III. Classical Field Theory and Canonical Quantization

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In the previous chapter, we saw how to construct Lorentz covariant free field equations using group theoretic techniques applied to the Lorentz group. However, we still lack a dynamical principle which allows for the construction of the Poincaré generators including the Hamiltonian in terms of the operator fields nor a means to give a physical interpretation to the solution to the these field equations. In this chapter, we begin to address these issues. The canonical quantization approach to constructing a quantum field theory starts with a classical field theory whose dynamics is governed by classical field equations which are obtained by applying (a generalized) Hamilton's principle to an action functional of a Lagrange density (Lagrangian). The classical fields are then promoted to become quantum operator fields satisfying certain equal time canonical commutation or anti-commutation relations. Using this canonical formalism, the various symmetry generators, including the Hamiltonian, can be identified and constructed in terms of the field operators. As we shall see, a major advantage of basing a dynamical principle on an action is that it is Lorentz invariant while the Hamiltonian is not.

Hamilton's Principle and the Euler-Lagrange Equations

The classical Lagrangian (density) characterizing the underlying dynamics is assumed to be a function of the classical fields, $\varphi_s(\vec{r},t)$, and their first space-time derivatives, $\partial_\mu\varphi_s(\vec{r},t)$:

$$\mathcal{L} = \mathcal{L}(\varphi_s(\vec{r},t), \partial_\mu\varphi_s(\vec{r},t)) \,. \tag{1}$$

The action functional for the classical field theory is then defined as

$$S[\varphi] = \int_\Omega d^4x \mathcal{L}(\varphi_s(t,\vec{r}), \partial_\mu\varphi_s(t,\vec{r}))\,, \tag{2}$$

where Ω is the space-time volume. Here the subscript s is used to distinguish different fields. The (generalized) Hamilton principle states that the action is stationary with respect to variations of classical field, $\delta\varphi_s(\vec{r},t)$, which vanish on the boundary of Ω but are otherwise arbitrary.[1]

[1]Rather than employing this generalized Hamilton's principle, one could divide 3-space

Using functional derivatives, Hamilton's principle takes the form

$$
\begin{aligned}
0 &= \delta\varphi_r(x)\frac{\delta S[\varphi]}{\delta\varphi_r(x)} \\
&= \delta\varphi_r(x)\int_\Omega d^4y\left(\frac{\partial\mathcal{L}}{\partial\varphi_s(y)}\frac{\delta\varphi_s(y)}{\delta\varphi_r(x)}+\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi_s(y)}\frac{\delta\partial^y_\mu\varphi_s(y)}{\delta\varphi_r(x)}\right) \\
&= \delta\varphi_r(x)\int_\Omega d^4y\left(\frac{\partial\mathcal{L}}{\partial\varphi_s(y)}\frac{\delta\varphi_s(y)}{\delta\varphi_r(x)}+\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi_s(y)}\partial^y_\mu\frac{\delta\varphi_s(y)}{\delta\varphi_r(x)}\right) \\
&= \delta\varphi_r(x)\int_\Omega d^4y\left(\frac{\partial\mathcal{L}}{\partial\varphi_s(y)}-\partial^y_\mu\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi_s(y)}\right)\frac{\delta\varphi_s(y)}{\delta\varphi_r(x)} \\
&\quad+\delta\varphi_r(x)\int_\Omega d^4y\partial^y_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi(y)}\frac{\delta\varphi_s(y)}{\delta\varphi_r(x)}\right) \\
&= \delta\varphi_r(x)\int_\Omega d^4y\left(\frac{\partial\mathcal{L}}{\partial\varphi_s(y)}-\partial^y_\mu\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi_s(y)}\right)\delta_{rs}\delta^4(x-y) \\
&\quad+\delta\varphi_r(x)\int_\Omega d^4y\partial^y_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi(y)}\delta_{rs}\delta^4(x-y)\right) \\
&= \delta\varphi_r(x)\int_\Omega d^4y\left(\frac{\partial\mathcal{L}}{\partial\varphi_s(y)}-\partial^y_\mu\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi_s(y)}\right)\delta_{rs}\delta^4(x-y) \\
&\quad+\delta\varphi_r(x)n_\mu\frac{\partial\mathcal{L}}{\partial\partial^y_\mu\varphi(y)}\delta_{rs}\delta^4(x-y)|_{y\in\Sigma} \\
&= \delta\varphi_r(x)\left(\frac{\partial\mathcal{L}}{\partial\varphi_r(x)}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r(x)}\right) \qquad (3)
\end{aligned}
$$

