Math 558 - Applied Nonlinear Dynamics

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Consider a vector-valued function $y : \mathbb{R} \to \mathbb{R}^d$ and let $y'(t) = f(t, y(t))$, $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Let f be supplied with initial condition $y(t_0) = \alpha \in \mathbb{R}^d$. Our focus will remain on these types of differential equations.

Remark: We can note that $y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$. We can introduce auxiliary variables $z_0(t) = y(t)$, $z_1(t) = y'(t)$, …, $z_{n-1}(t) = y^{(n-1)}(t)$. Then, we get the natural relations $z_0' = z_1$, $z_1' = z_2$, …, $z_{n-2}' = z_{n-1}$, and $z_{n-1}' = f(t, z_0, z_1, \ldots, z_{n-1})$, so in principle the aforementioned equation can be reduced into a system of first-order equations.

If $y' = f(y)$ (that is, f does not depend on t), then the equation is said to be *autonomous*. In general, the nonautonomous case can be reduced to the autonomous case by introducing the auxiliary variable z which satisfies $z' = 1$ and $z(t_0) = t_0$.

Now, let us consider the question of existence and uniqueness of solutions. We can consider the equation $y' = 1 + y^2$ with the initial condition $y(0) = 0$. Then, our equation can easily be solved using separation of variables, resulting in the equation $\arctan(y) = t + C$. By matching the boundary value conditions, we can deduce that $y(t) = \tan(t)$; however, this is only valid for $-\pi/2 < t < \pi/2$, so we may not have global existence for every ODE.

Now, let us consider the ODE $y' = \sqrt{|y|}$ with the initial condition $y(0) = 0$. By separating and integrating, we get that $y(t) = \frac{1}{4}t^2$ - which is a solution; however, there are uncountably many solutions. Chose any $a > 0$ and define

$$y_a(t) = \begin{cases} 0, & 0 \le t \le a \\ \dfrac{1}{4}(t-a)^2 & t \ge a \end{cases}$$

These functions are not twice-differentiable; however, $y \in C^{1,1}$ (that is, it's derivative is Lipshitz).

To formulate the Existence and Uniqueness theorems, we need the following results and definitions:

Contraction mapping principle (Banach fixed pt. theorem). Let X be a normed, complete vector space. let $S \subseteq X$ be a closed subset. Let $\Psi : S \to S$. Assume there is a scalar value $\theta \in [0, 1)$ such that $|\Psi(x) - \Psi(y)| \le \theta |x - y|$ for all $x, y \in S$. Then, Ψ has a fixed point $p \in S$. That is, there exists $p \in S$ such that $\Psi(p) = p$. Moreover, if $x_0 \in S$, the iteration $x_{n+1} = \Psi(x_n)$ converges to p: $\lim_{n\to\infty} x_n = p$. In fact, p is unique and

$$|x_n - p| \le \frac{\theta^n}{1-\theta} |x_1 - x_0|.$$

Proof. First, we can note that for any $j > 1$,

$$|x_{j+1} - x_j| = |\Psi(x_j) - \Psi(x_{j-1})| \le \theta |x_j - x_{j-1}| \le \theta^2 |x_{j-1} - x_{j-2}| \le \ldots \le \theta^j |x_1 - x_0|.$$

Now, we let $n \geq m$, with both n and m very large. By observing the natural telescoping sum, we get

$$x_n - x_m = \sum_{j=m}^{n-1} (x_{j+1} - x_j) \leq \sum_{j=m}^{n-1} \theta^j |x_1 - x_0| \leq \theta^m |x_1 - x_0| \sum_{j=0}^{\infty} \theta^j = \frac{\theta^m}{1-\theta} |x_1 - x_0|$$

by the geometric sum formula. This implies $\{x_n\}$ is Cauchy, so there is a limit $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Since S is closed, $p \in S$. By taking $n \to \infty$, we can deduce that

$$|p - x_m| \leq \frac{\theta^m}{1-\theta} |x_1 - x_0|.$$

Now, we can show the uniqueness of p. Suppose for the sake of contradiction that $p, q \in S$, $p \neq q$. Then $\Psi(p) = p$ and $\Psi(q) = q$. Then, $|p - q| = |\Psi(p) - \Psi(q)| \leq \theta |p - q|$, so $|p - q| = 0$, thus $p = q$. □

Definition. For some $A \subseteq \mathbb{R} \times \mathbb{R}^d$ and $f : A \to \mathbb{R}^d$ is *Lipshitz* if there exists a constant L such that

$$|f(t, y) - f(t, z)| \leq L|y - z|$$

for all $(t, y), (t, z) \in A$. If f is continuously differentiable with respect to y on A and A is compact, then f is Lipshitz.

Theorem (Hartman - Grobman) Define $Q_{a,b} = \{(t, y) : t_0 \leq t \leq t_0 + a, \alpha_j = b \leq y_j \leq \alpha_j + b\}$. Let $f : \mathbb{R} \times R^d$ be continuous and Lipshitz in y variable on $Q_{a,b}$. Then, there exists $\varepsilon > 0$ such that there is a solution to the ODE system $y' = f(t, y)$, $y(t_0) = \alpha$ for the time interval $t_0 \leq t \leq t_0 + \varepsilon$.

Proof. Let us define

$$M = \max_{\substack{t, y \in Q_{a,b} \\ j}} |f_j(t, y)| < \infty,$$

$$L = \max_{\substack{(t,y),(t,z) \in Q_{a,b} \\ y \neq z \\ j}} \frac{f_j(t, y) - f_j(t, z)}{|y - z|} < \infty.$$

Let $\varepsilon = \frac{1}{2d} \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$. Let $X_\varepsilon = C([t_0, t_0 + \varepsilon], \mathbb{R}^d)$ be a vector space. Furthermore, if $\phi \in X_\varepsilon$, we can define the norm of ϕ to be

$$||\phi|| = \max_{\substack{t_0 \leq t \leq t_0 + \varepsilon \\ j = 1, \ldots, d}} |\phi_j(t)|.$$

Since the uniform limit of continuous functions is continuous, the space X_ε is complete. Let $S = \{\phi \in X_\varepsilon : (t, \phi(t)) \in Q_{\varepsilon,b} \text{ for } t_0 \leq t \leq t_0 + \varepsilon\}$. Now, let us define the mapping Ψ by

$$\Psi[y](t) = \alpha + \int_{t_0}^{t} f(s, y(s))\, \mathrm{d}s.$$

Clearly, $\Psi : X_\varepsilon \to X_\varepsilon$. Now, we can note that

$$\begin{aligned} |\Psi[\phi_j](t) - \alpha_j| \leq \int_{t_0}^{t} |f_j(s, \phi(s))|\, \mathrm{d}s &\leq (t - t_0)M \\ &\leq \varepsilon M \leq b. \end{aligned}$$

Thus, we get $\Psi : S \to S$. Finally, we need to show that Ψ is a contraction. Consider $\phi, \psi \in S$. Now, we can note that

$$\begin{aligned} |\Psi[\phi_j](t) - \Psi[\psi_j](t)| &\leq \int_{t_0}^{t} |f_j(s, \phi(s)) - f_j(s, \psi(s))|\, \mathrm{d}s \\ &\leq \int_{t_0}^{t} L|\phi(s) - \psi(s)|\, \mathrm{d}s \\ &\leq \max_{t_0 \leq s \leq t_0 + \varepsilon} |\phi(s) - \psi(s)| \cdot L\varepsilon. \end{aligned}$$

For $t_0 \le t \le t_0 + \varepsilon$, we have

$$\begin{aligned} ||\Psi[\phi] - \Psi[\psi]|| &= \max_{t_0 \le t \le t_0+\varepsilon} |\Psi[\phi]_j(t) - \Psi[\psi]_j(t)| \\ &\le L\varepsilon \cdot \max_{t_0 \le s \le t_0+\varepsilon} |\phi(s) - \psi(s)| \\ &\le \frac{1}{2} \max_{t_0 \le s \le t_0+\varepsilon} |\phi(s) - \psi(s)| \\ &\le \frac{1}{2} \max_{\substack{t_0 \le s \le t_0+\varepsilon \\ 1 \le j \le d}} |\phi_j(s) - \psi_j(s)|. \end{aligned}$$

