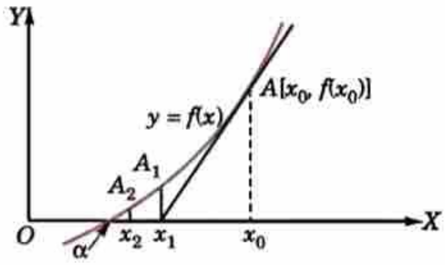

Fig. 28.3

Example 28.7. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.* (*J.N.T.U., 2008 ; Madras, 2006*)

Solution. Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 = -\text{ve}, f(2) = 16 - 2 - 10 = 4 = +\text{ve}$

$\therefore$ a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ...(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (i), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Here $x_2 = x_3$. Hence the desired is 1.856 correct to three decimal places.

Example 28.8. *Find the Newton's method, the real root of the equation $3x = \cos x + 1$.* (*V.T.U., 2009 ; S.V.T.U., 2007*)

Solution. Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}.$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad ...(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6) \sin (0.6) + \cos (0.6) + 1}{3 \sin (0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \quad \text{Clearly, } x_1 = x_2.$$

Hence the desired root is 0.6071 correct to four decimal places.

Example 28.9. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* (V.T.U., 2005 ; Mumbai, 2004 ; Burdwan, 2003)

Solution. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -\text{ve}, f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ ve}$

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ ve}$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429} \quad ...(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) *Iterative formula to find $1/N$ is* $\quad \boldsymbol{x_{n+1} = x_n (2 - Nx_n)}$

(2) *Iterative formula to find $\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find $1/\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)}$

(4) *Iterative formula to find $\sqrt[k]{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x^n} - N\right) x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n)$$ *(Madras, 2006)*

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(*i*) $1/31$ (*ii*) $\sqrt{5}$ (Anna, 2007)

(*iii*) $1/\sqrt{14}$ (*iv*) $\sqrt[3]{24}$ (Madras, 2003) (*v*) $(30)^{-1/5}$.

Solution. (*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then $x_1 = x_0(2 - 31x_0) = 0.03\,(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2\,(2 - 31x_2) = 0.032257\,(2 - 31 \times 0.032257) = 0.03226$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

Then $x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$

$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$

$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$

Then $x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$

$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$

$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $$x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then $$x_1 = \frac{x_0}{5}(6 - 30\, x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30\, x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30\, x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 28.1

1. Find a root of the following equations, using the bisection method correct to three decimal places :
(*i*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*) (*ii*) $x^3 - x^2 - 1 = 0$ (*J.N.T.U., 2009*)
(*iii*) $x^3 - x - 11 = 0$ which lies between 2 and 3 (*iv*) $2x^3 + x^2 - 20x + 12 = 0$.
2. Using the bisection method, find a real root of the following equations correct to three decimal places :
(*i*) $\cos x = xe^x$ (*Mumbai, 2004*) (*ii*) $x \log_{10} x = 1.2$ lying between 2 and 3
(*iii*) $e^x - x = 2$ lying between 1 and 1.4 (*iv*) $e^x = 4 \sin x$.
3. Find a real root of the following equations correct to three decimal places by the method of false position :
(*i*) $x^3 + x - 1 = 0$ (*ii*) $x^3 - 4x - 9 = 0$ (*V.T.U., 2007*)
(*iii*) $x^3 + x - 1 = 0$ near $x = 1$ (*iv*) $x^6 - x^4 - x^3 - 1 = 0$. (*Nagarjuna, 2001*)
4. Using regula-falsi method, compute the real root of the following equations correct to three decimal places :
(*i*) $xe^x = 2$ (*S.V.T.U., 2007*) (*ii*) $\cos x = 3x - 1$ (*iii*) $x \tan x - 1 = 0$
(*iv*) $2x - \log x = 7$ (*J.N.T.U., 2006*) (*v*) $xe^x = \sin x$. (*P.T.U., 2005*)
5. Find the fourth root of 12 correct to three decimal places using the method of false position.
6. Find by Newton's method, a root of the following equations correct to 3 decimal places :
(*i*) $x^3 - 3x + 1 = 0$ (*Bhopal, 2009*) (*ii*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*)
(*iii*) $x^3 - 5x + 3 = 0$ (*Mumbai, 2004*)
(*iv*) $3x^3 - 9x^2 + 8 = 0$ lying between 1 and 2. (*Madras, 2003*)
7. Find a root of the following equations correct to three significant figures using Newton's iterative method :
(*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3 (*Madras, 2003*)
(*ii*) $x^5 - 5x^2 + 3 = 0$.
8. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.
9. Using Newton-Raphson method, find a root of the following equations correct to the three decimal places :
(*i*) $xe^x - 2 = 0$ (*V.T.U., 2005*) (*ii*) $x^2 + 4 \sin x = 0$ (*Hazaribagh, 2009*)
(*iii*) $x \tan x + 1 = 0$ which is near $x = \pi$ (*J.N.T.U., 2006 ; V.T.U., 2006*)
(*iv*) $e^x = x^2 + \cos 25x$ which is near $x = 4.5$. (*V.T.U., 2007*)
10. Find by Newton's method, the root of the equations :
(*i*) $\cos x = xe^x$ (*J.N.T.U., 2009 ; V.T.U., 2003*) (*ii*) $x \log_{10} x = 12.34$ (*Anna, 2004*)
(*iii*) $10^x + x - 4 = 0$ (*iv*) $x + \log_{10} x = 3.375$ (*Rohtak, 2003*)
11. Develop a recurrence formula for finding $\sqrt{N}$, using Newton-Raphson method and hence compute to three decimal places
(*i*) $\sqrt{13}$ (*U.P.T.U., 2008*) (*ii*) $\sqrt{10}$ (*J.N.T.U., 2008*)

12. Find the cube root of 41, using Newton-Raphson method. *(Madras, 2003)*
13. Develop an algorithm using N-R method, to find the fourth root of a positive number N and hence find $(32)^{1/4}$. *(W.B.T.U., 2005)*
14. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method :
    (*i*) 1/18 *J.N.T.U., 2004)* (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

28.4 APPROXIMATE SOLUTION OF EQUATIONS—HORNER'S METHOD

This is the best method of finding approximate values of both rational and irrational roots of a numerical equation. Horner's method consists in diminution of the root of an equation by successive digits occurring in the roots.

