Math 075 Homework 4

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Exercise 2.3.7

Problem

Part (a)

If A^2 can be formed, what can be said about the size of A?

Solution (a)

If A^2 can be formed, this implies that a matrix must be square, since the number of rows and columns match.

Part (b)

If AB and BA can both be formed, what does this say about the sizes of bot A and B?

Solution (b)

When two matrices are multiplied together, the order in which they are multiplied affects the result, due to the fact that the dimensions must be matched. When AB is equal to BA, it means that A and B are the same size.

Part (c)

If ABC can be formed, $A = 3 \times 3$, and $C = 5 \times 5$, what size is B?

Solution (c)

For two matrices A and B to be multiplied together, the number of columns in A must be equal to the number of rows in B. For ABC to be a valid matrix multiplication, the number of columns in A must be equal to the number of rows in B, and the number of columns in B must be equal to the number of rows in C. Therefore, B has dimensions 3×5.

Exercise 2.3.9

Problem

Write P $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and let A be $3 \times n$ and B be $m \times 3$.

Part (a)

Describe PA in terms of the rows of A.

Solution (a)

For all PA where P is size 3×3, and A is size $3 \times n$, PA would have dimensions $3 \times n$.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \end{bmatrix}$$

$$PA = \begin{bmatrix} pa_{11} & pa_{12} & pa_{13} & \cdots & pa_{1n} \end{bmatrix}, \quad pa_{ij} = \sum_{k=1}^{n} p_{ik} \cdot a_{kj}$$

Part (b)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} \end{bmatrix}$$

$$PB = \begin{bmatrix} pb_1 & pb_2 & pb_3 \end{bmatrix}, \quad pb_i = \sum_{k=1}^{3} \sum_{j=1}^{m} p_{ik} \cdot b_{kj}$$

Exercise 2.4.2

Problem

Find the inverse of each of the following matrices.

Part (c)

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Solution (c)

$$A \cdot A^{-1} = I_n$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad A^{-1} = RREF\left(\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \right)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 3 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

Exercise 2.4.4

Problem

We are given the matrix $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$.

Part (a)

Solve the system of equations $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

Solution (a)

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1} \cdot A\mathbf{x} = I_n\mathbf{b}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = I_n\mathbf{b}$$

$$1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} 11 \\ 17 \\ -2 \end{bmatrix}$$

Part (c)

Find a matrix C such that $CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution (c)

$$CAA^{-1} = CI$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} = C$$

$$C = \begin{bmatrix} 1 - 1 + 3 & 4 + 0 + 10 & 1 - 1 + 0 \\ 3 - 3 + 9 & 2 + 0 + 5 & -1 + 1 + 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 14 & 0 \\ 9 & 7 & 0 \end{bmatrix}$$

Exercise 2.4.5

Problem

Find A.

Part (e)

$$\left(A\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}$$

Solution (e)

$$A\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}^{-1}$$

$$A\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \frac{1}{2-3}\begin{bmatrix}1 & -3\\ -1 & 2\end{bmatrix}$$

$$A\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = -1\begin{bmatrix}1 & -3\\ -1 & 2\end{bmatrix}$$

$$A = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}^{-1} \cdot -1\begin{bmatrix}1 & -3\\ -1 & 2\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}-1 & 3\\ 1 & -2\end{bmatrix}$$

$$A = \begin{bmatrix}-1+3 & 1-2\\ 0+0 & 1-2\end{bmatrix}$$

$$A = \begin{bmatrix}2 & -1\\ 0 & -1\end{bmatrix}$$

Part (g)

$$\left(A^T - 2I\right)^{-1} = 2\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$$

Solution (g)

$$A^T - 2I = 2\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}^{-1}$$

$$A^T - 2I = 2 \cdot \frac{1}{3-2}\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}$$

$$A^T = 2\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix} + 2I$$

$$A^T = 2\begin{bmatrix}5 & -1\\ -2 & 3\end{bmatrix}$$

$$A = 2\begin{bmatrix}5 & -1\\ -2 & 3\end{bmatrix}^T$$

$$A = 2\begin{bmatrix}5 & -2\\ -1 & 3\end{bmatrix}$$