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A study of electrostatic force on the walls of *N*-regular polygon-multifin line

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Abstract

By means of conformal transformations, the electrostatic energy of a charged *N*-regular polygon-multifin transmission line is achieved first. The electrostatic force exerted on each side of the outer wall is computed by using the principle of virtual work. Regarding the properties of the field, the electrostatic force exerted on each side of the inner fin could be calculated. The dependent variable and invariant quantities are analyzed as the cross section of the line undergoes a geometric equiform conversion. The cases for the triangular line, the square line and the hexagonal line are taken as typical examples. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Computation of electrostatic forces on the walls of a coaxial transmission line is a new research field in electrostatics. In Ref. [1], the electrostatic forces exerted on the walls of a charged square coaxial line have been calculated [1]. Furthermore, the case of a charged rectangular coaxial line has been investigated in Ref. [2]. The prime motivation of this paper is to compute the electrostatic forces on the outer and inner walls of a charged *N*-regular polygon-multifin coaxial line that is widely used in microwave engineering [3]. To achieve this aim the conformal mapping method, the

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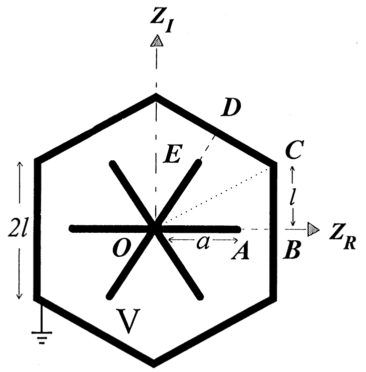

Fig. 1. The cross section of a charged N-regular polygon-multifin line (the z-plane).

principle of virtual work and the basic properties of the electrostatic field are utilized. The dependent variable and invariant in the condition of equiform conversion for the cross section of the line are discussed. The triangular line, the square line and the hexagonal line are examined quantitatively.

2. Transformation of the field region

Fig. 1 represents the cross section of a coaxial transmission line consisting of an outer N-regular polygon and an inner N-regular multifin. The side width of the outer wall is $2l$ and the fin width of the inner wall is a. Voltage V is applied across the two walls, with the outer wall grounded. We suppose that the line is so long that the electrostatic field might be treated as a two-dimensional problem in the cross section.

Due to the symmetry we only map $OABCDEO$, one Nth of the field region in the cross section. Using the Schwarz–Christoffel transformation

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = At^{(2/N)-1}\left(t^2 - \frac{1}{k^2}\right)^{-1/2}, \tag{1}$$

the region $OABCDEO$ in the Z-plane is mapped into the upper t-plane of Fig. 2. Integration of Eq. (1) gives

$$Z = A\int_0^t t^{(2/N)-1}\left(t^2 - \frac{1}{k^2}\right)^{-1/2}\,\mathrm{d}t. \tag{2}$$

Denote b as the distance from the center to a side of the outer N-regular polygon, namely

$$b = l\cot\frac{\pi}{N}. \tag{3}$$

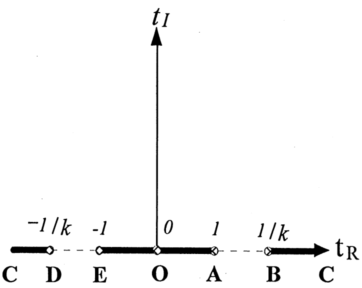

Fig. 2. The t-plane.

Consider the corresponding points of A and B in Figs. 1 and 2. Putting their coordinates into Eq. (2) we have, respectively,

$$a = A\int_0^1 t^{(2/N)-1}\left(t^2 - \frac{1}{k^2}\right)^{-1/2} \mathrm{d}t, \tag{4}$$

$$b = A\int_0^{1/k} t^{(2/N)-1}\left(t^2 - \frac{1}{k^2}\right)^{-1/2} \mathrm{d}t. \tag{5}$$