If the root of an equation lies between a and $a + 1$, then the value of this root will be $a \,.\, bcd$ ..., where b, c, d ... are digits in its decimal part. To obtain these, we proceed as follows :

(*i*) Diminish the roots of the given equation by a so that the root of the new equation is $0 \,.\, bcd$ ...

(*ii*) Then multiply the roots of the transformed equation by 10 so that the root of the new equation is $b \,.\, cd$...

(*iii*) Now diminish the root by b and multiply the roots of the resulting equation by 10 so that the root is $c.\,d$...

(*iv*) Next diminish the root by c and so on. By continuing this process, the root may be evaluated to any desired degree of accuracy digit by digit. The method will be clear from the following example.

Example 28.11. *Find by Horner's method, the positive root of the equation $x^3 + x^2 + x - 100 = 0$ correct to three decimal places.*

Solution. *Step I.* Let $f(x) = x^3 + x^2 + x - 100$

By Descartes' rule of signs, there is only one positive root. Also $f(4) = -$ ve and $f(5) = +$ ve, therefore, the root lies between 4 and 5.

Step II. Diminish the roots of given equation by 4 so that the transformed equation is

$$x^3 + 13x^2 + 57x - 16 = 0 \qquad ...(i)$$

Its root lies between 0 and 1. (We draw a zig-zag line above the set of figures 13, 57, − 16 which are the coefficients of the terms in (*i*) as shown below. Now multiply the roots of (*i*) by 10 for which multiply the second term by 10, the third term by 100 and the fourth term by 1000 (*i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term). Then we get the equation

$$f_1(x) = x^3 + 130x^2 + 5700x - 16000 = 0 \qquad ...(ii)$$

1	1	1	− 100	(4.264
	4	20	84	
	5	21	− 16000	
	4	36	11928	
	9	5700	− 4072000	
	4	264	3788376	
	130	5964	− 283624000	
	2	268		
	132	623200		
	2	8196		
	134	631396		
	2	8232		
	1360	63962800		
	6			
	1366			
	6			
	1372			
	6			
	13780			

Its root lies between 0 and 10.

Clearly $\quad f_1(2) = -\text{ ve}, f_1(3) = +\text{ ve}$

$\therefore$ the root of (*ii*) lies between 2 and 3 *i.e.*, first figure after decimal is 2.

Step III. Diminish the roots of $f_1(x) = 0$ by 2 so that the next transformed equation is

$$x^3 + 136x^2 + 6232x - 4072 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw the second zig-zag line above the set of figures 136, 6232, – 4072). Multiply the roots of (*iii*), by 10, *i.e.* attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 1360x^2 + 623200x - 4072000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{4072000}{623200} = 6$

Hence second figure after decimal place is 6.

Step IV. Diminish the roots of $f_2(x) = 0$ by 6, so that the transformed equation is

$$x^3 + 1378x^2 + 639628x - 283624 = 0.$$

Its root lies between 0 and 1. (We draw the third zig-zag line above the set of figures 1378, 639628, – 283624.) As before multiply its roots by 10, *i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then the equation becomes

$$f_3(x) = x^3 + 13780x^2 + 63962800x - 283624000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{283624000}{63962800} = 4$. Thus the roots of $f_3(x) = 0$ are to be diminished by 4 *i.e.* the third figure after decimal place is 4. But there is no need to proceed further as the root is required correct to three decimal places only. Hence the root is 4.264.

Obs. 1. *After two steps of diminishing, we apply the* **principle of trial divisor** *in which we divide the last coefficient by last but one coefficient to get the next integer by which the roots are to be diminished. These last two coefficients should have opposite signs.*

Obs. 2. *At any stage if the trial divisor suggests the next integer to be zero, then we should again multiply the roots by 10 and write zero in decimal place of the root.*

Example 28.12. *Find the cube root of 30 correct to 3 decimal places, using Horner's method.*

Solution. *Step I.* Let $x = \sqrt[3]{30}$ *i.e.* $f(x) = x^3 - 30 = 0$

Now $\quad f(3) = -3\ (-\text{ ve}), \quad f(4) = 34\ (+\text{ ve})$

$\therefore$ the root lies between 3 and 4.

Step II. Diminish the roots of the given equation by 3 so that the transformed equation is

$$x^3 + 9x^2 + 27x - 3 = 0 \qquad ...(i)$$

Its roots lies between 0 and 1. (We draw a zig-zag line above the set of numbers 9, 27, – 3 which are the coefficients of the terms in (*i*)). Now multiply the roots of (*i*) by 10 for which attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then we get the equation

$$f_1(x) = x^3 + 90x^2 + 2700x - 3000 = 0 \qquad ...(ii)$$

Its roots lies between 0 and 10.

Clearly $\quad f_1(1) = -\text{ ve}, \quad f_2(2) = +\text{ ve}$

$\therefore$ the root of (*ii*) lies between 1 and 2 *i.e.*, first figure after decimal place is 1.

Step III. Diminish the roots of $f_1(x) = 0$ by 1, so that the next transformed equation is

$$x^3 + 93x^2 + 2883x - 209 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw a second zig-zag line above the set of figures 93, 2883, – 209). Multiply the roots of (*iii*) by 10 *i.e.*, attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 930x^2 + 288300x - 209000 = 0$$

Its root lies between 0 and 10, which is nearly

$$= 209000/288300 = 0.724 > 0 \text{ and } < 1.$$

Hence second figure after decimal place is 0.

1	0	0	– 30	(3.107
	3	9	27	
	3	9	– 30000	
	3	18	2791	
	6	2700	– 209000000	
	3	91		
	90	2791		
	1	92		
	91	28830000		
	1			
	92			
	1			
	9300			

Step IV. Diminish the root of $f_2(x) = 0$ by 0 and then multiply its roots by 10 so that
$f_3(x) = x^3 + 9300x^2 + 28830000x - 209000000 = 0.$

Its root lies between 0 and 10, which is nearly $= 209000000/28830000 = 7.2 > 7$ and < 8. Thus the roots of $f_3(x) = 0$ are to be diminished by 7 *i.e.*, the third figure after decimal is 7. Hence the required root is 3.107.

