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Further study on electrostatic capacitance of an inclined plate capacitor

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ABSTRACT

Developing the expression for calculating the capacitance of an inclined plate capacitor is investigated. From the result when the intersection line of the two planes containing the two electrodes separately lies outside the electrodes, the situation that the intersection line locates on an electrode is treated as a combination of two capacitors. Then the capacitance is achieved in term of complete elliptic integrals. The perpendicular-plate capacitor in such a case is taken as an example for capacitance calculation.

1. Introduction

Determining the exact electrostatic capacitance of an inclined plate capacitor in general is complicated due to the dimension and position of the electrode as well as the fringing effect. Recently, Xiang has done such work in Ref. [1]. The result of the capacitance was achieved in a broad manner with the aid of elliptic functions by using the conformal mapping method. That formula can be used for any width of plates. However, there is some restriction on the relative position of two plates. Upon closer inspection, it may be found that the result in Ref. [1] is valid only for the following case: the intersection line of the two planes, containing the two electrodes separately, lies outside the electrodes. If that intersection line does locate on one plate, the capacitance expression will be incomplete. As a short communication, this paper aims at that goal. Developing the results obtained in Ref. [1], the capacitance of two inclined plates in the latter case is calculated. Special cases are analyzed and conclusions are drawn.

2. Capacitance in the case studied

Fig. 1 shows the cross-section of an inclined plate capacitor with angle φ between two plates AB and CD of lengths l_1 and l_2. That is the case researched in Ref. [1]. The extended lines BA and DC intersect at the point O that locates outside the two electrodes. The distances from O to A and C are r_1 and r_2, respectively. The longitudinal dimension of the plates is sufficient that the problem can be simplified as two-dimensional in the z-plane.

Making use of the conforming mapping technique for that case, Ref. [1] gives the inner capacitance inside $\angle AOC$ per unit longitudinal length as

$$C_{\text{in}} = \varepsilon_0 \frac{K'(k_{\text{in}})}{K(k_{\text{in}})} \tag{1}$$

Here $K(k)$ is the complete elliptic integral of the first kind [2], and the modulus k_{in} is

$$k_{\text{in}} = \sqrt{\frac{\left(r_1^{(\pi/\varphi)} + r_2^{(\pi/\varphi)}\right)\left((r_1+l_1)^{(\pi/\varphi)} + (r_2+l_2)^{(\pi/\varphi)}\right)}{\left(r_1^{(\pi/\varphi)} + (r_2+l_2)^{(\pi/\varphi)}\right)\left(r_2^{(\pi/\varphi)} + (r_1+l_1)^{(\pi/\varphi)}\right)}} \tag{2}$$

The outer capacitance outside $\angle AOC$ per unit longitudinal length is

$$C_{\text{out}} = \varepsilon_0 \frac{K'(k_{\text{out}})}{K(k_{\text{out}})} \tag{3}$$

where

$$k_{\text{out}} = \sqrt{\frac{\left(r_1^{(\pi/(2\pi-\varphi))} + r_2^{(\pi/(2\pi-\varphi))}\right)\left((r_1+l_1)^{(\pi/(2\pi-\varphi))} + (r_2+l_2)^{(\pi/(2\pi-\varphi))}\right)}{\left(r_1^{(\pi/(2\pi-\varphi))} + (r_2+l_2)^{(\pi/(2\pi-\varphi))}\right)\left(r_2^{(\pi/(2\pi-\varphi))} + (r_1+l_1)^{(\pi/(2\pi-\varphi))}\right)}} \tag{4}$$

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Thus, the resulting electrostatic capacitance per unit longitudinal length is

$$C = C_{\text{in}} + C_{\text{out}} = \varepsilon_0 \left(\frac{K'(k_{\text{in}})}{K(k_{\text{in}})} + \frac{K'(k_{\text{out}})}{K(k_{\text{out}})} \right) \tag{5}$$

It should be emphasized that Eq. (5) is effective only in the case that the intersection point O lies beyond AB and CD, as shown in Fig. 1. That means the intersection line of the two planes containing the two electrodes must be outside the electrodes. Otherwise, Eq. (5) is invalid.

