Finite Difference Methods 5
(Advection Equations)

Advection

Scalar advection equation takes the (general) form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad \textbf{(18.1)}$$

where a is a constant. For the Cauchy* problem, the equation is completed with initial condition

$$u(x,0) = \eta(x) \qquad \textbf{(18.2)}$$

This is the simplest example of a hyperbolic equation, where its exact solution is:

$$u(x,t) = \eta(x - at) \qquad \textbf{(18.3)}$$

* Cauchy problem: on $-\infty < x < \infty$ (without boundary)

Advection

If the spatial dependence is approximated using the central difference and the temporal dependence is approximated using the forward difference, we get

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = -\frac{a}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right) \quad \textbf{(18.4)}$$

which can be rewritten as

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right) \quad \textbf{(18.5)}$$

If we replace $U_{i,j}$ on the right-hand side of eqn. (18.5) by the average

$$\frac{1}{2}\left(U_{i+1,j} + U_{i-1,j}\right)$$

Advection

then we obtain the **Lax-Friedrichs method**:

$$U_{i,j+1} = \frac{1}{2}\left(U_{i+1,j} + U_{i-1,j}\right) - \frac{a\Delta t}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right) \tag{18.6}$$

This method is not commonly used in practice because of its low accuracy. This method is convergent only if the following requirement is fulfilled:

$$\left|\frac{a\Delta t}{\Delta x}\right| \leq 1 \tag{18.7}$$

This stability restriction allows us to use a time step $\Delta t = \mathcal{O}(\Delta x)$ although the method is explicit.

Advection

The basic reason is that advection equation involves only the first order derivative of u_x rather than u_{xx}, so the difference equation involves $1/\Delta x$ rather than $1/\Delta x^2$.

Unlike the heat/diffusion equation, the advection equation is not stiff. This is a fundamental difference between hyperbolic equations (such as the advection equation) and parabolic equations (such as the diffusion equation). The hyperbolic equations are typically solved with explicit methods, while the efficient solution of parabolic equations generally requires implicit methods.

Method of Lines Discretization

To obtain a system of equations with finite dimension, we must solve the equation on some bounded domain rather than solving the Cauchy problem. In a bounded domain, say, $0 \leq x \leq 1$ the advection equation can have a boundary condition specified on only one of the two boundaries.

If ***a* > 0**, then we need a boundary condition at $x = 0$, say,

$$u(0,t) = g_0(t) \tag{18.8}$$

which is the inflow boundary. The boundary at $x = 1$ is the outflow boundary and the solution at this boundary is completely determined by what is advecting to the right from the interior.

Method of Lines Discretization

If $a < 0$, then we need a boundary condition at $x = 1$, which is the inflow boundary in this case.

If we consider the special case of periodic boundary conditions,

$$u(0,t) = u(1,t) \quad \text{for} \quad t \geq 0 \tag{18.9}$$

where these conditions say that whatever flows out at the outflow boundary flows back in at the inflow boundary.

In this case, the value $U_0(t) = U_{M+1}(t)$ along the boundaries is another unknown, where we must introduce one of these into the vector $U(t)$.

Method of Lines Discretization

The vector of grid values

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{M+1}(t) \end{bmatrix} \qquad \textbf{(18.9)}$$

For $2 \leq i \leq M$ we have the ordinary differential equation

$$\frac{dU_i}{dt} = -\frac{a}{2\Delta x}(U_{i+1} - U_{i-1})$$

where the first and last equations are modified using the periodicity, giving

Method of Lines Discretization

$$\frac{dU_1}{dt} = -\frac{a}{2\Delta x}(U_2 - U_{M+1})$$

$$\frac{dU_{M+1}}{dt} = -\frac{a}{2\Delta x}(U_1 - U_M)$$

which can written as

$$\frac{dU}{dt} = \mathbf{A}U \qquad \textbf{(18.10)}$$

where matrix A is represented as

Method of Lines Discretization

$$\mathbf{A} = -\frac{a}{2\Delta x}\begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{bmatrix} \in \mathbb{R}_{(M+1)\times(M+1)} \qquad \textbf{(18.11)}$$

Note that this matrix is skew-symmetric $(A^T = -A)$ and so its eigenvalues must be pure imaginary:

$$\lambda_p = -\frac{\operatorname{Im} a}{\Delta x}\sin(2\pi p \Delta x) \qquad \textbf{(18.12)}$$

for $p = 1, 2, 3, \ldots, M+1$.

Method of Lines Discretization

The corresponding eigenvector u^p has components

$$u_i^p = e^{2\pi \, \mathrm{Im} \, p i \Delta x} \qquad \textbf{(18.13)}$$

for $i = 1, 2, 3, \ldots, M+1$.

