Math 556 - Applied Functional Analysis

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Introduction

Why do we look at spaces of functions? First, we can consider the geometric view. That is, we look at functions as elements of a function space in a manner analogous to the way we view vectors in a vector space. Just like a vector space, function spaces are *linear spaces*. Furthermore, we can quantify the distances between functions using a *metric*. For instance, consider the metric

$$\rho(f,g) = \sup_x |f(x) - g(x)|.$$

There are several natural choices for distances. We could consider something of the form

$$\rho(f,g) = \int |f(x) - g(x)| \,\mathrm{d}x,$$

or perhaps

$$\rho^p(f,g) = \left(\int |f(x) - g(x)|^p \,\mathrm{d}x \right)^{1/p}.$$

These two metrics define different function spaces.

Another purpose of functional analysis is to "generalize useful concepts." For instance, spectral methods can be used to solve systems of linear equations and ODEs. Let A be a $n \times n$ symmetric amtrix with real entries. We can consider the system of equations $Ax = y$. If A is invertible, the solution to the system is unique and $x = A^{-1}y$. There is a natural parallel to the existance and uniqueness theorem from the study of ODEs.

But suppose we want an explicit form of x. To acheive this information, we need more information on A^{-1}, so we consider the eigenproblem for A. That is we find the vectors u_i, $i \in \{1, \dots, n\}$ satisfying $Au_i = \lambda_i u_i$ for some scalarvalues u_i. Since A is symmetric, the eigenvalues are real and the eigenvectors are orthogonal to each other. Furthermore, we can choose u_i so that $\langle u_i, u_j \rangle = \delta_{i,j}$, where the inner product $\langle x, y \rangle = \sum_i x_i y_i$ in this case. These vectors form an orthonormal basis of $\mathbb{R}^n$, and we can expand x and y by

$$x = \sum_i a_i u_i, \quad y = \sum_i b_i u_i,$$

so, we write

$$Ax = \sum_i a_i \lambda_i u_i$$

and by combining a few equations, we can write down

$$\sum_i (a_i \lambda_i - b_i) u_i = 0.$$

Since $\{u_1, \ldots, u_n\}$ form an orthonormal basis, we can deduce that $a_i = b_i/\lambda_i$, and thus

$$x = \sum_i \frac{b_i u_i}{\lambda_i}.$$

We can note that $b_i = \langle u_i, y \rangle$, so we get

$$x = \sum_i \frac{1}{\lambda_i} \langle u_i, y \rangle u_i,$$

so we arrive at our spectral representation of x. Furthermore, we can find the n-th coordinate by observing

$$\begin{aligned} x_n &= \sum_i \frac{1}{\lambda_i} \langle u_i, y \rangle (u_i)_n \\ &= \sum_m \left(\sum_i \frac{1}{\lambda_i} (u_i)_m (u_i)_n \right) y_m. \end{aligned}$$

From this, since $x_n = \sum_m A^{-1}_{n,m} y_m$, from the solution, we can deduce that

$$A^{-1}_{n,m} = \sum_i \frac{1}{\lambda_i} (u_i)_n (u_i)_m,$$

which is called the spectral decomposition of A^{-1}. We will generalize this notion to functional spaces in this course.

Now, let us consider the ODE $u'' + k^2 u = f$ with the boundary condition $u(0) = u(\pi) = 0$ for some constant $k \in \mathbb{R}$. This is equivalent to solving the eigenproblem $-\phi'' = k\phi$ with $\phi(0) = \phi(\pi) = 0$. The general solution of the eigenproblem is given by $\phi(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$, with A and B constant. We can use the boundary condition , adn we can also normalize by the condition

$$\int_0^\pi |\phi(x)| \, \mathrm{d}x = 1.$$

Doing these computations can be taken as an exercise. In general, we will show in this course that

$$u(x) = \int G(x,y) f(y) \, \mathrm{d}y,$$

where

$$G(x,y) = \sum_n \frac{\phi_n(x)\phi_n(y)}{k^2 - \lambda_n^2}.$$

$G(x,y)$ is called *Green's function.* In fact, from this differential equation, we can derive

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) G(x-y) = \delta(x-y),$$

where δ is the *Dirac Distribution.*