Eq. (4) divided by Eq. (5) yields

$$\frac{a}{b} = \frac{\int_0^1 t^{(2/N)-1}\,(1/k^2 - t^2)^{-1/2}\,\mathrm{d}t}{\int_0^{1/k} t^{(2/N)-1}\,(1/k^2 - t^2)^{-1/2}\,\mathrm{d}t}. \tag{6}$$

If we may determine k after integrating Eq. (6), k must be a function of a/b. We denote it as

$$k = k\left(\frac{a}{b}\right). \tag{7}$$

Now, we take the transformation

$$t = sn(u, k) \tag{8}$$

in which $sn(u, k)$ is a Jacobian elliptic function, to map the upper half of the t-plane into the interior of the rectangle in the u-plane of Fig. 3, where the electrostatic field is uniformly distributed.

3. Electrostatic energy

From the uniformity of the field confined in the rectangle $ABDE$ of Fig. 3, it will be seen that the capacitance per unit length of the transmission line is

$$C = N\varepsilon\frac{2K(k)}{K'(k)} = 2N\varepsilon\frac{K(k)}{K'(k)}, \tag{9}$$

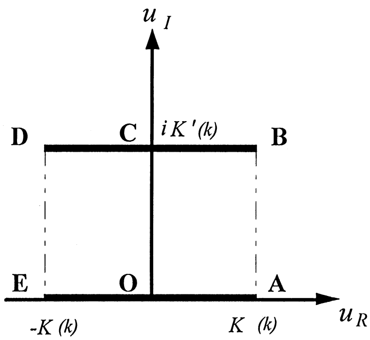

Fig. 3. The u-plane.

where $K(k)$ is the first complete elliptic integral and

$$K'(k) = K(k'), \tag{10}$$

k' is the complementary modulus of k, i.e.

$$k^2 + k'^2 = 1. \tag{11}$$

So the electrostatic energy per unit length of the line could be expressed by

$$W = \frac{1}{2} C V^2 = N\varepsilon \frac{K(k)}{K'(k)} V^2. \tag{12}$$

4. Electrostatic force on the outer wall

Regarding b as a generalized coordinate in space, we employ the principle of virtual work to calculate the electrostatic force on the outer wall [2]. Thus, the electrostatic force on each side of the outer wall per unit length of the line is given by

$$F_{2l} = \frac{1}{N} \frac{\partial W}{\partial b} \bigg|_{V=c}. \tag{13}$$

Substituting Eq. (12) into the above equation leads to

$$F_{2l} = \varepsilon V^2 \frac{\partial}{\partial b} \left(\frac{K(k)}{K'(k)} \right) = \varepsilon V^2 \frac{\partial}{\partial k} \left(\frac{K(k)}{K'(k)} \right) \cdot \frac{\partial k}{\partial b}. \tag{14}$$

Using derivative of the complete elliptic integrals we have [4]

$$\frac{\partial}{\partial k} \left(\frac{K(k)}{K'(k)} \right) = \frac{\pi}{2kk'^2 K'^2(k)} \tag{15}$$

Differentiating Eq. (7) with respect to b yields

$$\frac{\partial k}{\partial b} = -\frac{a}{b^2}Dk\left(\frac{a}{b}\right), \tag{16}$$

where Dk represents derivative of k with respect to a/b. Apparently, Dk is also a function of a/b.

Substituting Eqs. (15) and (16) into Eq. (14) we get the electrostatic force on each side of the outer wall per unit length of the coaxial line as

$$F_{2l} = -\frac{a\pi\varepsilon V^2}{2b^2kk'^2K'^2(k)}Dk, \tag{17}$$

where the minus indicates that the force points to the center O.