In obtaining the penultimate equality, we used the divergence theorem with Σ the boundary of Ω having unit (space-like) normal n^μ ($n^\mu n_\mu = 1$), where the last equality follows since the field variations vanish on Σ.
Since, other than being constrained to vanish on the boundary, each variation

into discrete cells (a lattice) and decompose the field which depends on the continuum of spatial variables into a discretely numerated number of degrees of freedom, one associated with each lattice cell, all depending only on time. The Lagrange function and its associated action can then be constructed. The ordinary Hamilton's principle which states that the action is stationary with respect to variations which vanish at the temporal endpoints, but are otherwise arbitrary, can then be applied. The resultant Euler-Lagrange equations so obtained reproduces the Euler-Lagrange equations obtained with the generalized Hamilton's principle once the continuum limit is taken.

$\delta\varphi_r$ is arbitrary within Ω, it follows that

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi_r(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r(x)} \tag{4}$$

which are the Euler-Lagrange equations.

The canonically conjugate momentum classical field, $\pi_s(x)$, is defined as

$$\pi_s(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_s(x)} \, . \tag{5}$$

Noether's Theorem and Symmetries

Poincaré invariance restricts the form of the field equations and consequently the form of the actions from which they are derived. In addition, there may be additional symmetries of the system which will also restrict the form of the action. We shall formalize these restrictions and also establish the connection between the presence of a continuous symmetry, the existence of a conserved current and its associated time independent charge. This is embodied in Noether's theorem.

Let $\varphi_r(x)$ denote a multicomponent field labeled by r. A transformation which changes the space-time point (eg. translations, Lorentz transformations) is referred to as a space-time transformation while a transformation which leaves the space-time point unchanged is referred to as an internal transformation.

Consider the transformation parametrized by Υ which is a symmetry of the action and is continuously connected to the identity. The classical field theory transformation law is

$$\varphi_r(x) \to \varphi_r^\Upsilon(x^\Upsilon) \;\; = \;\; \mathcal{D}_{rs}(\Upsilon)\varphi_s(x) \tag{6}$$

so that

$$\varphi^\Upsilon(x) \;\; = \;\; \mathcal{D}_{rs}(\Upsilon)\varphi_s(x^{\Upsilon^{-1}}) \tag{7}$$

where $\mathcal{D}(\Upsilon)$ is a matrix representation of the transformation. Here x^Υ is the transformed space-time point (if the transformation is internal, then

$x_\mu^\Upsilon = x_\mu$).

For infinitesimal transformations, parametrized by real υ_A (cryptically $\Upsilon = 1 + \upsilon$) with $|\upsilon_A| << 1$, the space-time point changes as

$$\Delta(\upsilon)x^\mu = (x^\upsilon)^\mu - x^\mu = \sum_A \upsilon_A \Delta_A x^\mu \tag{8}$$

If the transformation is internal transformation, then $\Delta(\upsilon)x^\mu = 0$.

For infinitesimal transformations, the matrix representation can be written

$$\mathcal{D}_{rs}(I + \upsilon) \quad = \quad \delta_{rs} + i \sum_A \upsilon_A (d^A)_{rs} \tag{9}$$

with $d_{rs}^{(A)}$ being υ independent numbers. The intrinsic transformation of the field is

$$\varphi_r(x) \to \varphi_r(x) + \delta(\upsilon)\varphi_r(x) \tag{10}$$

where

$$\begin{aligned}
\delta(\upsilon)\varphi_r(x) &= \varphi_r^\upsilon(x) - \varphi_r(x) \\
&= \mathcal{D}_{rs}(I + \upsilon)\varphi_s(x - \Delta(\upsilon)x) - \varphi(x) \\
&= \left(\delta_{rs} + i \sum_A \upsilon_A (d^A)_{rs}\right) \varphi_s(x - \Delta(\upsilon)x) - \varphi_r(x) \\
&= -\Delta x^\mu(\upsilon)\partial_\mu \varphi_r(x) + i \sum_A \upsilon_A (d^A)_{rs} \varphi_s(x) \\
&= \sum_A \upsilon_A \left(-\delta_{rs}\Delta_A x^\mu \partial_\mu + i(d^A)_{rs}\right) \varphi_s(x) \\
&= \sum_A \upsilon_A \delta_A \varphi_r(x)
\end{aligned} \tag{11}$$

which identifies

$$\delta_A \varphi_r(x) \quad = \quad \left(-\delta_{rs}\Delta_A x^\mu \partial_\mu + i(d^A)_{rs}\right) \varphi_s(x) \tag{12}$$

while

$$\delta(\upsilon)\partial_\mu \varphi_r(x) \quad = \quad \partial_\mu \Big(\delta(\upsilon)\varphi_r(x)\Big) \quad \Rightarrow \quad \delta_A \partial_\mu \varphi_r(x) = \partial_\mu \delta_A \varphi_r(x) \tag{13}$$

