Linear Algebra

Matrices application in communication theory, network analysis, theory of structures, quantum mechanics....
Determinants solving simultaneous linear eqns, for testing consistency of a system of linear eqns.

Matrices:

Elementary Transformations:

The foll elementary aparations are known as elementary transformation.

1) Interchanging any 2 rows & columns
2) Multiplying every element of a row or a column by a non zero constant.
3) To the elements of a row or a column adding k times elements of anyother row or column where k is a non-zero constant.

Elementary transformations are applied to both row & column & respectively termed as row transformation & column transformation.

Following symbols are employed for E.Ts.

Symbol	Meaning
$R_i \leftrightarrow R_j$	Interchanging ith row & j^{th} row
$C_i \leftrightarrow C_j$	— " — " — ith column & j^{th} column.
$R_i \to kR_i$	Multiplying each element of i^{th} row by a non zero const k
$C_i \to kC_i$	— " — " — " — i^{th} column
$R_i \to R_i + kR_j$	Changing ith row to sum of i^{th} row & k times j^{th} row.
$C_i \to C_i + kC_j$	" — " — j^{th} column & k times i^{th} column.

Eg: consider

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 6 & 4 & 6 \\ 7 & -3 & 8 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$C_1 \leftrightarrow C_2$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 1 & 6 \\ 6 & 4 & 6 \\ 7 & -3 & 8 \end{bmatrix} . \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 6 \\ 4 & 6 & 6 \\ -3 & 7 & 8 \end{bmatrix} \quad R_3 \to R_3/2$$

$$D = \begin{bmatrix} 7 & 4 & 4 \\ 1 & 4 & 6 \\ 2 & 3 & 3 \\ -3 & 7 & 8 \end{bmatrix}$$

Here matrix D is obtained by successive E Ts on matrix A.

Defns:

1) <u>Equivalent Matrices</u>:

Two matrices are said to be equivalent matrices if one of them can be obtained by application of successive E Ts on other. If A & B are 2 equivalent matrices, it is denoted by $A \sim B$. $\sim \to$ equivalent to

$\therefore$ in above eg $A \sim D$.

2) <u>Zero row</u>:

A row in a matrix is said to be 'zero row' if every element of the row is zero.

3) <u>Non-zero row</u>:

A row in a matrix is said to be 'non-zero row' if there exists atleast one non-zero element in that row.

4) <u>Leading entry or leading term</u>:

In a matrix, in every row, 1st non-zero element is known as leading entry or leading term.

By defn, elementary transformations does not increase nos. of rows or column. ie., order of matrix is unaltered by elementary transformations.

ET are used to find rank of a matrix by reducing it into following forms:

1) Echelon form

2) Normal form

Reducing a matrix into echelon form or row echelon form

A matrix is said to be in echelon form if

1) leading entry in each row is 1.
2) all the entries below the leading entry are zeros.
3) Nos. of zeroes before the leading entry in any row exceeds nos. of zeroes before the leading entry in previous row.
4) all the zero rows are below non zero rows

Given a matrix A, a matrix E in echelon form which is equivalent to A is obtained by successive Row transformations on A.

1) Reduce the foll matrices into echelon form

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \quad R_3 \to R_3 - 2R_1 \quad \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1/2 \\ R_2 \to R_2/2 \end{matrix} \quad \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) $$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \quad \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad R_3 \to R_3 \to R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3) $$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_4 \quad \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \quad \begin{matrix} R_2/-3 \\ \\ R_4 \to R_4 - R_3 \end{matrix}$$

3) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & \\ 0 & 1 & -3 & \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Minors of a Matrix:

It is the determinant obtained by deleting few rows & columns of the matrix.

Eg: $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}_{3\times 4}$ We have minors of order 3 & 2

Minors of order 3, $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 5 & 7 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{vmatrix} \ldots$

2, $\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 7 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} \ldots$

Rank of a Matrix

1: **Basic Defn:** Let A be a non-zero $m \times n$ matrix. A +ve integer r is said to be rank of matrix A if

(i) A has atleast one non-zero minor of order r.

(ii) Every minor of A of order greater than r is equal to zero

* In otherwords rank of the matrix is the order of the largest non-zero minor of the matrix.

Rank of matrix A is denoted by $\rho(A)$.