This implies $||\Psi[\phi] - \Psi[\phi]|| \le \frac{1}{2}||\phi - \psi||$. Thus, by the contraction mapping principle, there exists a $y \in S$ such that $\Psi[y] = y$ such that

$$y(t) = \alpha + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s.$$

By differentiating component-wise, we can deduce that $y'(t) = f(t, y(t))$ with $y(t_0) = \alpha$ for $t_0 \le t \le t_0{+}\varepsilon$. ☐

Example. The following example demonstrates fixed point iteration. Consider the differential equation $y' = y$ with the initial condition $y(0) = 1$. Clearly, $y(t) = e^t$, but we will show this using fixed-point iteration. Let $y_0(t) = 1$. Then, we can note that

$$y_1(t) = \Psi[y_0]t = 1 + \int_0^t y_0(s)\,\mathrm{d}s = 1 + \int_0^t 1\,\mathrm{d}s = 1 + t.$$

Doing this agian, we can deduce

$$y_2(t) = \Psi[y_1](t) = 1 + \int_0^t (1+s)\,\mathrm{d}s = 1 + t + \frac{t^2}{2}.$$

Continuing onwards, we arrive at the Taylor series expansion for e^t (which is easy to show).

Lemma. *Gromwall Inequality:* Let $y(t), g(t)$ be continuous, non-negative functions for $t = t_0$. Let $A \ge 0$. If

$$y(t) \le A + \int_{t_0}^{t} g(s)y(s)\mathrm{d}s$$

for $t \ge t_0$, then it follows that

$$|y(t)| \le A \exp\left(\int_{t_0}^{t} g(s)\,\mathrm{d}s\right)$$

for all $t \ge t_0$.

Proof. Let

$$z(t) = A + \int_{t_0}^{t} g(s)y(s)\,\mathrm{d}s.$$

Then, $y(t) \le z(t)$. And furthermore, $z'(t) = g(t)y(t) \le g(t)z(t)$. For now, let us assume that $A > 0$. Thus, we get $z'(t)/z(t) \le g(t)$, and since z never vanishes, we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\log(z(t)) - \int_{t_0}^{t} g(s)\,\mathrm{d}s\right\} = 0.$$

Thus,

$$\log(z(t)) - \int_{t_0}^{t} g(s)\,\mathrm{d}s \le \log(A).$$

From this, finally arrive at

$$A \exp\left(\int_{t_0}^{t} g(s) \mathrm{d}s\right) \geq z(t) \geq y(t).$$

It is easy to verify this is true even when $A = 0$. $\square$

Note: if

$$y(t) = y(t_0) + \int_{t_0}^{t} g(s)y(s) \mathrm{d}s,$$

it follows that $y'(t) = g(t)y(t)$. This is an easy differential equation to solve, and it turns out that

$$y(t) = y(t_0) \exp\left(\int_{t_0}^{t} g(s)\, \mathrm{d}s\right),$$

which is precisely the bound we were looking for.

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Theorem (Uniqueness of Solutions and Continuous Dependence on Initial Disc). The solution guaranteed by Hartman-Grobman is unique.

Proof. Asume $y(t)$ and $z(t)$ are two solutions:

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \leq t \leq T \\ y(t_0) = \alpha \end{cases}$$

and

$$\begin{cases} z'(t) = f(t, z(t)), & t_0 \leq t \leq T \\ z(t_0) = \beta. \end{cases}$$

Assume that $f(t, \xi)$ is continuous, and also that $\frac{\partial f}{\partial \xi}$ is also continuous. Then,

$$y(t) = \alpha + \int_{t_0}^{t} f(s, y(s)), \mathrm{d}s$$
$$z(t) = \beta + \int_{t_0}^{t} f(s, y(s)), \mathrm{d}s.$$

Thus, we can conclude that

$$|y(t) - z(t)| \leq |\alpha - \beta| + \int_{t_0}^{t} |f(s, y(s)) - f(s, z(s))|\, \mathrm{d}s \leq |\alpha - \beta| + \int_{t_0}^{t} L|y(s) - z(s)|\, \mathrm{d}s.$$

where we used

$$L = \max_{\substack{(t,y),(t,z) \in B \\ y \notin z}} \frac{|f(t, y) - f(t, z)|}{|y - z|}.$$

By Gronwall inequality, we can deduce that

$$|y(t) - z(t)| \leq A \exp(L(t - t_0)),$$

which is true for $t_0 \leq t \leq T$. $\square$

Given a solution to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = \alpha, \end{cases}$$

f is continuous and $\frac{\partial f}{\partial y}$ is continuous. Let the maximum time of existence, t_*, be defined as

$$t_* = \sup\{T > t_0 : \text{ a solution exists on } t \in [t_0, T)\}.$$

Claim. If $t_* < \infty$, then the solution cannot be bounded on $[t_0, t_*)$.

Proof. Suppose the solution remains bounded on $[t_0, t_*)$ (that is, $\sup |y(t)| = M < \infty$ on $t_0 \le t < t_*$), and that $t_* < \infty$. Let $B = t_0 \le t \le t_*$ and $|y| \le M$. Furthermore, let $A = \max_{(t,y)\in B} |f(t,y)|$. We claim that $\lim_{t\to t_*^-} y(t)$ exists. Take $t_1 > t_2 \to t_*^{-1}$. Then,

$$|y(t_1) - y(t_2)| \le \int_{t_1}^{t_2} |f(s, y(s))| \,\mathrm{d}s \le A(t_1 - t_2) \to 0$$

as $t_1, t_2 \to t_*^-$. Now, we apply the existence theorem for $y' = f(t,y)$, $y(t_*) = \lim_{t\to t_*^-} y(t)$ $\square$

Example. **C**onsider the equations

$$\begin{aligned} y_1' &= -y_2^2 \\ y_2' &= y_1 y_2 - y_2. \end{aligned}$$

We now discuss how long solutions exist in terms of the initial data

$$\begin{cases} y_1(0) = \alpha_1 \\ y_2(0) = \alpha_2. \end{cases}$$

Suppose solutions exists on $[0, T)$. By adding up the solutiosn and cancelling, we can deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(y_1^2 + y_2^2) = -y_2^2 \le 0,$$

so $y_1^2 + y_2^2$ is a decreasing function on $|y_1|, |y_2| \le \sqrt{\alpha_1^2 + \alpha_2^2}$ on $[0, T)$. This implies that all solutions exist for all time.

Now, let us consider a general linear system $y'(t) = A(t)y(t) + f(t)$. $A(t)$ is a $n \times n$ matrix and $f(t)$ is a $n \times 1$ vector. Both are continuous.

Claim. For systems of the form depicted above, all solutions exist for all time.

Proof. Note that $A(t)$ and $f(t)$ being continuous suffices since continuity implies Lipshitz continuity along any bounded interval. Let a solution $y(t)$ exist on $[t_0, T)$. Then,

$$y(t) = y(t_0) + \int_{t_0}^{t} |A(s)y(s) + f(s)| \,\mathrm{d}s \le |y(t_0)| + \int_{t_0}^{t} ||A(s)||_2 |y(s)|_2 + |f(s)|_2 \,\mathrm{d}s.$$

From this, we can conclude that

$$|y(t)|_2 \le |y(t_0)|_2 + \int_{t_0}^{T} |f(s)|_2 \,\mathrm{d}s + \int_{t_0}^{t} ||A(s)||_2 |y(s)|_2 \,\mathrm{d}s.$$

By Gronwall inequality, we have

$$|y(t)|_2 \le \left(|y(t_0)|_2 + \int_{t_0}^{T} |f(s)|_2 \,\mathrm{d}s\right) \exp\left(\int_{t_0}^{t} ||A(s)||_2 \,\mathrm{d}s\right).$$

$\square$

Linear Algebra Digression.