PROBLEMS 28.2

1. Find by Horner's method, the root (correct to three decimal places) of the equation
   (*i*) $x^3 - 3x + 1 = 0$ which lies between 1 and 2
   (*ii*) $x^3 + x - 1 = 0$ (*Coimbatore, 1997*) (*iii*) $x^3 - 6x - 13 = 0$
   (*iv*) $x^3 - 3x^2 + 2.5 = 0$ which lies between 1 and 2. (*Madras, 2000 S*)
2. Using Horner's method, find the largest real root of $x^3 - 4x + 2 = 0$ correct to three decimal places.
3. Show that the root of the equation $x^4 + x^3 - 4x^2 - 16 = 0$ lies between 2 and 3. Find its value correct to two decimal places by Horner's method.
4. Find the negative root of the equation $x^3 - 9x^2 + 18 = 0$ correct to two decimal places by Horner's method.
5. Find the cube root of 25 by Horner's method correct to 3 decimal places.

28.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method (§ 2.10). But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. We now explain some direct and iterative methods of solution.

28.6 DIRECT METHODS OF SOLUTION

(1) Gauss elimination method*. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad ...(1)$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

*See footnote p. 37.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad ...(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad ...(3)$$

Here the second equation is the *pivotal equation* and b_2' is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Obs. 1. On writing the given equations as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ *i.e.*, $AX = D$, this method consists in ***transforming the coefficient matrix A to upper triangular matrix*** by elementary row transformations only.

Obs. 2. Clearly the method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Obs. 3. *Partial and complete pivoting.* In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the *second pivot* by interchanging the second equation with the equation having the largest coefficient of y'. This process is continued till we arrive at the equation with the single variable. This modified procedure is called *partial pivoting.*

If we are not taken about the elimination of x, y, z in a specified order, then we choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting.* It is more complicated and does not appreciably improve the accuracy.

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* *(Mumbai, 2009)*

Solution. We have

	Check sum	
$x + 4y - z = -5$	-1	...(*i*)
$x + y - 6z = -12$	-16	...(*ii*)
$3x - y - z = 4$	5	...(*iii*)

Step I. Operate (*ii*) – (*i*) and (*iii*) – 3(*i*) to eliminate x :

	Check sum	
$-3y - 5z = -7$	-15	...(*iv*)
$-13y + 2z = 19$	8	...(*v*)

Step II. Operate (*v*) – $\frac{13}{3}$ (*iv*) to eliminate y :

	Check sum	
$\frac{71}{3}z = \frac{148}{3}$	73	...(*vi*)

Step III. By back-substitution, we get

From (*vi*) : $z = \dfrac{148}{71} = 2.0845$

From (*iv*) : $y = \dfrac{7}{3} - \dfrac{5}{3}\left(\dfrac{148}{71}\right) = -\dfrac{81}{71} = -1.1408$

From (*i*) : $x = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71} = 1.6479$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise : We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3} R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$

and $$x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$.

Example 28.14. *Solve $10x - 7y + 3z + 5u = 6, -6x + 8y - z - 4u = 5, 3x + y + 4z + 11u = 2, 5x - 9y - 2z + 4u = 7$ by Gauss elimination method.* (*S.V.T.U., 2007*)

Solution. We have

	Check sum	
$10x - 7y + 3z + 5u = 6$	17	...(*i*)
$-6x + 8y - z - 4u = 5$	2	...(*ii*)
$3x + y + 4z + 11u = 2$	21	...(*iii*)
$5x - 9y - 2z + 4u = 7$	5	...(*iv*)

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \frac{3}{10}(i)\right], \left[(iv) - \frac{5}{10}(i)\right]$:

	Check sum	
$3.8y + 0.8z - u = 8.6$	12.2	...(*v*)
$3.1y + 3.1z + 9.5u = 0.2$	15.9	...(*vi*)
$-5.5y - 3.5z + 1.5u = 4$	-3.5	...(*vii*)

Step II. To eliminate y, operate $\left[(vi) - \frac{3.1}{3.8}(v)\right], \left[(vii) - \left(\frac{-5.5}{3.8}\right)(v)\right]$:

$$2.4473684z + 10.315789u = -6.8157895 \quad \text{...(viii)}$$

$$-2.3421053z + 0.0526315u = 16.447368 \quad \text{...(ix)}$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684}\right)(viii)\right]$:

$$9.9249319u = 9.9245977$$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

(2) Gauss-Jordan method*. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately **reducing the system to a diagonal matrix form *i.e.,*** *each equation involving only one unknown. From these equations the unknowns x, y, z can be obtained readily.*

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

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*See footnote p. 37.

Example 28.15. *Apply Gauss-Jordan method to solve the equations*

$x + y + z = 9 \,;\, 2x - 3y + 4z = 13 \,;\, 3x + 4y + 5z = 40.$ (V.T.U., 2009 ; P.T.U., 2005)

Solution. We have

$$x + y + z = 9 \quad ...(i)$$
$$2x - 3y + 4z = 13 \quad ...(ii)$$
$$3x + 4y + 5z = 40 \quad ...(iii)$$

Step I. Operate $(ii) - 2(i)$ and $(iii) - 3(i)$ to eliminate x from (ii) and (iii).

$$x + y + z = 9 \quad ...(iv)$$
$$-5y + 2z = -5 \quad ...(v)$$
$$y + 2z = 13 \quad ...(vi)$$

Step II. Operate $(iv) + \frac{1}{5}(v)$ and $(vi) + \frac{1}{5}(v)$ to eliminate y from (iv) and (vi) :

$$x + \frac{7}{5}z = 8 \quad ...(vii)$$
$$-5y + 2z = -5 \quad ...(viii)$$
$$\frac{12}{5}z = 12 \quad ...(ix)$$

Step III. Operate $(vii) - \frac{7}{12}(ix)$ and $(viii) - \frac{5}{6}(ix)$ to eliminate z from (vii) and $(viii)$:

$$x = 1$$
$$-5y = -15$$
$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise : Rewriting the equations as $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 3R_1$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$

Operate $R_3 + \frac{1}{5}R_2$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$

Operate $-R_2, 5R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$

Operate $R_2 + \frac{1}{6}R_3, \frac{1}{12}R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$

Operate $\frac{1}{5}R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$

Operate $R_1 - R_2 - R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Hence, $x = 1, y = 3, z = 5$.

