Math 558 - Applied Nonlinear Dynamics

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Consider a vector-valued function $y : \mathbb{R} \to \mathbb{R}^d$ and let $y'(t) = f(t, y(t))$, $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Let f be supplied with initial condition $y(t_0) = \alpha \in \mathbb{R}^d$. Our focus will remain on these types of differential equations.

Remark: We can note that $y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$. We can introduce auxiliary variables $z_0(t) = y(t)$, $z_1(t) = y'(t)$, ..., $z_{n-1}(t) = y^{(n-1)}(t)$. Then, we get the natural relations $z_0' = z_1$, $z_1' = z_2$, ..., $z_{n-2}' = z_{n-1}$, and $z_{n-1}' = f(t, z_0, z_1, \ldots, z_{n-1})$, so in principle the aforementioned equation can be reduced into a system of first-order equations.

If $y' = f(y)$ (that is, f does not depend on t), then the equation is said to be *autonomous*. In general, the nonautonomous case can be reduced to the autonomous case by introducing the auxiliary variable z which satisfies $z' = 1$ and $z(t_0) = t_0$.

Now, let us consider the question of existence and uniqueness of solutions. We can consider the equation $y' = 1 + y^2$ with the initial condition $y(0) = 0$. Then, our equation can easily be solved using separation of variables, resulting in the equation $\arctan(y) = t + C$. By matching the boundary value conditions, we can deduce that $y(t) = \tan(t)$; however, this is only valid for $-\pi/2 < t < \pi/2$, so we may not have global existence for every ODE.

Now, let us consider the ODE $y' = \sqrt{|y|}$ with the initial condition $y(0) = 0$. By separating and integrating, we get that $y(t) = \frac{1}{4}t^2$ - which is a solution; however, there are uncountably many solutions. Chose any $a > 0$ and define

$$y_a(t) = \begin{cases} 0, & 0 \le t \le a \\ \dfrac{1}{4}(t-a)^2 & t \ge a \end{cases}$$

These functions are not twice-differentiable; however, $y \in C^{1,1}$ (that is, it's derivative is Lipshitz).

To formulate the Existence and Uniqueness theorems, we need the following results and definitions:

Contraction mapping principle (Banach fixed pt. theorem). Let X be a normed, complete vector space. let $S \subseteq X$ be a closed subset. Let $\Psi : S \to S$. Assume there is a scalar value $\theta \in [0, 1)$ such that $|\Psi(x) - \Psi(y)| \le \theta|x - y|$ for all $x, y \in S$. Then, Ψ has a fixed point $p \in S$. That is, there exists $p \in S$ such that $\Psi(p) = p$. Moreover, if $x_0 \in S$, the iteration $x_{n+1} = \Psi(x_n)$ converges to p: $\lim_{n\to\infty} x_n = p$. In fact, p is unique and

$$|x_n - p| \le \frac{\theta^n}{1-\theta}|x_1 - x_0|.$$

Proof. First, we can note that for any $j > 1$,

$$|x_{j+1} - x_j| = |\Psi(x_j) - \Psi(x_{j-1})| \le \theta|x_j - x_{j-1}| \le \theta^2|x_{j-1} - x_{j-2}| \le \ldots \le \theta^j|x_1 - x_0|.$$

Now, we let $n \geq m$, with both n and m very large. By observing the natural telescoping sum, we get

$$x_n - x_m = \sum_{j=m}^{n-1} (x_{j+1} - x_j) \leq \sum_{j=m}^{n-1} \theta^j |x_1 - x_0| \leq \theta^m |x_1 - x_0| \sum_{j=0}^{\infty} \theta^j = \frac{\theta^m}{1-\theta} |x_1 - x_0|$$

by the geometric sum formula. This implies $\{x_n\}$ is Cauchy, so there is a limit $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Since S is closed, $p \in S$. By taking $n \to \infty$, we can deduce that

$$|p - x_m| \leq \frac{\theta^m}{1-\theta} |x_1 - x_0|.$$

Now, we can show the uniqueness of p. Suppose for the sake of contradiction that $p, q \in S$, $p \neq q$. Then $\Psi(p) = p$ and $\Psi(q) = q$. Then, $|p - q| = |\Psi(p) - \Psi(q)| \leq \theta |p - q|$, so $|p - q| = 0$, thus $p = q$. □

Definition. **F**or some $A \subseteq \mathbb{R} \times \mathbb{R}^d$ and $f : A \to \mathbb{R}^d$ is *Lipshitz* if there exists a constant L such that

$$|f(t, y) - f(t, z)| \leq L|y - z|$$

for all $(t, y), (t, z) \in A$. If f is continuously differentiable with respect to y on A and A is compact, then f is Lipshitz.

Theorem (Hartman - Grobman) Define $Q_{a,b} = \{(t, y) : t_0 \leq t \leq t_0 + a, \alpha_j = b \leq y_j \leq \alpha_j + b\}$. Let $f : \mathbb{R} \times R^d$ be continuous and Lipshitz in y variable on $Q_{a,b}$. Then, there exists $\varepsilon > 0$ such that there is a solution to the ODE system $y' = f(t, y)$, $y(t_0) = \alpha$ for the time interval $t_0 \leq t \leq t_0 + \varepsilon$.

