PDEs
Fourier Series

Partial Differential Equations (PDEs)

Let the *n*-dimensional Euclidean space be donated by $\mathbb{R}^n$.
A point in $\mathbb{R}^n$ has *n* coordinates $x_1, x_2, \ldots, x_n$.
Let *u* be a function having *n* coordinates, hence

$$u = u(x_1, x_2, \ldots, x_n)$$

For *n* = 2 or *n* = 3 we may also have different notation, for example:

$u(x, y)$ for functions in $\mathbb{R}^2$
$u(x, y, z)$ for functions in $\mathbb{R}^3$

Sometimes we also write $u(r, \theta, \phi)$ for spherical coordinates.
Sometimes we have a "time" coordinate *t*, in which case
$u(t, x_1, x_2, \ldots, x_n)$ for functions in $\mathbb{R}^{1+n}$

Partial Differential Equations (PDEs)

There are many different notation for partial derivatives. For example, the partial derivatives of a function in $\mathbb{R}^2$ space and time $u(t, x, y)$:

$$\frac{\partial u}{\partial t} = u_t = \partial_t u$$

$$\frac{\partial u}{\partial x} = u_x = \partial_x u \,, \quad \frac{\partial u}{\partial y} = u_y = \partial_y u$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = u_{xy} = u_{yx} = \partial_x \partial_y u = \partial_y \partial_x u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = u_{xx} = \partial_{xx} u$$

Physical Examples of PDEs

(1) Wave equation, second-order, linear, homogeneous:

$$-\partial_t^2 u + \partial_x^2 u = 0$$

(2) Heat equation, second-order, linear, homogeneous:

$$-\partial_t u + \partial_x^2 u = 0$$

(3) Laplace's equation, second-order, linear, homogeneous:

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0$$

(4) Poisson's equation with source function f, second-order, linear, inhomogeneous:

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = f(x, y, z)$$

(5) Transport equation, first-order, linear, homogeneous:

$$\partial_t u + \partial_x u = 0$$

Physical Examples of PDEs

(6) Burger's equation, first-order, nonlinear, homogeneous:

$$\partial_t u + u\,\partial_x u = 0$$

(7) Schrödinger equation, second-order, linear, homogeneous:

$$i\,\partial_t u + \partial_x^2 u = 0$$

(8) Maxwell's equations in a vacuum, first-order, linear, homogeneous:

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = 0\,, \quad \nabla \cdot \mathbf{E} = 0$$

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0\,, \quad \nabla \cdot \mathbf{B} = 0$$

Grad, Div, Curl

Define $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ and $u = u(x, y, z)$

$$\operatorname{grad} u = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\operatorname{div} u = \nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$\begin{aligned} \operatorname{curl} u = \nabla \times u &= \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & u & u \end{pmatrix} \\ &= \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}$$

Grad, Div, Curl

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot (\nabla u) = \nabla^2 u = \Delta u \quad \text{(Laplacian)}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

IBVP PDEs

PDEs are IBVPs (Initial Boundary Value Problems).

Suppose we are interested in studying the diffusion of heat in a body that occupies a bounded region D of **x**-space. That is, we are interested in solving the heat equation:

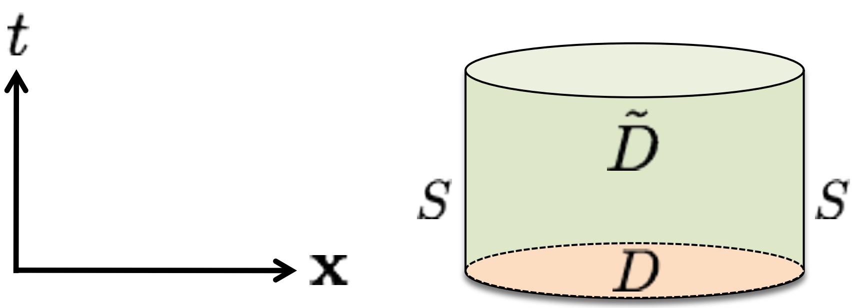
$$u_t = k\, u_{xx} \tag{13.1}$$

in the region:

$$\tilde{D} = \{(t, \mathbf{x}) : \mathbf{x} \in D, \quad t > 0\}$$

of $(t, \mathbf{x})$-space subject to the initial condition:

$$u(0, \mathbf{x}) = f(\mathbf{x}) \tag{13.2}$$



$f(\mathbf{x})$ is temperature distribution at time $t = 0$.