Since $\mathcal{L} = \mathcal{L}(\varphi_r, \partial_\mu \varphi_r)$, it follows that

$$\begin{aligned}
\delta(\upsilon)\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi_r}\delta(\upsilon)\varphi_r + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\delta(\upsilon)(\partial_\mu \varphi_r) \\
&= \frac{\partial \mathcal{L}}{\partial \varphi_r}\delta(\upsilon)\varphi_r + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\partial_\mu \delta(\upsilon)\varphi_r \\
&= \frac{\partial \mathcal{L}}{\partial \varphi_r}\delta(\upsilon)\varphi_r + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\delta(\upsilon)\varphi_r\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\right)\delta(\upsilon)\varphi_r \\
&= \left(\frac{\partial \mathcal{L}}{\partial \varphi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\right)\delta(\upsilon)\varphi_r + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\delta(\upsilon)\varphi_r\right) \\
&= \frac{\delta S}{\delta \varphi_r}\delta(\upsilon)\varphi_r + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r}\delta(\upsilon)\varphi_r\right) \qquad (14)
\end{aligned}$$

or

$$\frac{\delta S}{\delta \varphi_r}\delta(\upsilon)\varphi_r = \delta(\upsilon)\mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r(x)}\delta(\upsilon)\varphi_r(x)\right) . \qquad (15)$$

which is an algebraic identity and constitutes Noether's theorem. This identity holds for an field transformation where it is a symmetry of action or not.

If the transformation is a symmetry of the classical action so that the action is invariant under the transformation,,[2] then the Lagrangian is either invariant or transforms into a total divergence. As we shall see, for space-time symmetries, the Lagrangian transforms into a total divergence so that

$$\delta(\upsilon)\mathcal{L} = \partial_\mu \Lambda^\mu(\upsilon) \qquad (18)$$

[2]The action is invariant provided it algebraically (independent of field equations) satisfies

$$\delta(\upsilon)S[\varphi] = 0 \qquad (16)$$

where

$$\begin{aligned}
\delta(\upsilon) &= \int d^4x \, (\delta(\upsilon)\varphi_r(x)) \, \frac{\delta}{\delta \varphi_r(x)} \\
&= \int d^4x \left(\sum_A \upsilon_A \left(-\delta_{rs}\Delta_A x^\mu \partial_\mu + i(d^A)_{rs}\right) \varphi_s(x)\right) \frac{\delta}{\delta \varphi_r(x)} \qquad (17)
\end{aligned}$$

while for internal symmetries, it is invariant ($\Lambda^\mu(\upsilon) = 0$). In either case, the RHS of Noether's theorem is the divergence of a current

$$J^\mu(\upsilon) \;=\; \Lambda^\mu(\upsilon) - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r(x)} \delta(\upsilon)\varphi_r(x) \sum_A \upsilon_A J^\mu_A(x) \tag{19}$$

and Noether's theorem takes the form

$$\frac{\delta S[\varphi]}{\delta \varphi_r(x)} \delta(\upsilon)\varphi_r(x) \;=\; \partial_\mu J^\mu(\upsilon) \tag{20}$$

or equivalently

$$\frac{\delta S[\varphi]}{\delta \varphi_r(x)} \delta_A \varphi_r(x) \;=\; \partial_\mu J^\mu_A(x) \tag{21}$$

After application of the Euler-Lagrange field equations: $\frac{\delta S[\varphi]}{\delta \varphi_r(x)} = 0$, we see that that the current $J^\mu(\upsilon)$ or $J^\mu_A(x)$ is conserved: $\partial_\mu J^\mu(\upsilon) = 0 = \partial_\mu J^\mu_A(x)$.

For such a conserved current, we can define charges

$$Q(\upsilon) \;=\; \int d^3x J^0(\upsilon) = \sum_A \upsilon_A Q_A \tag{22}$$

such that

$$\frac{dQ(\upsilon)}{dt} \;=\; \int d^3x \partial_0 J^0(\upsilon) = -\int d^3x \partial_i J^i(\upsilon) = 0 \;\;\Rightarrow\;\; \frac{dQ_A}{dt} = 0 \tag{23}$$

where we have used the divergence theorem and set the surface terms at infinity to zero.
Thus, after application of the Euler-Lagrange field equations, the Noether constructed charges $Q(\upsilon)$ or Q_A are time independent. These charges will be identified with the generators of the associated symmetries.