Now, let us consider $A \in \mathbb{C}^{d\times d}$ and let λ be an eigenvalue of A. Recall that if v is an eigenvector associated with λ if $v \neq 0$ and $(A-\lambda I)v = 0$.

Definition. $v \neq 0$ is a *generalized eigenvector* associated with the eigenvalue λ if

$$(A-\lambda I)^n v = 0$$

for some $n \in \mathbb{N}$.

Definition. If $v \neq 0$ is a generalized eigenvector with ccorresponding eigenvalue λ, let

$$\text{Index}(v) = \text{ Smallest } n \text{ for which } (A-\lambda I)^n v = 0.$$

Claim. Let v be a generalized eigenvector with eigenvalue λ. Assume Index$(v) = n$. Then, v, $(A\lambda I)v$, $(A-\lambda I)^2 v$, ..., $(A-\lambda I)^{n-1}v$ are linearly independent.

Proof. Assume they are not linearly independent. We can find scalars $c_0, \ldots, c_{n-1}$, with not all 0, such that

$$\sum_{j=0}^{n-1} c_j (A-\lambda I)^j v = 0.$$

Let c_i be the first c that is not zero. That is, $c_0 = c_1 = \ldots = c_{i-1} = 0$, but $c_{i\neq 0}$. Then,

$$\sum_{j=i}^{n-1} c_j (A-\lambda I)^j v = 0.$$

Now, we multiply both sides by $(A-\lambda I)^{n-i-1}$. Then, it follows that

$$c_i(A-\lambda I)^{n-1}v + \text{ a bunch of terms which vanish } = 0,$$

which is a clear contradiction since v has index n. □

Corollary. Index$(v) \leq d$.

Definition. For λ an eigenvalue of A, let $V(\lambda)$ be the *generalized eigenspace* corresponding with λ. In particular,

$$V(\lambda) = \text{Span}\{\text{all generalized eigenvectors with eigenvalue}\lambda\}.$$

Definition. Index$(V(\lambda))$ =Largest index of any $v \in V(\lambda), v \neq 0$.

Some notation: r(λ) = Index$(V(\lambda))$.

Claim. Range$((A-\lambda I)^{r(\lambda)}) \cap \ker((A-\lambda I)^{r(\lambda)}) = \{0\}$.

Proof. Assume $v \in \text{Range} \cap \ker$. Then, $v = (A-\lambda I)^{r(\lambda)}v = (A-\lambda I)^{2r(\lambda)}y = 0$. Suppose $v \neq 0$. This would imply $y \neq 0$. But $y \in V(\lambda)$, and yet $(A-\lambda I)^{r(\lambda)}y \neq 0$. And thus we have arrived at a contradiction. □

Claim. $\mathbb{C}^d = \text{Range}((A-\lambda I)^{r(\lambda)}) \oplus \ker((A-\lambda I)^{r(\lambda)})$.

Proof. $\dim(\text{Range}) + \dim(\ker) = d$. □

Claim. $\text{Range}((A-\lambda I)^{r(\lambda)})$ and $\ker((A-\lambda I)^{r(\lambda)})$ are invariant subspaces for A and any polynomial $p(A)$ in A.

Proof. If $x \in \text{Range}(A-\lambda I)^{r(\lambda)}$ and $x = (A-\lambda I)^{r(\lambda)}y$ for some y, then

$$p(A)x = p(A)(A-\lambda I)^{r(\lambda)}y = (A-\lambda I)^{r(\lambda)}p(A)y.$$

The same proof works for the kernal. □

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Claim. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $V(\lambda)$. Complete that to a basis $\{v_1, v_2, \ldots, v_d\}$ of $\mathbb{C}^d$. Let $M = (v_1, v_2, \ldots, v_d)$. Then

$$M^{-1}AM = \begin{pmatrix} A_1 & 0 \\ 0 & A_2, \end{pmatrix}$$

where A_1 is a $k \times k$ matrix and A_2 is $(d-k)\times(d-k)$. Therefore, $p_A(\xi)$ = characteristic polynomial of $A = p_{A_1}(\xi)\cdot p_{A_2}(\xi)$.

Proof. If $1 \le j \le k$, then $Av_j \in V(\lambda)$. Then, $M^{-1}Av_j = (a_1, \ldots, a_k, 0, 0, \ldots, 0)^T$, where the first k entries may be nonzero. If $k < j$, then $(0, \ldots, 0, a_{k+1}, \ldots, a_d)^T$. To show the second part, we can conclude that

$$\begin{aligned} \det(A-\lambda I) &= \det\big(M^{-1}(A-\lambda I)M\big) \\ &= \det\big(M^{-1}AM - \lambda I\big). \end{aligned}$$

From this point, the statement in the claim follows almost immediately from the definition of a determinant. □

Claim. Let $v \neq 0$, $Au = \mu v$, and $\mu \neq \lambda$, where λ is also an eigenvalue of A. Then, $u \notin V(\lambda)$.

Proof. We have $Au = \mu u$. Suppose $u \in V(\lambda)$, then $(A-\lambda I)^{r(\lambda)}u = 0$. Thus, $(\mu-\lambda)^{r(u)}u = 0$, so $\mu = \lambda$. □

Claim. Let λ, μ be eigenvalues of A, $\lambda \neq \mu$. Then, $V(\lambda) \cap V(\mu) = \{0\}$.

Proof. Suppose $u \neq 0$, $u \in V(\lambda) \cap V(\mu)$. Then, $(A-\lambda I)^{r(u)-1}u$ is an eigenvector with eigenvalue λ. This is a contradiction by a previous claim. □

Claim. Let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues of A, with multiplicities $m_1, \ldots, m_n$ so that $m_1 + m_2 + \ldots + m_n = d$. Then,

$$\mathbb{C}^d = \bigoplus_{i=1}^{n} V(\lambda_i)$$

That is, $\dim V(\lambda_j) = m_j$.

Proof. Exercise. □

Back to Differential Equations

Consider a system of the form

$$\begin{cases} y' = Ay \\ y(t_0) = \alpha, \end{cases}$$

where A is a constant $d \times d$ matrix. We look for a change of variables which changes this system into a decoupled system. If A is diagonalizable, $A = MDM^{-1}$, so our equation reduces to

$$\begin{cases} (M^{-1}y)' = D(M^{-1}y) \\ y(t_0) = \alpha, \end{cases}$$

and if we set $z = M^{-1}y$, our system becomes entirely decoupled, which is easy to solve. In general, we can write $z_j(t) = c_j e^{\lambda_j(t-t_0)}$ where $z_j(t_0) = c_j = (M^{-1}\alpha)_j$.

Definition. If $A \in \mathbb{C}^{d\times d}$,

$$||A||_2 = \max_{x\neq 0} \frac{|Ax|_2}{|x|_2},$$

and let us introduce the Frobenius norm

$$||A||_F = \sqrt{\sum_{i,j=1}^{d} |A_{i,j}|^2}.$$

Definition. For $A \in \mathbb{C}^{d\times d}$, we can write

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

We can observe that the series defined above will converge for any matrix A with respect to either of the norms defined above. Indeed,

$$\left\|\sum_{n=N}^{\infty} \frac{A^n}{n!}\right\| \leq \sum_{n=N}^{\infty} \frac{||A^n||}{n!} \leq \sum_{n=N}^{\infty} \frac{||A||^n}{n!} \to 0 \text{ as } N \to \infty$$

Exercises.

(1) Show $\exp(A)\exp(B) = \exp(A+B)$ if A and B commute.

(2) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(At) = \lim_{h\to 0} \frac{\exp(A(t+h)) - \exp(A)}{h} = A\exp(At).$$

(3) If $A = MBM^{-1}$, then

$$\exp(A) = M\exp(B)M^{-1}.$$

(4) If D is diagonal, $\exp(D)$ is a diagonal matrix where each diagonal entry in D is exponentiated.

