ODE-Basic Algorithms

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1 Euler Method

1.1 Forward Derivative

1. The formal definition of the derivative is,

$$f^{'}(t) = lim_{\tau \to 0} \frac{f(t+\tau) - f(t)}{\tau}$$

2. From the definition of Taylor's theorem, we can write:

$$f^{'}(t) = \frac{f(t+\tau) - f(t)}{\tau} - \frac{1}{2}\tau f^{''}(\zeta)$$

   where $t \leq \zeta \leq t + \tau$. This is the *right derivative* or *forward derivative formula*. The last term is the truncation error which is of the order of τ here.

1.2 Euler's Method

Consider the equations of motion here, which I want to solve numerically,

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(\mathbf{r}, \mathbf{v})$$

$$\frac{dr}{dt} = \mathbf{v}$$

Using the forward derivative equation, we can write these as,

$$\mathbf{v}(t+\tau) = \mathbf{v}(t) + \tau\mathbf{a}(\mathbf{r}(t), \mathbf{v}(t)) + \mathcal{O}(\tau^2)$$
$$\mathbf{r}(t+\tau) = \mathbf{r}(t) + \tau\mathbf{v}(t) + \mathcal{O}(\tau^2)$$

Our notation will be, $f_n = f(t_n)$, $t_n = (n-1)\tau$
The Euler method equations become,

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau\mathbf{a}_n$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\mathbf{v}_n$$

1.3 Euler-Cromer Method

Instead of v_n in the quation, we put the modified v_{n+1}

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau \mathbf{a}_n$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau \mathbf{v}_{n+1}$$

The truncation is still of $\mathcal{O}(\tau^{\in})$.

1.4 Midpoint Method

We can have the midpoint of velocities between vn and vn+1

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau \mathbf{a}_n$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau \frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2}$$

Plugging the velcoity equation into the position equation, we see that

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau \mathbf{v}_n + \frac{1}{2} \mathbf{a}_n \tau^2$$

The order is $\mathcal{O}(\tau^3)$
global error = N_τ x (local error) = (T/τ)$\mathcal{O}(\tau^n) = T\mathcal{O}(\tau^{n-1})$

Centered derivative formulas

The Euler Method was based on the right derivative formulation for df.dt. Now we can use the centered derivative formula on ODE algorithms.

$$f^{'}(t) = \lim_{\tau \to 0} \frac{f(t+\tau) - f(t-\tau)}{2\tau}$$

Using the taylor series expansion of $f(t+\tau)$ and $f(t-\tau)$, we ger

$$f^{'}(t) = \frac{f(t+\tau) - f(t-\tau)}{2\tau} - \frac{1}{6}\tau^2 f^3(\zeta)$$

where $t - \tau \leq \zeta \leq t + \tau$.

In this *centered first derivative approximation* the truncation error is now quadratic in τ, which is better when compared to the basi Euler algorithm.

We also see that, we can build the second derivative in a similar fashion

$$f^{''}(t) = \frac{f(t+\tau) - f(t-\tau) - 2f(t)}{\tau^2} - \frac{1}{12}\tau^2 f^4(\zeta)$$

where$t - \tau \leq \zeta \leq t + \tau$.

Leap-Frog method

Starting from the equations of motion,

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \mathbf{a}(\mathbf{r}(t)) \\
\frac{d\mathbf{r}}{dt} &= \mathbf{v}(t)
\end{aligned}
$$

Now,

$$\frac{\mathbf{v}(t+\tau) - \mathbf{v}(t-\tau)}{2\tau} + \mathcal{O}(\tau^2) = \mathbf{a}(\mathbf{r}(t))$$

For position, we are centering it between $t + 2\tau$ and t. The reason will be apparent soon.

$$\frac{\mathbf{r}(t+2\tau) - \mathbf{r}(t-\tau)}{2\tau} + \mathcal{O}(\tau^2) = \mathbf{v}(t+\tau)$$

Rearranging the terms and using our previous notation,

$$
\begin{aligned}
\mathbf{v}_{n+1} &= \mathbf{v}_{n-1} + 2\tau a(\mathbf{r}_n) + \mathcal{O}(\tau^3) \\
\mathbf{r}_{n+2} &= \mathbf{r}_n + 2\tau \mathbf{v}_{n+1} + \mathcal{O}(\tau^3)
\end{aligned}
$$

We are advancing in steps of 2τ hence the name 'leap-frog'.

Verlet Method

Using the central difference formulas for first and second derivatives, we have

$$
\begin{aligned}
\frac{r_{n+1} - r_{n-1}}{2\tau} + \mathcal{O}(\tau^2) &= v_n \\
\frac{r_{n+1} - r_{n-1} - 2r_n}{\tau^2} + \mathcal{O}(\tau^2) &= a_n
\end{aligned}
$$

Rearranging terms,

$$
\begin{aligned}
\mathbf{v}_n &= \frac{\mathbf{r}_{n+1} - \mathbf{r}_{n-1}}{2\tau} + \prime(\tau^2) \\
\mathbf{r}_{n+1} &= 2\mathbf{r}_n - \mathbf{r}_{n-1} + \tau^2 \mathbf{a}_n + \mathcal{O}(\tau^4)
\end{aligned}
$$