Obs. Here the process of elimination of variables amounts to *reducing the given coefficient metric to a* **diagonal matrix** *by elementary row transformations only.*

Example 28.16. *Solve the equations of example 28.14, by Gauss-Jordan method.*

Solution. We have

$$10x - 7y + 3z + 5u = 6 \quad ...(i)$$
$$-6x + 8y - z - 4u = 5 \quad ...(ii)$$
$$3x + y + 4z + 11u = 2 \quad ...(iii)$$
$$5x - 9y - 2z + 4u = 7 \quad ...(iv)$$

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \left(\frac{3}{10}\right)(i)\right], \left[(iv) - \left(\frac{5}{10}\right)(i)\right]$:

$$10x - 7y + 3z + 5u = 6 \quad ...(v)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(vi)$$
$$3.1y + 3.1z + 9.5u = 0.2 \quad ...(vii)$$
$$-5.5y - 3.5z + 1.5u = 4 \quad ...(viii)$$

Step II. To eliminate y, operate $\left[(v) - \left(\frac{-7}{3.8}\right)(vi)\right], \left[(vii) - \left(\frac{3.1}{3.8}\right)(vi)\right], \left[(viii) - \left(\frac{-5.5}{3.8}\right)(vi)\right]$:

$$10x + 4.4736842z + 3.1578947u = 21.842105 \quad ...(ix)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(x)$$
$$2.4473684z + 10.315789u = -6.8157895 \quad ...(xi)$$
$$-2.3421053x + 0.0526315u = 16.447368 \quad ...(xii)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{4.473684}{2.4473684}\right)(xi)\right]$,

$$\left[(x) - \left(\frac{0.8}{2.4473684}\right)(xi)\right], \left[(xii) - \left(\frac{-2.3421053}{2.4473684}\right)(xi)\right]:$$

$$10x - 15.698923u = 34.301075$$
$$3.8y - 4.3720429u = 10.827957$$
$$2.4473684z + 10.315789u = -6.8157895$$
$$9.9247309u = 9.9245975$$

Step IV. From the last equation $u = 1$ nearly.

Substitution of $u = 1$ in the above three equations gives $x = 5, y = 4, z = -7$.

(3) Factorization method*. This method is based on the fact that every matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, *i.e.*, if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as
$$AX = B \quad ...(1)$$

where
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let
$$A = LU, \quad ...(2)$$

where
$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

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*Another name given to this discomposition is **Doolittle's method.**

Then (1) becomes $\quad LUX = B$ ...(3)

Writing $\quad UX = V,$ ...(4)

(3) becomes $\quad LV = B$ which is equivalent to the equations

$$v_1 = b_1 \,;\, l_{21}v_1 + v_2 = b_2 \,;\, l_{31}v_1 + l_{31}v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3, we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1 \,;\, u_{22}x_2 + u_{23}x_3 = v_2 \,;\, u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by *back-substitution.*

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(*i*) $u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$

(*ii*) $l_{21}u_{11} = a_{21}$ or $l_{21} = a_{21}/a_{11}$

$l_{31}u_{11} = a_{31}$ or $l_{31} = a_{31}/a_{11}$

(*iii*) $l_{21}u_{12} + u_{22} = a_{22}$ or $u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}\, a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}$ or $u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}}\, a_{13}$

(*iv*) $l_{31}u_{12} + l_{33}u_{22} = a_{32}$ or $l_{32} = \dfrac{1}{u_{22}}\left[a_{32} - \dfrac{a_{31}}{a_{11}}a_{12}\right]$

(*v*) $l_{31}u_{13} + l_{33}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order :

(*i*) First row of U, (*ii*) First column of L,

(*iii*) Second row of U, (*iv*) Second column of L, (*v*) Third row of U.

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and **for finding the inverse of a matrix.** Among the direct methods, Factorization method is also preferred as the software for computers.

Example 28.17. *Apply factorization method to solve the equations :*

$3x + 2y + 7z = 4$; $2x + 3y + z = 5$; $3x + 4y + z = 7$. *(Madras, 2000 S)*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (*i.e.* A),

so that

(*i*) $u_{11} = 3, u_{12} = 2, \quad u_{13} = 7.$

(*ii*) $l_{21}u_{11} = 2, \quad \therefore\ l_{21} = 2/3$

$l_{31}u_{11} = 3, \quad \therefore\ l_{31} = 1.$

(*iii*) $l_{21}u_{12} + u_{22} = 3, \quad \therefore\ u_{22} = 5/3,$

$l_{21}u_{13} + u_{23} = 1, \quad \therefore\ u_{23} = -11/3.$

(*iv*) $l_{31}u_{12} + l_{32}u_{22} = 4, \quad \therefore\ l_{32} = 6/5.$

(*v*) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$\therefore \quad u_{33} = -8/5$

Thus $$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $UX = V$, the given system becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Solving this system, we have $v_1 = 4$,

$$\frac{2}{3}v_1 + v_2 = 5 \qquad \text{or} \qquad v_2 = \frac{7}{3}$$

$$v_1 + \frac{6}{5}v_2 + v_3 = 7 \qquad \text{or} \qquad v_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e., $$3x + 2y + 7z = 4 \,;\ \frac{5}{3}y - \frac{11}{3}z = \frac{7}{3} \,;\ -\frac{8}{5}z = \frac{1}{5}$$

By back-substitution, we have $z = -1/8$, $y = 9/8$ and $x = 7/8$.

Example 28.18. *Solve the equations of Example 28.14 by factorization method.*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{13} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$ (*i.e.*, *A*)

so that

(*i*) R_1 of $U : u_{11} = 10, u_{12} = -7, u_{13} = 3, u_{14} = 5$

(*ii*) C_1 of $L : l_{21} = -0.6, l_{31} = 0.3, l_{41} = 0.5$

(*iii*) R_2 of $U : u_{22} = 3.8, u_{23} = 0.8, u_{24} = -1$

(*iv*) C_2 of $L : l_{32} = 0.81579, l_{42} = -1.44737$

(*v*) R_3 of $U : u_{33} = 2.44737, u_{34} = 10.31579$

(*vi*) C_3 of $L : l_{43} = -0.95699$

(*vii*) R_4 of $U : u_{44} = 9.92474$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix}\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6,\ v_2 = 8.6,\ v_3 = -6.81579,\ v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e., $$10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6,$$
$$2.44737z + 10.31579u = -6.81579,\ u = 1.$$

By back-substitution, we get $u = 1, z = -7, y = 4, x = 5$.

