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Short communication

Non-linear oscillation of the fluid in a plate capacitor

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Abstract

Employing the principle of virtual work the equation of motion for the liquid between two electrode walls of a charged parallel plate capacitor is established. Analysis in phase plane stands out the motion is non-linear oscillation. The phase orbit is determined then the oscillation period is expressed with elliptic integrations. Besides the intrinsic parameters in the system, the initial disturbing conditions affect the oscillation period, which readily demonstrates the non-linearity of oscillation. The characters of the period are discussed and the approximate computation is presented. Further, the threshold of control voltage under which the results can be still used for the capacitor with an inclined plate is achieved.

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Keywords: Plate capacitor; The principle of virtual work; Non-linear oscillation; Elliptic integration; Control voltage

1. Introduction

In engineering, most circuits comprise a number of capacitors. For a capacitor, it is often of significance to maintain high conductivity and high dielectric voltage at high temperatures for a long time. It can be done by putting the electrodes in a liquid shoes where water is used as a main solvent and some acid or salt as a solute [1]. How to control the capacitance of such

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capacitor sensitively in engineering is a problem to be considered [2]. Besides, to measure the permittivity of a liquid, a simple and convenient way is to set a capacitor in that liquid then test the capacitance as the liquid locates between two electrodes [3].

The typical electrodes are parallel plates. If a parallel plate capacitor is partly filled with liquid, the capacitance consequentially varies with the motion of the liquid. The non-linearity from capacitors is an essential factor to control the motion in some engineering advices such as micro-electro-mechanical system [4]. The prime motivation of this paper is to study the non-linear oscillation of the liquid between the two electrode plates. With the aid of the principle of virtual work the equation of motion is established. The oscillation solution is obtained and the oscillation period is given with elliptic integrations. The non-linear characters of the oscillation are stated. For convenience of computation, the approximate calculation for the period is stood out. More generally, the results can be used for a capacitor with an inclined plate provided the control voltage is under a threshold value.

2. Motion equation

Fig. 1 shows a charged parallel plate capacitor. Two plates having width b and separation distance d are dipped in a liquid. The plate height above the liquid surface is l. The voltage across the two electrode plates is U. The liquid density is ρ.

Due to the electric force the liquid between the two plates of the capacitor raises up. The elevation of the liquid inside the capacitor is denoted as h. The mass of the liquid raised between the two plates is

$$m = \rho b d h \tag{1}$$

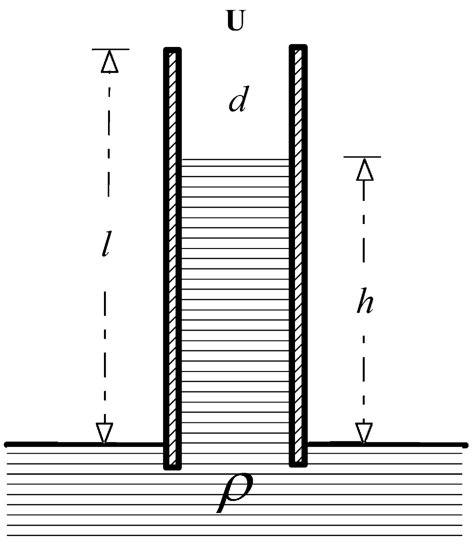

Fig. 1. The liquid in a charged parallel plate capacitor.

The system above the liquid surface could be treated as two capacitors connected in parallel. So the capacitance is

$$C = \frac{\varepsilon_0 b(l-h)}{d} + \frac{\varepsilon b h}{d} \tag{2}$$

where ε_0 and ε denotes permittivity of the air and the liquid respectively. The electric potential energy of the system has the form

$$W = \frac{1}{2}CU^2 = \frac{bU^2}{2d}[\varepsilon_0(l-h) + \varepsilon h] \tag{3}$$

According to the principle of the virtual work [5], the electric force upwards exerting on the liquid is

$$F = \left.\frac{\partial W}{\partial h}\right|_{U=c.} = \frac{1}{2}U^2\frac{\partial C}{\partial h} = \frac{bU^2}{2d}(\varepsilon - \varepsilon_0) \tag{4}$$

Therefore, the equation of motion for the liquid inside the capacitor is

$$F - mg = \frac{\mathrm{d}(mv)}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} \tag{5}$$