5. Electrostatic force on the inner wall

Reconsider the region $OABCO$ in the z-plane of Fig. 1. According to the symmetry, the direction of the electrostatic field between A and B must be on the line AB while that between C and O must be on the line CO. Hence, from Gauss's law $\oiint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = 0$,

$$\int_a E_a\,\mathrm{d}a = -\int_l E_l\,\mathrm{d}l. \tag{18}$$

Due to the mean theorem of integration Eq. (18) may be rewritten as

$$a\bar{E}_a = -l\bar{E}_l. \tag{19}$$

Denoting the force on each side of the inner fin per unit length of the line as F_a, we have

$$F_a = \tfrac{1}{2}|q|\bar{E}_a; \qquad F_l = \tfrac{1}{2}|q|\bar{E}_l, \tag{20}$$

where $|q|$ is the absolute magnitude of the charge on each side of the inner fin or on half side of outer wall per unit length of the line. Combining Eqs. (19) and (20) yields

$$aF_a = -lF_l. \tag{21}$$

Therefore, the electrostatic force on each side of the inner fin per unit length of the line is computed as

$$F_a = -\frac{l}{a}F_l = -\frac{l}{2a}F_{2l} = -\frac{b}{2a}F_{2l}\tan\frac{\pi}{N} = \frac{\pi\varepsilon V^2}{4bk\,k'^2K'^2(k)}Dk\tan\frac{\pi}{N}. \tag{22}$$

Eqs. (17) and (22) tell us: $2lF_{2l}$, the product of the force F_{2l} and the side width $2l$ of the outer wall, and aF_a, the product of the force F_a and the fin width a of the inner wall, do not vary as the cross section of the line undergoes a geometric equiform conversion.

6. Typical cases

6.1. Triangular line

Putting $N = 3$ in Eq. (3), we find the solution of k to be

$$k = \left(\frac{\sqrt{3}(M-1)}{M+1} + 1\right)^{3/2}, \tag{23}$$

where

$$M = cn\left[\left(1 - \frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right], \tag{24}$$

$cn(u, k)$ and $dn(u, k)$ are Jacobian elliptic functions. $cn^{-1}(u, k)$ is the inverse function of $cn(u, k)$.

Recalling that D is represented as differentiation with respect to a/b, we have

$$Dk = \frac{\partial k}{\partial M} DM. \tag{25}$$

Differentiating Eq. (23) with respect to M yields

$$\frac{\partial k}{\partial M} = \frac{3\sqrt{3}k^{1/3}}{(M+1)^2}. \tag{26}$$

Also, we can write down by differentiating Eq. (24) with respect to a/b, that

$$DM = cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\, sn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right]$$
$$\times\, dn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right]. \tag{27}$$

Putting the above results into Eqs. (17) and (22) we obtain, respectively,

$$F_{2l} = -\frac{3\sqrt{3}\pi\varepsilon a V^2}{2b^2 k^{2/3}\, k'^2 K'^2(k)(M+1)^2} cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)$$
$$\times\, sn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right]$$
$$\times\, dn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right], \tag{28}$$

$$F_a = \frac{9\pi\varepsilon V^2}{4bk^{2/3}\,k'^2\,K'^2(k)(M+1)^2} cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)$$

$$\times sn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right]$$

$$\times dn\left[\left(1-\frac{a}{b}\right)cn^{-1}(2-\sqrt{3}, \sqrt{2+\sqrt{3}}/2)\right] \tag{29}$$

6.2. Square line

Letting $N = 4$ in Eq. (3) and solving for k give

$$k = cn^2\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right] \tag{30}$$

so that

$$Dk = 2K\left(\frac{1}{\sqrt{2}}\right)sn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]cn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]$$

$$\times dn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]. \tag{31}$$

Thus, Eqs. (17) and (22) become

$$F_{2l} = -\frac{a\pi\varepsilon V_0^2}{b^2k\,k'^2K'^2(k)}K\left(\frac{1}{\sqrt{2}}\right)sn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]$$

$$\times cn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]dn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right], \tag{32}$$

$$F_a = \frac{\pi\varepsilon V_0^2}{2bk\,k'^2K'^2(k)}K\left(\frac{1}{\sqrt{2}}\right)sn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]$$

$$\times cn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]dn\left[\left(1-\frac{a}{b}\right)K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right]. \tag{33}$$

