Dynamic Programming

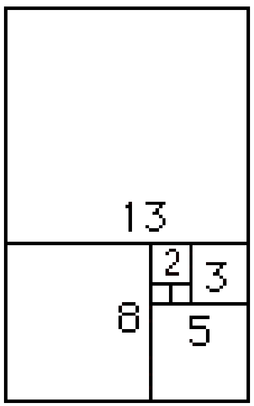
- Like divide and conquer, DP solves problems by combining solutions to subproblems.
- Unlike divide and conquer, subproblems are not unique.
  - Subproblems may share subsubproblems,
  - However, solution to one subproblem may not affect the solutions to other subproblems of the same problem.
- Key: Determine structure of optimal solutions

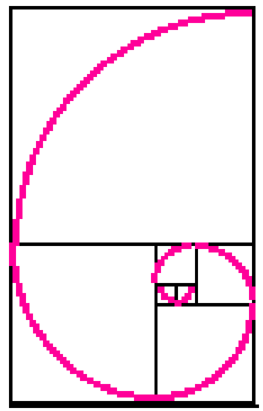
DP Examples

- Fibonacci
- Knapsack
- Rod Cutting
- Longest Common Subsequence
- Longest Increasing Subsequence
- Chain Matrix Multiplication
- CYK Algorithm
- Subset Sum

Fibonacci Sequence

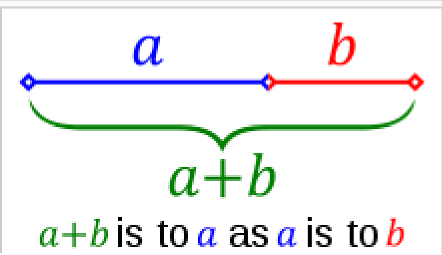
- 0,1,1,2,3,5,8,13,21,34,...





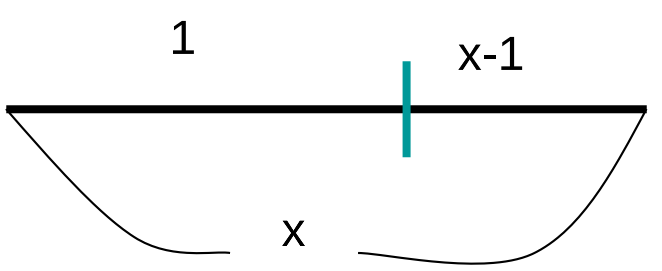
Fibonacci Number and Golden Ratio

0,1,1,2,3,5,8,13,21,34,...



$$
\begin{cases}
f_n = 0 & \text{if } n = 0 \\
f_n = 1 & \text{if } n = 1 \\
f_n = f_{n-1} + f_{n-2} & \text{if } n \geq 2
\end{cases}
$$

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \frac{1+\sqrt{5}}{2} = \text{Golden Ratio} = \phi = 1.61803..$$



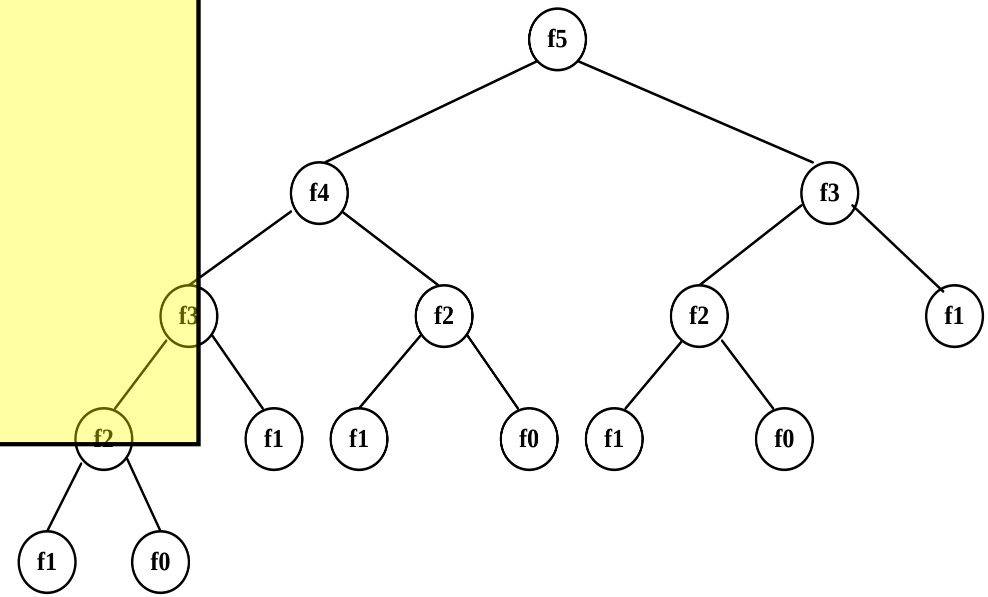
$$\frac{x}{1} = \frac{1}{x-1}$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1+\sqrt{5}}{2}$$