Substituting Eqs. (1) and (4) into Eq. (5) leads to

$$\frac{U^2}{2\rho d^2}(\varepsilon - \varepsilon_0) - gh = h\frac{\mathrm{d}v}{\mathrm{d}t} + v^2 \tag{6}$$

where we have used

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v \tag{7}$$

Under the equilibrium condition of

$$v = 0; \quad \frac{\mathrm{d}v}{\mathrm{d}t} = 0 \tag{8}$$

we derive the raising elevation of liquid, from Eq. (6)

$$h = \frac{U^2}{2\rho g d^2}(\varepsilon - \varepsilon_0) = h_0 \tag{9}$$

Combining Eqs. (6), (8) and (9) we arrive at

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\left(\frac{h_0}{h} - 1\right) - \frac{v^2}{h} \tag{10}$$

3. Equilibrium point in phase plane

Note

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}h} = v\frac{\mathrm{d}v}{\mathrm{d}h} \tag{11}$$

Then Eq. (10) could be rewritten as

$$g(h_0 - h) = hv\frac{dv}{dh} + v^2 \tag{12}$$

Obviously, Eq. (12) is a non-linear differential equation which determines the path of motion in the phase plane. The equilibrium point in the phase plane is $(h_0, 0)$. To show the property of the equilibrium point, the dynamics system of Eqs. (7) and (10) is examined. Its Jacobian matrix is [6]

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{1}{h^2}(gh_0 - v^2) & -\frac{2v}{h} \end{bmatrix}_{(h=h_0, v=0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{h_0} & 0 \end{bmatrix} \tag{13}$$

Hence, the eigenvalue equation is

$$|J - \lambda E| = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{h_0} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{h_0} = 0 \tag{14}$$

Since $h_0 > 0$, the eigenvalues are pure imagines

$$\lambda = \pm i\sqrt{\frac{g}{h_0}} \tag{15}$$

Thus, the equilibrium point $(h_0, 0)$ is a center [7]. The phase orbit of Eq. (12) is closed. The system has periodical oscillation around the equilibrium point.

4. Analytic solution

Taking a transformation

$$u = v^2 \quad \text{and} \quad r = \ln h \tag{16}$$

Eq. (12) then becomes

$$\frac{du}{dr} + 2u = 2g(h_0 - e^r) \tag{17}$$

which is a non-homogeneous linear differential equation. It is not difficult to obtain the solution

$$u = ce^{-2r} + g\left(h_0 - \frac{2}{3}e^r\right) \tag{18}$$

where c is an integration constant to be determined. Combination of Eqs. (16) and (18) yields

$$v^2 = \frac{c}{h^2} + g\left(h_0 - \frac{2}{3}h\right) \tag{19}$$

Suppose the initial conditions are

$$h|_{t=0} = h_0 \quad \text{and} \quad v|_{t=0} = v_0 \tag{20}$$

Substituting Eqs. (20) into Eq. (19) we have

$$c = h_0^2\left(v_0^2 - \frac{1}{3}gh_0\right) \tag{21}$$

Thus, Eq. (19) becomes

$$v^2 = \frac{-2gh^3 + 3gh_0h^2 + 3h_0^2v_0^2 - gh_0^3}{3h^2} \tag{22}$$

The phase orbit is represented by Eq. (22). Under the condition of $v_0^2 = gh_0/4$ it is depicted in Fig. 2. It is seen that the maximum of speed is at $h < h_0$ rather than $h = h_0$.

Due to the symmetry of the phase orbit, our attention is focused on the half orbit in the upper phase plane. From Eq. (22) we derive

$$v = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\sqrt{-2gh^3 + 3gh_0h^2 + 3h_0^2v_0^2 - gh_0^3}}{\sqrt{3}h} \tag{23}$$

Therefore

$$\frac{h\,\mathrm{d}h}{\sqrt{-(h-h_1)(h-h_2)(h-h_3)}} = \sqrt{\frac{2g}{3}}\mathrm{d}t \tag{24}$$

where h_1, h_2 and h_3 are the tree roots of the following equation:

$$h^3 - \frac{3}{2}h_0h^2 - \frac{1}{2}h_0^2\left(\frac{3v_0^2}{g} - h_0\right) = 0 \tag{25}$$