Proof. Let us define

$$M = \max_{\substack{t,y \in Q_{a,b} \\ j}} |f_j(t, y)| < \infty,$$

$$L = \max_{\substack{(t,y),(t,z) \in Q_{a,b} \\ y \neq z \\ j}} \frac{f_j(t, y) - f_j(t, z)}{|y - z|} < \infty.$$

Let $\varepsilon = \frac{1}{2d} \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$. Let $X_\varepsilon = C([t_0, t_0 + \varepsilon], \mathbb{R}^d)$ be a vector space. Furthermore, if $\phi \in X_\varepsilon$, we can define the norm of ϕ to be

$$||\phi|| = \max_{\substack{t_0 \leq t \leq t_0 + \varepsilon \\ j = 1, \ldots, d}} |\phi_j(t)|.$$

Since the uniform limit of continuous functions is continuous, the space X_ε is complete. Let $S = \{\phi \in X_\varepsilon : (t, \phi(t)) \in Q_{\varepsilon,b}$ for $t_0 \leq t \leq t_0 + \varepsilon\}$. Now, let us define the mapping Ψ by

$$\Psi[y](t) = \alpha + \int_{t_0}^{t} f(s, y(s))\, \mathrm{d}s.$$

Clearly, $\Psi : X_\varepsilon \to X_\varepsilon$. Now, we can note that

$$\begin{aligned} |\Psi[\phi_j](t) - \alpha_j| \leq \int_{t_0}^{t} |f_j(s, \phi(s))|\, \mathrm{d}s &\leq (t - t_0) M \\ &\leq \varepsilon M \leq b. \end{aligned}$$

Thus, we get $\Psi : S \to S$. Finally, we need to show that Ψ is a contraction. Consider $\phi, \psi \in S$. Now, we can note that

$$\begin{aligned} |\Psi[\phi_j](t) - \Psi[\psi_j](t)| &\leq \int_{t_0}^{t} |f_j(s, \phi(s)) - f_j(s, \psi(s))|\, \mathrm{d}s \\ &\leq \int_{t_0}^{t} L|\phi(s) - \psi(s)|\, \mathrm{d}s \\ &\leq \max_{t_0 \leq s \leq t_0 + \varepsilon} |\phi(s) - \psi(s)| \cdot L\varepsilon. \end{aligned}$$

For $t_0 \le t \le t_0 + \varepsilon$, we have

$$\begin{aligned} ||\Psi[\phi] - \Psi[\psi]|| &= \max_{t_0 \le t \le t_0+\varepsilon} |\Psi[\phi]_j(t) - \Psi[\psi]_j(t)| \\ &\le L\varepsilon \cdot \max_{t_0 \le s \le t_0+\varepsilon} |\phi(s) - \psi(s)| \\ &\le \frac{1}{2} \max_{t_0 \le s \le t_0+\varepsilon} |\phi(s) - \psi(s)| \\ &\le \frac{1}{2} \max_{\substack{t_0 \le s \le t_0+\varepsilon \\ 1 \le j \le d}} |\phi_j(s) - \psi_j(s)|. \end{aligned}$$

This implies $||\Psi[\phi] - \Psi[\phi]|| \le \frac{1}{2}||\phi - \psi||$. Thus, by the contraction mapping principle, there exists a $y \in S$ such that $\Psi[y] = y$ such that

$$y(t) = \alpha + \int_{t_0}^t f(s, y(s))\,\mathrm{d}s.$$

By differentiating component-wise, we can deduce that $y'(t) = f(t, y(t))$ with $y(t_0) = \alpha$ for $t_0 \le t \le t_0 + \varepsilon$. □

Example. The following example demonstrates fixed point iteration. Consider the differential equation $y' = y$ with the initial condition $y(0) = 1$. Clearly, $y(t) = e^t$, but we will show this using fixed-point iteration. Let $y_0(t) = 1$. Then, we can note that

$$y_1(t) = \Psi[y_0]t = 1 + \int_0^t y_0(s)\,\mathrm{d}s = 1 + \int_0^t 1\,\mathrm{d}s = 1 + t.$$

Doing this agian, we can deduce

$$y_2(t) = \Psi[y_1](t) = 1 + \int_0^t (1+s)\,\mathrm{d}s = 1 + t + \frac{t^2}{2}.$$

Continuing onwards, we arrive at the Taylor series expansion for e^t (which is easy to show).

Lemma. *Gromwall Inequality:* Let $y(t), g(t)$ be continuous, non-negative functions for $t = t_0$. Let $A \ge 0$. If

$$y(t) \le A + \int_{t_0}^t g(s)y(s)\mathrm{d}s$$

for $t \ge t_0$, then it follows that

$$|y(t)| \le A \exp\left(\int_{t_0}^t g(s)\,\mathrm{d}s\right)$$

for all $t \ge t_0$.

Proof. Let

$$z(t) = A + \int_{t_0}^t g(s)y(s)\,\mathrm{d}s.$$

Then, $y(t) \le z(t)$. And furthermore, $z'(t) = g(t)y(t) \le g(t)z(t)$. For now, let us assume that $A > 0$. Thus, we get $z'(t)/z(t) \le g(t)$, and since z never vanishes, we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\log(z(t)) - \int_{t_0}^t g(s)\,\mathrm{d}s\right\} = 0.$$

Thus,

$$\log(z(t)) - \int_{t_0}^t g(s)\,\mathrm{d}s \le \log(A).$$

From this, finally arrive at

$$A\exp\left(\int_{t_0}^{t} g(s)\mathrm{d}s\right) \geq z(t) \geq y(t).$$

It is easy to verify this is true even when $A = 0$. $\square$

Note: if

$$y(t) = y(t_0) + \int_{t_0}^{t} g(s)y(s)\mathrm{d}s,$$

it follows that $y'(t) = g(t)y(t)$. This is an easy differential equation to solve, and it turns out that

$$y(t) = y(t_0)\exp\left(\int_{t_0}^{t} g(s)\,\mathrm{d}s\right),$$

which is precisely the bound we were looking for.