Boundary Conditions for PDEs

Equation (13.2) is a condition on u on the "horizontal" part of the boundary of $\tilde{D}$, but it is not enough to specify u completely; we also need a boundary condition on the "vertical" part of the boundary to tell what happens to the heat when it reaches the boundary surface S of the spatial region D.

(1) One assumption is that S is held at a constant temperature u_0, for example by immersing the body in a bath of ice water. This is called **Dirichlet condition**, where:

$$u(t, \mathbf{x}) - u_0 = 0 \quad \text{for } \mathbf{x} \in S, \quad t > 0 \qquad \textbf{(13.3)}$$

Boundary Conditions for PDEs

(2) Another assumption is that D is insulated, so that no heat can flow in or out across S. This is called **homogeneous Neumann condition** or **zero-flux condition**. Mathematically, this condition amounts to requiring the normal derivative of u along the boundary S to vanish:

$$(\nabla u \cdot \mathbf{n})(t, \mathbf{x}) = 0 \quad \text{for } \ \mathbf{x} \in S\,, \quad t > 0 \tag{13.4}$$

(3) **Robin condition** is a condition when the region outside D is held at a constant temperature u_0, and the rate of heat flow across the boundary S is proportional to the difference in temperatures on the two sides:

$$(\nabla u \cdot \mathbf{n})(t, \mathbf{x}) + a(u(t, \mathbf{x}) - u_0) = 0 \tag{13.5}$$

$$\text{for } \ \mathbf{x} \in S\,, \quad t > 0$$

Boundary Conditions for PDEs

Case (3) is also called Newton's law of cooling, and $a > 0$ is the proportionality constant. The conditions (13.3) – (13.4) may be regarded as the limiting case of (13.5) as $a \to \infty$ or $a \to 0$.

For wave equation, which is second-order in the time variable t,

$$u_{tt} = c^2 u_{xx} \tag{13.6}$$

the conditions required to solve such problem consist of

-Initial values of u: $u(0, \mathbf{x}) = f(\mathbf{x})$

-Initial velocity: $u_t(0, \mathbf{x}) = g(\mathbf{x})$

-Boundary conditions: $(\nabla u \cdot \mathbf{n})(t, \mathbf{x}) + a(u(t, \mathbf{x}) - u_0) = 0$

Solving a Heat/Diffusion Equation

Consider a circular metal rod of length L, insulated along its curved surface so that heat can enter or leave only at the ends. Suppose that both ends are held at temperature zero. The 1-dimensional heat equation with boundary conditions:

$$u_t = k\, u_{xx}\,, \quad u(t,0) = u(t,L) = 0 \tag{13.7}$$

and the initial condition

$$u(0,x) = f(x) \tag{13.8}$$

Using a method of <u>separation of variables</u>, we try to find solutions of u of the form

$$u(x,y) = X(x)Y(y) \tag{13.9}$$

Solving a Heat/Diffusion Equation

If we substitute $u(t,x) = T(t)X(x)$ into (13.7) we obtain

$$T'(t)X(x) = k\,T(t)X''(x) \qquad \textbf{(13.10)}$$

$$X(0) = X(L) = 0 \qquad \textbf{(13.11)}$$

The variables in (13.10) may be separated by dividing both sides by $k\,T(t)X(x)$ yielding

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)}$$

Now the left side depends only on t, whereas the right side depends only on x. Since they are equal, they both must be equal to a constant A:

$$T'(t) = A\,k\,T(t)\,, \qquad X''(x) = A\,X(x)$$

Solving a Heat/Diffusion Equation

These are simple ODEs for T and X that can be solved by elementary methods.