Introducing a transformation

$$h = x + \frac{1}{2}h_0 \tag{26}$$

Eq. (25) becomes

$$x^3 + px + q = 0 \tag{27}$$

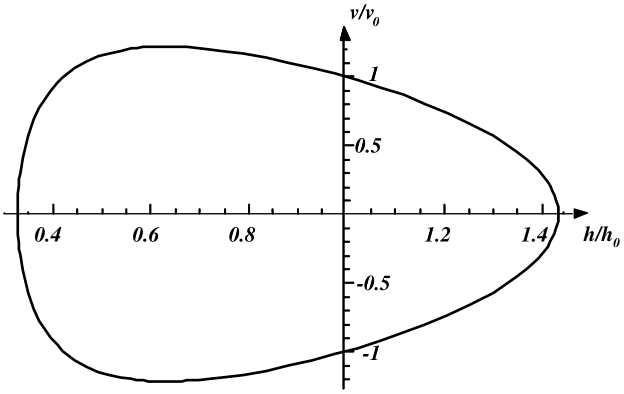

Fig. 2. The phase orbit under the condition of $v_0^2 = gh_0/4$.

where

$$p = -\frac{3}{4}h_0^2; \quad q = -\frac{1}{4}h_0^2\left(\frac{6v_0^2}{g} - h_0\right) \tag{28}$$

Under the condition of

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0 \tag{29}$$

Eq. (27) has three different real roots. The solution of the above equation is

$$0 < v_0 < \sqrt{\frac{gh_0}{3}} \tag{30}$$

Considering the relations between the roots and the coefficients in Eq. (25), the tree roots must be

$$h_1 > h_2 > 0; \quad h_3 < 0 \tag{31}$$

Integrating Eq. (24) under the initial conditions of Eqs. (20) leads to

$$\int_{h_0}^{h} \frac{h\,\mathrm{d}h}{\sqrt{-(h-h_1)(h-h_2)(h-h_3)}} = \sqrt{\frac{2g}{3}}t \tag{32}$$

The integration result on the left side of Eq. (32) could be given by $F(\varphi,k)$ and $E(\varphi,k)$, the elliptic integration of the first and second kind [8]. Since $h_2 < h < h_1$, from Eq. (32) we obtain

$$2\sqrt{\frac{1}{h_1-h_3}}[(h_1-h_3)(E(\varphi_0,k)-E(\varphi,k)) + h_3(F(\varphi_0,k)-F(\varphi,k))] = \sqrt{\frac{2g}{3}}t \tag{33}$$

where

$$\varphi = \arcsin\sqrt{\frac{h_1-h}{h_1-h_2}} \tag{34}$$

$$\varphi_0 = \arcsin\sqrt{\frac{h_1-h_0}{h_1-h_2}} \tag{35}$$

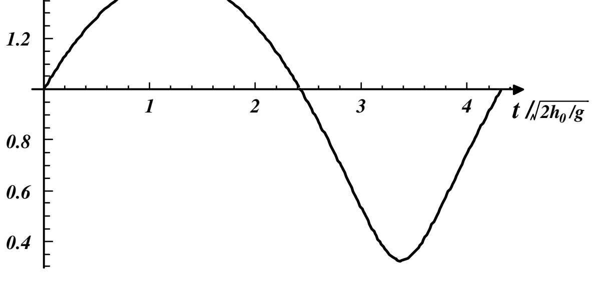

Fig. 3. The function curve of h/h_0 versus $t/\sqrt{2h_0/g}$.

and

$$k = \sqrt{\frac{h_1 - h_2}{h_1 - h_3}} \tag{36}$$

Eq. (33) indicates the elevation of liquid surface in the capacitor oscillates with time. The oscillation curve is shown in Fig. 3.

5. Oscillation period and the discussion

Now the oscillation period of the liquid surface between the two conducting plates of the capacitor is calculated. Considering the symmetry of the phase orbit in Fig. 2, the oscillation period is written as

$$T = \sqrt{\frac{6}{g}} \int_{h_2}^{h_1} \frac{h\,\mathrm{d}h}{\sqrt{-(h-h_1)(h-h_2)(h-h_3)}} \tag{37}$$

Using the result of Eq. (33) we have

$$T = 2\sqrt{\frac{6}{g}}\left[\sqrt{h_1 - h_3}E(k) + \frac{h_3}{\sqrt{h_1 - h_3}}K(k)\right] \tag{38}$$

where $K(k)$ and $E(k)$ are the complete elliptic integration of the first and second kind respectively [9].