The upshot of all this information is that if we go back to the system $y' = Ay$ with the initial condition $y(t_0) = \alpha$, then we can write $y(t) = \exp(A(t-t_0))\alpha$. The only question is how do we compute the exponential of a defective matrix.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of A. Let $m_1, \ldots, m_n$ be their multiplicities. Find a basis for each generalized eigenspace $V(\lambda_j)$: call it $\{y_{j,1}, \ldots, y_{j,m_j}\}$. Observe that if $v \in V(\lambda_j)$, then

$$\exp(A(t-t_0))v = \exp((A-\lambda_j I)(t-t_0))\exp(\lambda_j I(t-t_0))v = \exp((A-\lambda_j I)(t-t_0))\exp(\lambda_j(t-t_0))v.$$

Rewriting this in terms of the definition, we can conclude that

$$\begin{aligned}\exp((A-\lambda_j I)(t-t_0))\exp(\lambda_j(t-t_0))v &= \exp(\lambda_j(t-t_0))\sum_{k=0}^{\infty}\frac{(A-\lambda_j I)^k(t-t_0)^k}{k!}v \\ &= \exp(\lambda_j(t-t_0))\sum_{k=0}^{m_j}\frac{(A-\lambda_j I)^k(t-t_0)^k}{k!}v.\end{aligned}$$

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Abel's Formula.
If $\Phi(t)$ is a FSM of a differential equation, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det\Phi(t)) = \operatorname{tr}(A(t))(\det\Phi(t)).$$

Proof. Suppose

$$\Phi(t) = \begin{pmatrix} -- & \Phi_{1,:}(t) & -- \\ -- & \Phi_{2,:}(t) & -- \\ & \vdots & \\ -- & \Phi_{n,:}(t) & -- \end{pmatrix}$$

is a FSM. Then, we can denote that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \det\begin{pmatrix} -- & \Phi'_{1,:}(t) & -- \\ -- & \Phi_{2,:}(t) & -- \\ & \vdots & \\ -- & \Phi_{n,:}(t) & -- \end{pmatrix} + \ldots + \det\begin{pmatrix} -- & \Phi_{1,:}(t) & -- \\ -- & \Phi_{2,:}(t) & -- \\ & \vdots & \\ -- & \Phi'_{n,:}(t) & -- \end{pmatrix}.$$

Since

$$\Phi'(t) = A(t)\Phi(t),$$

we can conclude that

$$\Phi'_{i,:}(t) = \sum_k A_{ik}(t)\Phi_{k,:}(t).$$

We look at the i-th summand of the expression. We can conclude that

$$\det\begin{pmatrix} -- & \Phi_{1,:}(t) & -- \\ -- & \Phi_{2,:}(t) & -- \\ -- & \sum_k A_{ik}(t)\Phi_{k,:}(t) & -- \\ -- & \Phi_{i+1,:} & -- \\ & \vdots & \\ -- & \Phi_{d,:}(t) & -- \end{pmatrix} = A_{ii}(t)\det\Phi(t).$$

The desired result follows quickly from this point. ☐

Corollary. If

$$\frac{1}{\det\Phi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi(t) = \operatorname{tr}(A(t)),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(\det\Phi(t)) = \operatorname{tr}A(t) \implies \det\phi(t) = \det\phi(t_0)\exp\left(\int_{t_0}^{t}\operatorname{tr}(A(s))\,\mathrm{d}s\right).$$

We can note that if $y' = A(t)y$, let $\Phi(t)$ be a FSM. then, any solution of (*) can be expressed as

$$y = \Phi(t)C,$$

where C is a constant vector. If $y(t_0) = \alpha$, then

$$\Phi(t_0)C = \alpha \implies C = (\Phi(t_0))^{-1}\alpha.$$

Thus,

$$y(t) = \Phi(t)(\Phi(t))^{-1}\alpha.$$

Now, we can consider the inhomogenous equation

$$y' = A(t)y + h(t).$$

If y_1 and y_2 are solutions to the above differential equation, $y_1 = y_2$ solves $y' = A(t)y$. Thus,

$$y_1(t) = y_2(t) = \Phi(t)C,$$

where $\Phi(t)$ is a FSM of the homogenous system $y' = A(t)y$. If $y_p(t)$ is any solution of the inhomogeneous system, then the general form of the solution of the inhomogeneous system is given by

$$y(t) = y_p(t) + \Phi(t)C.$$

Finally, we can discuss variation of parameters. To find the particular solution y_p, you look for a solution $y_p(t) = \Phi(t)C(t)$. Then,

$$y'(t) = \Phi'(t)c(t) + \Phi(t)c'(t) = A(t)\Phi(t)c(t) + h(t),$$

so since $\Phi'(t)c(t) = A(t)\Phi(t)c(t)$, it follows that $\Phi(t)c'(t) = h(t)$, so $c'(t) = \Phi^{-1}(t)h(t)$. From here, we can find $c(t)$. It follows that

$$y(t) = \Phi(t)C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s,$$

where we set

$$\Phi^{-1}(t_0)\alpha = C,$$

where $y(t_0) = \alpha$.

Floquet Theory.

First, let us consider when $A \in \mathbb{C}^{d\times d}$. Given that

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

given $B \in \mathbb{C}^{d\times d}$ invertible, when can we write $B = \exp(A)$ for some A? That is, we want to define $\log(B)$.

First, let us recall that $\exp\left(MAM^{-1}\right) = M\exp(A)M^{-1}$. Then, $M^{-1}BM = \exp A$, so $B = M\exp(A)M^{-1} = \exp\left(MAM^{-1}\right)$. If Σ is a diagonal matrix, it is easy to define $\log(\Sigma)$ just by taking the logarithm of the diagonal entries.

Given $B \in \mathbb{C}^{d\times d}$ invertible that is possibly defective, let $\lambda_1, \ldots, \lambda_n$ be its distinct eigenvalues with multiplicities $m_1, \ldots, m_n$. Let $V(\lambda_j)$ be the generalized eigenspace associated with λ_j. $\mathbb{C}^d = \bigoplus_{k=1}^{n} V(\lambda_k)$. $V(\lambda_j) = \ker(A - \lambda_j I)$. Given that $V(\lambda_j) = \ker((B - \lambda_j I)^{m_i})$, we can find a basis $\phi_{j,1}, \ldots, \phi_{j,m_j}$. Let $M = (\phi_{1,1}, \ldots, \phi_{1,m_1}, \phi_{2,1}, \ldots, \phi_{2,m_2}, \ldots, \phi_{n,1}, \ldots, \phi_{n,m_n})$. Then, $M^{-1}BM$ is in block diagonal form with blocks $B_1, \ldots, B_n$ satisfying $(B_j - \lambda_j I)^{m_j} = 0$.

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Example. Consider $y'' = -(1+a(t))y$. Suppose $a(t)$ is continuous and ω periodic. Then, we can write the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -(1+a(t_1))y_1, \end{aligned}$$

or in other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1+a(t)) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (*)$$

Let $\Phi(t)$ be a FSM of (U) such that $\Phi(0) = I$. We can look at the eigenvalues of $\Phi(\omega)$, which are the multipliers of the system (call them λ_1, λ_2). By Abel's theorem, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\Phi(t) = \operatorname{tr}(A(t))\det\Phi(t),$$

so we get

$$\det\Phi(t) + \det\Phi(0)\exp\left(\int_0^t \operatorname{tr} A(s)\,\mathrm{d}s\right) = 1$$

for all time. This implies that $\lambda_1 \cdot \lambda_2 = 1$.

Claim: For $\max\limits_{0\le t\le\omega}|a(t)|$ small enough, all solutions of Hill's equation are bounded, provided that $\omega \neq 2\pi n$.

