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Parameter Estimation

Q1 $(x_1, x_2, \ldots, x_n) \rightarrow$ random sample.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

MLE of 2 parameters

$$L(x_1 \ldots x_n) = L(x_1) \times L(x_2) \ldots L(x_n)$$

$$= \frac{1}{\sqrt{2\pi\theta^2}} e^{(-x-\mu)/2\theta_1} \times \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(x-\mu)^2}{2\theta_2}}$$

$$= \frac{1}{\sqrt{2\pi\theta^2}} \prod_{i=1}^{n} e^{-\frac{(x-\mu)^2}{\sigma^2}}$$

Taking log on both sides

$$\ln(L(x_1, x_2 \ldots x_n)) = \ln\left(\frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_1-\theta_1)^2}{2\pi\theta_2}} \times \ldots \ldots \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_n-\theta_1)^2}{2\theta_2}}\right)$$

$$\Rightarrow \ln\left(\frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_1-\theta_1)^2}{2\pi\theta_2}}\right) + \ln\left(\frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_2-\theta_1)^2}{2\theta_2}}\right)^2 \ldots$$

$$\ldots \ldots \ln\left(\frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_n-\theta_1)^2}{2\theta_2}}\right)$$

$\therefore \ln\left(\frac{1}{\sqrt{2\pi\theta_2}}\right) + \ln\left(e^{-(x_1-\theta_1)^2/2\theta_2}\right)$ can be written as

$$\Rightarrow \ln\left(\frac{1}{\sqrt{2\pi\theta_2}}\right) + \ln\left(e^{-(x_1-\theta_1)^2/2\theta_2}\right) \rightarrow 2$$

$$\Rightarrow \ln\left(\frac{1}{\sqrt{2\pi\theta_2}}\right) \Rightarrow \ln\left((2\pi\theta_2)^{-1/2}\right)$$

Simplifying 2 and substituting $\theta_2 = \sigma^2$

$$-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x_1-\theta_1)^2}{2\theta_2}\ln(e)$$

$$\Rightarrow -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\theta_2) - \frac{(x_1-\theta_1)}{2\theta_2} \Rightarrow [\ln e = 1]$$

$$\Rightarrow -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\theta_2 - \frac{(x-\theta_1)^2}{2\theta_2}$$

Similarly, we can get this expression for n-1 terms

$$\therefore L(x_1, x_2 \ldots x_n) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta_2 - \frac{(x_1-\theta_1)^2}{2\theta_2} \ldots \frac{(x_n-\theta_1)^2}{2\theta_2}$$

Taking $\frac{\delta}{\delta\theta_1}$ on both sides.

$$\frac{\delta(L(x_1 \ldots x_n))}{\delta\theta_1} \Rightarrow 0 - 0 + \frac{x_1-\theta_1}{\theta_2} + \ldots \frac{x_n-\theta_1}{\theta_2}$$

$$\Rightarrow \frac{1}{\theta_2}(x_1 - \theta_1 \ldots x_n - \theta_1)$$

$$\frac{\delta(L(x_1 - x_n))}{\delta\theta_1} = \frac{1}{\theta_2}\left[(x_1 + x_2 \ldots x_n) - n\theta_1\right]$$

Taking $\frac{\delta}{\delta \theta_2}$

$$\frac{\delta L(x_1 + \dots x_n)}{\delta \theta_2} = 0 - \frac{n}{2} \times \frac{1}{\theta_2} + \frac{(x_1 - \theta_1)^2}{2\theta_2^2} + \frac{(x_2 - \theta_1)^2}{2\theta_2^2} \dots \frac{(x_n - \mu)^2}{2\theta_2^2}$$

$$-\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2}((x_1 - \theta_1)^2 \dots (x_n - \theta_1)^2)$$

For MLE over θ_1 put $\frac{\delta L(x_1 \dots x_n)}{\delta \theta_1} = 0$

$$0 = \frac{1}{\theta_2}\left[(x_1 \dots x_n) - n\theta_1\right].$$

$$\frac{x_1 + x_2 + \dots x_n}{n} = \theta_1$$

$$\Rightarrow \theta_1 = \text{Mean} = \frac{x_1 + x_2 + \dots x_n}{n}$$

For MLE over θ_2 put $\frac{\delta L(x_1 \dots x_n)}{\delta \theta_2} = 0$

$$0 = \frac{-n}{2\theta_2} + \frac{1}{2\theta_2^2}((x_1 - \theta_1)^2 \dots (x_n - \theta_1)^2)$$

$$0 = -n + \frac{1}{\theta_2}((x_1 - \theta_1)^2 \dots (x_n - \theta_1)^2)$$

$$n\theta_2 = (X_1-\theta_1)^2 + \dots + (X_n-\theta_1)^2$$

$$\boxed{\theta_2 = \frac{(X_1-\theta_1)^2 + \dots + (X_n-\theta_1)^2}{n}}$$

↳ Variance for measurement

Q2 $X_1 \dots X_n \in B(m, \theta)$

$\theta \in \Theta(0,1)$

⇒ This is a binomial distribution

⇒ m is +ve integer

→ MLE for θ parameters

$$P(x,\theta) = {}^{m}C_{x}\, \theta^{x}(1-\theta)^{m-x}$$

$$L(x_1 \dots x_n) = \prod_{i=1}^{n} P(x_i;\theta)$$

Taking log on both sides

$$\ln L(x_1 \cdots x_n) = \sum_{i=1}^{n} \ln({}^{m}C_{x_i}) + x_i \ln\theta + (m-x_i)\log(1-\theta)$$

Now do $\frac{\delta(\ln(x_1 \dots x_n))}{\delta\theta}$

$$\frac{\delta \ln L(x_1 \dots x_n)}{\delta\theta} \Rightarrow \sum_{i=1}^{n} 0 + \frac{x_i}{\theta} \; \frac{-m-x_i}{1-\theta}$$

$$= \sum_{i=1}^{n} \frac{x_i (1-\theta) - (m + x_i)\theta}{\theta(1-\theta)}$$

$$= \sum_{i=1}^{n} \frac{x_i - n\theta}{\theta(1-\theta)} \quad \text{— (1)}$$

To minimize, put eq (1) = 0

$$\sum_{i=1}^{n} x_i - m\theta = 0$$

$$\sum_{i=1}^{n} x_i = m\theta n$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{mn}$$

$$\Rightarrow \boxed{\theta = \frac{\bar{x}}{m}}$$

$$\therefore \bar{x} = m\theta$$

$\therefore$ MLE for $\theta = \dfrac{\bar{x}}{m}$