The eigenvalues lie on the imaginary axis between $-\mathrm{Im}\; a/\Delta x$ and $\mathrm{Im}\; a/\Delta x$.

Leapfrog Method

If we discretize time by using the midpoint method

$$U_{j+1} = U_{j-1} + 2\Delta t \mathbf{A} U_j$$

it gives us the **leapfrog method** for the advection equation

$$U_{i,j+1} = U_{i,j-1} - \frac{a\Delta t}{\Delta x}(U_{i+1,j} - U_{i-1,j}) \qquad \textbf{(18.14)}$$

This is a 3-level explicit method and is second order accurate in both space and time. This method is stable on the advection equation provided that

$$\left|\frac{a\Delta t}{\Delta x}\right| < 1 \qquad \textbf{(18.15)}$$

Leapfrog Method

This leapfrog method is only marginally stable, meaning there is no growth but also no decay of any eigenmode. The difference equation is said to be non-dissipative. The true advection equation is also non-dissipative, and any initial condition simply translates unchanged, no matter how oscillatory. Leapfrog method captures this qualitative behavior well.

However, there are problems with this. All modes translate without decay, but they do not all propagate at the correct velocity. As a result initial data that contains high wave number components (*e.g.*, if the data contains steep gradients) will disperse and can result in highly oscillatory numerical approximations.

Lax-Friedrichs Method

Using the fact that

$$\frac{1}{2}\left(U_{i+1,j}+U_{i-1,j}\right)=U_{i,j}+\frac{1}{2}\left(U_{i+1,j}-2U_{i,j}+U_{i-1,j}\right) \quad \textbf{(18.16)}$$

we rewrite equation (18.16) to get

$$\begin{aligned} U_{i,j+1} = U_{i,j} &- \frac{a\Delta t}{2\Delta x}\left(U_{i+1,j}-U_{i-1,j}\right) \\ &+ \frac{1}{2}\left(U_{i+1,j}-2U_{i,j}+U_{i-1,j}\right) \end{aligned} \quad \textbf{(18.17)}$$

The right hand side vanishes as Δt, $\Delta x \to 0$, assuming $\Delta t/\Delta x$ is fixed.

Lax-Friedrichs Method

Rearranging gives

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} + a\left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}\right) = \frac{\Delta x^2}{2\Delta t}\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}\right) \tag{18.18}$$

Equation (18.17) looks like discretization of the advection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$$

where $D = \dfrac{\Delta x^2}{2\Delta t}$.

Lax-Friedrichs Method

Equation (18.17) can be viewed as a forward Euler discretization of the system of ODEs

$$\frac{d\mathbf{U}}{dt} = \mathbf{A}_D\mathbf{U} \qquad \textbf{(18.19)}$$

where

$$\mathbf{A}_D = -\frac{a}{2\Delta x}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix} + \frac{D}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

The method is stable provided that $\left|\frac{a\Delta t}{\Delta x}\right| \leq 1$.

Lax-Wendroff Method

The Lax-Wendroff method is given by

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right) \tag{18.20}$$

Upwind Methods

The upwind methods with forward differencing in time and backward and forward in space are given by

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{\Delta x}\left(U_{i,j} - U_{i-1,j}\right) \qquad \textbf{(18.21)}$$

or

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{\Delta x}\left(U_{i+1,j} - U_{i,j}\right) \qquad \textbf{(18.22)}$$

These methods are first order accurate in both space and time. If $a > 0$ the solution moves to the right, while if $a < 0$ the solution moves to the left.

Upwind Methods

The choice between the two methods (18.21) and (18.22) should be dictated by the sign of a.

The method in (18.21) is stable only if

$$0 \leq \frac{a\Delta t}{\Delta x} \leq 1 \qquad \textbf{(18.23)}$$

which can only be used if $a > 0$. And conversely, the method in (18.22) is stable only if

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 0 \qquad \textbf{(18.24)}$$

which can be used only if $a < 0$.

Beam-Warming Method

The Beam-Warming method is second order accurate.
For $a > 0$, the method takes the form

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{2\Delta x}\left(3U_{i,j} - 4U_{i-1,j} + U_{i-2,j}\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(U_{i,j} - 2U_{i-1,j} + U_{i-2,j}\right) \tag{18.25}$$

For $a < 0$, the method takes the form

$$U_{i,j+1} = U_{i,j} - \frac{a\Delta t}{2\Delta x}\left(-3U_{i,j} + 4U_{i+1,j} - U_{i+2,j}\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(U_{i,j} - 2U_{i+1,j} + U_{i+2,j}\right) \tag{18.26}$$

References

[1] Finite Difference Methods for Differential Equations, Randall J. LeVeque

End of Lecture 18