Project Notes

June 1, 2020

LISA Verification Binaries

The Laser Interferometer Space Antenna(LISA) will be the first gravitational wave observatory in space. LISA will be operating in the low frequency part of the gravitational wave spectrum(10^4–1 Hz). In this range, we expect to observe lots of ultracompact binaries with orbital periods shorter than few hours. Out of these UCBs, AM CVn type binaries are of particular interest. Due to therir strong GW signals, they are guaranteed to be detetcted on LISA band. These are termed 'verification binaries'.

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{4G\mu v^2}{c^4 D} \begin{pmatrix} \frac{1}{2}(1+\cos^2\iota)\cos(\omega t) \\ \cos\iota\sin(\omega t) \end{pmatrix} \tag{1}$$

$$ds^2 = -c^2dt^2 + dz^2 + (1 + h\cos 2\psi)dx^2 + (1 - h\cos 2\psi)dy^2 - h\sin 2\psi dx\, dy \tag{2}$$

$$h(t) = \frac{4G\mu v^2}{c^4 D}\left(\tfrac{1}{2}(1+\cos^2\iota) + i\cos\iota\right)\exp(i\omega t) \tag{3}$$

$$\frac{\delta L}{L} = \tau(\omega, \theta, L)\, \cos 2\psi\, h(t) \tag{4}$$

where

$$\begin{aligned} \tau(\omega,\theta,L) \equiv \tfrac{1}{2}\sin^2\theta\Big(& \operatorname{sinc}(\alpha(1-\cos\theta))\, \exp(-i\alpha(3+\cos\theta)) + \\ & \operatorname{sinc}(\alpha(1+\cos\theta))\, \exp(-i\alpha(1+\cos\theta))\Big) \end{aligned} \tag{5}$$

and $\alpha \equiv (\omega L)/(2c)$.

LISA Response Function

LISA's response function can be calculated by imagining it as a two arm Michelson interferometer. We first derive the response of a single arm and then cross correlate with the other to measure the total strain.

Consider two LISA spacecraft, one ar the origin and another at a distance L away from it. We choose the coordinate axes such that the spacecrafts lie in the x-z plane. The vector $\mathbf{u} = \hat{x}\ \sin\theta + \hat{z}\ \cos\theta$ describes the single arm here.

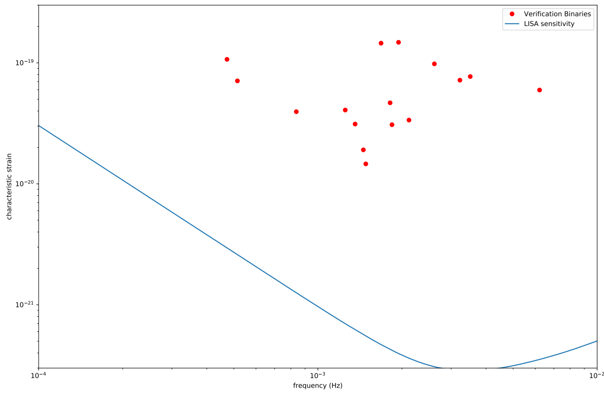

Figure 1: The calculated strains due to LISA verification binaries and the sensitivity curve of LISA

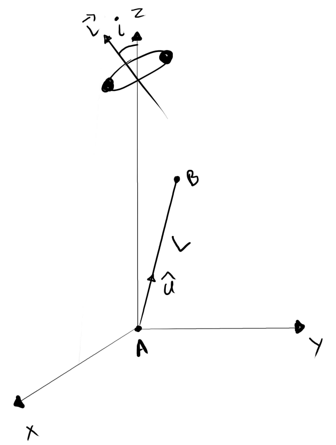

Figure 2: The one arm arrangement with the source in the z-axis with angle of inclination i and the arm along the vector $\mathbf{u}$ which lies in the x-z plane.

The spacecraft at origin sends a series of photons while the weak plane gravitational wave passes through space in the +z direction. Later we will generalize the source to be in a random direction.