PROBLEMS 28.3

Solve the following equations by Gauss elimination method :

1. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(P.T.U., 2005)*
2. $2x + 2y + z = 12$; $3x + 2y + 2z = 8$; $5x + 10y - 8z = 10$. *(W.B.T.U., 2004)*
3. $2x - y + 3z = 9$; $x + y + z = 6$; $x - y + z = 2$. *(Bhopal, 2009)*
4. $2x_1 + 4x_2 + x_3 = 3$; $3x_1 + 2x_2 - 2x_3 = -2$; $x_1 - x_2 + x_3 = 6$. *(Marathwada, 2008)*
5. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$;
$x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Solve the following equations by Gauss-Jordan method :

6. $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$. *(V.T.U., 2010)*
7. $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$. *(Kerala, 2003)*
8. $x + 3y + 3z = 16$; $z + 4y + 3z = 18$; $x + 3y + 4z = 19$. *(Anna, 2005)*
9. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(V.T.U., 2008)*
10. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$;
$3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$.

Solve the following equations by factorization method :

11. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 14$. *(Andhra, 2004 ; P.T.U., 2003)*
12. $x + 2y + 3z = 14$; $2x + 3y + 4z = 20$; $3x + 4y + z = 14$.
13. $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.
14. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 5$.
15. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ by Crout's method.

28.7 ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods :

(1) Jacobi's iteration method*. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \ldots(1)$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned}\right\} \quad \ldots(2)$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z. Substituting these on the right, we get the first approximations $x_1 = k_1, y_1 = k_2, z_1 = k_3$.

Substituting these on the right-hand sides of (2), the second approximations are given by

$$\begin{aligned} x_2 &= k_1 - l_1y_1 - m_1z_1 \\ y_2 &= k_2 - l_2x_1 - m_2z_1 \\ z_2 &= k_3 - l_3x_1 - m_3y_1 \end{aligned}$$

This process is repeated till the difference between two consecutive approximations is negligible.

*See footnote p. 215.

Example 28.19. *Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places.* *(Anna, 2007)*

Solution. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z),\ y = \frac{1}{10}(28.08 - x - z),\ z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration $\quad x_1 = \frac{11.19}{10} = 1.119,\ y_1 = \frac{28.08}{10} = 2.808,\ z_1 = \frac{35.61}{10} = 3.561$

Second iteration $\quad x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration $\quad x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$$y_3 = \frac{1}{10}(28.03 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration $\quad x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$$y_4 = \frac{1}{10}(28.03 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration $\quad x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$.

Example 28.20. *Solve, by Jacobi's iteration method, the equations*

$$20x + y - 2z = 17;\ 3x + 20y - z = -18;\ 2x - 3y + 20z = 25.$$ *(Bhopal, 2009)*

Solution. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \text{...}(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.
Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85;\ y_1 = -\frac{18}{20} = -0.9;\ z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.1515$$

Substituting these values in the right sides of the equations (*i*), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0134$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0032$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0009$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9993$$

Putting these values, we have

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.9996$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 1.0000$$

The values in the 5th and 6th iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

(2) Gauss-Seidel iteration method*. This is a modification of the Jacobi's iteration method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0, z = z_0$ in the first of the equations (2) on page 837, we get

$$x_1 = k_1$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2) on page 837, we have

$$y_1 = k_2 - l_2x_1 - m_2z_0$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2) on page 837, we obtain

$$z_1 = k_3 - l_3x_1 - m_3y_1$$

and so on, *i.e.*, as soon as new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Obs 1. Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Obs 2. Gauss-Sedial method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

*See footnote p. 37. *After Philipp Ludwig Von Seidel* (1821–1896) who also suggested a similar method.

Example 28.21. *Apply Gauss-Seidel iteration method to solve the equations of Ex. 28.20.*
(V.T.U., 2011 ; Rohtak, 2005 ; Madras, 2003)

Solution. We write the given equation in the form

$$x = \frac{1}{20}(17 - y + 2z)\ ;\ y = \frac{1}{20}(-18 - 3x + z)\ ;\ z = \frac{1}{20}(25 - 2x + 3y) \qquad ...(i)$$

We start from the approximation $x_0 = y_0 = z_0 = 0$. Substituting $y = y_0$, $z = z_0$ in the right side of the first of equations (i), we get

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1$, $z = z_0$ in the second of the equations (i), we have

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Putting $x = x_1$, $y = y_1$ in the last of the equations (i), we obtain

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

For the second iteration, we have

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

For the third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 3x_3 + 2y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

Example 28.22. *Solve the equations :*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidal iteration method. *(Bhopal, 2009 ; J.N.T.U., 2004)*

Solution. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \qquad ...(i)$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \qquad ...(ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \qquad ...(iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \qquad ...(iv)$$

First iteration

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (i), we get $x_1 = 0.3$

Putting $x_1 = 0.3, x_3 = 0, x_4 = 0$ in (ii), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3, x_2 = 1.56, x_4 = 0$ in (iii), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3, x_2 = 1.56, x_3 = 2.886$ in (iv), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56, x_3 = 2.886, x_4 = -0.1368$ in (*i*), we obtain

$x_1 = 0.8869$

Putting $x_1 = 0.8869, x_3 = 2.886, x_4 = -0.1368$ in (*ii*), we obtain

$x_2 = 1.9523$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_4 = -0.1368$ in (*iii*), we have

$x_3 = 2.9566$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_3 = 2.9566$ in (*iv*), we get

$x_4 = -0.0248.$

Third iteration

Putting $x_2 = 1.9523, x_3 = 2.9566, x_4 = -0.0248$ in (*i*), we obtain

$x_1 = 0.9836$

Putting $x_1 = 0.9836, x_3 = 2.9566, x_4 = -0.0248$ in (*ii*), we obtain

$x_2 = 1.9899$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_4 = -0.0248$ in (*iii*), we get

$x_3 = 2.9924$

Putting $x_1 = 0.9836, x_1 = 1.9899, x_3 = 2.9924$ in (*iv*), we get

$x_4 = -0.0042.$

Fourth iteration. Proceeding as above

$x_1 = 0.9968, x_2 = 1.9982, x_3 = 2.9987, x_4 = -0.0008.$

Fifth iteration is

$x_1 = 0.9994, x_2 = 1.9997, x_3 = 2.9997, x_4 = -0.0001.$

Sixth iteration is

$x_1 = 0.9999, x_2 = 1.9999, x_3 = 2.9999, x_4 = -0.0001.$

Hence the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0$.

(3) **Relaxation method*.** Consider the equations

$$a_1x + b_1y + c_1z = d_1\,;\ a_2x + b_2y + c_2z = d_2\,;\ a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z\,;\ R_y = d_2 - a_2x - b_2y - c_2z\,;\ R_z = d_3 - a_3x - b_3y - c_3z \qquad ...(1)$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step by giving increments to the variables. For this purpose, we construct the following *operation table :*

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; *e.g.*, to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Example 28.23. *Solve, by Relaxation method, the equations :*

$$9x - 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19. \qquad \textit{(Madras, 2000 S)}$$

*This method was originally developed by *R.V. Southwell* in 1935, for application to structural engineering problems.