3. Capacitance in the supplementary case

Now we investigate the case where the intersection line of the two planes containing the two electrodes locates on an electrode. In the cross-sectional view, the intersection point O is on AB or CD. The latter case of CD is illustrated in Fig. 2.

To calculate the capacitance, we consider the capacitor in such case to be a system of two capacitors connected in parallel [3]. One capacitor consists of plates DO and AB, and the other of CO and AB. In this way, the capacitance of each of the two separate capacitors can be calculated based on the results of Section 2.

For the capacitor consisting of plates DO and AB, substituting $r_2 = 0$ and $l_2 = l_{2\text{R}}$ into Eqs. (2), (4) and (5), we derive its capacitance per unit longitudinal length as

$$C_{\text{R}} = C_{\text{R in}} + C_{\text{R out}} = \varepsilon_0 \left(\frac{K'(k_{\text{R in}})}{K(k_{\text{R in}})} + \frac{K'(k_{\text{R out}})}{K(k_{\text{R out}})} \right) \tag{6}$$

where

$$k_{\text{R in}} = \sqrt{\frac{r_1^{(\pi/\varphi)}\left((r_1+l_1)^{(\pi/\varphi)} + l_{2\text{R}}^{(\pi/\varphi)}\right)}{(r_1+l_1)^{(\pi/\varphi)}\left(r_1^{(\pi/\varphi)} + l_{2\text{R}}^{(\pi/\varphi)}\right)}} \tag{7}$$

and

$$k_{\text{R out}} = \sqrt{\frac{r_1^{(\pi/(2\pi-\varphi))}\left((r_1+l_1)^{(\pi/(2\pi-\varphi))} + l_{2\text{R}}^{(\pi/(2\pi-\varphi))}\right)}{(r_1+l_1)^{(\pi/(2\pi-\varphi))}\left(r_1^{(\pi/(2\pi-\varphi))} + l_{2\text{R}}^{(\pi/(2\pi-\varphi))}\right)}} \tag{8}$$

Similarly, substituting $r_2 = 0$, $l_2 = l_{2\text{L}}$ and $\varphi \to \pi - \varphi$ into Eqs. (2), (4) and (5), the capacitance per unit longitudinal length for the capacitor consisting of plates CO and AB is obtained as

$$C_{\text{L}} = C_{\text{L in}} + C_{\text{L out}} = \varepsilon_0 \left(\frac{K'(k_{\text{L in}})}{K(k_{\text{L in}})} + \frac{K'(k_{\text{L out}})}{K(k_{\text{L out}})} \right) \tag{9}$$

where

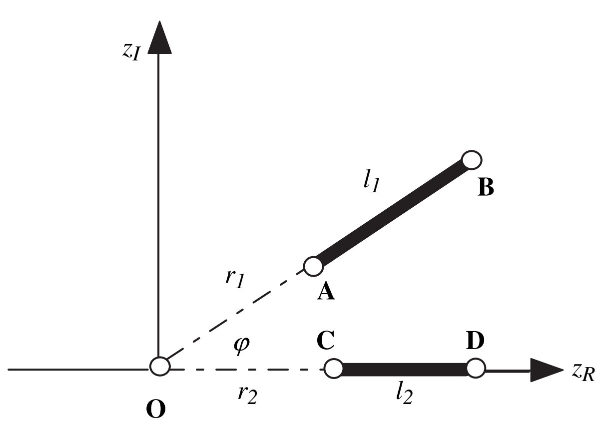

Fig. 1. The cross-section of an inclined plate capacitor for the case studied.

