PH509 Computational Physics

Assignment 3: Mini-projects on Planetary Motion

This assignment is based on Chapter 4 of the course text (Planetary Motion and Few–Body Problems). Each problem is a mini-project. Submit clean code, relevant plots, and concise answers to the analytical questions. Attempt Projects 1 and any one of the remaining mini projects (2, 3 or 4).

Mini-project 1: Closed and Open Orbits in Modified Gravity

Goal: Study how small deviations from the Newtonian inverse–square force destroy closed Keplerian orbits.

Important: Use correct physical scales. As discussed in Section 4.2.4 of the chapter (pp. 206–207), use:

- distance unit: 1 AU,
- time unit: 1 year,
- gravitational parameter: $GM = 4\pi^2$ in these units,
- typical orbital speeds: a few AU/year.

These scales allow stable long-time integrations of Earth–Sun motion and provide physically meaningful comparisons when the force law is modified.

1. Implement the Earth–Sun system using the standard leapfrog integrator and the force

$$\mathbf{F}(r) = -\frac{GM}{r^3}\mathbf{r}.$$

   Use AU–year units and initial conditions as in the Earth example from the text (pp. 198–200).

2. Verify that the orbit is closed and that energy and angular momentum are conserved to numerical accuracy. Plot the trajectory in the xy-plane over several periods and overlay multiple orbital cycles to show that they coincide.

3. Modify the force law to

$$\mathbf{F}(r) \propto \frac{1}{r^{2+\delta}}, \qquad \delta = 0.05,\ 0.10.$$

   Produce trajectory plots and demonstrate that the orbit is no longer closed. Show how the orientation of the orbit slowly rotates with each revolution.

4. Estimate the ratio of the radial oscillation frequency n_r to the angular sweep frequency n_θ by:

   - counting maxima (or minima) of $r(t)$ over a long integration time,
   - counting the number of full angular revolutions in $\theta(t)$ over the same interval.

   Comment on why a rational n_r/n_θ leads to closed orbits and why irrational values produce open orbits.

5. **Open-ended:** Repeat the experiment for a Hooke-like force $F(r) = -kr$. For several choices of initial conditions, check whether all bounded orbits are closed and compare with the analytic statement that only $1/r$ and r potentials support closed bounded orbits.

Mini-project 2: Precession of Mercury

We study relativistic perihelion precession using both a direct geometrical method and the Runge–Lenz vector. Use Mercury's parameters from the table in the text (semimajor axis, eccentricity, period, etc.) in AU–year units.

A. GR-corrected force and time-transformed leapfrog

1. Implement the GR-corrected central force

$$\mathbf{F}(r) = -\frac{k}{r^3}\mathbf{r}\left(1 + \frac{\lambda}{r^2}\right),$$

   where $k = GM$ in the chosen units and λ is a small dimensionless parameter.

2. Use the leapfrog method with time transformation and

$$\Omega(r) = \frac{1}{r}.$$

   Evolve the orbit for at least 100 years of physical time. Verify that the orbit is bound and that the energy and angular momentum are approximately conserved.

B. Measuring precession without the Runge–Lenz vector

In this part, do *not* use the Runge–Lenz vector.

1. At each timestep, compute

$$r(t) = \sqrt{x(t)^2 + y(t)^2}, \qquad \theta(t) = \mathrm{atan2}(y(t), x(t)).$$

2. Detect successive perihelia by identifying local minima of $r(t)$:

$$r_{n-1} > r_n < r_{n+1}.$$

   Record the angle $\theta_{\mathrm{peri},n}$ at each detected perihelion.

3. For an artificially large relativistic parameter $\lambda = 0.01$, show that the sequence $\theta_{\mathrm{peri},n}$ increases monotonically. Plot $\theta_{\mathrm{peri},n}$ versus orbit index n.

4. For the physical value $\lambda = 1.1 \times 10^{-8}$, run the simulation for ~ 100 years, extract several perihelia, and estimate the average precession rate in arcseconds per century.

5. (Optional) Fit a quadratic to $r(t)$ around each minimum to refine the perihelion position.

C. Measuring precession using the Runge–Lenz vector

1. Implement the Runge–Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\,\hat{\mathbf{r}},$$

   where $\mathbf{p} = m\mathbf{v}$ and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

2. Track the angle

$$\theta_A(t) = \mathrm{atan2}(A_y, A_x).$$

3. For $\lambda = 0.01$, plot $\theta_A(t)$ versus time to illustrate the rotation and any oscillatory behaviour.

4. For $\lambda = 1.1 \times 10^{-8}$, compute the change in θ_A over 100 years and convert it to arcseconds per century. Compare with the expected $\sim 43''$/century.

D. Short comparison (write-up)

In 5–10 sentences, compare:

- direct perihelion tracking (Part B), and
- Runge–Lenz vector tracking (Part C),

in terms of numerical stability, sensitivity to timestep, and practical detectability of small precession angles.

E. Time scales and detectability

1. Using Mercury's orbital period $T_{\rm orb} \approx 0.24$ yr and the GR precession rate $43''$/century, estimate:
   - the precession per orbit (in arcseconds and radians),
   - the number of orbits required for a full 2π precession,
   - the corresponding physical time (in years).
2. Suppose your scheme can resolve angular changes down to $\sim 10^{-5}$ rad. Estimate how many years of evolution are needed for the accumulated precession to reach 10^{-4} rad. Compare with the 100 yr simulation.
3. Estimate the angular change $\Delta\theta$ per timestep near aphelion and perihelion. Compare with the GR precession per orbit and discuss why direct visual inspection of a single orbit is insufficient.
4. Discuss which modelling assumptions in your simulation allow you to "compress" centuries of dynamics into a short numerical run, and which assumptions would matter if simulating a full 2π precession time.