Solution. The residuals are given by

$R_x = 50 - 9x + 2y - z$; $R_y = 18 - x - 5y + 3z$; $R_z = 19 + 2x - 2y - 7z$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	3	-7

The relaxation table is

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	...(*i*)
$\delta x = 5$	5	13	29	...(*ii*)
$\delta z = 4$	1	25	1	...(*iii*)
$\delta y = 5$	11	0	-9	...(*iv*)
$\delta x = 1$	2	-1	-7	...(*v*)
$\delta z = -1$	3	-4	0	...(*vi*)
$\delta y = -0.8$	1.4	0	1.6	...(*vii*)
$\delta z = 0.23$	1.17	0.69	$-.09$	...(*viii*)
$\delta x = 0.13$	0	0.56	0.17	...(*ix*)
$\delta y = 0.112$	0.224	0	-0.054	...(*x*)

$\Sigma\delta x = 6.13$, $\Sigma\delta y = 4.31$, $\Sigma\delta z = 3.23$

Thus $x = 6.13, y = 4.31, z = 3.23.$

[**Explanation.** In (*i*), the largest residual is 50. To reduce it, we give an increment $dx = 5$ and the resulting residuals are shown in (*ii*). Of these $R_x = 29$ is the largest and we given an increment $\delta z = 4$ to get the results in (*iii*). In (*vi*), $R_y = -4$ is the (numerically) largest and we give an increment $\delta y = -4/5 = -0.8$ to obtain the results in (*vii*). Similarly the other steps have been carried out.]

Example 28.24. *Solve by Relaxation method, the equations :*

$10x - 2y - 3z = 205$; $-2x + 10y - 2z = 154$; $-2x - y + 10z = 120$. (V.T.U., 2011 S ; Rohtak, 2005)

Solution. The residuals are given by

$R_x = 205 - 10x + 2y + 3z$; $R_y = 154 + 2x - 10y + 2z$; $R_z = 120 + 2x + y - 10z$.

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-10	2	2
$\delta y = 1$	2	-10	-1
$\delta z = 1$	3	2	-10

The relaxation table is :

	R_x	R_y	R_z
$x = y = z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta z = 1$	-2	10	-1
$\delta y = 1$	0	0	0

$\Sigma\delta x = 32$, $\Sigma y = 26$, $\Sigma z = 21$.

Hence $x = 32, y = 26, z = 21.$

PROBLEMS 28.4

1. Solve by Jacobi's method, the equations : $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$. Start with the solution (2, 3, 0).
2. Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.
   by (*a*) Jacobi's method (*b*) Gauss-Seidel method. (*Anna, 2006*)

   Solve the following equations by Gauss-Seidel method :
3. $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$.
4. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$. (*Mumbai, 2009*)
5. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 104$. (*V.T.U., MCA, 2007*)
6. $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$. (*Hazaribagh, 2009*)
7. $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$; $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$; $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$ (*Mumbai, 2004*)
8. $1.2x + 2.1y + 4.2z = 9.9$; $5.3x + 6.1y + 4.7z = 21.6$; $9.2x + 8.3y + z = 15.2$.
9. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

   Solve by Relaxation method, the following sets of equations :
10. $3x + 9y - 2z = 11$; $4x + 2y + 13z = 24$; $4x - 4y + 3z = -8$. (*Bhopal, 2002*)
11. $10x - 2y - 2z = 6$; $-x + 10y - 2z = 7$; $-x - y + 10z = 8$.
12. $-9x + 3y + 4z + 100 = 0$; $x - 7y + 3z + 80 = 0$; $2x + 3y - 5z + 60 = 0$.
13. $54x + y + z = 110$; $2x + 15y + 6z = 72$; $-x + 6y + 27z = 85$. (*Bhopal, 2003*)

28.8 SOLUTION OF NON-LINEAR SIMULTANEOUS EQUATIONS—NEWTON-RAPHSON METHOD

Consider the equations

$$f(x, y) = 0, g(x, y) = 0 \qquad ...(1)$$

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be obtained as follows :

Let $x_1 = x_0 + h, y_1 = y_0 + k$, so that $f(x_0 + h, y_0 + k) = 0, g(x_0 + h, y_0 + k) = 0$ ...(2)

Expanding each of the functions in (2) by Taylor's series to first degree terms, we get approximately

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \end{aligned}\right\} \qquad ...(3)$$

where $f_0 = f(x_0, y_0), \ \dfrac{\partial f}{\partial x_0} = \left(\dfrac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.

Solving the equations (3) for h and k, we get a new approximation to the root as

$$x_1 = x_0 + h, y_1 = y_0 + k$$

This process is repeated till we get the values to the desired accuracy.

Example 28.25. *Solve the system of non-linear equations :*

$$x^2 + y = 11, y^2 + x = 7. \qquad (Pune, 2000)$$

Solution. An initial approximation to the solution is obtained from a rough graph of the given equations, as $x_0 = 3.5$ and $y_0 = -1.8$.

We have $f = x^2 + y - 11$ and $g = y^2 + x - 7$ so that

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = 1, \frac{\partial g}{\partial y} = 2y.$$

Then Newton-Raphson's equations (3) above will be

$$7h + k = 0.55, h - 3.6k = 0.26$$

Solving these, we get $h = 0.0855, k = -0.0485$

$\therefore$ the better approximation to the root is

$$x_1 = x_0 + h = 3.5855, y_1 = y_0 + k = -1.8485$$

Repeating the above process, replacing (x_0, y_0) by (x_1, y_1), we obtain $x_2 = 3.5844$, $y_2 = -1.8482$.

PROBLEMS 28.5

1. Solve the equations $x^2 + y = 5$; $y^2 + x = 3$.
2. Solve the non-linear equations $x = 2(y + 1)$, $y^2 = 3xy - 7$ correct to three decimals.
3. Use Newton-Raphson method to solve the equations $x = x^2 + y^2$, $y = x^2 - y^2$ correct to two decimals, starting with the approximation (0.8, 0.4).
4. Solve the non-linear equations $x^2 - y^2 = 4$, $x^2 + y^2 = 16$ numerically with $x_0 = y_0 = 2.828$ using N.R. method. Carry out two iterations. *(V.T.U., MCA, 2007)*
5. Solve the equations $2x^2 + 3xy + y^2 = 3$; $4x^2 + 2xy + y^2 = 30$. Correct to three decimal places, using Newton-Raphson method, given that $x_0 = -3$, and $y_0 = 2$.