We can write the space-time metric in the transverse-traceless gauge (TT-Gauge). Here ψ refers to the polarization angle.

$$ds^2 = -c^2dt^2 + (1 + h_+ \cos 2\psi + h_x \sin 2\psi)dx^2 + (1 - h_+ \cos 2\psi - h_x \sin 2\psi)dy^2 - 2(h_+ \sin 2\psi - h_x \cos 2\psi)dxdy + dz^2 \quad (6)$$

Using this metric we can find the photon path similar to [1]. Using that we compute the round trip journey from spacecraft 1 to spacecraft 2 and back again.

$$\delta l(t_2) = \frac{L \sin^2 \theta}{2} \, \tau(\cos \theta, f) \left[h_+(t_2) \cos 2\psi + h_x(t_2) \sin 2\psi \right] \quad (7)$$

τ is the transfer function as defined in eq(7) of [1]

Now from we consider the plus and cross polarisation amplitudes of a binary black hole with an angle of inclination i with the line of sight $\hat{z}$ (3.27a and 3.27b) [2].

Substituting the h_+ and h_x expressions, eq [7] becomes,

$$\delta l = \frac{A(t) L \sin^2 \theta \tau(\cos \theta, f)}{2d} \left[\frac{1 + \cos^2 i}{2} \cos 2\psi \ \cos \Phi(t) + \cos i \ \sin 2\psi \ \sin \Phi(t) \right] \quad (8)$$

$$= \frac{A(t) L \sin^2 \theta \tau(\cos \theta, f)}{2d} \left(\sqrt{\frac{(1 + \cos^2 i)^2}{4} \cos^2 2\psi + \cos^2 i \sin^2 2\psi} \right) \cos(\Phi(t) - \Psi) \quad (9)$$

Where $\Phi(t) = 2\pi f t + \Phi_0$. The total strain therefore depends on the angle of orientation of the arm θ, the angle of inclination of the orbit i, the location of the source(in this case we took it as z-axis) and the polarization angle ψ

A general coordinate free formula is given by eq(8) in [1].

For a general location of the binary ($\hat{\Omega}$) specified by θ_s, ω in spherical coordinate system, we can derive the antenna beam patterns.[2]

$$F_+(\theta_s, \phi, \psi, i) = \frac{(1 + \cos^2 i)}{2} \left[\frac{1}{2}(1 + \cos^2 \theta_s) \cos 2\phi \cos \psi - \cos \theta_s \sin 2\phi \sin 2\psi \right]$$

$$F_\times(\theta_s, \phi, \psi, i) = \cos i \left[\frac{1}{2}(1 + \cos^2 \theta_s) \cos 2\phi \cos \psi + \cos \theta \sin 2\phi \cos 2\psi \right]$$

If we put back $\theta_s = 0$ and $\phi = 0$ in these equations we get back the responses in eq [4].

We can form an interferometer by introducing a third spacecraft at a distance L from the corner spacecraft and subrating the outputs of the two arms [1] to get the total strain.

$$s(\hat{\Omega}, f, \mathbf{x}, t) = \frac{\delta l_u(t) - \delta l_v(t)}{l} \quad (10)$$

