Finite Difference Methods 6 (ADI Scheme)

Method Of Lines

In MATLAB, use **del2** to discretize Laplacian in 2D space.

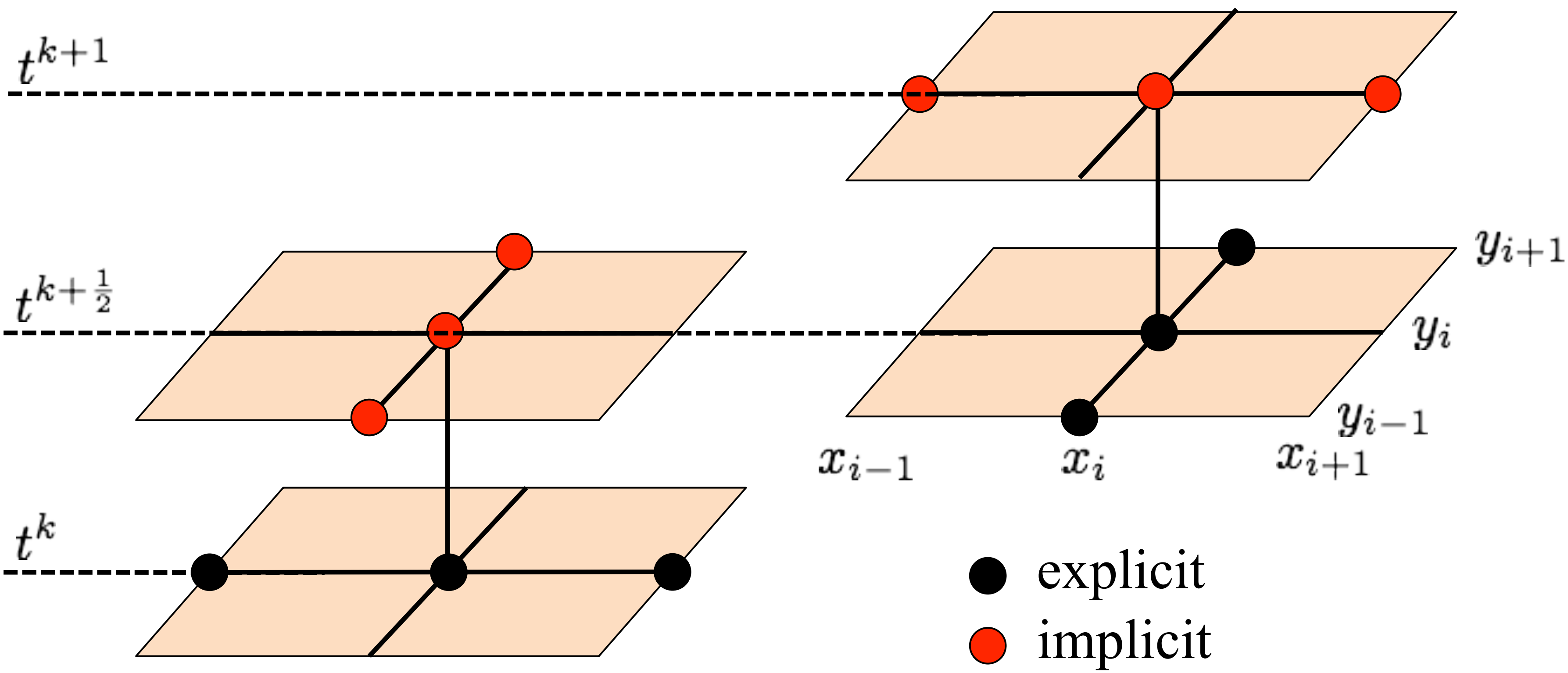
If the matrix U is regarded as a function $u(x,y)$ evaluated at the point on a square grid, then 4*del2(U) is a finite difference approximation of Laplace's differential operator applied to u, that is

$$l = \frac{\nabla^2 u}{4} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

ADI Scheme

The alternating-direction implicit, or ADI, scheme provides a means for solving parabolic equations in 2-spatial dimensions using tri-diagonal matrices.

To do this, each time increment is executed in two steps.



ADI Scheme

For the first step, the diffusion equation

$$u_t = D\,\nabla^2 u \tag{19.1}$$

is approximated by

$$\frac{U_{i,j}^{k+\frac{1}{2}} - U_{i,j}^{k}}{\Delta t/2} = D\left(\frac{U_{i+1,j}^{k} - 2U_{i,j}^{k} + U_{i-1,j}^{k}}{\Delta x^2} + \frac{U_{i,j+1}^{k+\frac{1}{2}} - 2U_{i,j}^{k+\frac{1}{2}} + U_{i,j-1}^{k+\frac{1}{2}}}{\Delta y^2}\right) \tag{19.2}$$

For the case $\Delta x = \Delta y = h$ and letting $r = \dfrac{\Delta t D}{2h^2}$ the eqn can be expressed as

$$-rU_{i,j-1}^{k+\frac{1}{2}} + (1+2r)U_{i,j}^{k+\frac{1}{2}} - rU_{i,j+1}^{k+\frac{1}{2}} = rU_{i-1,j}^{k} + (1-2r)U_{i,j}^{k} + rU_{i+1,j}^{k} \tag{19.3}$$

which is explicit.