28.9 DETERMINATION OF EIGEN VALUES BY ITERATION

In § 2.14, we came across equations of the type

$$\left.\begin{array}{r} (a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 = 0 \end{array}\right\} \quad ...(1)$$

which in matrix form, may be written as $[A - \lambda I] X = 0$ or $AX = \lambda X$ ...(2)

where $A = [a_{ij}]$ and X is the column matrix $[x_i]$.

Equation (1) will have a non-trivial solution if the coefficient matrix vanishes *e.g.*,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

This gives a cubic in λ whose roots are *eigen values* of (2) and corresponding to each *eigen value*, we have a non-zero solution $X = [x_1, x_2, x_3]$ which is called an *eigen vector*. Such an equation can ordinarily be solved easily.

In some applications, it is required to compute the numerically largest *eigen value* and the corresponding *eigen vector*. In such cases, the following iterative method is more convenient which is also well-suited for computing machines.

If X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$, then an arbitrary column vector can be written as $X = k_1X_1 + k_2X_2 + k_3X_3$

Then $\quad AX = k_1AX_1 + k_2AX_2 + k_3AX_3 = k_1\lambda_1X_1 + k_2\lambda_2X_2 + k_3\lambda_3X_3$

Similarly $\quad A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + k_3\lambda_3^2X_3$

and $\quad A^rX = k_1\lambda_1^r X_1 + k_2\lambda_2^rX_2 + k_3\lambda_3^rX_3$

If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus we start with a column vector X which is as near the solution as possible and evaluate AX which is written as $\lambda^{(1)} X^{(1)}$ after normalisation. This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector. Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the second approximation. We repeat this process till $[X^{(r)} - X^{(r-1)}]$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value of (1) and $X^{(r)}$, the corresponding eigen vector.

*This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.**

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*After the English mathematician and physicist John William Strut known as *Lord Rayleigh* (1842–1919) who made important contributions to the theory of waves, elasticity and hydrodynamics. He was professor at Cambridge and London.

Example 28.26. *Determine the largest eigen value and the corresponding eigen vector of the matrices using the power method :*

$$(i)\ A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \qquad (ii)\ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ (V.T.U., 2007)

Solution. (*i*) Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Thus the second approximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen-vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

Now $$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

(*ii*) Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)}X^{(1)}.$$

So the first approximation to the eigen value is $\lambda^{(1)} = 2$ and the corresponding eigen vector

$$X^{(1)} = [1, -0.5, 0]'.$$

Hence $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)}X^{(2)}.$$

Repeating the above process, we get

$$AX^{(2)} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)}X^{(3)}\ ;\ AX^{(3)} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)}X^{(5)}\ ;\ AX^{(5)} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)}X^{(6)}\ ;\ AX^{(6)} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)}X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

PROBLEMS 28.6

1. Find by power method, the larger eigen-value of the matrices :

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *(Anna, 2005)* (b) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Obtain the largest eigen-value and the corresponding eigen-vector for the equations
$(2-\lambda)x_1 - x_2 = 0 \,;\, -x_1 + (2-\lambda)x_2 - x_3 = 0 \,;\, -x_2 + (2-\lambda)x_3 = 0$
by Rayleigh Quotient method.

3. Find the dominant eigen value and the corresponding eigen vector of the following matrices using the power method :

(a) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ *(V.T.U., 2011)* (b) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *(V.T.U., 2011 S)*

4. Find the largest eigen-value and the corresponding eigen-vector of the matrices :

(a) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ *(Anna, 2005)* (b) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ *(V.T.U., 2008)*

(c) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $[1, 1, 0]^T$. *(Madras, 2006)*

28.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 28.7

Fill up the blanks or select the correct answer to each of the following problems :

1. Out of Regula-falsi method and Newton-Rephson method, the rate of convergence is faster for ....
2. If x_n is the nth iterate, then the Newton-Raphson formula is ....
3. In the Regula-falsi method of finding the real root of an equation, the curve AB is replaced by ....
4. Newton's iterative formula to find the value of $\sqrt{N}$ is ....
5. Newton-Raphson formula converges when ....
6. In solving simultaneous equations by Gauss-Jordan method, the coefficient matrix is reduced to .... matrix.
7. In the case of bisection method, the convergence is
   (a) linear (b) quadratic (c) very slow.
8. The order of convergence in Newton-Raphson method is
   (a) 2 (b) 3 (c) 0 (d) none.
9. The Newton-Raphson algorithm for finding the cube root of N is ....
10. The bisection method for finding the root of an equation $f(x) = 0$ is ....
11. In Regula-falsi method, the first approximation is given by ....
12. The order of convergence in Newton-Raphson method is
    (a) 2 (b) 3 (c) 0 (d) none.
13. The iterative formula for finding the reciprocal of N is $x_{n+1} =$ ....
14. As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
    (a) Gauss-Jordan method (b) Gauss-Seidal method
    (c) Jacobi's method (d) Relaxation method.
15. Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for ....
16. The difference between direct and iterative methods of solving simultaneous linear equations is ....
17. To which form the coefficient matrix is transformed when $AX = B$ is solved by Gauss elimination method.
18. Jacobi's iteration method can be used to solve a system of non-linear equations. (True or False)
19. The convergence in the Gauss-Seidal method is thrice as fast as in Jacobi's method. (True or False)
20. By Gauss elimination method, solve $x + y = 2$ and $2x + 3y = 5$. *(Anna, 2007)*

6. Not significant at 1% level and just significant at 5% level as $F = 2$, $F_{0.01} = 2.62$ and $F_{0.05} = 1.98$
7. $F = 1.49$, Not-significant
8. $F = 1.025$; Yes

Problems 27.6, page 917

1. § 27.3 (3)
2. § 27.15
3. § 27.3 (2)
4. We are testing the hypothesis that one process is better than another
5. § 27.11
6. 1
7. 50
8. $(0, \infty)$
9. II
10. 8 ; 16
11. $r = n - 1$
12. False
13. Less than 30
14. § 27.17
15. $-\infty < t < \infty$
16. (*ii*)
17. True

