Calculus

Today's Agenda:

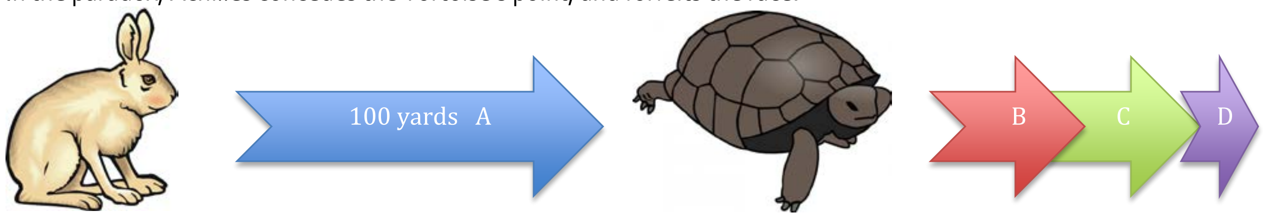
- Limits
- HW review

Zeno’s paradox of Achilles and the Tortoise

In a race, Achilles agrees to give the tortoise a 100 yard headstart (to point A). The tortoise suggests that the race is futile for Achilles, since Achilles cannot win:

- While Achilles is running the initial distance, the tortoise will have moved farther ahead, to point B
- Achilles must then travel from A to B but by then the tortoise will have moved to point C
- By the time Achilles get to point C, the tortoise will already be up to point D, etc.

In the paradox, Achilles concedes the Tortoise’s point, and forfeits the race!



Commented [BB1]: The idea here is just to engage them in a quick discussion about getting close to something. While technically not a limit example (a moving target isn’t like a limit) it should engage them quickly.

Key point to focus on is actually the distance between the two of them – if we were to graph (T – A), it would be increasingly close to zero even using the Tortoise logic!

Limits:

The big idea here is that limits are all about the ***approach*** – we don't care about the destination as long as the approach is leading somewhere!

Example – as *x* gets really large, what happens to the value of *1/x*?

x	*1/x*
1	1
2	½
3	1/3
4	¼
10	
100	
1000	
10000	
100000	
…	

What value of x will make 1/x ***equal*** zero?

Commented [BB2]: Since the two sections are both informal/computational approaches to limits, it's good to start with a basic idea (and here I'm trying to use AP's concept of multiple representations).

The second question is actually the key point – while 1/x never gets to zero as x gets really large, it doesn't need to for us to call it a limit.

Limit informal definition: There is a limit if the information from the function in the immediate vicinity (the "neighborhood") of a specific point suggests that the value of the function can be projected to be a certain number. Limits can be two-sided or one-sided.

Limit notation:

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a+} f(x) = L$$

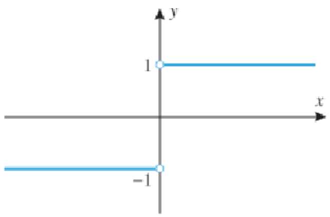
Commented [BB3]: I like the idea of emphasizing the "neighborhood" idea – we do not care if the value of the function exists at the limit point or not, but we do care about the neighborhood around it.

Emphasis here is on what each of the pieces represent in the equation. Also, a chance to remind them the f(x) does not ever need to equal L, it just has to be approaching L as x approaches a.

What are situations where there is no limit?

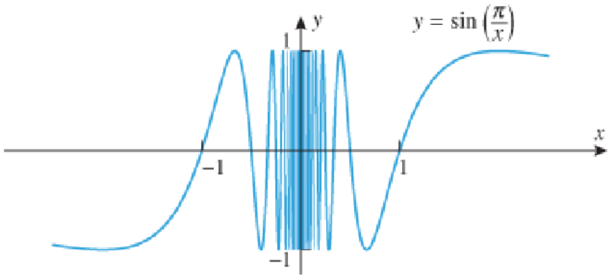
NO LIMITS…

1. **Discontinuity:** $\lim_{x \to 0} \frac{|x|}{x}$ does not exist (DNE)



$y = \frac{|x|}{x}$

2. **Oscillation:** $\lim_{x \to 0} \sin \frac{\pi}{x}$ does not exist (DNE)

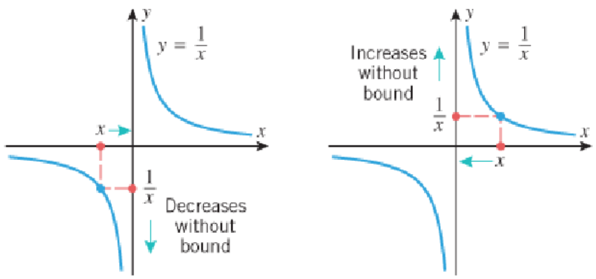


Commented [BB4]: This is a quick discussion of the three situations where there are no limits

For the first one, emphasize that the only place on this graph where there is NOT a limit is when x approaches 0; everywhere else it's either 1 or -1. Further, there is a one-sided limit from both directions here.

On the second one, it might be good to come back to Zeno's paradox – the reason that there's no limit (from either side) as x approaches zero is that it never converges on a single point, because you can always squeeze in another oscillation.

3. **Vertical asymptotes (and other infinite limits):** $\lim_{x \to 0+} \frac{1}{x}$ does not exist (DNE)



note – these are not actually limits because infinity is a quantitative concept but not a specific number; it is acceptable in these cases to write $\lim_{x \to 0+} \frac{1}{x} = \infty$, but these are technically DNE

Commented [BB5]: This one's a bit counterintuitive, especially because convention has us show a "number" of infinity as the limit. We should emphasize that while the limit doesn't actually exist, we can say that for any very small arbitrary value of x the limit would be an extremely large number, and for a slightly smaller x-value it would be an even larger number. So infinity is a good way to describe the kinds of numbers we're talking about very close to the y-axis, but there's no actual number "infinity".