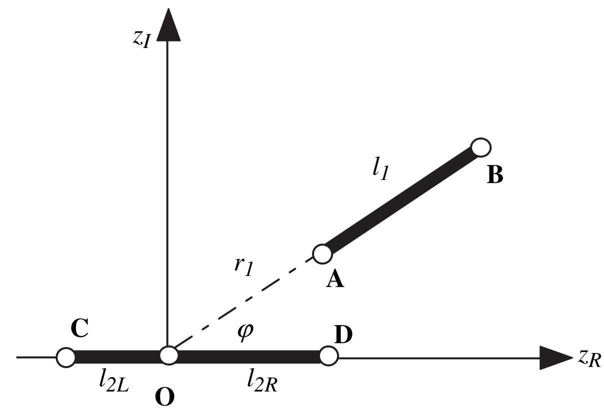

Fig. 2. The supplementary case.

$$k_{\text{L in}} = \sqrt{\frac{r_1^{(\pi/(\pi-\varphi))}\left((r_1+l_1)^{(\pi/(\pi-\varphi))} + l_{2\text{L}}^{(\pi/(\pi-\varphi))}\right)}{(r_1+l_1)^{(\pi/(\pi-\varphi))}\left(r_1^{(\pi/(\pi-\varphi))} + l_{2\text{L}}^{(\pi/(\pi-\varphi))}\right)}} \tag{10}$$

and

$$k_{\text{L out}} = \sqrt{\frac{r_1^{(\pi/(\pi+\varphi))}\left((r_1+l_1)^{(\pi/(\pi+\varphi))} + l_{2\text{L}}^{(\pi/(\pi+\varphi))}\right)}{(r_1+l_1)^{(\pi/(\pi+\varphi))}\left(r_1^{(\pi/(\pi+\varphi))} + l_{2\text{L}}^{(\pi/(\pi+\varphi))}\right)}} \tag{11}$$

The capacitor of plates AB and CD is equivalent to the combination of the above two capacitors connected in parallel. Therefore, the total capacitance per unit longitudinal length is

$$\begin{aligned} C &= C_{\text{R}} + C_{\text{L}} \\ &= \varepsilon_0 \left(\frac{K'(k_{\text{R in}})}{K(k_{\text{R in}})} + \frac{K'(k_{\text{R out}})}{K(k_{\text{R out}})} + \frac{K'(k_{\text{L in}})}{K(k_{\text{L in}})} + \frac{K'(k_{\text{L out}})}{K(k_{\text{L out}})} \right) \end{aligned} \tag{12}$$

4. Perpendicular electrode plates

We now turn our attention to the case of the capacitance of two perpendicular electrode plates. For this special case φ is right angle, as depicted in Fig. 3.

Letting $\varphi = (\pi/2)$ in Eqs. (7), (8), (10) and (11) leads to

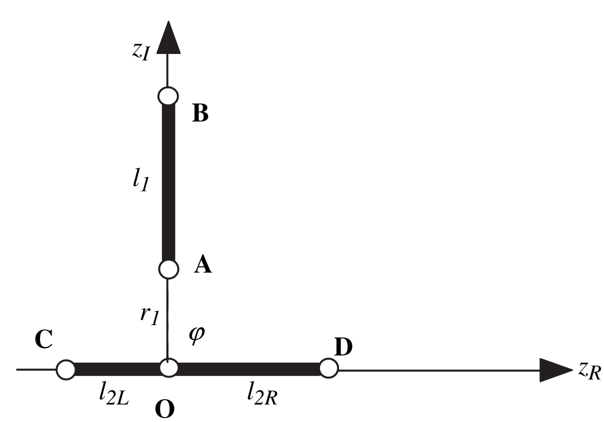

Fig. 3. Two perpendicular electrode plates.