Problems 28.1, page 926

1. (*i*) 2.687, (*ii*) 1.46, (*iii*) 2.375, (*iv*) 2.875
2. (*i*) 0.519, (*ii*) 2.875, (*iii*) 1.146, (*iv*) 0.367
3. (*i*) – 0.686, (*ii*) 2.7065, (*iii*) 0.686, (*iv*) 1.4036
4. (*i*) 0.853, (*ii*) 0.607, (*iii*) 2.798, (*iv*) 3.789, (*v*) – 0.134
5. 1.861
6. (*i*) 1.532, (*ii*) 2.095, (*iii*) 1.834, (*iv*) 1.226
7. (*i*) 1.855 (*ii*) 2.198 (*iii*) 1.662
8. – 16.56
9. (*i*) 0.853, (*ii*) – 1.9338, (*iii*) 2.7985, (*iv*) 4.545
10. (*i*) 0.518, (*ii*) 0.052, (*iii*) 0.695, (*iv*) 2.911
11. $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$; (*i*) 3.605 (*ii*) 3.162
12. 3.4482
13. 2.3784
14. (*i*) 0.055 (*ii*) 0.258 (*iii*) 0.4347

Problems 28.2, page 929

1. (*i*) 1.532, (*ii*) 0.684, (*iii*) 3.18, (*iv*) 1.168
2. 1.674
3. 2.231
4. – 1.328
5. 2.924

Problems 28.3, page 936

1. $x = 7, y = -9, z = 5$
2. $x = -51/4, y = 115/8, z = 35/4$
3. $x = 1, y = 2, z = 3$
4. $x_1 = 2, x_2 = -1, x_3 = 3$
5. $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$
6. $x = 1, y = 3, z = 5$
7. $x = 8.7, y = 5.7, z = -1.3$
8. $x = 1, y = 2, z = 3$
9. $x = 7, y = -9, z = 5$
10. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$
11. $x = y = z = 1$
12. $x = 1, y = 2, z = 3$
13. $x = 35/18, y = 29/18, z = 5/18$
14. $x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$
15. $\begin{bmatrix} 1.2 & -0.4 & 0.2 \\ -0.2 & -0.1 & 0.3 \\ -0.4 & 0.3 & 0.1 \end{bmatrix}$.

Problems 28.4, page 942

1. $x = 2.556, y = 1.722, z = -1.055$
2. (*a*) $x = 2.426, y = 3.573, z = 1.926$ (*b*) $x = 2.426, y = 3.573, z = 1.926$
3. $x = 1, y = 1, z = 1$
4. $x = 0.998, y = 1.723, z = 2.024$
6. $x = 1.052, y = 1.369, z = 1.962$
8. $x = -13.223, y = 16.766, z = -2.306$
9. $x = 1, y = 2, z = 3, u = 4$
10. $x = 1.36, y = 2.103, z = 2.845$
11. $x = y = z = 1$
12. $x = 52.5, y = 44.5, z = 59.7$
13. $x = 1.93, y = 3.57, z = 2.43$

Problems 28.5, page 943

1. $x = 2, y = 1$

2. $x = -1.853, y = -1.927$

3. $x = 0.7974, y = 0.4006$

5. $x = -3.131, y = 2.362$

4. $x = 3.162, y = 6.45$

Problems 28.6, page 945

1. (*a*) 5.38, $\begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$; (*b*) 4.418, $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$

2. 3.41 ; [0.74, – 1, 0.67]′

3. (*a*) 6, [1, 1, – 1]′ (*b*) 8, [1, – 0.5, 0.5]′

4. (*a*) 7 ; [2.099/7, 0.467/7, 1] (*b*) 25.182, [1, 0.045, 0.068]′ (*d*) 11.66 [0.025, 0.422, 1.000].

Problems 28.7, page 945

1. Newton-Raphson method

2. $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$

3. Chord AB

4. $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$

5. initial approximation x_0 is chosen sufficiently close to the root

6. diagonal

7. (*c*)

8. (*a*)

9. $x_{n+1} = \dfrac{1}{3}(2x_n + N/x_n^2)$

10. $x_{n+1} = \dfrac{1}{2}(x_n + x_{n-1})$

11. $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$

12. (*a*)

13. $x_{n+1} = x_n(2 - Nx_n)$

14. (*b*)

15. Newton-Raphson method

16. § 28.6

17. Upper triangular matrix

18. False

19. True

20. $x = 1, y = 1$.

Problems 29.1, page 952

1. 0.4

2. – 7459

5. 239

6. 4.68, 2.68, 55.8, 99.88

8. (*i*) $1 - 2\sin(x + 1/2)\sin 1/2$; (*ii*) $\tan^{-1}(1/2n^2)$;
(*iii*) $192[x(x+4)(x+8)(x+12)(x+16)]$ (*iv*) $-2/[(x+2)(x+3)(x+4)]$

9. (*i*) $e^{3x}[e^3\log(1 + 1/x) + (e^3 - 1)\log 2x]$ (*ii*) $2^x(1 - x)/(1 + x)$
(*iii*) $(a - 1)^n a^x$; (*iv*) $(-1)^n n!/[x(x+1)(x+2)\ldots(x+n)]$.

12. (*i*) – 36 ; (*ii*) $24 \times 2^{10} \times 10!$

14. $u = [x]^4 - 6[x]^3 + 13[x]^2 + x + 9$

15. $4x^3 - 12x^2 + 8x + 1$; $12x(x - 1)$

16. $\dfrac{1}{2}[x]^4 + 3[x]^3 + 4[x] + c$

17. $y(4) = 74, y(6) = 261$

19. 15.

Problems 29.2, page 957

1. $\left(\dfrac{\Delta^2}{E}\right)u_x = u_{x+h} - 2u_x + u_{x-h}$; $\dfrac{\Delta^2 u_x}{Eu_x} = \dfrac{u_{x+2h} - 2u_{x+h} + u_x}{u_{x+h}}$

2. (*i*) $2(\cos h - 1)\sin x$; (*ii*) $6x$; (*iii*) $2(\cos h - 1)[\sin(x + h) + 1]$; (*iv*) 8

8. *Error* = 10

9. 31

10. $f(1.5) = 0.222, f(5) = 22.022$

11. $y(4) = 74, y(6) = 261$

12. – 99

13. $y_4 = 1$ approx

15. (*i*) $n(3n^2 + 6n + 1)$; (*ii*) $\dfrac{n(n+1)(n+2)(n+3)}{4}$

16. $2/(1 - x)^3$.