(1) If A is positive, the general solutions of the equations for T and X are

$$T(t) = C_0\, e^{Akt} \quad \textbf{(13.12a)}$$

$$X(x) = C_1 e^{\sqrt{A}\,x} + C_2 e^{-\sqrt{A}\,x} \quad \textbf{(13.12b)}$$

(2) If A is negative, the general solutions of the equations for T and X are

$$T(t) = C_0\, e^{-Akt} \quad \textbf{(13.13a)}$$

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \quad \textbf{(13.13b)}$$

where $\lambda = \sqrt{A}\,.$

Solving a Heat/Diffusion Equation

Choosing case (2), in equation (13.13):

- the condition $X(0) = 0$ forces $C_1 = 0$
- the condition $X(L) = 0$ forces $C_2 \sin(\lambda L) = 0.$

We take $C_2 \neq 0$ thence $\sin(\lambda L) = 0,$ which means that $\lambda L = n\pi$ for some integer *n*. In other words,

$$A = \left(\frac{n\pi}{L}\right)^2$$

Taking $C_0 = C_2 = 1,$ for every positive integer *n* we have obtained a solution $u_n(t, x)$ of (13.7):

$$u_n(t, x) = \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right) \qquad \textbf{(13.14)}$$

$n = 1, 2, 3, \ldots$

Solving a Heat/Diffusion Equation

We obtain more solutions by taking linear combinations of the u_n's, and then passing to infinite linear combinations, where the solutions now

$$u(t,x) = \sum_{1}^{\infty} a_n u_n = \sum_{1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right) \quad \textbf{(13.15)}$$

Applying the initial condition in (13.8) to (13.15) we get

$$f(x) = \sum_{1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \textbf{(13.16)}$$

Solving a Heat/Diffusion Equation

If we consider other boundary conditions, such as zero-flux condition, the problem now becomes

$$u_t = k\, u_{xx}\,, \quad u_x(t,0) = u_x(t,L) = 0 \qquad \textbf{(13.17)}$$

We use the same technique as before, but the conditions in (13.11) are replaced by

$$X'(0) = X'(L) = 0 \qquad \textbf{(13.18)}$$

Differentiating (13.13b) and plugging the conditions:

$$X'(x) = -\lambda\, C_1 \sin(\lambda x) + \lambda\, C_2 \cos(\lambda x)$$

$$X'(0) = 0 + \lambda\, C_2 \cos(\lambda x) = 0 \quad \longrightarrow \quad C_2 = 0$$

$$X'(L) = -\lambda\, C_1 \sin(\lambda L) = 0 \quad \longrightarrow \quad \lambda L = n\pi$$

Solving a Heat/Diffusion Equation

From which we obtain the sequence of solutions

$$u_n(t,x) = \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right) \qquad \textbf{(13.19)}$$

$$n = 1, 2, 3, \ldots$$

which can be combined to form the series

$$u(t,x) = \sum_0^\infty a_n u_n = \sum_0^\infty a_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right) \qquad \textbf{(13.20)}$$

Example

Solve the following 1D heat/diffusion equation

$$u_t = \frac{1}{10} u_{xx}\,, \qquad u_x(t,0) = u_x(t,\pi) = 0 \quad \textbf{(13.21)}$$

$$u(0,x) = 3 - 4\cos(2x) \qquad (0 < x < \pi)$$

<u>Solution</u>:

We use the results described in equation (13.19) for the heat equation with homogeneous Neumann boundary condition as in (13.17). From $\lambda L = n\pi$ where $L = \pi$, we get $\lambda = n$. Applying equation (13.20) we obtain the general solution

$$u(t,x) = \sum_{0}^{\infty} a_n \exp\left(-\frac{n^2 t}{10}\right) \cos(nx)$$

Example

From the initial condition

$$u(0,x) = \sum_{0}^{\infty} a_n \exp\left(-\frac{n^2 0}{10}\right)\cos(nx) = 3 - 4\cos(2x)$$