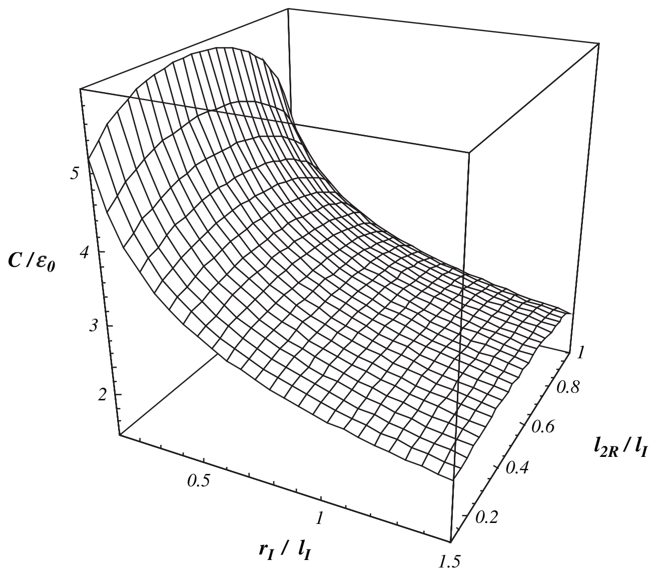

Fig. 4. The ratio (C/ε_0) vs. l_{2R}/l_1 and (r_1/l_1) for the case of perpendicular electrodes $(l_{2R}+l_{2L}=l_1)$.

$$k_{\mathrm{R\,in}} = \frac{r_1}{r_1+l_1}\sqrt{\frac{(r_1+l_1)^2+l_{2R}^2}{r_1^2+l_{2R}^2}} \tag{13}$$

$$k_{\mathrm{R\,out}} = \sqrt{\frac{r_1^{(2/3)}\left((r_1+l_1)^{(2/3)}+l_{2R}^{(2/3)}\right)}{(r_1+l_1)^{(2/3)}\left(r_1^{(2/3)}+l_{2R}^{(2/3)}\right)}} \tag{14}$$

$$k_{\mathrm{L\,in}} = \frac{r_1}{r_1+l_1}\sqrt{\frac{(r_1+l_1)^2+l_{2L}^2}{r_1^2+l_{2L}^2}} \tag{15}$$

and

$$k_{\mathrm{L\,out}} = \sqrt{\frac{r_1^{(2/3)}\left((r_1+l_1)^{(2/3)}+l_{2L}^{(2/3)}\right)}{(r_1+l_1)^{(2/3)}\left(r_1^{(2/3)}+l_{2L}^{(2/3)}\right)}} \tag{16}$$

Obviously, the capacitance C of two perpendicular plates is affected by variables l_{2R}, l_{2L}, l_1 and r_1. Under the condition $l_{2R}+l_{2L}=l_1$ which is the common case $l_2=l_1$, the value of C normalized to ε_0 as a function of relative dimensions (l_{2R}/l_1) and (r_1/l_1) is plotted in Fig. 4.

Taking a close look at Eqs. (12)–(16), we find that the function manifold of C has a symmetry about $l_{2R}=l_{2L}$ and reaches its maximum value at that position. Fig. 4 does demonstrate that apparent characteristic.

A more special case occurs when the two points O and C in Fig. 3 coincide with each other, i.e. when $l_{2L}=0$. Then Eqs. (15) and (16) give

$$k_{\mathrm{L\,in}} = k_{\mathrm{L\,out}} = 1 \tag{17}$$

Since $K(k)\to\infty$, but $K'(k)\to(\pi/2)$ as $k\to 1$ [4], C_L in Eq. (12) becomes

$$C_L = \varepsilon_0\left(\frac{K'(k_{\mathrm{L\,in}})}{K(k_{\mathrm{L\,in}})}+\frac{K'(k_{\mathrm{L\,out}})}{K(k_{\mathrm{L\,out}})}\right) = 0 \tag{18}$$

Hence the total capacitance becomes just

$$C = C_R = \varepsilon_0\left(\frac{K'(k_{\mathrm{R\,in}})}{K(k_{\mathrm{R\,in}})}+\frac{K'(k_{\mathrm{R\,out}})}{K(k_{\mathrm{R\,out}})}\right) \tag{19}$$

The corresponding plot of (C/ε_0) vs. (r_1/l_1) under the condition $l_2=l_1$ can be found on the function surface in Fig. 4, where we let (l_{2R}/l_2) be equal to 0 or 1.

5. Conclusion

For an inclined plate capacitor, the capacitance expression for the case where the intersection line of the two planes containing the electrodes lies outside the electrodes provides a solid foundation for calculating the capacitance in the more general case. As the intersection line lies on one of the electrode plates, the capacitance can be obtained conveniently so long as we treat the system as two capacitors connected in parallel.

Acknowledgment

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