We can note that for the system $z'' = -z$, the FSM is

$$\tilde{\Phi}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Let $\phi_1(t)$ and $\phi_2(t)$ be two solutions of Hill's equation with solutions $\phi_1(0) = 1$, $\phi_1'(0) = 1$, $\phi_2(0) = 0$, $\phi_2'(0) = 1$. Then,

$$\Phi(\omega) = \begin{pmatrix} \Phi_1(\omega) & \Phi_2(\omega) \\ \Phi_1'(\omega) & \Phi_2'(\omega). \end{pmatrix}$$

To find the eigenvalues of Φ, we first evaluate the characteristic polynomial, and we note that

$$\begin{aligned} p_a(\lambda) &= (\Phi_1(\omega)-\lambda)(\Phi_2'(\omega)-\lambda) - \Phi_2(\omega)\Phi_1'(\omega) \\ &= \lambda^2 - \underbrace{(\Phi_1(\omega)+\Phi_2'(\omega))}_{\approx\cos(\omega)+\cos\omega=2\cos\omega}\lambda + \underbrace{\Phi_1(\omega)\Phi_2'(\omega) - \Phi_2(\omega)\Phi_1'(\omega)}_{=1\text{ since determinant}}, \end{aligned}$$

where we used the fact that $\Phi_1 \approx \cos t$, $\Phi_2 \approx \sin t$ for $0 \le t \le \omega$. Thus, we can deduce that

$$p(\lambda) = \lambda^2 - \beta\lambda + 1$$

where $\beta \approx 2\cos\omega$. Since $\omega \neq n(2\pi)$, we can assert that $|\beta| < 2$. Solving for λ, one can see that both eigenvalues must be complex. Since λ_1 and λ_2 are complex conjugates of each other, they must both have magniutudes equal to 1, so solutions to Hill's equations are bounded by Floquet's theorem.

Example. Consider the equation $y'' = -\frac{1}{4}y$. Multiplying both sides by y', we get

$$y'y'' = -\frac{1}{4}yy',$$

so we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(y')^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}(y^2) = 0,$$

so the energy $E = (y')^2 + \frac{1}{4}y^2$ is conserved. All trajectories are ellipses with the same eccentricities. Similarly, if we consider the equation $y'' = -4y$, by mutliplying both sides by y' and then finding the conserved quantity, we get

$$E = (y')^2 + 4y^2$$

and the corresponding phase portrait paths are elongated ellipses. If we denote the first and second system of differential equations as disussed above, we can write down

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1/4 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}.$$

Now, let us consider a new system, where

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = B_t \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where we define

$$B(t) = \begin{cases} A_1, & 0 \leq t \leq \pi \\ A_2, & \pi \leq t \leq \frac{5\pi}{/}4. \end{cases}$$

Extend $B(t)$ periodically ($\omega = \frac{5\pi}{4}$). It is easy to see that the phase portraits spin out into oblivion.

Scalar First-Order Nonlinear Equations

First, we can consider the phase line for calar, first order, autonomous equations. Consider an equation of the form

$$y' = f(y).$$

We can find stationary solutions $f(y) = 0$.

Example. Consider the equation $y' = y(1 - y)$. This is the logistic equation. The critical points are located at $y = 0$ and $y = 1$.

Take f to be continuously differentiable. Now, we can discuss the stability of the critical points. Let p be an isolated critical point (or equilibrium solution) of $y' = f(y)$ and that there exists $r > 0$ such that $|x - p| < r$ and $f(x) = 0$ implies $x = p$. Then, the sign of f is constant in $(p, p + r)$ and $(p - r, p)$.If $f > 0$ on $(p, p+r)$ and $f < 0$ on $(p - r, p)$, then p is unstable. Likewise, one can analyze the other cases.

Claim. Suppose $y' = f(y)$ has $p < q$ as equilibrium solutions and $f(p) = f(q) = 0$, and $f(x) \neq 0$ for any $x \in (p, q)$. Then, there are two cases. Either $f > 0$ on (p, q) or $f < 0$ on (p, q). For the first case, if $x \in (p, q)$ and $y(0) = x$, then $\lim_{t\to\infty} y(t) = q$. The analogous result holds for the other case.

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Derivative Test for Stability. Let P be an isolated critical point for $u' = f(u)$. If $f'(p) > 0$, then p is unstable. If $f'(p) < 0$, then p is an asymptotically stable equilibrium point. If $f'(p) = 0$, then our test is inconclusive.

Consider the example of logistic growth given by $u' = u(1-u) = u - u^2$. Draw the phase portrait, and we know that if $u'(t) < 0$ for all $t \geq 0$ if $u(0) < 0$. Furthermore, by comparing with the equation $v' = -v^2$, we can see that u goes down to $-\infty$ in finite time. I'f we have

$$u' = f(t,u), \quad u(0) = \alpha \qquad (*)$$

and f is periodic in the t variable with period ω, then we consider $\phi(t, \alpha)$ which represents the solution of $(*)$ at time t. An important observation: If u solves $u' = f(t, \alpha)$, then so does $u(t-\omega)$. Consider the map

$$\phi(\omega, \alpha) : D \to \mathbb{R},$$

and let $p(\alpha) = \phi(\omega, \alpha)$. p is known as the Poincare map. An ω-periodic solution of $(*)$ corresponds to a fixed point α of p. That is, $p(\alpha) = \alpha$.

To illustrate this idea, consider a logistic equation with periodic harvesting. That is, we can consider

$$u' = u(1-u) - h(1 + sin(2\pi t)).$$

For what values of h do we have sustainable harvesting (that is, a periodic solution with period 1)? We can write

$$\partial_t \phi(t, \alpha) = f(t, \phi(t, \alpha))$$

for all $t \geq 0$. The claim is that $\partial_\alpha \phi(t, \alpha)$ is a solution to another differential equation. In particular, we can note that

$$\begin{aligned}
\partial_t(\partial_\alpha \phi(t, \alpha)) &= \partial_\alpha \partial_t \phi(t, \alpha) \\
&= \partial_\alpha f(t, \phi(t, \alpha)) \\
&= \frac{\partial f}{\partial u}(t, \phi(t, \alpha)) \partial_\alpha \phi(t, \alpha).
\end{aligned}$$

Solving, we can write down

$$\partial_\alpha \phi(t, \alpha) = \partial_\alpha \phi(0, \alpha) \exp\left(\int_0^t \frac{\partial f}{\partial u}(s, \phi(s, \alpha)) \, \mathrm{d}s \right).$$

We get that

$$\partial_\alpha \phi(0, \alpha) = \frac{\partial \alpha}{\partial \alpha} = 1,$$

so it follows that

$$\partial_a \phi(t, \alpha) = \exp\left(\int_0^1 \frac{\partial f}{\partial u}(s, \phi(s, \alpha)) \, \mathrm{d}s \right) > 0,$$

so $\phi'(\alpha) > 0$ for all α. To continue on, we can see what happens with the second derivative of α. In particular, we have

$$\begin{aligned}
\partial_t \partial_\alpha^2 \phi(t, \alpha) &= \partial_\alpha^2 \partial_t \phi(t, \alpha) \\
&= \partial_\alpha^2 f(t, \phi(t, \alpha)) \\
&= \partial_\alpha \left[\frac{\partial f}{\partial u}(t, \phi(t, \alpha)) \partial_\alpha \phi(t, \alpha) \right] \\
&= \frac{\partial^2 f}{\partial u^2}(t, \phi(t, \alpha))(\partial_\alpha \phi)^2(t, \alpha) + \frac{\partial f}{\partial u}(t, \phi(t, \alpha)) \partial_\alpha^2 \phi(t, \alpha).
\end{aligned}$$

Recall that if $y' = a(t)y + b(t)$, then we have

$$y(t) = \exp\left(\int_0^t a(s) \, \mathrm{d}s \right) \left[y(0) + \int_0^t b(s) \exp\left(- \int_0^s a(\xi) \, \mathrm{d}\xi \right) \mathrm{d}s \right],$$

Using the fact that $\partial_\alpha^2 \phi(0,\alpha) = 0$, then we can conclude that

$$\partial_\alpha^2 \phi(t,\alpha) = \underbrace{\exp\left(\int_0^t a(s)\,\mathrm{d}s\right)}_{>0} \int_0^t \frac{\partial^2 f}{\partial u^2}\,(s,\phi(s,\alpha))\,\underbrace{(\partial_\alpha \phi)^2}_{>0}(s,\alpha)\,\underbrace{\exp\left(-\int_0^s \alpha(\xi) d\xi\right)}_{>0}\,\mathrm{d}s.$$

Since

$$f(t,u) = u(1-u) - h(1+\sin(2\pi t)),$$

it follows that

$$\frac{\partial^2 f}{\partial u^2} = -2,$$

so $p(\alpha)$ is strictly concave, so there are at most two periodic solutions. In fact, if $x \in \text{domain}(p)$, if $u' = f(t,u)$, f is smooth and ω-periodic in t, then $\tilde{x} \in B_\varepsilon(p)$ also belongs to domain(p). By uniqueness, it follows that p is strictly increasing.

