Finite Difference Methods (FDMs) 3

Explicit and Implicit Methods

Approximations of u at arbitrary grid point (x_i, t_j) :

$$u = U_{i,j}$$

$$\frac{\partial u}{\partial t} = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} \qquad \text{(explicit/forward)}$$

or

$$\frac{\partial u}{\partial t} = \frac{U_{i,j} - U_{i,j-1}}{\Delta t} \qquad \text{(implicit/backward)}$$

$$\frac{\partial u}{\partial x} = \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}$$

Crank-Nicolson Method

Approximations of u and its derivatives:

$$u \approx \frac{U_{i,j} + U_{i,j+1}}{2}$$

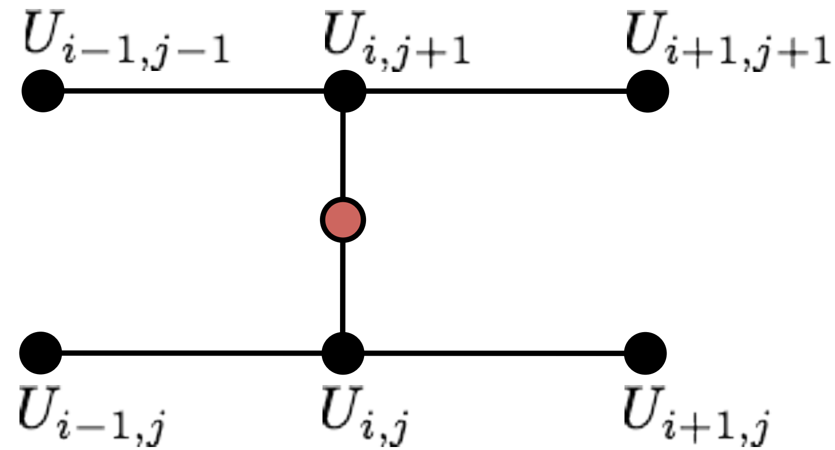
$$\frac{\partial u}{\partial t} \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

$$\frac{\partial u}{\partial x} \approx \frac{U_{i+1,j+1} - U_{i-1,j+1} + U_{i+1,j} - U_{i-1,j}}{4\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{2\Delta x^2} + \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{2\Delta x^2}$$

Crank-Nicolson Method

Numerical stencil for illustrating the Crank-Nicolson method.



Crank-Nicolson method is the recommended approximation algorithm for most problems because it has the virtues of being unconditionally stable. This method also is second order accurate in both the x and t directions, where we still can get a given level of accuracy with a coarser grid in the time direction, hence less computation cost.

Crank-Nicolson Method

In Crank-Nicolson method, the partial derivatives are centered around $\left(x_i, t_{j+\frac{\Delta t}{2}}\right)$ rather than around (x_i, t_j).

$$u = \frac{U_{i,j} + U_{i,j+1}}{2}$$

$$\frac{\partial u}{\partial t} = \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{U_{i+1,j+1} - U_{i-1,j+1}}{2\Delta x}\right) + \frac{1}{2}\left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\Delta x^2}\right) + \frac{1}{2}\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}\right)$$

Crank-Nicolson Method

For a 1D diffusion equation

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

the approximation using Crank-Nicolson method is given by

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = \frac{D}{2\Delta x^2}\big(U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1} \\ + U_{i-1,j} - 2U_{i,j} + U_{i+1,j}\big)$$

Substituting $\dfrac{D\Delta t}{2\Delta x^2}$ with r and rearranging gives the implicit difference formula

$$-rU_{i-1,j+1} + (1+2r)U_{i,j+1} - rU_{i+1,j+1} = \\ rU_{i-1,j} + (1-2r)U_{i,j} + rU_{i+1,j}$$

Crank-Nicolson Method

If Neumann BC is imposed along $x = 0$ where $\partial_n u = -\partial_x u = g$, the difference formula is given by

$$\begin{aligned}(1+2r)U_{0,j+1} - 2rU_{1,j+1} = (1-2r)U_{0,j} \\ +\ 2rU_{1,j} + 4r\Delta x g_{0,j}\end{aligned}$$

If Neumann BC is imposed along $x = L$ or at $i = M$, where $\partial_n u = \partial_y u = f$, the difference formula is given by

$$\begin{aligned}-2rU_{M-1,j+1} + (1+2r)U_{M,j+1} = 2rU_{M-1,j} \\ +\ (1+2r)U_{M,j} + 4r\Delta x f_{M,j}\end{aligned}$$

MATLAB WORK 1

Solve numerically the following diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0 \qquad\qquad u_x(1,t) = 0$$

$$u(x,0) = \sin\left(\frac{x\pi}{2}\right) \qquad 0 < x < 1$$

using implicit/backward and Crank-Nicolson methods.

If the exact solution is $u(x,t) = e^{-(\pi^2/4)t} \sin\left(\frac{x\pi}{2}\right)$

calculate the absolute error at $x = 0.5$

Method of Lines (MOL)

The method of lines (MOL) is a technique for solving time-dependent PDEs by replacing the spatial derivatives with algebraic approximations and letting the time variable remain independent variable. Thus, we have a system of ODEs that approximate the original PDE.

For example, a diffusion equation

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

is approximated using central finite difference to

$$\frac{dU_i}{dt} = D\,\frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} \qquad i = 1, 2, \ldots, M \quad \textbf{(16.1)}$$

Method of Lines (MOL)

If Dirichlet BC is imposed at $x = 0$, where $u(0, t) = g$, then

$$U_1 = g$$

Therefore the ODE of equation (16.1) for $i = 1$ is not required and the ODE for $i = 2$ becomes

$$\frac{dU_2}{dt} = D\,\frac{U_3 - 2U_2 + g}{\Delta x^2}$$

At $i = M$, we have

$$\frac{dU_M}{dt} = D\,\frac{U_{M+1} - 2U_M + U_{M-1}}{\Delta x^2}$$

The value at $i = M+1$ is obtain from the boundary.

MATLAB WORK 2

Solve the following reaction diffusion equation using MOL

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + \lambda u(1-u)$$

$$\lambda = 1 \qquad D = 1$$

$$u(x,0) = \sin(\pi x) \qquad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0$$

$$\Delta x = 0.1$$

MATLAB WORK 3

Solve the following reaction diffusion equation using MOL

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + \lambda u(1-u)(u-a)$$

$$D = 1 \qquad \lambda = 1 \qquad a = 0.2$$

$$u(x,0) = \sin(\pi x) \qquad\qquad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0$$

$$\Delta x = 0.1$$

Crank-Nicolson Method

http://www.dynamicearth.de/compgeo/

http://www.scholarpedia.org/article/Method_of_lines