Poincarè Symmetries

We now apply Noether's theorem to the space-time symmetries associated with the Poincaré transformations.

Space-Time Translations: Canonical Energy-Momentum Tensor

The infinitesimal space-time translation is given by

$$\Delta(\epsilon)x^\mu \;\; = \;\; \epsilon^\mu \tag{24}$$

with $|\epsilon^\mu| << 1$. Using $\Delta(\epsilon)x^\mu = \sum_A \upsilon_A \Delta_A x^\mu = \epsilon^\mu = \epsilon^\nu g^\mu_\nu$, we can identify (c.f. Eq.(8)) $A \to \mu$, $\upsilon_A \to \epsilon^\mu$ and $\Delta_A x^\mu \to g^\mu_\nu$.
The classical field infinitesimal intrinsic variation is

$$\delta(\epsilon)\varphi_r(x) \;\; = \;\; \varphi_r(x-\epsilon) - \varphi_r(x) = -\epsilon^\mu \partial_\mu \varphi_r(x) \tag{25}$$

Similarly, for a translation invariant action[3], the Lagrangian transforms as

$$\delta(\epsilon)\mathcal{L} \;\; = \;\; -\epsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu(-\epsilon_\nu g^{\mu\nu}\mathcal{L}) \tag{28}$$

so that

$$\Lambda^\mu(\epsilon) \;\; = \;\; -\epsilon_\nu g^{\mu\nu}\mathcal{L} \tag{29}$$

Combining these results gives the conserved Noether current

$$\begin{aligned} J^\mu(\epsilon) &= \Lambda^\mu(\epsilon) - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r(x)}\Big(\delta(\epsilon)\varphi_r(x)\Big) \\ &= -\epsilon_\nu \left(g^{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r(x)} \partial_\nu \varphi_r(x) \right) \\ &= -\epsilon_\nu T^{\mu\nu} \end{aligned} \tag{30}$$

where we have introduced the canonical energy-momentum tensor as

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r} \partial^\nu \varphi_r \tag{31}$$

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[3]The action is invariant under space-time translations provided it algebraically (i.e. not using field equations) satisfies

$$\delta(\epsilon)S[\phi] = 0\,. \tag{26}$$

where

$$\delta(\epsilon) = \int d^4x \delta(\epsilon)\phi_r(x) \frac{\delta}{\delta \phi_r(x)} = -\epsilon^\mu \int d^4x (\partial_\mu \phi_r(x)) \frac{\delta}{\delta \phi_r(x)}\,. \tag{27}$$

which, after applications of the field equations, is conserved as a consequence of Noether's theorem:

$$0 = \frac{\delta S}{\delta \varphi_r} \partial^\nu \varphi_r = \partial_\mu T^{\mu\nu} \tag{32}$$

The canonical energy-momentum tensor is the Noether "current" associated with space-time translation invariance ($J_A^\mu \to T^{\mu\nu}$).
Note the conservation is on the first index of the canonical energy-momentum tensor.

The charges which are identified with the space-time translation generators have the Noether construction

$$P^\nu = \int d^3x T^{0\nu}(t, \vec{r}) \tag{33}$$

and are time independent $\frac{dP^\nu}{dt} = 0$.
These charges are identified as the Hamiltonian

$$\begin{aligned} H = P^0 &= \int d^3x T^{00}(t, \vec{r}) \\ &= \int d^3x(-\mathcal{L}(x) + \pi_r(x)\partial_0\varphi_r(x)) \end{aligned} \tag{34}$$

which generates time translations and the linear momentum

$$\begin{aligned} P^i &= \int d^3x T^{0i}(t, \vec{r}) \\ &= -\int d^3x \pi_r(x)\partial_i\varphi_r(x) \end{aligned} \tag{35}$$

which generates space translations.

Note that the Hamiltonian so constructed is a 3-space integral, $H = \int d^3x\mathcal{H}$, of the Hamiltonian density $\mathcal{H}$ which is the Legendre transform of the Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi_r, \partial_\mu\phi_r)$.
As such, the Hamiltonian density, $\mathcal{H} = \mathcal{H}(\varphi_r, \pi_r)$, is a natural function of the fields, φ_r and their conjugate momenta fields, π_r.

Lorentz Transformations: Improved Energy-Momentum Tensor

The infinitesimal Lorentz transformation is

$$\Delta(\omega)x_\mu = \omega_{\mu\nu}x^\nu \tag{36}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $|\omega_{\mu\nu}| << 1$.