6.3. Hexagonal line

Similarly, for the case of $N = 6$ we obtain the results as follows:

$$k = \left[\frac{(\sqrt{3}-1)\left[1 - cn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right)\right]}{2\left[(2-\sqrt{3})cn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right)+1\right]}\right]^{3/2}, \tag{34}$$

$$\begin{aligned} Dk = 3\sqrt{3}k^{1/3} & \frac{2-\sqrt{3}}{\left[(2-\sqrt{3})cn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right)+1\right]^2} \\ & \times K(\sqrt{2-\sqrt{3}}/2)sn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right) \\ & \times dn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right), \end{aligned} \tag{35}$$

$$\begin{aligned} F_{2l} = & -\frac{3\sqrt{3}(2-\sqrt{3})\pi\varepsilon a V^2}{2\left[(2-\sqrt{3})cn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right)+1\right]^2 k^{2/3}\, k'^2 b^2 K'^2(k)} \\ & \times K(\sqrt{2-\sqrt{3}}/2)sn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right) \\ & \times dn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right), \end{aligned} \tag{36}$$

$$\begin{aligned} F_a = & \frac{3(2-\sqrt{3})\pi\varepsilon V^2}{4\left[(2-\sqrt{3})cn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right)+1\right]^2 k^{2/3}\, k'^2 b K'^2(k)} \\ & \times K(\sqrt{2-\sqrt{3}}/2)sn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right) \\ & \times dn\left(\frac{2a}{b}K(\sqrt{2-\sqrt{3}}/2), \sqrt{2-\sqrt{3}}/2\right). \end{aligned} \tag{37}$$

We define the normalized products $2lf_{2l}$ and af_a as

$$2lf_{2l} = \frac{-2lF_{2l}}{\pi\varepsilon V^2} = \frac{a}{bk\,k'^2K'^2(k)}Dk\tan\frac{\pi}{N} = \frac{4aF_a}{\pi\varepsilon V^2} = 4af_a. \tag{38}$$

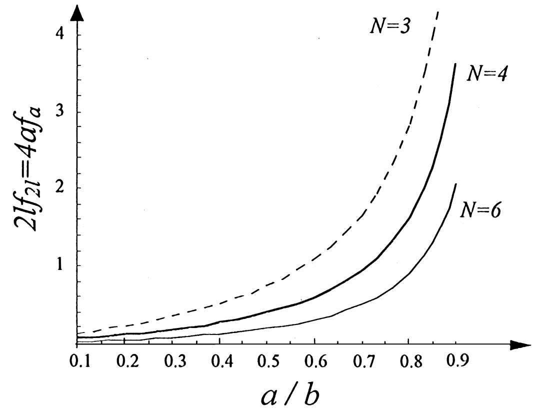

Fig. 4. The curves of $2lf_{2l} = 4af_a$ versus a/b.

Table 1
Some numerical values of $2lf_{2l} = 4af_a$

N	a/b								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.13843	0.24040	0.36504	0.52875	0.75734	1.10120	1.67731	2.83652	6.33343
4	0.06734	0.12119	0.18964	0.28200	0.41325	0.61264	0.94794	1.62191	3.64767
6	0.02913	0.05461	0.08886	0.13740	0.20939	0.32248	0.51691	0.91105	2.09080

The curves of $2lf_{2l} = 4af_a$ as the function of a/b for $N = 3$, 4 and 6 are plotted in Fig. 4. Table 1 gives some numerical values.

7. Conclusion

The electrostatic energy of a charged N-regular polygon-multifin line can be readily obtained by mapping the field region through conformal transformations. The principle of virtual work contributes a convenient method to calculate the electrostatic force on each side of the outer wall; while the basic properties of the field provide an effective way to determine the force on each side of the inner fin. When the line cross section is changed proportionally, these two forces also vary, but the products $2lF_{2l}$ and aF_a remain invariant.

Although today one possibly will solve this type of problem approximately by using numerical methods on a computer, this paper may find the greatest utility in providing analytic solutions that can be applied to check numerical methods and computer solutions.

Acknowledgements

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