$$\sum_{0}^{\infty} a_n \cos(nx) = 3 - 4\cos(2x)$$

and from matching with the initial condition, we get

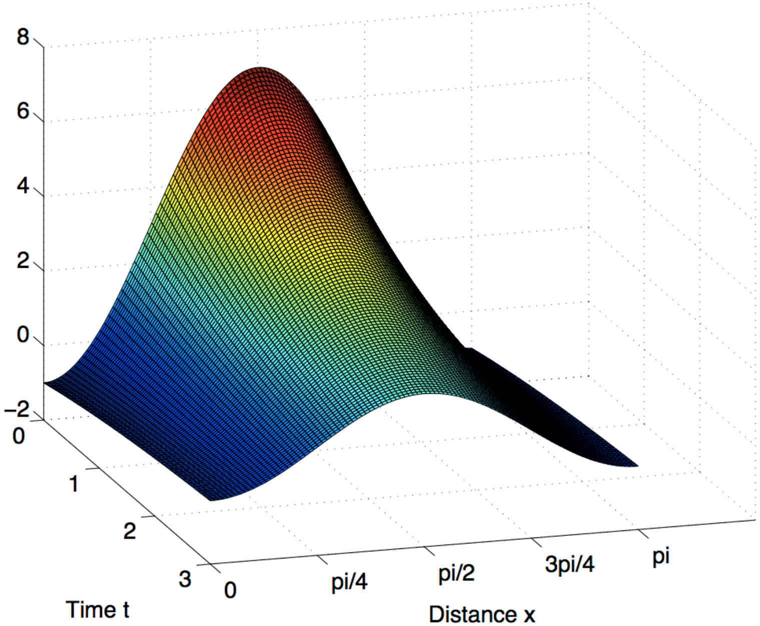
$$n = 2 \;\rightarrow\; \cos(nx) = \cos(2x)$$

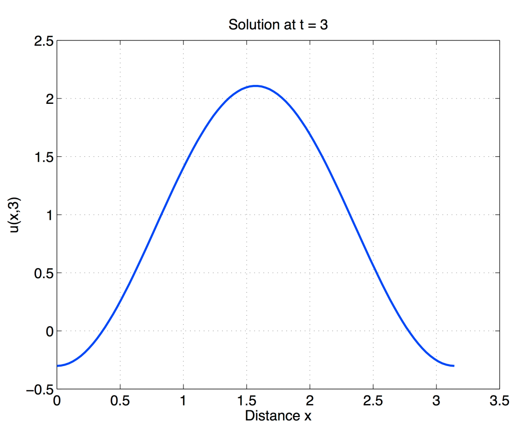
Substituting these into the general solution, we obtain the particular solution:

$$u(t,x) = \exp\left(-\frac{2t}{5}\right)(3 - 4\cos(2x))$$

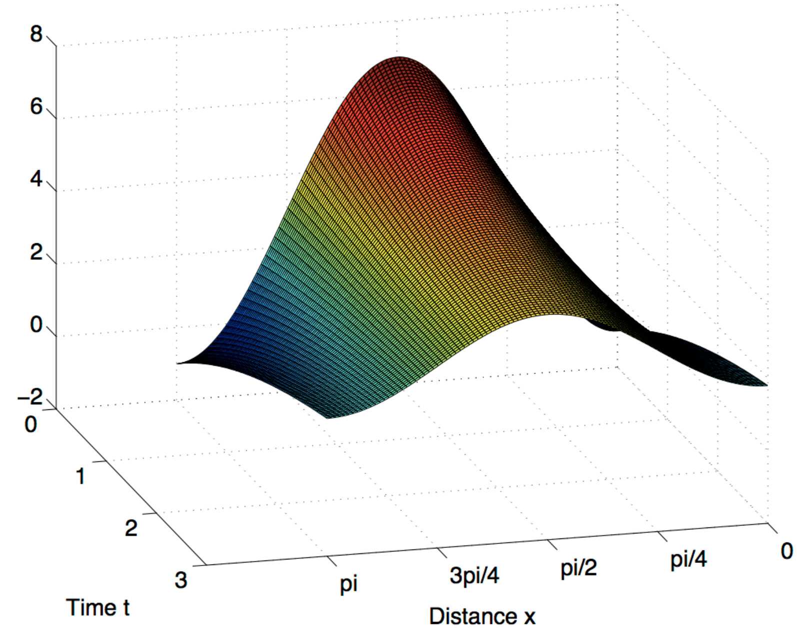
Example

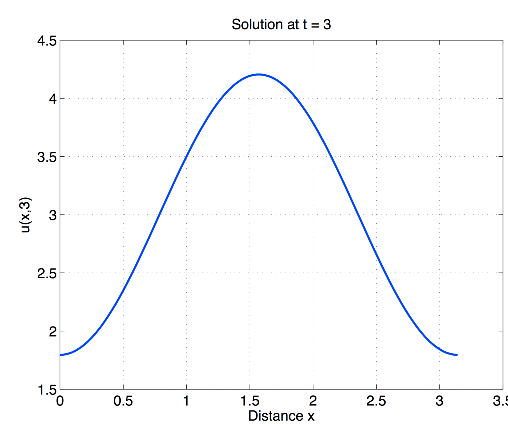
Analytical Solution





MATLAB's pdepe function





Fourier Series of Periodic Functions

A function $f(t)$ is said to be periodic with period T if there is a number $T > 0$ such that

$$f(t+T) = f(t)$$

for all t. Every integer multiple of the period is also a period:

$$f(t+nT) = f(t)\,, \quad n = 0, \pm 1, \pm 2, \ldots$$

Consider the function

$$f(t) = \cos(2\pi t) + \frac{1}{2}\cos(4\pi t)$$

The individual terms are periodic with periods 1 and ½, respectively, but the sum is periodic with period 1.

Fourier Series of Periodic Functions

$$
\begin{aligned}
f(t+1) &= \cos(2\pi(t+1)) + \frac{1}{2}\cos(4\pi(t+1)) \\
&= \cos(2\pi t + 2\pi) + \frac{1}{2}\cos(4\pi t + 4\pi) \\
&= \cos(2\pi t)\cos(2\pi) - \sin(2\pi t)\sin(2\pi) \\
&\quad + \frac{1}{2}[\cos(4\pi t)\cos(4\pi) - \sin(4\pi t)\sin(4\pi)] \\
&= \cos(2\pi t) + \frac{1}{2}\cos(4\pi t) \\
&= f(t)
\end{aligned}
$$

Fourier Series of Periodic Functions

Suppose that $f(t)$ is a function defined on the real line such that $f(t+T) = f(t)$ for all t. Such functions are said to be periodic with period T, or T-periodic. A continuous **Fourier series** of a function with period T can be written:

$$\begin{aligned} f(t) = a_0 + a_1\cos(\omega t) + b_1\sin(\omega t) + a_2\cos(2\omega t) + \\ b_2\sin(2\omega t) + a_3\cos(3\omega t) + b_3\sin(3\omega t) + \ldots \end{aligned}$$

or more concisely,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega t) + b_n\sin(n\omega t)\right) \quad \textbf{(13.21)}$$

where $\omega = \dfrac{2\pi}{T}$ is called the fundamental frequency and its constant multiples 2ω, 3ω, etc are called harmonics.

Fourier Series of Periodic Functions

Recall the formulas

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Hence

$$\cos(n\omega t) = \frac{e^{in\omega t} + e^{-in\omega t}}{2} \text{ and } \sin(n\omega t) = \frac{e^{in\omega t} - e^{-in\omega t}}{2i}$$

Also the formulas

$$e^{in\omega t} = \cos(n\omega t) + i\sin(n\omega t)$$

$$e^{-in\omega t} = \cos(n\omega t) - i\sin(n\omega t)$$

Fourier Series of Periodic Functions

The series in (13.21) can be rewritten as

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{in\omega t} + e^{-in\omega t}}{2} \right) + b_n \left(\frac{e^{in\omega t} - e^{-in\omega t}}{2i} \right) \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[e^{in\omega t} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) + e^{-in\omega t} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[e^{in\omega t} \left(\frac{a_n - ib_n}{2} \right) + e^{-in\omega t} \left(\frac{a_n + ib_n}{2} \right) \right]$$

Now, letting

$$c_0 = a_0 \,, \quad c_n = \frac{a_n - ib_n}{2} \,, \quad c_{-n} = \frac{a_n + ib_n}{2} \qquad \textbf{(13.22)}$$

Fourier Series of Periodic Functions

The expression for $f(t)$ becomes

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\omega t} + c_{-n} e^{-in\omega t} \right)$$

The coefficients c_n are complex numbers and they satisfy

$$c_{-n} = \overline{c_n}$$

Notice that when $n = 0$ we have $c_0 = \overline{c_0}$ which implies that c_0 is a real number.