Eg: $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 0 & -1 & 2 & -4 \end{bmatrix}_{3 \times 4}$

Minors of order 3 are

$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ -1 & 2 & -4 \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & -1 & -4 \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 0 & 2 & -4 \end{vmatrix} = 0$

Minors of matrix 2 are $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} = -2 \neq 0$

$\therefore$ Rank is 2

$\rho(A) = 2$

2: **Defn:** The no. of non-zero rows in the row echelon form of a given matrix is rank of A.

3: **Defn:** If A is in the normal form $[I_r, 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$\rho(A) = r$

Note:

1) If A is a $m \times n$ matrix then $\rho(A)$ cannot exceed m or n

ie., $\rho(A) \leq \min(m, n)$

2) If I_n is the unit matrix of order n, then $\rho(I_n) = n$

3) A matrix & its transpose will have same rank

4) Equivalent matrix will have same rank.

Find the rank of foll matrices using E Tn.

1) $A = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$ $R_1 \to -R_1$ $\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$ $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 5R_1$

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$ $R_4 - R_3$ $\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Any minor of A of order 4 & 3 are zero.

$\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} = -1 \quad \therefore \rho(A) = 2$

2) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_3 - 3R_1$, $R_4 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \therefore \rho(A) = 2$

Find the rank of foll matrices by reducing them into echelon form:

1) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 2$ (No of nonzero rows)

2) $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 2 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 13/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 2$

3) $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 1 & 3 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 1 & -3/2 & -5/2 \\ 0 & 0 & 1 & 5/2 \end{bmatrix}$

$\rho(A) = 3$

Find rank of foll matrix by reducing them into normal form:

6) 1), 7) 2), 8) 3) 9) 4)

10) $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 0 & 3 & 5 & 1 \end{bmatrix}$ $\rho(A) = 2$.

$\begin{matrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{matrix}$

$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

Solution to a System of linear eqns.

Consider system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{(3 eqn in 3 unknowns)} \quad \text{—— ①}$$

In matrix notation, these eqns can be written as,

$$\begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{—— ②}$$

or $AX = B$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

is called coefficient matrix, is column matrix of unknowns, is column matrix of constants

If $(d_1 = d_2 = d_3 = 0)$ then $B = 0$, matrix eqn ② reduces to

$$AX = 0 \quad \text{—— ③}$$

Such a system of eqn is called a system of **homogeneous linear eqn**.

If atleast one of d_1, d_2, d_3 is non-zero, then $B \neq 0$ such a system of eqn is called a system of **non-homogeneous linear eqn**.

Solving ① means finding unknowns x, y, z which is nothing but finding X i.e., finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ Then $x = \alpha$, $y = \beta$, $z = \gamma$

The system of eqn ① need not always have a soln. It may have (i) no soln (ii) a unique soln or (iii) infinite no. of soln.

A system of eqn having no soln is called inconsistent system of eqn.

A system of eqn having one or more soln is called consistent system of eqn.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix} \quad — ④$$

Method of soln

Here the matrix $[A : B]$ in which the elements of A & B are written side by side is called the augmented matrix.

For a system of non-homogeneous linear eqn $AX = B$

(i) if $\rho[A:B] \neq \rho(A)$, no soln, system is inconsistent

(ii) if $\rho[A:B] = \rho(A) =$ no. of unknowns, the system has a unique soln.

(iii) if $\rho[A:B] = \rho(A) <$ no. of unknowns, the system has an infinite no. of soln.

For a system of homogeneous linear eqn $AX = 0$

(i) $X = 0$ is always a soln. This soln in which each unknown has the value zero is called null soln or trivial soln. Thus a homogeneous system is always consistent.

∴ A system of homogeneous linear eqn has either trivial soln or an infinite no. of soln.

(ii) if $\rho(A) =$ no. of unknowns the system has only trivial soln

(iii) if $\rho(A) <$ no. of unknowns, the system has an infinite no. of non-trivial soln.

[If A is a non-singular matrix then matrix eqn $AX = B$ has a unique soln.

Given eqn is $AX = B$ —①

∵ A is non singular A^{-1} exists

Pre multiplying both sides of ① by A^{-1} we get $A^{-1}AX = A^{-1}B$ or $(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$ or $X = A^{-1}B$ which is required unique soln ∵ A^{-1} is unique

Method of solution:

The process of finding a soln of ① is called 'solving the sysm'. Here the method employed is Gauss' elimination method also called direct method.