Using $\Delta(\omega)x^\mu = \sum_A \upsilon_A \Delta_A x^\mu = \omega^{\mu\nu}x_\nu = \frac{1}{2}\omega_{\rho\sigma}(g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho)$, we can identify $A \to \mu\nu$ and $\upsilon_A \to \frac{1}{2}\omega_{\mu\nu}$, while $\Delta_A x^\mu \to g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho$.

The representation matrices are denoted $d^{(A)} \to S^{\mu\nu} = -S^{\nu\mu}$ so that for infinitesimal Lorentz transformations For infinitesimal transformations,

$$\mathcal{D}_{rs}(I+\omega) = \delta_{rs} + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{rs} \tag{37}$$

The classical field infinitesimal transformation is thus

$$\delta(\omega)\varphi_r(x) = \frac{\omega^{\mu\nu}}{2}\Big(\delta_{rs}(x_\mu\partial_\nu - x_\nu\partial_\mu) + i(S_{\mu\nu})_{rs}\Big)\varphi_s(x) \tag{38}$$

Since the action is a Lorentz invariant[4], the Lagrangian carries the trivial representation ($S^{\mu\nu} = 0$) and transforms as

$$\begin{aligned}\delta(\omega)\mathcal{L} &= \frac{\omega^{\mu\nu}}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{L} \\ &= \partial_\mu(-\omega^{\mu\nu}x_\nu\mathcal{L}) \\ &= \partial_\mu\Lambda^\mu(\omega)\end{aligned} \tag{41}$$

with

$$\begin{aligned}\Lambda^\mu(\omega) &= -\omega^{\mu\nu}x_\nu\mathcal{L} \\ &= \frac{\omega^{\rho\sigma}}{2}(g^\mu_\sigma x_\rho - g^\mu_\rho x_\sigma)\mathcal{L}\end{aligned} \tag{42}$$

[4]The action is Lorentz invariant provided that it algebraically (i.e. not using field equations) satisfies

$$\delta(\omega)S[\phi] = 0 \tag{39}$$

where

$$\begin{aligned}\delta(\omega) &= \int d^4x\,\delta(\omega)\phi_r(x)\frac{\delta}{\delta\phi_r(x)} \\ &= \int d^4x\;\frac{\omega^{\mu\nu}}{2}\Big(\left(\delta_{rs}(x_\mu\partial_\nu - x_\nu\partial_\mu) + i(S_{\mu\nu})_{rs}\right)\phi_s(x)\Big)\frac{\delta}{\delta\phi_r(x)}\end{aligned} \tag{40}$$

The conserved Noether current for Lorentz transformations then takes the form

$$\begin{aligned} J^\mu(\omega) &= \Lambda^\mu(\omega) - \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r(x)}\delta(\omega)\varphi_r(x) \\ &= \frac{\omega^{\rho\sigma}}{2}\Big((g^\mu_\sigma x_\rho - g^\mu_\rho x_\sigma)\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r(x)}\Big(\delta_{rs}(x_\rho\partial_\sigma - x_\sigma\partial_\rho) + i(S_{\rho\sigma})_{rs}\Big)\varphi_s(x)\Big) \\ &= \frac{\omega^{\rho\sigma}}{2}\Big(x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r(x)}(S_{\rho\sigma})_{rs}\varphi_s(x)\Big) \end{aligned} \tag{43}$$

The term containing the $S_{\rho\sigma}$ matrix can be rewritten by adding a term symmetric under the $\rho \leftrightarrow \sigma$ interchange. Such a term gives no contribution since it is contracted with $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$. Thus we can write

$$\tfrac{\omega_{\rho\sigma}}{2}\, i\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r}(S^{\rho\sigma})_{rs}\varphi_s = \tfrac{\omega^{\rho\sigma}}{2}\,\Big(i(\varphi_s\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r}S^{\rho\sigma})_{rs} + i\tfrac{\partial\mathcal{L}}{\partial\partial_\rho\varphi_r}(S^{\sigma\mu})_{rs}\varphi_s + i\tfrac{\partial\mathcal{L}}{\partial\partial_\sigma\varphi_r}(S^{\rho\mu})_{rs}\varphi_s\Big)$$