Theorem. Let p be a continuous, strictly increasing function. Let $\alpha_1 < \alpha_2$ be two fixed points of p such that $p(\xi) \neq \xi$ for any $\alpha_1 < \xi < \alpha_2$. We can claim that

(1) p is a bijection from $[\alpha_1, \alpha_2] \to [\alpha_1, \alpha_2]$.

(2) Either $p(\alpha) > \alpha$ for all $\alpha \in (\alpha_1, \alpha_2)$ or $p(\alpha) < \alpha$ for all $\alpha \in (\alpha_1, \alpha_2)$.

(3) If $p(\alpha) > \alpha$ for all $\alpha \in (\alpha_1, \alpha_2)$, then $p^n(\alpha) = \underbrace{p \circ p \circ \ldots \circ p}_{n \text{ times}}(\alpha) \to \alpha_2$ as $n \to \infty$.

Proof. If $\alpha_1 \le \alpha \le \alpha_2$, then $\alpha_1 = p(\alpha_1) \le p(\alpha) \le p(\alpha_2) = \alpha_2$. If $\alpha_1 \le \alpha < \tilde{\alpha} \le \alpha_2$, then $p(\alpha) < p(\tilde{\alpha})$, so p is injective. Furthermore, take $\xi \in (\alpha_1, \alpha_2)$, and look at $p(\alpha) - \xi\big|_{\alpha=\alpha_1} = \alpha_1 - \xi < 0$. Likewise, $p(\alpha) - \xi\big|_{\alpha=\alpha_2} = \alpha_2 - \varepsilon > 0$. Item (2) follows immediately from IVT. To show (3), we note that $p^n(\beta) \in (\alpha_1, \alpha_2]$ for all n due to (1) and $p^{n+1}(\beta) \ge p^n(\beta)$ for all n, so

$$\{p^n(\beta)\}_{n=1}^\infty$$

is an increasing sequence bounded from above by α_2. Thus, it follows that

$$\lim_{n\to\infty} p^n(\beta) = L \in (\alpha_1, \alpha_2].$$

Note that

$$L = \lim_{n\to\infty} p(p^n(\beta)) = p(\lim_{n\to\infty} p^n(\beta)) = p(L),$$

so it follows that $L = \alpha_2$. □

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Claim. If $p(\alpha) = \alpha$ and $p'(\alpha) < 1$, then α is asymptotically stable. Furthermore, if $p'(\alpha) > 1$, then α is unstable.

From this, we can conclude that if $p(\alpha) = \alpha$ and $p'(\alpha) < 1$, then the periodic solution $y' = f(t,y)$ with the initial condition $y(0) = \alpha$ is periodic and asymptotically stable.

Now, let us consider Autonomous systems. Namely, we have

$$u' = f(u) \qquad (*)$$

where $u : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, which is a vector field. Assume it is at least Lipshitz (we will assume smooth for now). If $u(t)$ solves $(*)$. then $u(t-c)$ also solves $(*)$. One of the most important feature of autonomous equations are stationary solutions (also known as stationary solutions or critical points)¿ These are solutions of the form $f(p) = 0$, so the constant $u(t) = p$ is a solution.

Example. Consider the system

$$u_1' = u_1(1-u_2)$$
$$u_2' = (u_1-5)(u_2-3).$$

Consider the vector field

$$f(u_1,u_2) = (u_1(1-u_1),(u_1-5)(u_2-3)),$$

so we can conclude that $f(u_1,u_2)=0$ precisely when $(u_1,u_2)=(0,3)$ or $(u_1,u_2)=(5,1)$.

To inspect the solutions to $(*)$ near a critial point $u=p$, we replace f by its linearization at p. This means do the Taylor Expansion of f. Namely,

$$f(u) = f(p) + Df(p)(u-p) + O(|u-p|^2)$$

as $u \to p$. We can write

$$(v-p)' = Df(p)(v-p),$$

so just by defining $w = v-p$, we can write down

$$w' = Df(p)w. \qquad (**)$$

This is a first order $n\times n$ homogeneous linear system. Now, we can ask questions about the behavior of the system. The question is now if the stability/instability of $w=0$ for $(**)$ predict that of $u=p$ for system $(*)$? Not always necessarily.

Example. We can linearize (u_1,u_2) in the previous example at $(0,3)$. We can note that

$$Df(u_1,u_2) = \begin{pmatrix} 1-u_1 & -u_1 \\ u_2-3 & u_1-5 \end{pmatrix}.$$

Evaluating at the critical point $(0,3)$, we can note that our system reduces to

$$w' = \begin{pmatrix} -2 & 0 \\ 0 & -5 \end{pmatrix},$$

so $w=0$ is asymptotically stable with a stable node. Now the question is whether the linearization implies asymptotical stability of the original equaiton.

Claim. Consider a system $u' = Au + f(t,u)$ where $(u:\mathbb{R}\to\mathbb{R}^n)$ where the constant $n\times n$ matrix A has eigenvalues with negative real parts. Suppose f is smooth and globally Lipshitz in the u variable. Then there exists an $\varepsilon>0$ such that if $|f(t,u)|<\varepsilon|u|$ for all t and u, then $u=0$ is asymptotically stable. then $u=0$ is asymptotically 0

Proof. Variation of parameters formula (Duhamel) tells us that

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}f(s)u(s)\,\mathrm{d}s.$$

The fact that A has eigenvalues with strictly negative real parts implies that there exists a constant M and $\delta>0$ such that $||e^{At}|| < Me^{-\delta t}$. Thus,

$$\begin{aligned}|u(t)| &\le ||e^{At}|||u(0)| + \int_0^t ||e^{A(t-s)}|||f(s)||u(s)|\,\mathrm{d}s \\ &\le Me^{-\delta t}|u(0)| + \int_0^t \varepsilon M e^{-\delta(t-s)}|u(s)|\,\mathrm{d}s.\end{aligned}$$

Multiplying both sides by $e^{-\delta t}$, we can write down

$$\underbrace{e^{\delta t}|u(t)|}_{=\phi(t)} \leq M|u(0)| + \int_0^t \varepsilon M \underbrace{e^{\delta s}|u(s)|}_{\phi(s)} \, \mathrm{d}s \implies \phi(t) \leq M|u(0)| + \int_0^t \varepsilon M \phi(s) \, \mathrm{d}s.$$

By Gronwall inequality applied to ϕ, we have

$$e^{\delta t}|u(t)| = \phi(t) \leq M|u(0)| \exp(\varepsilon M t),$$

so we get

$$|u(t)| \leq M|u(0)| \exp((\varepsilon M - \delta)t).$$

Choose $0 < \varepsilon < \delta/M$. Thus, we immediately get stability and asymptotical stability for $u = 0$. $\square$