For any value of n, the magnitudes of c_n and c_{-n} are equal:

$$|c_n| = |c_{-n}|$$

Fourier Series of Periodic Functions

Then the series

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

can be written as

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega t} \qquad \textbf{(13.23)}$$

where, they are both called the **Fourier series** of *f*.

Fourier Series of Periodic Functions

How can the coefficients c_n be calculated in terms of $f(t)$? Let us multiply both sides of (13.22) by $e^{-ik\omega t}$ (*k* being an integer) and integrate from 0 to *T*:

$$\int_0^T f(t)e^{-ik\omega t}\,dt = \sum_{-\infty}^{\infty} c_n \int_0^T e^{i(n-k)\omega t} dt$$

If $n \neq k$, the integral term on the right hand side:

$$\int_0^T e^{i(n-k)\omega t} dt = \frac{1}{i(n-k)\omega t} e^{i(n-k)\omega t}\Big|_0^T$$

$$= \frac{e^{i2\pi(n-k)} - e^{0(n-k)}}{i(n-k)\omega}$$

Fourier Series of Periodic Functions

$$\int_0^T e^{i(n-k)\omega t}dt = \frac{e^{i2\pi}e^{(n-k)} - e^0 e^{(n-k)}}{i(n-k)\omega}$$

$$= \frac{e^{(n-k)} - e^{(n-k)}}{i(n-k)\omega}$$

$$= 0 \quad \text{if} \;\; n \neq k$$

and the case if $n = k$:

$$\int_0^T e^{i(n-k)\omega t}dt = \int_0^T dt = T$$

Fourier Series of Periodic Functions

Hence, the only term in the series that survives the integration is the term with $n = k$, and we obtain

$$\int_0^T f(t)e^{-ik\omega t}dt = c_n T$$

In other words, relabeling the integer k as n, we have the formula for the coefficients c_n:

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt} \qquad \textbf{(13.24)}$$

which are the coefficients for the series in (13.23). It is then straightforward to find the coefficients if the series is in the form expressed in (13.21).

Fourier Series of Periodic Functions

From the expressions in (13.22) where $a_0 = c_0$

$$\boxed{a_0 = \frac{1}{T}\int_0^T f(t)\,dt} \qquad \textbf{(13.25)}$$

and for $n = 1, 2, 3, \ldots$

$$a_n = c_n + c_{-n} = \frac{1}{T}\int_0^T f(t)\left(e^{-in\omega t} + e^{in\omega t}\right)\,dt$$

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt} \qquad \textbf{(13.26)}$$

$$b_n = i(c_n - c_{-n}) = \frac{i}{T}\int_0^T f(t)\left(e^{-in\omega t} - e^{in\omega t}\right)\,dt$$

Fourier Series of Periodic Functions

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt} \qquad \textbf{(13.27)}$$

The coefficients in (13.24) – (13.27) are called the **Fourier coefficients** of f.

Fourier Series of Periodic Functions

A useful observation is

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ is even} \\ 0 & \text{if } F \text{ is odd} \end{cases}$$

Function *F* is **even** if *F*(-*x*) = *F*(*x*) and **odd** if *F*(-*x*) = -*F*(*x*).

Since $\cos(n\omega t)$ is even and $\sin(n\omega t)$ is odd, we have:

if *f* is even: $a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{and} \quad b_n = 0$

if *f* is odd: $a_n = 0 \quad \text{and} \quad b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt$

Example

Use the (continuous) Fourier series to approximate the square or rectangular wave function

$$f(t) = \begin{cases} -1 & \text{for } -T/2 < t < -T/4 \\ 1 & \text{for } -T/4 < t < T/4 \\ -1 & \text{for } T/4 < t < T/2 \end{cases}$$

Solution:
We first calculate

$$a_0 = \frac{1}{T}\left(\int_{-T/2}^{-T/4} -dt + \int_{-T/4}^{T/4} dt - \int_{T/4}^{T/2} dt\right) = 0$$

Example

Then the coefficient a_n:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$= \frac{2}{T}\left[-\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt + \int_{-T/4}^{T/4}\cos(n\omega t)\,dt\right.$$

$$\left.-\int_{T/4}^{T/2}\cos(n\omega t)\,dt\right]$$

Solving term by term:

$$-\frac{2}{T}\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt = -\frac{2}{T}\cdot\frac{1}{n\omega}\Big[\sin(n\omega t)\Big]_{-T/2}^{-T/4}$$