Then defining

$$h^{\mu\rho\sigma} = \tfrac{i}{2}\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r}(S^{\rho\sigma})_{rs}\varphi_s + \tfrac{i}{2}\tfrac{\partial\mathcal{L}}{\partial\partial_\rho\varphi_r}(S^{\sigma\mu})_{rs}\varphi_s + \tfrac{i}{2}\tfrac{\partial\mathcal{L}}{\partial\partial_\sigma\varphi_r}(S^{\rho\mu})_{rs}\varphi_s = -h^{\rho\mu\sigma}$$

where the last equality follows from the antisymmetry of $S_{\mu\nu}$, we have

$$\tfrac{\omega_{\rho\sigma}}{2}\, i\tfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_r}(S_{\rho\sigma})_{rs}\varphi_s = \omega_{\rho\sigma}h^{\mu\rho\sigma}$$

and the conserved Noether current for Lorentz transformations reads

$$J^\mu(\omega) = \frac{\omega_{\rho\sigma}}{2}\Big(x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - h^{\mu\rho\sigma} + h^{\mu\sigma\rho}\Big) \tag{44}$$

where we have antisymmetrized $h^{\mu\rho\sigma}$ in $\rho \leftrightarrow \sigma$ since it is contracted with $\omega^{\rho\sigma}$.

Now define the improved (Belinfante) energy-momentum tensor as

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda h^{\mu\lambda\nu}\,. \tag{45}$$

Since, by construction, $h^{\mu\lambda\nu} = -h^{\lambda\mu\nu}$, it follows that algebraically $\partial_\mu\partial_\lambda h^{\mu\lambda\nu} = 0$ and consequently, after application of the field equations, that

$$\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0 \tag{46}$$

so the improved energy-momentum tensor is conserved on the first index. Moreover, using that
$x^\rho\Theta^{\mu\sigma} - x^\sigma\Theta^{\mu\rho} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - h^{\mu\rho\sigma} + h^{\mu\sigma\rho} + \partial_\lambda(x^\rho h^{\mu\lambda\sigma} - x^\sigma h^{\mu\lambda\rho})$
along with $h^{\mu\lambda\sigma} = -h^{\lambda\mu\sigma}$, it follows that

$$\begin{aligned} \frac{\omega_{\rho\sigma}}{2}\partial_\mu(x^\rho\Theta^{\mu\sigma} - x^\sigma\Theta^{\mu\rho}) &= \frac{\omega_{\rho\sigma}}{2}\partial_\mu(x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - h^{\mu\rho\sigma} + h^{\mu\sigma\rho}) \\ &= \partial_\mu J^\mu(\omega) \end{aligned} \tag{47}$$

where $J^\mu(\omega)$ is the conserved Noether current for Lorentz transformations. Thus we can define the improved, conserved current for Lorentz transformations as

$$M^\mu(\omega) = \frac{\omega_{\rho\sigma}}{2}\partial_\mu M^{\rho\mu\sigma} \tag{48}$$

with the conserved angular momentum tensor

$$M^{\rho\mu\sigma} = x^\rho\Theta^{\mu\sigma} - x^\sigma\Theta^{\mu\rho} = -M^{\sigma\mu\rho} \tag{49}$$

satisfying $\partial_\mu M^{\rho\mu\sigma} = 0$ after application of the field equations. Using this conservation law in conjunction with the conservation of the improved energy-momentum tensor gives

$$\begin{aligned} 0 &= \partial_\lambda M^{\mu\lambda\nu} = \partial_\lambda(x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu}) \\ &= \Theta^{\mu\nu} + x^\mu\partial_\lambda\Theta^{\lambda\nu} - \Theta^{\nu\mu} - x^\nu\partial_\lambda\Theta^{\lambda\mu} \\ &= \Theta^{\mu\nu} - \Theta^{\nu\mu} \end{aligned} \tag{50}$$

so that the improved energy-momentum tensor is symmetric:

$$\Theta^{\mu\nu} = \Theta^{\nu\mu} \tag{51}$$

and is thus conserved on both indices.
Note that if only scalar fields are present, then $S^{\mu\nu} = 0 \Rightarrow h^{\mu\lambda\nu} = 0$. Consequently, in this case, $\Theta^{\mu\nu} = T^{\mu\nu}$ and $T^{\mu\nu} = T^{\nu\mu}$.

The angular momentum generating the Lorentz transformations is obtained via the Noether construction as

$$J^{\mu\nu} = \int d^3x M^{\mu 0\nu} \tag{52}$$

and is time independent: $\frac{dJ^{\mu\nu}}{dt} = 0$.