The three roots h_1, h_2 and h_3 in Eq. (25) are

$$\begin{aligned} h_1 = \frac{h_0}{2}\Bigg[1 &+ \left(\frac{-1 - \mathrm{i}\sqrt{3}}{2}\right)\sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 - \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1} \\ &+ \left(\frac{-1 + \mathrm{i}\sqrt{3}}{2}\right)\sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 + \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1}\Bigg] \end{aligned} \tag{39}$$

$$\begin{aligned} h_2 = \frac{h_0}{2}\Bigg[1 &+ \left(\frac{-1 + \mathrm{i}\sqrt{3}}{2}\right)\sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 - \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1} \\ &+ \left(\frac{-1 - \mathrm{i}\sqrt{3}}{2}\right)\sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 + \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1}\Bigg] \end{aligned} \tag{40}$$

$$h_3 = \frac{h_0}{2}\left[1 + \sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 - \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1} + \sqrt[3]{\frac{6v_0^2}{h_0 g}\left(1 + \sqrt{1 - \frac{gh_0}{3v_0^2}}\right) - 1}\right] \tag{41}$$

All of them are real under the condition of Eq. (30). Obviously, they are determined by h_0 and v_0. Eq. (38) turns out the oscillation period is the function of not only the intrinsic parameters in the

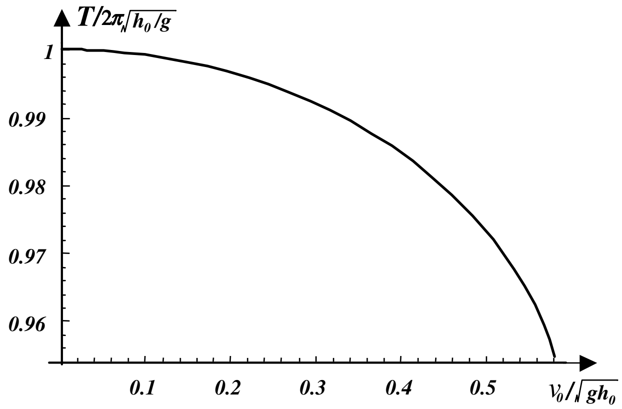

Fig. 4. The function curve of $\frac{T}{2\pi\sqrt{h_0/g}}$ versus $\frac{v_0}{\sqrt{gh_0}}$.

system but also the initial conditions, which is the essential character of non-linear oscillation [10]. The function curve of T versus and v_0 is plotted in Fig. 4.

It should be possible to following the procedures outlined bellow to compute the approximate value of T. Expanding $K(k)$ and $E(k)$ as

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(2n)!(2n)!}{2^{4n}(n!)^4}k^{2n} \tag{42}$$

$$E(k) = \frac{\pi}{2}\left[1 - \sum_{n=1}^{\infty}\frac{(2n-2)!(2n)!}{2^{4n-1}(n-1)!(n!)^3}k^{2n}\right] \tag{43}$$

From $h_1 > h_2$ and $h_3 < 0$ we know the modulus k in Eq. (36) is much less than 1. So merely truncating two items in Eqs. (39) and (40) we approximately have

$$K(k) = \frac{\pi}{2}\left[1 + \frac{h_1 - h_2}{4(h_1 - h_3)}\right] \tag{44}$$

$$E(k) = \frac{\pi}{2}\left[1 - \frac{h_1 - h_2}{4(h_1 - h_3)}\right] \tag{45}$$

Substituting Eqs. (44) and (45) into Eq. (38) the approximate expression for the oscillation period could be expressed as

$$T = \pi h_1\sqrt{\frac{6}{g(h_1 - h_3)}}\left[1 - \frac{(h_1 - h_2)(h_1 - 2h_3)}{4h_1(h_1 - h_3)}\right] \tag{46}$$

6. Effect of inclination of plate

In engineering practice, the two electrode plates of a capacitor are not strictly parallel to each other. Referring Fig. 5, suppose a plate is inclined with a small slope angle α and d represents the distance between the two button ends.

