CS 325 - NP-Complete Practice -Solutions

1. Is the 3-SAT problem known to be NP-complete, or is that only conjectured? Is it known whether the 3-SAT problem is in the class P? Is it known whether the 3-SAT problem is in the class NP?

    - The 3-SAT problem is known to be NP-complete,
    - It is also known that 3-SAT is in NP.
    - It is not known whether the 3-SAT problem is in P. It is in P if and only if P = NP.

2. What is the definition of the class P?

    P is the class of all decision problems that can be solved (deterministically) in polynomial time.

3. What is the definition of the class NP?

    NP is the class of all decision problems that can be solved nondeterministically in polynomial time. That is, it is the problems that possess nondeterministic polynomial time algorithms. Another way of defining NP problems is the set of decision problems for which a solution certificate can be verified in polynomial time.

4. What is the definition of a polynomial time transformation from *A* to *B*?

    Let *A* and *B* be two decision problem sets. A polynomial time tranformation from *A* to *B* is a function *F* that has both of the following characteristics.

    a. *F* is computable in polynomial time.
    b. For every *x*, *x* is in *A* if and only if *F*(*x*) is true in *B*. That is, if x is an input to A with a yes output then F(x) must be an input to B with a yes output and vis-a-versa.

5. What is the definition of an NP-complete problem? How do you prove that a problem is NP-complete?

    A problem *B* is NP-complete if both of the following are true.

    1) *B* is in NP.
    2) For every *X* in NP, there exists a polynomial time transformation from *X* to *B*. That is $X \le_p B$ for all X in NP..

    To prove that a problem *B* is NP-complete

    1) Prove that B is in NP by showing that a certificate solution can be verified in polynomial time.
    2) Show that a known NP-complete problem A can be reduced to B in polynomial time using some “function” f. That is $A \le_p B$ for some A in NP-complete. Prove that an instance x of problem A is true if and only if an instance f(x) is true for problem B.

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6. Prove that SET-PARTITION is in NP-complete

To show that SET-PARTITION is NP-Complete, we need to show

(1) That SET-PARTITION ∈ NP. Given a partition X of set S we can verify in polynomial time that the two subsets X and S-X have equal sums by adding the values in each set. This clearly takes time O(n) where n is the number of elements in S. We also must verify that each element in S belongs to exactly one partition which takes O(n).

(2) That some NP-Complete problem X can be reduced to SET-PARTITION in polynomial time and the original problem X has a yes solution if and only if SET-PARTITION has a yes solution.

We will select X to be SUBSET-SUM which has been proven to be in NP-Complete. We will need to show a polynomial time reduction from SUBSET-SUM to SET-PARTITION:

SUBSET-SUM is de fined as follows: Given a set S of integers and a target number t, find a subset X ⊆ S such that the members of X add up to exactly t. Let s be the sum of numbers in S. Feed S' = S ∪ {s − 2t} into SET-PARTITION and answer YES if and only if SET-PARTITION answers YES. This reduction clearly works in polynomial time since all we did was add a single element, s − 2t to S. To calculate s-2t we must computer the sum of all numbers in S which takes O(n) time.

We must show that <S, t> ∈ SUBSET-SUM iff <S'> ∈ SET-PARTITION. In other words there exists a subset of S that sums to t if and only if there exists a set partition of S' with equal sums. Note that the sum of numbers in S' = sum(S) + (s-2t) = s + (s-2t) = 2s − 2t = 2(s – t).

a) If there exists a subset of S that sums to t then there exists a SET-PARTITION of S'. If there exists a subset Y of numbers in S that sum to t, then the remaining numbers in S-Y sum to s − t. Now S' = (S-Y) ∪ Y ∪ {s-2t} and we can partition S' into two sets (S-Y) and Y ∪ {s-2t} such that each partition sums to s − t.

b) If there exists a SET-PARTITION of S' then there exists a subset of S that sums to t. If there exists a partition of S' into two sets X and S'-X such that the sum over each set is s−t and one of these sets say X must contain the number s−2t. Removing this number, we get a set X' = X-{s-2t} of numbers whose sum is (s – t) - (s-2t) = t, and since S' = S ∪ {s − 2t} all of the elements in X' are in S. Therefore there exists a subset X' of S that sums to t.

Since, SET-PARTITION ∈ NP and SUBSET-SUM $\leq_p$ SET-PARTITION, SET-PARTITION is in NP-Complete.