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Can the origin be stable for the linearized system, but unstable for the nonlinear system? Yes. Consider the system

$$\begin{aligned} x' &= y + x^3 \\ y' &= -x + y^3. \end{aligned}$$

The linearization of the system at the point $(0, 0)$ is merely

$$w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w$$

and the eigenvalues are both complex with zero real part, so the origin is a stable fixed point. To solve the general linear system, we can rewrite the equation as

$$\begin{aligned} xx' &= x(y + x^3) \\ yy' &= y(-x + y^3), \end{aligned}$$

so by aadding the two equations together, we can write down

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{1}{2}x^2 + \frac{1}{2}y^2 \right\} = x^4 + y^4 \geq 0,$$

and is identically zero at zero. By examining the trajectories in phase space, we can establish that the nonlinear system is unstable at the origin. On a similar note, the zero solution of

$$\begin{aligned} x' &= y - x^3 \\ y' &= -x - y^3 \end{aligned}$$

is asymptotically stable. Now, what if the linearized system is unstable at the origin? Then, it is still possible for the nonlinear system to be unstable. Consider the equation

$$x'' = -x^3.$$

Then, we can write down

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{(x')^2}{2} + \frac{1}{4}x^4 \right) = 0$$

which implies that

$$\frac{(x')^2}{2} + \frac{1}{4}x^4$$

is constant along any solution. Thus, if $y = x'$, the energy term

$$E(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4$$

is constant along trajectories. Since the level curves are convex shapes, it is easy to see that the solutions are indeed stable (but not asymptotically stable). Despite this, the linearized equation

$$w' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has linearly growing terms (so must be unstable). What about if you have a matrix with strictly positive real parts and strictly negative real parts. In the linearized case, this implies we have a saddle.

We now specify the stable and unstable manifold theorem (which is called the stable and unstable curve theorem in 2D).

Stable and Unstable Curve Theorem. Let us consider the equation

$$y' = f(y), \quad f(0) = 0,$$

and suppose the matrix $Df(0)$ has two real eigenvalues $-\mu < 0 < \lambda$, where $\mu, \lambda > 0$. Then, in a small neighborhood of the origin, there is a curve Γ such that

(1) Γ contains $(0, 0)$.

(2) Any trajectory that starts on Γ stays on Γ and goes to the origin as $t \to \infty$.

(3) Γ is tangent to the eigenvector corresponding to the eigenvalue $-\mu$ at the origin.

(4) Any trajectory that starts off Γ exits the neighborhood in finite time.

Proof. Suppose $w' = Aw$ where $A = MDM^{-1}$. Thus, we can write

$$(M^{-1}w)' = DM^{-1}w,$$

so $u' = Du$. Thus, $u = M^{-1}w$, and $w = Mu$. Make this change of variables in the nonlinear system. Namely, $y = Mu$, so $Mu' = f(Mu)$ so $u' = M^{-1}f(Mu)$. Linearizing in u, we get $u' = Du$. We can assume that the eigenvectors of $Df(0)$ are $(1, 0)^T$ for λ, and $(0, 1)^T$ for μ. We can write

$$\begin{aligned} x' &= \lambda x + f_1(x, y), \\ y' &= -\mu y + f_2(x, y) \end{aligned}$$

where $|f_1(x, y)|, |f_2(x, y)| \leq C(x^2 + y^2)$ for $x^2 + y^2 \leq r^2$. Now, let us define $C_M = \{|y| \geq M|x|\}$ and we can consider the intersection of this region with $Q_\varepsilon = \{|x| \leq \frac{\varepsilon}{2} \text{ and } |y| \leq \frac{\varepsilon}{2}\}$. First, we can note that for $\varepsilon > 0$ small enough and $M > 0$ large enoug, trajectories in $Q_\varepsilon \cap \{y > 0\}$ are travelling down. In particular,

$$\begin{aligned} y' = -\mu y + f_2(x, y) &\leq -\mu y + |f_2(x, y)| \\ &\leq -\mu y + C(x^2 + y^2) \\ &\leq -\mu y + C\left(\frac{y^2}{M^2} + y^2\right) \\ &\leq -\mu y + Cy^2\left(1 + \frac{1}{M^2}\right) \\ &\leq -\mu y + C\varepsilon\left(1 + \frac{1}{M^2}\right)y < 0, \end{aligned}$$

if $y > 0$. Let $\ell^+ = \left\{(x,y) : y = \varepsilon, -\frac{\varepsilon}{M} < x < \frac{\varepsilon}{M}\right\}$. Along the right boundary of $C_M^+ \cap Q_\varepsilon$, we can note that

$$\begin{aligned} x' &= \lambda x + f_1(x,y) \\ &\geq \lambda x - L(x^2 + y^2) \\ &= \lambda x - L(x^2 + M^2 x^2) \\ &= \lambda x - xL(1 + M^2)x \\ &\geq \lambda x - L(1 + M^2)\varepsilon x \\ &= \{\lambda - L(1 + M^2)\varepsilon\}x. \end{aligned}$$

Choose $\varepsilon > 0$ so small that $L(1+M^2)\varepsilon < \lambda$. Call the two endpoints of ℓ^+ p and q. There is a neighborhood $B_\delta(q)$ such that if a trajectory starts on $B_\delta(q) \cap \ell^+$, then the trajecctory must exist $C_M^+ \cap Q_\varepsilon$ from the right boundary. Let us take S_R^+ to be the set of all points $\xi \in \ell^+$: trajectories starting at ξ exit $C_M^+ \cap Q_\varepsilon$ in finite time from the right boundary. Take S_L^+ to be defined analogously. By continuous dependence on initial data, S_R^+ and S_L^+ are open, nonempty, and disjoint. As a conclusion, $S_R^+ \cap S_L^+ \neq \ell^+$, so there exists a point $\eta \in \ell^+ \setminus (S_L^+ \cup S_L^+)$ that remains in $C_M^+ \cap Q_\varepsilon$. Trajectories that start at $\eta \in \ell^+$ has to converge to $(0,0)$. In fact, there is only one point $\eta \in \ell^+$ such that the trajectory starting at η goes to $(0,0)$. We will show this next.

Note that $u' = F(u)$ where $F : \mathbb{R}^d \to \mathbb{R}^d$, we can reparametrize u so that $v' = \Phi(v)F(v)$ where $\Phi : \mathbb{R}^d \to \mathbb{R}$, with $\phi \neq 0$. With this, we can reparametrize our original system

$$\begin{aligned} x' &= \lambda x + f_1(x,y) \\ y' &= -\mu y + f_2(x,y), \end{aligned}$$

so we can reparametrize our system as

$$\begin{aligned} \overline{x}' &= \frac{\lambda \overline{x} + f_1(\overline{x}, \overline{y})}{\mu - f_2(x,y)/y} =: H(\overline{x}, \overline{y}). \\ \overline{y}' &= -\overline{y}. \end{aligned}$$

(x,y) and $(\overline{x}, \overline{y})$ systems have the same shaped trajectories. Now, we claim that

$$\frac{\partial H}{\partial x} \geq 0 \text{ for } (\overline{x}, \overline{y}) \in C_M^+ \cap Q_\varepsilon.$$

The implication is that $(\overline{x}, \overline{y})|_{t=0} = y$, and $(\overline{\mathbf{x}}, \overline{\mathbf{y}})|_{t=0} = y'$, so the two solutions $\overline{x}(t)$ and $\overline{\mathbf{x}}(t)$ are getting further apart. With this, we can prove the claim. In particular, we have

$$H(x,y) = \frac{\lambda xy + yf_1(x,y)}{\mu y - f_2(x,y)},$$

so we can see that

$$\frac{\partial H}{\partial x} = \frac{\left(\lambda y + y\frac{\partial f_1}{\partial x}\right)(\mu y - f_2(x,y)) + (\lambda xy + yf_1(x,y))\frac{\partial f_2}{\partial y}}{(\mu y - f_2(x,y))^2} \approx \frac{\lambda \mu y^2}{\mu^2 y^2} = \frac{\lambda}{\mu} > 0.$$