Example

$$-\frac{2}{T}\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt = -\frac{2}{nT\omega}\left[\sin\left(-\frac{n\omega T}{4}\right) - \sin\left(-\frac{n\omega T}{2}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\sin\left(-\frac{n\pi}{2}\right) - \sin(-n\pi)\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

from which, the solutions for each sequence:

$$n = 1 \rightarrow \frac{1}{\pi} \qquad n = 3 \rightarrow -\frac{1}{3\pi} \qquad n = 5 \rightarrow \frac{1}{5\pi} \qquad n = 7 \rightarrow -\frac{1}{7\pi}$$

$$n = 2 \rightarrow 0 \qquad n = 4 \rightarrow 0 \qquad n = 6 \rightarrow 0 \qquad n = 8 \rightarrow 0$$

Example

The middle term gives

$$\frac{2}{T}\int_{-T/4}^{T/4}\cos(n\omega t)\,dt = \frac{2}{nT\omega}\left[\sin\left(\frac{n\omega T}{4}\right) - \sin\left(-\frac{n\omega T}{4}\right)\right]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Checking the solutions for each n:

$n = 1 \rightarrow \frac{2}{\pi}$ $\quad n = 3 \rightarrow -\frac{2}{3\pi}$ $\quad n = 5 \rightarrow \frac{2}{5\pi}$ $\quad n = 7 \rightarrow -\frac{2}{7\pi}$

$n = 2 \rightarrow 0$ $\quad n = 4 \rightarrow 0$ $\quad n = 6 \rightarrow 0$ $\quad n = 8 \rightarrow 0$

Example

And the last term gives

$$-\frac{2}{T}\int_{T/4}^{T/2}\cos(n\omega t)\,dt = -\frac{2}{nT\omega}\left[\sin\left(\frac{n\omega T}{2}\right) - \sin\left(\frac{n\omega T}{4}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Checking the solutions for each *n*:

$$n = 1 \rightarrow \frac{1}{\pi} \qquad n = 3 \rightarrow -\frac{1}{3\pi} \qquad n = 5 \rightarrow \frac{1}{5\pi} \qquad n = 7 \rightarrow -\frac{1}{7\pi}$$

$$n = 2 \rightarrow 0 \qquad n = 4 \rightarrow 0 \qquad n = 6 \rightarrow 0 \qquad n = 8 \rightarrow 0$$

Example

Collecting all solutions from all terms gives us

$$\text{for } n = 1 \longrightarrow \qquad a_1 = \frac{1}{\pi} + \frac{2}{\pi} + \frac{1}{\pi} = \frac{4}{\pi}$$

$$\text{for } n = 3 \longrightarrow \qquad a_3 = -\frac{1}{3\pi} - \frac{2}{3\pi} - \frac{1}{3\pi} = -\frac{4}{3\pi}$$

$$\text{for } n = 5 \longrightarrow \qquad a_5 = \frac{1}{5\pi} + \frac{2}{5\pi} + \frac{1}{5\pi} = \frac{4}{5\pi}$$

$$\text{for } n = 7 \longrightarrow \qquad a_7 = -\frac{1}{7\pi} - \frac{2}{7\pi} - \frac{1}{7\pi} = -\frac{4}{7\pi}$$

$$\text{for } n = 9 \longrightarrow \qquad a_9 = \frac{1}{9\pi} + \frac{2}{9\pi} + \frac{1}{9\pi} = \frac{4}{9\pi}$$

etc.

