Solving ordinary differential equations numerically

Pranav Satheesh

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1 First order ODE

A general initial value problem of first order ODE are of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with the inital value at a point specified $y(x_0) = y_0$.

There are two types of methods used to get a solution of first-order equations:

- A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.
- A set of tabulated values of x and y.

i) - Taylor and Picard ii) Euler, Runge-Kutta, Adams-Bashforth - called step-by-step methods

Problems in which all the initial conditions are specified at the initial point only are called initial value problems. A differential equation of the nth order will requre n initial conditions.

First order - one conditions Problems involving second-and higher-order differential equations, we may prescribe the conditions at two or more points. Such problems are called boundary value problems.

2 Taylor's Series

For the general first-order ODE $y^{'} = f(x, y)$ with the initial condition $y(x_0) = y_0$ we can give a solution by taylor expanding $y(x)$ around $x = x_0$.

$$y(x) = y_0 + (x - x_0)y_0^{'} + \frac{(x - x_0)^2}{2!}y_0^{''} + \cdots \tag{2}$$

We now need to evalue the higher orders $y_0^{'}, y_0^{''}, \cdots$.

$$y^{''} = f_x + ff_y \tag{3}$$
$$y^{''} = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2 \tag{4}$$
$$\tag{5}$$

3 Euler's Method

The methods previously discussed give power series solution. But if we require to have a table of (x, y) values by solving the ODE numericaly, we need to use step-by-step methods

We will solve by takinf steps of the form $x = x_r + rh$ where $r = 1, 2, \cdots$. Integrating we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \tag{6}$$

We will assume as an approximation that $f(x, y) = f(x_0, y_0)$ in the interval $x_0 \leq x \leq x_1$.

This gives us

$$y_1 \approxeq y_0 + hf(x_0, y_0) \tag{7}$$

Continuing the same way, we get the general formula for Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n),\ n = 0, 1, 2, \cdots \tag{8}$$

3.1 Error estimates for Euler Method

3.2 Modified Euler method

In the Euler's method we approximated $f(x, y)$ in the interval and now we approximate the integral (6) by trapezoidal rule

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \tag{9}$$

We get the general iteration formula

$$y_1^{(n+!)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})],\ n = 0, 1, 2. \cdots \tag{10}$$

whwre $y_1^{(n)}$ is the nth approximation to y_1 and y_1^0 is chosen from Euler's formula:

$$y_1^{(0)} = y_0 + hf(x_0, y_0) \tag{11}$$