We close this section by showing that for a classical field theory, the operator charges representing the Poincaré generators can constructed using the improved energy-momentum tensor.
To do so, define

$$
\begin{aligned}
P^{\mu}_{\Theta} &= \int d^3x \Theta^{0\mu} \\
P^{\mu}_{T} &= \int d^3x T^{0\mu} \\
J^{\mu\nu}_{\Theta} &= \int d^3x (x^{\mu}\Theta^{0\nu} - x^{\nu}\Theta^{0\mu}) \\
J^{\mu\nu}_{T} &= \int d^3x \left(x^{\mu}T^{0\nu} - x^{\nu}T^{0\mu} - i\frac{\partial\mathcal{L}}{\partial\partial_0\varphi_r}(S^{\mu\nu})_{rs}\varphi_s \right) \qquad (53)
\end{aligned}
$$

where

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \sum_a \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi_a}\partial^{\nu}\varphi_a \qquad (54)$$

and

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_{\rho}h^{\mu\rho\nu} \qquad (55)$$

with

$$h^{\mu\rho\nu} = \frac{i}{2}r\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi_r}S^{r\rho\nu}_{rs}\varphi_s + \frac{\partial\mathcal{L}}{\partial\partial_{\nu}\varphi_r}S^{\rho\mu}_{rs}\varphi_s + \frac{\partial\mathcal{L}}{\partial\partial_{\rho}\varphi_r}S^{\nu\mu}_{rs}\varphi_s \right) \qquad (56)$$

Now consider

$$P^{\mu}_{\theta} - P^{\mu}_{T} = \int d^3x(\Theta^{0\mu} - T^{0\mu}) = \int d^3x \partial_{\rho}h^{0\rho\mu} \qquad (57)$$

However since $h^{\mu\nu\rho} = -h^{\nu\mu\rho}$, it follows that $h^{00\rho} = 0$. Therefore,

$$P^{\mu}_{\theta} - P^{\mu}_{T} = \int d^3x \partial_i h^{0i\mu} \quad ; \quad i = 1, 2, 3 \qquad (58)$$

Using the divergence theorem and assuming the fields fall sufficiently fast at spatial infinity, the RHS vanishes and hence

$$P^{\mu}_{\theta} = P^{\mu}_{T} \equiv P^{\mu} \qquad (59)$$

Next consider

$$J_\Theta^{\mu\nu} - J_T^{\mu\nu} = \int d^3x \left(x^\mu \partial_\rho h^{0\rho\nu} - x^\nu \partial_\rho h^{0\rho\mu} + i \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_r} (S^{\mu\nu})_{rs} \varphi_s \right) \tag{60}$$

Integrate the first two terms on the RHS by parts and drop the surface terms giving

$$J_\Theta^{\mu\nu} - J_T^{\mu\nu} = \int d^3x \left(-h^{0\mu\nu} + h^{0\nu\mu} + i \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_r} (S^{\mu\nu})_{rs} \varphi_s \right) \tag{61}$$

Substituting the expression for $h^{\mu\nu\rho}$ gives

$$\begin{aligned}
J_\Theta^{\mu\nu} - J_T^{\mu\nu} &= \int d^3x \frac{i}{2} \Big(- \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_r} (S^{\mu\nu})_{rs} \varphi_s - \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} (S^{\mu 0})_{rs} \varphi_s - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r} (S^{\nu 0})_{rs} \varphi_s \\
&\quad + \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_r} (S^{\nu\mu})_{rs} \varphi_s + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r} (S^{\nu 0})_{rs} \varphi_s + \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi_r} (S^{\mu 0})_{rs} \varphi_s \\
&\quad + 2 \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_a} (S^{\mu\nu})_{rs} \varphi_a \Big) \\
&= \int d^3x \frac{i}{2} \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi_r} \Big((S^{\mu\nu})_{rs} \varphi_s + (S^{\nu\mu})_{rs} \varphi_s \Big)
\end{aligned} \tag{62}$$

But $S^{\mu\nu} = -S^{\nu\mu}$ and hence

$$J_\Theta^{\mu\nu} = J_T^{\mu\nu} \equiv J^{\mu\nu} \tag{63}$$

Canonical Quantization

To canonically quantize the classical field theory, we replace the classical field $\varphi_r(x)$ by operator fields $\phi_r(x)$ so that the quantum action functional is

$$S[\phi_s] = \int_\Omega d^4x \mathcal{L}(\phi_s(t,\vec{r}), \partial_\mu \phi_s(t,\vec{r})) \tag{64}$$

and the operator field equations are

$$0 = \frac{\delta S}{\delta \phi_s(x)} = \frac{\partial \mathcal{L}}{\partial \phi_s(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_s(x)} \tag{65}$$

In addition to the field equations, the operator fields and their canonically conjugate momentum operators are subjected to certain equal time commutation or equal time anti-commutation relations.