Example

In general

$$a_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n = 1, 5, 9, \ldots \\[2ex] -\dfrac{4}{n\pi} & \text{for } n = 3, 7, 11, \ldots \\[2ex] 0 & \text{for } n = \text{ even integers} \end{cases}$$

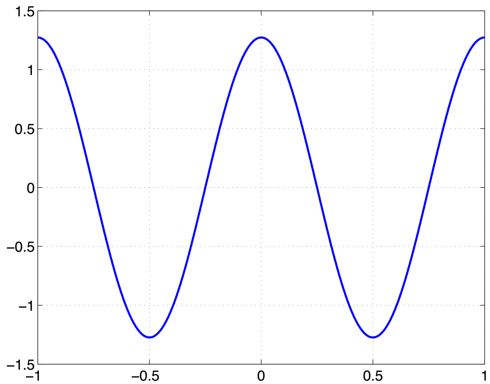
Similarly, it can be determined that all the b's = 0. Therefore the Fourier series approximation is

$$f(t) = \frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \frac{4}{5\pi}\cos(5\omega t) + \ldots$$

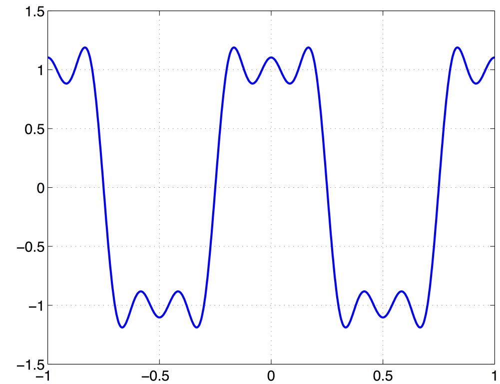
Example

$$f(t) = \frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \frac{4}{5\pi}\cos(5\omega t) + \ldots$$

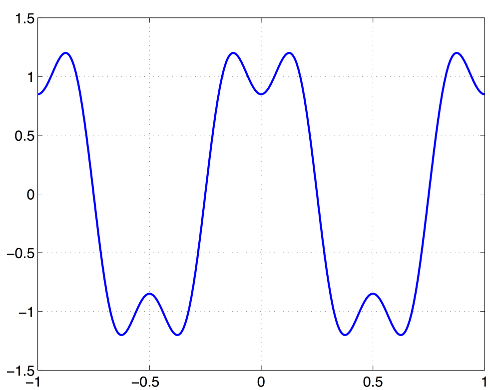
$n = 1$



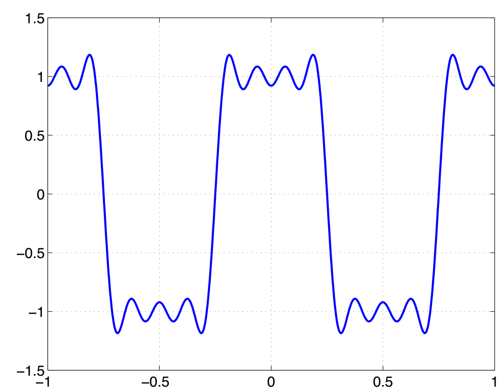
$n = 5$



$n = 3$



$n = 7$



Fourier Integral and Transform

Fourier integral is a tool used to analyze non-periodic waveforms or non-recurring signals, such as lightning bolts. Fourier integral formula is derived from Fourier series by allowing the period to approach infinity:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)\, e^{i\omega t}\, d\omega \qquad \textbf{(13.28)}$$

where the coefficients become a continuous function of the frequency variable ω, as in

$$F(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad \textbf{(13.29)}$$

Fourier Integral and Transform

The function $F(i\omega)$ is called the Fourier integral of $f(t)$. The function $F(i\omega)$ is also called the Fourier transform of $f(t)$. In the same spirit, $f(t)$ is referred to as the inverse Fourier transform of $F(i\omega)$.

The pair allows us to transform back and forth between the time and the frequency domains for a non-periodic signal.

Discrete Fourier Transform

For functions that are represented by finite sets of discrete values we apply Discrete Fourier Transform. For example, an interval 0 to t is to be divided into N equal subintervals with width $\Delta t = T/N$. The data points are specified at $n = 0$, 1, 2, …, N-1. The last value at $n = N$ is not included. The discrete Fourier transform is given by

$$F_k = \sum_{n=0}^{N-1} f_n \, e^{-ik\omega n} \qquad \textbf{(13.30)}$$

for $k = 0, 1, \ldots, N-1$ and the inverse Fourier transform is

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k \, e^{ik\omega n} \qquad \textbf{(13.31)}$$

$n = 0, 1, \ldots, N-1$

References

1) Fourier Analysis and Its Applications, Gerald B. Folland