Working rule:

1. Given system of linear eqn ① write down augmented matrix ④.
2. Reduce augm matrix [A : B] to echelon form, (or upper triangular matrix in case of square matrix) by employing elementary row transformations.
3. Find ranks of A & [A : B] by inspecting echelon forms of A decide the consistency.

   If 2 ranks not equal system - inconsistent.

   If $\rho(A) = \rho[A:B] = r = n$ (no. of unknowns) consistent unique soln

   $\rho(A) = \rho[A:B] = r < n$ consistent infinite no of soln.

4. If system is consistent write down linear eqn associated with echelon form obtained in step 3. If no. of eqns in the reduced system is r & $r < n$ then $n - r$ unknown chosen arbitrarily & possible solns of given system can be written.

   If ~~$r = n$ then unique soln~~.

   Unknowns are obtained by back substitution.

∴ Direct method of solving consisting & finding soln of system of eqns ① is termed as Gauss Elimination.

∴ If system is consistent writedown corresponding system of eqns from reduced form of [A : B]. Find the values of unknowns by Back substitution.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

in matrix notation, $AX = B$

where $A = [a_{ij}]_{m\times n}$ is called the co-efficient matrix

$X = [x_j]_{n\times 1}$ is column matrix of unknowns.

$B = [b_i]_{m\times 1}$, is column matrix of constants.

If $b_1 = b_2 = \dots = b_m = 0$, i.e. $B = 0$, $AX = 0$

is called a system of 'homogenious linear equations'

if $B \neq 0$, system is 'non-homogenious'.

A system of equations $AX = B$, having no solution is called as 'inconsistent'

A system of equations having one or more solutions is called as 'consistent'.

If $\rho[A|B] \neq \rho[A]$, the system is inconsistent

If $\rho[A|B] = \rho[A]$ = number of unknowns, the system has a unique solution.

If $\rho[A|B] = \rho[A]$ < number of unknowns, the system has infinite number of solutions.

The matrix $[A|B]$ is known as 'augmented matrix'

The system of homogenious equations $AX = 0$ is always consistent and has $X = 0$ as solution, called trivial or null solution.

Examples:

1. Solve $x_1 + 3x_2 + 2x_3 = 0$, $2x_1 - x_2 + 3x_3 = 0$,
$3x_1 - 5x_2 + 4x_3 = 0$, $x_1 + 17x_2 + 4x_3 = 0$.

$$[A|0] \sim \left[\begin{array}{ccc|c} 1 & -11 & 0 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \rho(A) = 2 < 3 \text{ no. of unknowns}$$

2. Determine the values of λ and μ such the following system of equations
$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$
has (i) no solution, (ii) unique solution, (iii) more than one solutions.

Consistency of system of Linear equations

- Equation
  - Homogeneous
    - Consistent
      - Solution
        - Trivial
          ($x=0, y=0, z=0 \dots$)
        - Nontrivial
          (Infinite)
          * Choose $(n-r)$ arbitrary Constants
  - Non homogeneous
    - Consistent
      [$\rho(A) = \rho(A:B)$]
      - Solution
        - Unique
          $\rho(A) = \rho(A:B) = n$
        - Infinite
          $\rho(A) = \rho(A:B) < n$.
    - Inconsistent
      [$\rho(A) \neq \rho(A:B)$]
      - No solution

* where $n \rightarrow$ no of unknowns
$r \rightarrow$ rank

(1)
$x + 2y + 3z = 0$
$3x + 4y + 4z = 0$
$7x + 10y + 12z = 0$

(2)
$4x + 2y + z + 3w$
$6x + 3y + 4z + 7w = 0$
$2x + y + w = 0$

(3)
$x_1 + 2x_2 - x_3 = 3$
$3x_1 - x_2 + 2x_3 = 1$
$2x_1 - 2x_2 + 3x_3 = 2$
$x_1 - x_2 + x_3 = -1$

(4)
$2x_1 + x_2 + 4x_3 = 12$
$8x_1 - 3x_2 + 2x_3 = 20$
$4x_1 + 11x_2 - x_3 = 33$

$5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$

Examples:

1) Test for consistency & solve,

$x+2y+2z=5, \quad 2x+y+3z=6, \quad 3x-y+2z=4, \quad x+y+z=-1$

Augmented matrix $[A:B]$,

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 2 & 1 & 3 & 6 \\ 3 & -1 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array} \sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & -7 & -4 & -11 \\ 0 & -1 & -1 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to 3R_3 - 7R_2 \\ R_4 \to 3R_4 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right] R_4 - 2R_3$$