Focusing on bosonic fields, we impose the equal time commutation relations

$$\begin{aligned} [\phi_r(t,\vec{x}), \phi_s(t,\vec{y})] &= 0 \\ [\pi_r(t,\vec{x}), \pi_s(t,\vec{y})] &= 0 \\ [\phi_r(t,\vec{x}), \pi_s(t,\vec{y})] &= i\delta_{rs}\delta^3(\vec{x}-\vec{y}) \end{aligned} \tag{66}$$

where the canonical momentum operator is

$$\pi_r(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_r(t,\vec{x})} \tag{67}$$

Note that as a consequence of these relations, it follows that

$$\begin{aligned} [\partial_i \phi_r(t,\vec{x}), \phi_s(t,\vec{y})] &= 0 \\ [\partial_{x^i} \phi_r(t,\vec{x}), \partial_{y^j} \phi_s(t,\vec{y})] &= 0 \end{aligned} \tag{68}$$

We reiterate that the quantum operator Lagrangian is obtained from the classical field theory Lagrangian by simply replacing the classical fields by the quantum operator fields. In fact, this seemingly innocuous step is actually highly non-trivial. This follows since $\mathcal{L}$ contains products of operator fields and their derivatives at the same space-time point where certain equal time commutation relations are non-zero and singular. Thus great care is actually required in attempting to define these products of fields at coincident space-time points. We shall have a great deal to say about how to define products of operator fields at coincident space-time points throughout the course. For the time being, we shall simply proceed in a somewhat cavalier fashion and when we encounter a problem resulting from this singular structure, we shall attempt to make a sensible interpretation.

In the canonical procedure, Noether currents and charges are also promoted to be quantum mechanical operators. The charges are identified with the generators of the respective symmetries. This identification needs to be checked model by model to insure that the Noether charges so constructed indeed have the proper commutation relations with the fields as required by the symmetry.

Since the momentum and Hamiltonian operators are time independent, we can evaluate all the fields appearing in their Noether construction to be at the same time as the field with which we are commuting. Proceeding formally, we have

$$\begin{aligned}
[P_i, \phi_r(t,\vec{x})] &= -\int d^3y [\pi_s(t,\vec{y})\partial_{y^i}\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \\
&= -\int d^4y (-i)\delta_{rs}\delta^3(\vec{x}-\vec{y})\partial_{y^i}\phi_s(t,\vec{y}) \\
&= i\partial_i\phi_s(t,\vec{x}) \qquad (69)
\end{aligned}$$

which is precisely what we expect the momentum operator to do.

Next consider

$$\begin{aligned}
[H, \phi_r(t,\vec{x})] &= \int d^3y [-\mathcal{L}(\phi(t,\vec{y}), \partial_\mu\phi(t,\vec{y})) + \pi_s(t,\vec{y})\partial_0\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \\
&= \int d^3y \Big(-\frac{\partial\mathcal{L}}{\partial\phi_s(t,\vec{y})}[\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \\
&\quad - \frac{\partial\mathcal{L}}{\partial_i^y\phi_s(t,\vec{y})}[\partial_i^y\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \\
&\quad - \frac{\partial\mathcal{L}}{\partial_0\phi_s(t,\vec{y})}[\partial_0\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \\
&\quad + [\pi_s(t,\vec{y}), \phi_r(t,\vec{x})]\partial_0\phi_s(t,\vec{y}) \\
&\quad + \pi_s(t,\vec{y})[\partial_0\phi_s(t,\vec{y}), \phi_r(t,\vec{x})] \Big) \\
&= \int d^3y \Big(-\pi_s(t,\vec{y})[\partial_0\phi_s(t,\vec{y}), \phi_r(t,\vec{x})]
\end{aligned}$$

$$
\begin{aligned}
&\quad + [\pi_s(t,\vec{y}), \phi_r(t,\vec{x})]\partial_0\phi_s(t,\vec{y}) \\
&\quad + \pi_s(t,\vec{y})[\partial_0\phi_s(t,\vec{y}), \phi_r(t,\vec{x})]\Big) \\
&= \int d^3y(-i)\delta_{rs}\delta^3(\vec{x}-\vec{y})\partial_0\phi_s(t,\vec{y}) \\
&= -i\partial_0\phi_r(t,\vec{x}) \qquad (70)
\end{aligned}
$$

which is recognized as the Heisenberg equation of motion and is precisely the relationship satisfied by the generator of time translations.