Mini-project 3: Radial-Velocity Method and Synthetic Exoplanet Data

Goal: Generate and fit synthetic radial-velocity (RV) data for a star–planet system, and connect this to how JWST studies exoplanets.

A. Two-body model and stellar wobble

1. Treat the system as a star of mass M and a planet of mass m in a Keplerian orbit. In the centre-of-mass frame:
$$\mathbf{v}_* = -\frac{m}{M+m}\,\mathbf{v}_{\rm rel}.$$
2. For a Sun–Jupiter-like system, simulate the relative orbit and plot one component of $\mathbf{v}_*(t)$ over several periods.

B. RV curve generation with eccentricity

1. Introduce eccentricity e and use Kepler's equation to obtain $\theta(t)$.
2. For inclination i and argument of perihelion ω, compute
$$v_z(t) = C + V\left[\cos(\theta(t)+\omega) + e\cos\omega\right].$$
3. Generate RV curves for a low-eccentricity case ($e = 0.05$) and a high-eccentricity case ($e = 0.6$). Comment on the differences.

C. Synthetic dataset with noise

1. Sample $v_z(t)$ at realistic observation times.
2. Add Gaussian noise to simulate measurement uncertainties.
3. Plot noisy datapoints and the noiseless model together.

D. Parameter recovery

1. Fit the synthetic dataset to recover (T, e, V, ω, C).
2. Compare the best-fit parameters with the true values. Comment on correlations or degeneracies.

E. Open-ended extension: multi-planet systems

1. Add a second planet and superpose the two reflex motions.
2. Explore a resonant configuration (e.g. 2:1) and a non-resonant case.
3. Comment on detectability and interpretation.

F. Connecting RV modelling to JWST exoplanet observations

JWST measures infrared spectra during transits and secondary eclipses, revealing atmospheric composition, cloud layers, and thermal structure. It does *not* measure radial velocities directly, but complements RV data strongly.

1. For an exoplanet with known $m \sin i$ from RV measurements, explain how JWST transit and eclipse observations constrain i and therefore the true mass m.
2. Choose one short-period planet ($T \sim 3$–10 days) and one long-period planet ($T \sim 200$–500 days) from your RV simulations. For each, discuss whether JWST could detect:
   - a primary transit,
   - a secondary eclipse,
   - a transmission spectrum,
   - a dayside emission spectrum.
3. JWST can detect transit depth variations at the $\sim$ 10–100 ppm level. Estimate the transit depth $\delta = (R_p/R_*)^2$ for Earth-sized, Neptune-sized, and Jupiter-sized planets around a Sun-like star. Comment on detectability.
4. Short reflection: explain how combining RV + JWST observations enables determination of:
   - mean density and bulk composition,
   - atmospheric scale height,
   - molecular species.

Suggested Reading for the JWST Exoplanet Component

- **Webb's Impact on Exoplanet Research** Overview of how JWST studies exoplanet atmospheres and dynamics. https://science.nasa.gov/mission/webb/science-overview/science-explainers/webbs-impact-on-exoplanet-research/
- **How JWST Studies Exoplanets** Introduction to transmission spectroscopy, secondary eclipses, and thermal emission. https://science.nasa.gov/mission/webb/science-overview/science-explainers/how-will-webb-study-exoplanets/
- **WASP-39b Transmission Spectra** Real JWST atmospheric spectra showing CO_2, H_2O, and other molecules. https://science.nasa.gov/asset/webb/exoplanet-wasp-39-b-transmission-spectra/
- **NASA's Webb Mission Overview** Mission details and instrument capabilities relevant for exoplanet work. https://science.nasa.gov/mission/webb/

These readings connect your RV modelling to how JWST characterises exoplanet atmospheres.

Mini-project 4: Simulating the JWST orbit near the Sun Earth L2 point

JWST operates near the Sun Earth L2 point. This is an equilibrium point in the rotating frame of the Sun Earth system, and the spacecraft follows a large amplitude Lissajous type orbit around it. In this problem you will build a simple model of the L2 region using the restricted three body problem and explore the motion of a test particle near L2.

1. Set up the restricted three body problem Treat the Sun and the Earth as massive bodies moving in circular orbits around their common centre of mass. Ignore the mass of the spacecraft. Use AU and year units with the combined gravitational parameter of Sun plus Earth equal to $4\pi^2$. Place the Earth at 1 AU from the Sun. Write the equations of motion for a test particle in the inertial frame.
2. Equations of motion in the rotating frame Introduce a frame that rotates with the angular speed of the Earth, so the Sun and Earth remain fixed. Include the Coriolis and centrifugal terms in the equations of motion. Write down the effective potential of the restricted three body problem and determine the approximate location of the L2 point along the Sun Earth axis.
3. Linearisation near L2 Expand the equations of motion for small displacements around the L2 point. Show that the linearised system has two oscillatory directions and one unstable direction. Estimate the characteristic frequencies of the oscillatory directions.
4. Numerical experiment Choose an initial position slightly displaced from L2 together with a small initial velocity that reduces growth along the unstable direction. Integrate the motion for several years in the rotating frame. Plot the trajectory in the rotating xy plane and identify the Lissajous type oscillations.
5. Transformation to the inertial frame Convert the rotating frame coordinates to inertial coordinates and plot the resulting spacecraft trajectory. Show that the motion is not a closed orbit around either the Sun or the Earth, but a large oscillatory path that remains roughly at the L2 distance from Earth.
6. Reflection Explain why station keeping is needed to prevent drift along the unstable direction of L2. Comment on why the L2 region provides a favourable thermal and pointing environment for a space telescope.