$-12+7$

$-33+28$

$-3+1$

$-3+4$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Here $\rho(A) = 3$

$\neq \rho[A:B] = 4$

$-1-2$

$\therefore$ System is inconsistent no soln

2) Test for consistency & solve:

$x+y+z=3, \quad x+2y+3z=4, \quad x+4y+9z=6$

Soln: $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right] R_3-3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\rho(A) = 3$

$z = 0$

$y + 2z = 1 \quad y = 1$

$x + y + z = 3$

$x = 2$

4. Solve the system of eqns

$2x_1 + x_2 + 2x_3 + x_4 = 6, \quad 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36,$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1, \quad 2x_1 + 2x_2 - x_3 + x_4 = 10$

$x_1 = 2$
$x_2 = 1$
$x_3 = -1$
$x_4 = 3$

$[A:B] = \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$

Solve the traffic flow problem represented by foll system of eqns where unknowns represent no. of cars entering & leaving junction

$$\left(\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right)$$

$3-6$
$14-18$
-4

~~1 3 2 : 13~~

~~1 1 2 :~~

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{array}\right]$$

$14-27$

$x + 2 + 6 = 9$
$x = 1$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right)$$

$y = 2$
$z = 3$

3. Test whether foll eqns are consistent. If found consistent find the soln.

$x+y+z=8$ $\qquad x=5,\ y=5/3,\ z=4/3$

$x-y+2z=6$

$3x+5y-7z=14$

1) Test for consistency & solve

$x+y+z=6$ $\qquad \rho(A)=2$

$x+2y+3z=10$ $\qquad \rho(A:B)=3$

$2x+4y+6z=8$ $\qquad$ inconsistent

. ST foll system do not have non trivial soln

$x+2y+z=0,\quad x-2z=0,\quad 2x+y-3z=0$

.) Solve $2x-y+3z=0,\ 3x+2y+z=0,\ x-4y+5z=0$

$x=-k,\ y=k,\ z=k$

7) $x+y-z+u=0,\quad x-y+2z-u=0,\quad 3x+y+u=0$

$z=k_1,\ u=k_2,\quad x=-\frac{1}{2}k_1,\ y=\frac{3}{2}k_1-k_2$

8) $4x-2y+6z=8,\quad x+y-3z=-1,\ 15x-3y+9z=21$

$x=1,\ y=3k-2,\ z=k$

9) Find the value of λ & μ for which system

$x+y+z=6,\quad x+2y+3z=10,\quad x+2y+\lambda z=\mu$

has a (i) unique soln (ii) infinitely many soln (iii) no soln.

$$[A:B]\sim\left[\begin{array}{ccc:c}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{array}\right]$$

(i) $\lambda\neq3$ unique soln ($\rho(A)=3$ $\quad\rho(A:B)=3$)

(ii) $\rho(A)=\rho(A:B)=2<n$, $\lambda=3,\ \mu=10$ infinite soln

(iii) $\lambda=3$ & $\mu\neq10$ $\quad\rho(A)\neq\rho[A:B]$ system inconsistent.

10) Find the values of λ for which system

$x+y+z=1,\quad x+2y+4z=\lambda,\ x+4y+10z=\lambda^2$

has a soln. Solve it in each case.

$$[A:B]\sim\left[\begin{array}{ccc:c}1&1&1&1\\0&1&3&\lambda-1\\0&0&0&\lambda^2-3\lambda+2\end{array}\right]$$

Here $\rho(A)=2$ $\qquad \rho(A:B)=\rho(A)$ only if $\lambda^2-3\lambda+2=0$

i.e, $\lambda=1$ or $\lambda=2$ system is consistent.

$\lambda=1$ $\qquad x+y+z=1$ $\qquad z=k,$

$\qquad y+3z=0$ $\qquad y=-3k,$

$\qquad x=2k,+1$

1. $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$

3. Determine the values of 'b' such that the rank of A is 3.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 0 & -2 \\ 0 & 0 & b+6 & 0 \end{bmatrix}$$

$b = 2, \quad b = -6 \qquad \rho(A) = 3$

4. $\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

6. Find all the values of μ for which rank of the matrix

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$ is equal to 3.

7. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$