Basic limit rules:

- For continuous functions, the limit approaching a point inside the domain is the value of the function
$$\lim_{x \to a} f(x) = f(a)$$
- The limit of the sum is the sum of the limits
$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a)$$
- The limit of the difference is the difference of the limits
$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a)$$
- The limit of the product is the product of the limits
$$\lim_{x \to a}[f(x)g(x)] = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x)) = f(a)g(a)$$
- The limit of a quotient is the quotient of the limits, provided that the limit of the denominators is not zero
$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$$
- The limit of an nth root is the nth root of the limit, provided that $f(a) > 0$ for even-numbered roots
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{f(a)}$$

In most cases, limits make sense!

Commented [BB6]: It's actually useful at this point for students to see that, in many mundane cases, limits are completely benign, and do what you'd expect them to do. I find that the "the limit of the …" phrases are good ways to trigger their memories in office hours, etc.

Special limit situations:

- Limits involving a rational function, where the limit of the denominator is zero:
  - Technically the limit does not exist!
  - We can describe the vertical asymptote as either one-sided or two-sided limits to $\pm\infty$
    - To do this, consider using a sign chart. Example: look at $f(x) = \frac{(x-2)(x+1)}{x-1}$

x-values	x < -1	-1 < x < 1	1 < x < 2	x >2
Signs	$\frac{(x-2)(x+1)}{x-1} == \frac{(-)(-)}{-} = (-)$			
Limit behavior	From the left side, the limit is negative infinity: $\lim_{x\to -1^-} f(x) = -\infty$			

- Limits involving radicals
  - If there is a radical in the denominator, consider using a conjugate to rewrite the limit:

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq 1)$$

- Piecewise-defined functions:
  - If finding a limit at a point where the formula changes, look at <u>both</u> sides of the point
    - If the limits match, ok to define as two-sided, otherwise you have two one-sided limits

Homework Questions?

Commented [BB7]: For the first one, it's probably useful to refer them back to the previous page – as the denominator gets really close to zero, the whole thing is acting like 1/x. However, the sign of the function can change as we move past the hole, so they need to think of the sign changes.

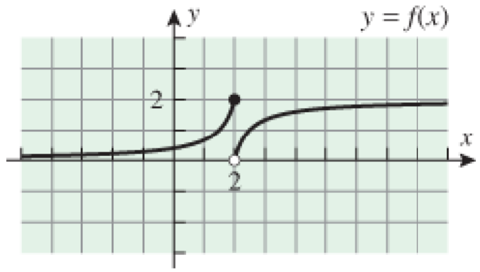
Sign charts are pretty useful tools (although AP wants us to emphasize that students need to be able to express what's going on and not just use a sign chart). I've shown them by having students first identify places where signs will change or where the function DNE, (therefore creating intervals with the same sign) and then pick a convenient number in the interval to use to check on the sign over that interval.

The radical example is a bit tricky, so you might want to walk them through it. A recurring theme in calculus is to look for ways to express an expression in a different form so that we can use our tools on it.

The piecewise idea will probably be pretty intuitive to people, but it's useful to emphasize the idea that both sides agreeing means that it's a one-sided limit (we'll have a few test problems along the way where they have to solve for a value that "connects" parts of a piecewise equation).

Breakout rooms

1. For the function *f* graphed in the accompanying figure, find:



(a) $\lim_{x \to 2^-} f(x)$

(b) $\lim_{x \to 2^+} f(x)$

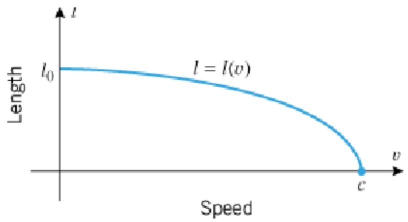
(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$.

Commented [BB8]: Answers:
a)2
b)0
c)DNE
d)2

2.

In the special theory of relativity the length l of a narrow rod moving longitudinally is a function $l = l(v)$ of the rod's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.



(a) What is the physical interpretation of l_0?

(b) What is $\lim_{v \to c^-} l(v)$? What is the physical significance of this limit?

Commented [BB9]: Answers:
a)Rest length
b)Limit is zero; as speed approaches c, length approaches zero

3. Find the limit:

$$\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 + x - 6}$$

Commented [BB10]: Answer: 0

4. Find the limit:

$$\lim_{t \to 2} \frac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$$

Commented [BB11]: Answer: 3/2

5. Find the limit:

$$\lim_{x \to 4^+} \frac{3-x}{x^2-2x-8}$$

Commented [BB12]: Answer: negative infinity

6. Find the limit:

$$\lim_{x \to 1} \frac{3x^2 - x - 2}{2x^2 + x - 3}$$

Commented [BB13]: Answer: 1

Commented [BB14]: The answer to this problem is 2. To solve it, multiply the first term by (x+1)/(x+1). Then, combine fractions; the numerator is (x + 1 – a). Since a denominator of (x+1) would not equal zero at x = 1, if a = 2 we can simplify to 1/(x+1) with a limit of ½.

7. Find all values of a such that

$$\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{a}{x^2 - 1} \right)$$

exists and is finite.