This implies that the curve is unique. □

8 October 2024

Suppose $y' = Ay + f(y)$ with $f(0) = 0$ and $Df(0) = 0$, with f being smooth. Let

$$\begin{aligned} E_S &= \mathbb{R}^d \cap \operatorname{Span}\left\{\mathrm{v} \in \mathbb{C}^{\mathrm{d}} : (\mathrm{A} - \lambda\mathrm{I})^{\mathrm{d}}\mathrm{v} = 0 \text{ where } \lambda \in \Lambda(\mathrm{A}) \text{ with } \operatorname{Re}(\lambda) < 0\right\} \\ E_U &= \mathbb{R}^d \cap \operatorname{Span}\left\{\mathrm{v} \in \mathbb{C}^{\mathrm{d}} : (\mathrm{A} - \lambda\mathrm{I})^{\mathrm{d}}\mathrm{v} = 0 \text{ where } \lambda \in \Lambda(\mathrm{A}) \text{ with } \operatorname{Re}(\lambda) > 0\right\} \\ E_C &= \mathbb{R}^d \cap \operatorname{Span}\left\{\mathrm{v} \in \mathbb{C}^{\mathrm{d}} : (\mathrm{A} - \lambda\mathrm{I})^{\mathrm{d}}\mathrm{v} = 0 \text{ where } \lambda \in \Lambda(\mathrm{A}) \text{ with } \operatorname{Re}(\lambda) = 0\right\}. \end{aligned}$$

Theorem. In a small neighborhood of 0, there exist invariant surfaces S, U, and C that contain 0 and are tangent to E_S, E_U, and E_C at 0 respectively.

If $E_U = \{0\}$ and and the projected ODE onto C is asymptotically stable (unstable), then $y = 0$ is asymptotically stable (unstable). Note that the dimensions of S, U, and C are that of E_S, E_U, and E_C.

Before proving this theorem, we must first define the porjected system onto C. First, we put the system into standard form - that is we do a linear change of variables

$$\begin{aligned} u' &= Mu + f(u,v) \\ v' &= Nu + g(u,v) \end{aligned}$$

where $\mathrm{Re}(\lambda) = 0$ if λ is an eigenvalue of M, and $\mathrm{Re}(\lambda) < 0$ if $\lambda \in \Lambda(N)$. The linear approximation of the system is now decoupled. Since C is tangent to $u_1, \ldots, u_m$ axis, C can be written as the graph of a function $(u_1, \ldots, u_m)$. That is

$$(v_1, \ldots, v_{d-m}) = h(u_1, \ldots, u_m), \quad h : \mathbb{R}^m \to \mathbb{R}^{d-m}.$$

Namely, we write

$$C = \{(u,v) \in \mathbb{R}^d : v = h(u)\}.$$

Since C is invariant, any trajectory that starts on C will forever live on C, so it follows that

$$u' = Mu + f(u, h(u)).$$

This is our projected ODE. All that remains is to find h. To do this, we note that

$$\begin{aligned} v &= h(u) \\ v' &= Dh(u)u' = Dh(u)\left\{Mu + f(u,h(u))\right\} = Nh(u) + g(u,h(u)). \end{aligned}$$

Thus, h solves a first-order Hamilton Jacobi PDE

$$Dh(u)\left\{Mu + f(u,h(u))\right\} - Nh(u) - g(u,h(u)) = 0. \qquad (*)$$

Note that $(*)$ is usually impossible to solve explicitly; however, we just need an approximation to $h(u)$.

Example. Consider

$$\begin{aligned} u' &= v \\ v' &= -v + u^2 + 3uv. \end{aligned}$$

The linearization of this ODE is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}^* = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}_{=A} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 + 3uv \end{pmatrix}$$

Now, we find the eigenvalues and eigenvectors of A. In particular, we have $\lambda_1 = 0$ with eigenvector $(1, 0)^T$ and $\lambda_2 = -1$ with eigenvector $(1, -1)^T$. We an note that

$$A = T \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} T^{-1},$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

.

After changing the basis, we get

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ u^2 + 3uv \end{pmatrix}.$$

It follows that

$$\begin{aligned} x' &= (x+y)^2 - 3(x+y)y \\ y' &= -y - (x+y)^2 + 3(x+y)y. \end{aligned}$$

This is in standard form, so $Ny = -y$ and $(x+y)^2 - 3(x+y)y = f(x,y)$ and $-(x+y)^2 + 3(x+y)y = g(x,y)$. On C, the PDE for h becomes

$$h'(x)\left\{(x+h(x))^2 - 3(x+h(x))h(x)\right\} + h(x) - (x+h(x))^2 - 3(x+h(x))h(x) = 0.$$

By noting that C passes through the origin and is tangent to the x-axis, we try an ansatz of the form $x^2 q(x)$, where $q(x)$ is analytic. Expand as a series and equate terms. Doing so, we can conclude that

$$h(x) = -x^2 + x^3 + \text{HOT}$$

for some b. So, our projected ODE looks like

$$\begin{aligned} x' = f(x, h(x)) &= (x+y)^2 - 3(x+y)y\Big|_{y=h(x)} \\ &= (x - x^2 + x^3 + \text{HOT})^2 - 3(x - x^2 + x^3 + \text{HOT})(x^2 + x^3 + \text{HOT}) \\ &= x^2 + \gamma x^3 + \text{HOT}. \end{aligned}$$

The projected ODE is one dimensional, so we can study the stability of the one dimensional system at the origin. By considering the one-dimensional phase portrait, it follows that the 0 solution is unstable.

10 October 2024

Consider the ODE system

$$u' = uv$$
$$v' = -v - u^2$$

Then, there is no need to make a preliminary linear change of variables, since the linear part is already decoupled. The reduced ODE on the center manifold is given by

$$u' = u \cdot O(u^2) = cu^3 + O(u^4).$$

This does not tell us anything about the stability of the solution, so we need more terms in our expansion. If we try $h(u) = au^2 + O(u^3)$, we get for our Hamilton-Jacobi equation

$$h'(u) \{0 + uh(u)\} + h(u) + u^2 = 0,$$

so by matching coefficients, we determine that $a = -1$, so by drawing the phase portrait of u, by central manifold theory, we determine that the origin must be asymptotically stable. We can demonstrate one more example of central manifold theory. In this case, take

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} xz \\ yz \\ -(x^2 + y^2) + z^2 \end{pmatrix}.$$

Our system is already decoupled to the extent to which we need it to be. The eigenvalues of A are $\pm i$ and -1.Since this equation is already in standard form, there is no need for the preliminary linear change of variables. As a conclusion, our center manifold C is tangent to the xy-plane. On C, $z = h(x, y)$, so we can attempt to find a reduced system that is 2 on the center manifold. Try $h = O(r^2) = O(x^2 + y^2)$. The projected ODE systmem is going to be

$$x' = -y + xO(r^2) = -y + O(r^3)$$
$$y' = x + yO(r^2) = x + O(r^3).$$

To examine the behavior of this system, we can consider

$$x' = -y \pm x^3$$
$$y' = x \pm y^3,$$

so we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2 + y^2\right) = \pm 2(x^4 + y^4),$$

which changes stability based on the sign, so our original ansatz was inconclusive. So, let us try

$$h(x, y) = ax^2 + bxy + cy^2 + O(r^3).$$

Then,

$$Dh(u) \{Mu + f(u, h(u))\} - Nh(u) - g(u, h(u)) = 0.$$

so we get some awful PDE which we then have to solve. Doing some algebraic stuff, we figure out that $a = -1$ and $c = -1$, so

$$h(x, y) = -x^2 - y^2 + O(r^3).$$

Our projected ODE is thus given by

$$x' = -y + x(-x^2 - y^2) + O(r^4)$$
$$y' = x + y(-x^2 + y^2) + O(r^4),$$

so we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right) &= x^2(-x^2-y^2)+O(r^5)+y^2(-x^2-y^2)+O(r^5)\\ &= -(x^2+y^2)^2+O(r^5),\end{aligned}$$

so in r, we write down

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}r^2 = -r^4+O(r^5),$$

so if we set $r^2 = R$, then we can conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}R = -R^2+O(R^{5/2}),$$

so as it turns out, our system is stable.