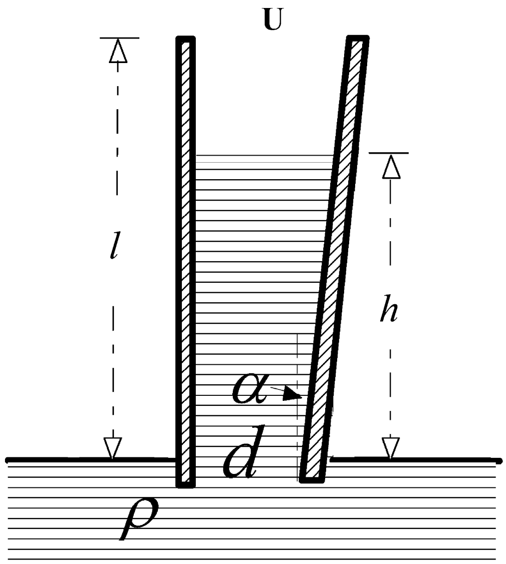

Fig. 5. In a charged capacitor with an inclined plate.

Modifications should be made in this case. First, the mass of the liquid raised between the two plates in Eq. (1) is replaced by

$$m = \rho b d\left(1 + \frac{h\alpha}{2d}\right)h \tag{47}$$

where we have used $\tan\alpha = \alpha$ due to the small angle α.

The capacitance for an inclined plate capacitor may be computed as [11]

$$C = \varepsilon_0 \frac{b}{\alpha}\ln\left(1 + \frac{l\alpha}{d}\right) \tag{48}$$

Eq. (2) should be modified as

$$C = \varepsilon_0 \frac{b}{\alpha}\ln\left(1 + \frac{(l-h)\alpha}{d + h\alpha}\right) + \varepsilon\frac{b}{\alpha}\ln\left(1 + \frac{h\alpha}{d}\right) \tag{49}$$

Then Eq. (4) becomes

$$F = \frac{1}{2}U^2\frac{\partial C}{\partial h} = \frac{bU^2}{2(d + h\alpha)}(\varepsilon - \varepsilon_0) \tag{50}$$

From Eq. (5) we have

$$\frac{U^2}{2\rho d^2\left(1 + \frac{h\alpha}{d}\right)}(\varepsilon - \varepsilon_0) - g\left(1 + \frac{h\alpha}{2d}\right)h = \left(1 + \frac{h\alpha}{2d}\right)h\frac{\mathrm{d}v}{\mathrm{d}t} + \left(1 + \frac{h\alpha}{d}\right)v^2 \tag{51}$$

Integration of Eq. (51) gives

$$v^2 = \frac{c}{h^2\left(2+\frac{\alpha}{d}h\right)^2} - \frac{U^2(\varepsilon-\varepsilon_0)\ln\left(1+\frac{\alpha}{d}h\right)}{\rho d^2\frac{\alpha}{d}h\left(2+\frac{\alpha}{d}h\right)} + \frac{2gh\left(20+15\frac{\alpha}{d}h+3\frac{\alpha^2}{d^2}h^2\right)}{15\left(2+\frac{\alpha}{d}h\right)^2} \tag{52}$$

Comparing Eqs. (51) and (52) with Eqs. (6) and (19), we can find the impact of inclination angle a. However, under the condition of

$$\frac{h\alpha}{d} \ll 1 \tag{53}$$

the results for parallel plate capacitor are still effective here. In Eq. (53), it is available to substitute h with h_0 for practice. Employing Eq. (9), we find the condition for applied voltage cross the two electrode plates is

$$U < d\sqrt{\frac{2\rho g d}{(\varepsilon-\varepsilon_0)\alpha}} \tag{54}$$

7. Conclusion

The liquid partly filled in the space between the walls of a charged parallel plate capacitor is in equilibrium due to the balance of the electrostatic field force and the gravitational force. If the equilibrium state is disturbed, the liquid undergoes a non-linear oscillation. The principle of virtual work may help us effectively to set up the oscillation equation. Treating this non-linear differential equation in the phase plane is convenient. After obtaining the phase orbit the expression of the oscillation period is represented with elliptic integrations. The result that the oscillation period is relative to both the intrinsic parameters in the system and the initial conditions reflects non-linearity of the oscillation. Under the threshold value of control voltage the results can be still used for the capacitor with an inclined plate.

Acknowledgement

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