ODE-Basic Algorithms

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1 Euler Method

1.1 Forward Derivative

1. The formal definition of the derivative is,

$$f^{'}(t) = lim_{\tau \to 0} \frac{f(t+\tau) - f(t)}{\tau}$$

2. From the definition of Taylor's theorem, we can write:

$$f^{'}(t) = \frac{f(t+\tau) - f(t)}{\tau} - \frac{1}{2}\tau f^{''}(\zeta)$$

where $t \leq \zeta \leq t + \tau$. This is the *right derivative* or *forward derivative formula*. The last term is the truncation error which is of the order of τ here.

1.2 Euler's Method

Consider the equations of motion here, which I want to solve numerically,

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(\mathbf{r}, \mathbf{v})$$

$$\frac{dr}{dt} = \mathbf{v}$$

Using the forward derivative equation, we can write these as,

$$\begin{aligned} \mathbf{v}(t+\tau) &= \mathbf{v}(t) + \tau\mathbf{a}(\mathbf{r}(t), \mathbf{v}(t)) + \mathcal{O}(\tau^2) \\ \mathbf{r}(t+\tau) &= \mathbf{r}(t) + \tau\mathbf{v}(t) + \mathcal{O}(\tau^2) \end{aligned}$$

Our notation will be, $f_n = f(t_n)$, $t_n = (n-1)\tau$
The Euler method equations become,

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + \tau\mathbf{a}_n \\ \mathbf{r}_{n+1} &= \mathbf{r}_n + \tau\mathbf{v}_n \end{aligned}$$

1.3 Euler-Cromer Method

Instead of v_n in the quation, we put the modified v_{n+1}

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau\mathbf{a}_n$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\mathbf{v}_{n+1}$$

The truncation is still of $\mathcal{O}(\tau^{\in})$.

1.4 Midpoint Method

We can have the midpoint of velocities between vn and vn+1

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau\mathbf{a}_n$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2}$$

Plugging the velcoity equation into the position equation, we see that

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\mathbf{v}_n + \tau\mathbf{v}_n + \frac{1}{2}\mathbf{a}_n\tau^2$$

The order is $\mathcal{O}(\tau^3)$
global error = N_τ x (local error) = (T/τ)$\mathcal{O}(\tau^n) = T\mathcal{O}(\tau^{n-1})$