GAUSSIAN ELIMINATION

$$Ax = W$$

* Starting of with this set of linear equations.
* We also assume A is non-singular.

After $n-1$ eliminations, we get a so-called upper triangular matrix

$$BX = Y$$

We see that,

$$x_m = \frac{1}{b_{mm}}\left(y_m - \sum_{k=m+1}^{n} b_{mk} x_k\right)$$

$$m = n-1, n-2, \ldots 1$$

The forward substitution method in gaussian elimination method converts the matrix coefficients A and W

$$a_{jk}^{(m+1)} = a_{jk}^{(m)} - \frac{a_{jm}^{(m)}\, a_{mk}^{(m)}}{a_{mm}^{(m)}}$$

$$j, k = m+1, \ldots n$$

Tridiagonal Matrices

Suppose we want to solve the following boundary value equation:

$$-\frac{d^2u(x)}{dx^2} = f(x, u(x))$$

with $x \in (a,b)$ and $U(a) = U(b) = 0$

being the Boundary condition

To solve this D.E we approximate the second derivative.

$$f'' = \frac{f_h - 2f_0 + f_{-h}}{h^2} + O(h^2)$$

$x \in (a,b) \longrightarrow$ We subdivide into n subintervals by setting $x_i = ih$, with $i = 0, 1, \ldots\ldots n+1$

Step size : $h = \frac{(b-a)}{n+1} \quad (n \in \mathbb{N})$

$\therefore$ The equation becomes,

$$u''(x_i) \approx \frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2}$$

$$u_i'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$-\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) = f(x_i, u(x_i))$$

$$i = 1, 2, \dots n$$

If we define a matrix

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

This is a trigonal matrix **A** and the corresponding

$$\mathbf{U} = (U_1, U_2, \dots U_n)^T$$

$$f(U) = f(x_1, x_2 \dots x_n, U_1, U_2, \dots U_n)^T$$

This is just a linear equation!

$$Au = f(u)$$

We can write it for a general trigonal matrix case.

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots \\ a_2 & b_2 & c_2 & \cdots & \cdots & \cdots \\ & a_3 & b_3 & c_3 & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{bmatrix}$$

$$= \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

The tridigonal system can be written as.

$$a_i u_{i-1} + b_i u_i + c_i u_{i+1} = f_i$$

for $i = 1, 2, \ldots n$

We also see that $a_1 = c_n = 0$

& u_{-1} & u_{n+1} are not required.

- In many cases the matrix is symmetric and we have $a_i = c_i$

The algorithm is just a normal Gaussian elimination one.

But due to its simplicity,

$$N(\text{float point}) = O(n)$$

while normally it is $\frac{2n^3}{3} + O(n^2)$

Forward Substitution

$$\tilde{b}_i = b_i - \frac{a_i c_{i-1}}{\tilde{b}_{i-1}}$$

& $$\tilde{f}_i = f_i - \frac{a_i \tilde{f}_{i-1}}{\tilde{b}_{i-1}}$$

Recall $\tilde{b}_1 = b_1$ & $\tilde{f}_1 = f_1$ always.

Backward Substitution:

$$u_{i-1} = \frac{\tilde{f}_{i-1} - c_{i-1} u_i}{\tilde{b}_{i-1}}$$

with $\quad u_n = \frac{\tilde{f}_n}{\tilde{b}_n} \quad$ when $i = n$

In the case of a D.E like this

$$\frac{d^2 u(x)}{dx^2} = f(x, u(x))$$

We can see that, $\quad A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$

$$a_i = -1 \quad (i = 2, \cdots)$$

$$c_{i-1} = -1 \quad (i = 2, \cdots)$$

$$\therefore \quad \tilde{b}_i = b_i - \frac{(-1)(-1)}{\tilde{b}_{i-1}}$$

$$= 2 - \frac{1}{\tilde{b}_{i-1}}$$

$$b_1 = 2 = \tilde{b}_1$$

$$\tilde{b}_2 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\tilde{b}_3 = 2 - \frac{1}{\left(\frac{3}{2}\right)} = 2 - \frac{2}{3} = \frac{4}{3}$$