ADI Scheme

For the second step from $t^{k+\frac{1}{2}}$ to t^{k+1} equation (19.1) is approximated by

$$\frac{U_{i,j}^{k+1} - U_{i,j}^{k+\frac{1}{2}}}{\Delta t/2} = D\left(\frac{U_{i+1,j}^{k+1} - 2U_{i,j}^{k+1} + U_{i-1,j}^{k+1}}{h^2} + \frac{U_{i,j+1}^{k+\frac{1}{2}} - 2U_{i,j}^{k+\frac{1}{2}} + U_{i,j-1}^{k+\frac{1}{2}}}{h^2}\right) \tag{19.4}$$

Rearranging gives

$$-rU_{i-1,j}^{k+1} + (1+2r)U_{i,j}^{k+1} - rU_{i+1,j}^{k+1} = rU_{i,j-1}^{k+\frac{1}{2}} + (1-2r)U_{i,j}^{k+\frac{1}{2}} + rU_{i,j+1}^{k+\frac{1}{2}} \tag{19.5}$$

which is implicit.

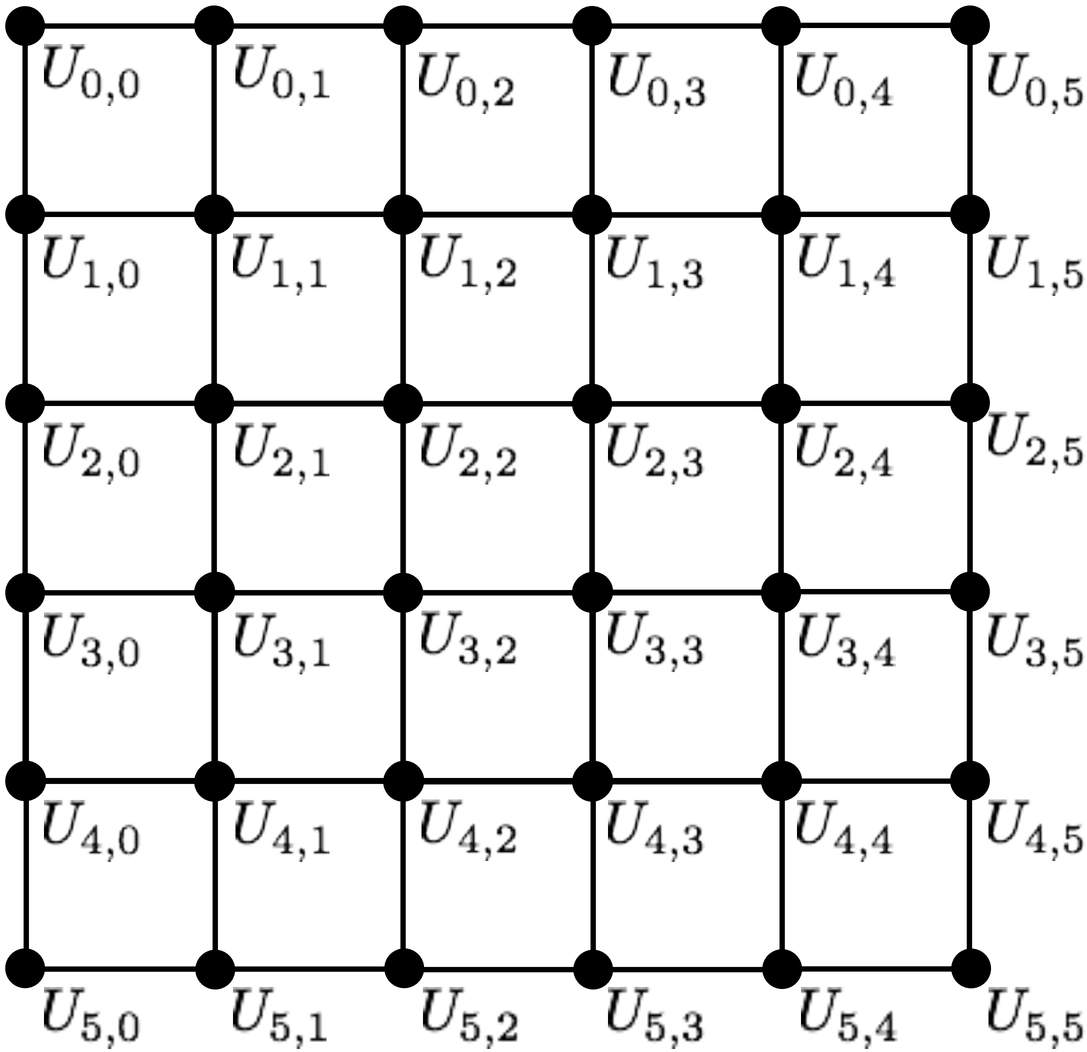
When writing for a 2-dimensional grid, the equation results in a tri-diagonal system.

ADI Scheme

To solve equation (19.3), fix $i = 1, 2, \ldots, M - 1$ and solve a tridiagonal system to get $U_{i,j}^{k+\frac{1}{2}}$ for $j = 1, 2, \ldots, N - 1$.

To solve equation (19.5), fix $j = 1, 2, \ldots, N - 1$ and solve a tridiagonal system to get $U_{i,j}^{k+1}$ for $i = 1, 2, \ldots, M - 1$.

End of Lecture 19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