PH509 Computational Physics
2025-26 Semester-II
Lab Sheet – 1

Instructions

Submit a `.ipynb` notebook or a well-documented `.py` script. Use only `numpy`, `scipy` (optional), and `matplotlib`. All plots must have labeled axes and legends where appropriate.

Goal. Implement Euler and RK2 methods for ODEs, compare with exact solutions, study error and stability, and understand step-size limitations.

Euler and RK2 Time-Stepping

Euler method (one-sided slope):

$$y_{n+1} = y_n + \Delta t\, f(y_n, t_n).$$

RK2 method (averaged slope):

$$\begin{aligned} y^* &= y_n + \Delta t\, f(y_n, t_n), \\ y_{n+1} &= y_n + \frac{\Delta t}{2}\left[f(y_n, t_n) + f(y^*, t_{n+1})\right]. \end{aligned}$$

Stability (operational definition). A numerical solution is *stable* if it remains bounded and qualitatively consistent with the expected physical behavior as time advances.

Problem 1: Radioactive Decay

$$\frac{dN}{dt} = -\frac{N}{\tau}, \qquad N(0) = N_0, \qquad N(t) = N_0 e^{-t/\tau}.$$

1. Implement Euler's method for $N_0 = 1$, $\tau = 1$, $t_{\max} = 5$.
2. Plot numerical and exact solutions for several Δt.
3. Repeat using RK2 and compare accuracy.
4. Plot the final-time error versus Δt on a log–log scale.
5. (2–3 lines) Why is it useful to solve a problem numerically even when the exact solution is known?
6. **Local truncation error (single-step error).** Using the exact solution of the decay equation, study how the error made in a *single time step* depends on Δt.
   (a) Start from the exact value $N(t_0)$ and advance the solution by one time step using Euler and RK2.
   (b) Compare the numerical result with the exact value $N(t_0 + \Delta t)$.
   (c) Repeat for several values of Δt.
   (d) Plot the single-step error versus Δt on a log–log scale and verify that it scales as $\mathcal{O}(\Delta t^2)$ for Euler and $\mathcal{O}(\Delta t^3)$ for RK2.
7. **Global error.** Fix a final time $t_{\max}$ and compute the numerical solution for several values of Δt.
   (a) Compute the error at $t_{\max}$ for each Δt.
   (b) Plot the error versus Δt on a log–log scale.
   (c) Infer how the global error scales with Δt for Euler and RK2.

Problem 2: Velocity-Dependent Drag

$$\frac{dv}{dt} = a - bv, \qquad v(0) = 0, \qquad a = 10,\ b = 1.$$

1. Solve using Euler's method and plot $v(t)$ for several Δt.
2. Estimate the terminal velocity numerically and verify it analytically.
3. Repeat using RK2 and compare with Euler for the same Δt.
4. For increasing values of Δt, describe how the numerical solution changes. At what point does the solution no longer resemble the expected physical behavior?
5. **Make it fail:** increase Δt until Euler's method produces oscillations, divergence, or other unphysical behavior. Does RK2 delay the onset of this behavior?

Problem 3: Coupled Radioactive Decay

$$\frac{dN_A}{dt} = -\frac{N_A}{\tau_A},$$
$$\frac{dN_B}{dt} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B},$$
$$N_A(0) = 1, \qquad N_B(0) = 0.$$

1. Solve numerically for $\tau_A = 1$ and several τ_B.
2. Compare with the analytical solution.
3. Study $\tau_B \ll \tau_A$, $\tau_B \sim \tau_A$, $\tau_B \gg \tau_A$.
4. Compare Euler and RK2 solutions for the same Δt. Which method better captures the expected qualitative behavior?
5. **Make it fail:** choose $\tau_B \ll \tau_A$ and increase Δt. Which variable (N_A or N_B) shows unphysical or erratic behavior first? What does this suggest about how the fastest physical process influences the choice of Δt?

Problem 4: Nonlinear Population Growth

$$\frac{dN}{dt} = aN - bN^2.$$

1. Set $b = 0$. Solve numerically and compare with the exact solution.
2. Solve with:
   - $a = 10$, $b = 3$ for small $N(0)$,
   - $a = 10$, $b = 0.01$ for $N(0) = 1000$.
3. Determine steady states and verify them numerically.
4. Compare Euler and RK2 for the same Δt.
5. **Make it fail:** identify time steps that produce unphysical behavior. Does RK2 delay this failure?
6. (2–3 lines) Why does the nonlinear term qualitatively change the long-time behavior?

Note. You are encouraged to deliberately choose parameters that cause a numerical method to fail and to explain why the failure occurs.