$$= \mathbf{D}(\hat{\Omega}, f) : h(\hat{\Omega}, f, \mathbf{x}, t) \quad (11)$$

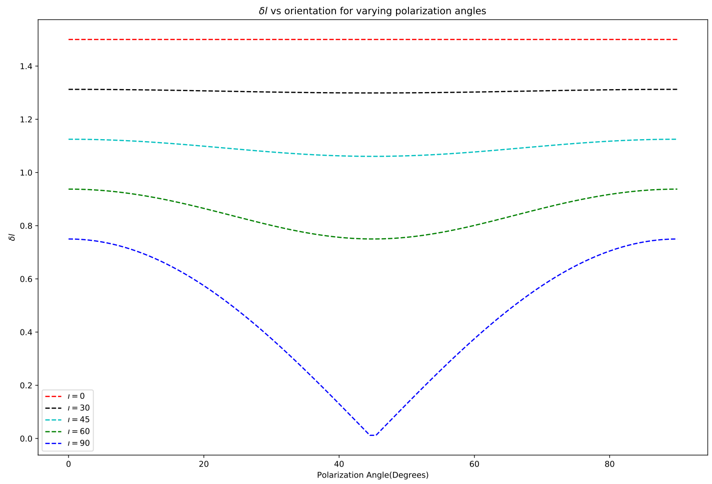

Figure 3: The response δl variying with the polarization angle Ψ for various inclination angle.

where

$$D(\hat{\Omega}, f) = \frac{1}{2}\left((u \otimes u)\tau - (v \otimes v)\tau\right)$$

is the detector response tensor. **u** and **v** are the unit vectors representing the direction of each interferometer arm. $\hat{\Omega}$ is the direction of propegation of the wave. We can use the orientation of LISA arms and compute the detetctor response functions F_+ and $F_\times$.

HP Lib Verfication Binary

We are interested in providing an independent prediction for i and Ψ for the AM CVn binary HP Lib. The binary consist of of on high mass white dwarf and a low mass star(brown dwarf).

Source	m1($M_\odot$)	m2($M_\odot$)	P_{orb}(sec)
HP Lib	0.49-0.80	0.048-0.08	1102.70

Table 1: The estimated values of mass and time period of HP Lib

Consider the verification binary HP Lib with masses m_1 and m_2 and period P . We can find the fraction of light recieved by the brown dwarf m_2 in the following way:

The flux from mass $m1$ obeys the inverse square law. Assuming a luminosity L, the flux at a distance d is given by $L/4\pi d^2$. The cross section area for brown dwarf is πR^2.

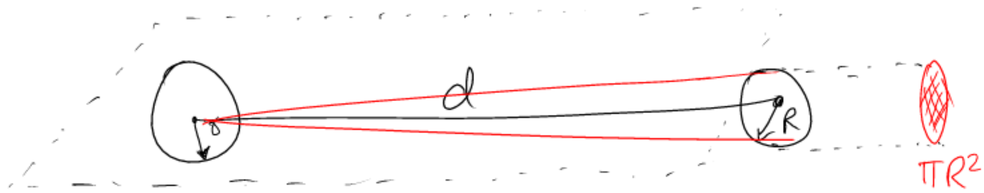

Figure 4: The binary system HP Lib

Therefore, the total flux recieved is:

$$\frac{L}{4\pi d^2}\,\pi R^2 = \frac{L}{4}\,\left(\frac{R}{d}\right)^2$$

where R is the radius of the brown dwarf and d is the seperation.

We can estimate d from the time period of the orbit. We know from Kepler's 3rd law that, $T^2 = \frac{4\pi^2}{G(m_1+m_2)}d^3$. We will denote $M = m_1 + m_2$

$$\begin{aligned}\frac{d}{c} &= \left(\frac{GM}{c^3}\right)^{1/3}\frac{T^{2/3}}{(4\pi^2)^{1/3}}\\ &= \frac{\left(\frac{GM}{c^3}\right)^{1/3}}{\left(\frac{GM_\odot}{c^3}\right)^{1/3}}\frac{T^{2/3}}{(4\pi^2)^{1/3}}(5\times 10^{-6})^{1/3}\end{aligned}$$

Now the fraction of the light recieved is,

$$\frac{1}{4}\left(\frac{R}{c}\right)^2\left(\frac{M}{M_\odot}\right)^{-2/3}\frac{T^{-4/3}}{(4\pi^2)^{-2/3}}(5\times 10^{-6})^{-2/3} \tag{12}$$

For the first estimate, we can take $R/c \approx 0.1$ which if we subtitute above give an estimate of the light received to be 0.01315

References

[1] https://arxiv.org/pdf/gr-qc/0103075.pdf

[2] https://dcc.ligo.org/public/0106/T1300666/003/Whelan_notes.pdf