Naive Recursive Algorithm

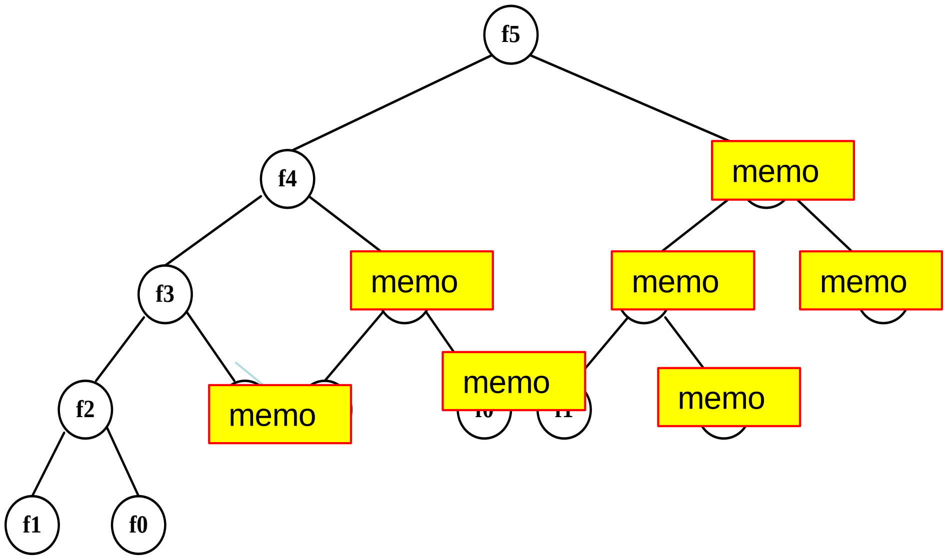
```
fib (n) {
  if (n = 0) {
     return 0;
  } else if (n = 1) {
     return 1;
  } else {
     return fib(n-1) + fib(n-2);
  }
}
```



- Solved by a recursive program
- Much replicated computation is done.
- Running time $\Theta(\phi^n)$ - exponential

Memoized DP Algorithm

```
memo = { }
fib (n) {
    if (n in memo) { return memo[n] }
    if (n <= 1) {
        f = n;
     } else {
        f = fib(n-1) + fib(n-2);
    }
    memo[n] = f;
    return f
}
```



- fib(k) only recurses the first time called only n nonmemoized calles
- Memorized calls “free” $\Theta(1)$.
- Time = #subproblems * time/subproblem

  $= n * \Theta(1)$
- Running time $\Theta(n)$ - linear

Bottom-up DP Algorithm

```
fib = { }
fib[0] = 0;
fib[1] = 1;
for k = 2 to n
      fib[k] =  fib[k-1] + fib[k-2];
return fib[n]
```

- Same as memoized DP with recursion “unrolled” into iteration.
- Practically faster since no recursion
- Analysis is more obvious
- Running time $\Theta(n)$ - linear

A Basic Idea of Dynamic Programming

- DP = recursion + memoization
  - Memoize = remember and reuse solutions to subproblems
- Bottom-Up Method stores all values in a table

Elements of Dynamic Programming

- Optimal Substructure
  - An optimal solution to a problem contains within it an optimal solution to subproblems
  - Optimal solution to the entire problem is built in a bottom-up manner from optimal solutions to subproblems
- Overlapping Subproblems
  - If a recursive algorithm revisits the same subproblems over and over $\Rightarrow$ the problem has overlapping subproblems

Dynamic Programming Algorithm

1. **Characterize** the structure of an optimal solution
2. **Recursively** define the value of an optimal solution
3. **Compute** the value of an optimal solution in a bottom-up fashion
4. **Construct** an optimal solution from computed information

Knapsack problem

Given a set of items, each with a weight and a value, pack a knapsack with a subset of items to achieve the maximum total value. Total weight that can be carried in the knapsack is no more than some fixed number W.

There are two versions:

1. "0-1 knapsack problem" use DP
   Items are indivisible: you either take an item or not.

2. "Fractional knapsack problem" Use a Greedy Method
   Items are divisible: you can take any fraction of an item

0-1 Knapsack problem:

This is a knapsack
Max weight: W = 13

Items	Weight w_i	value v_i
	2	3
	3	4
	4	5
	5	15
	10	16

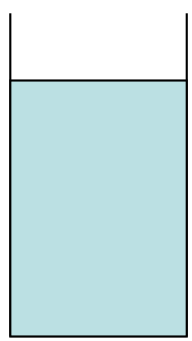
Value = 0
Weight = 0

0-1 Knapsack problem:

This is a knapsack
Max weight: $W = 13$

Items	Weight w_i	value v_i
	2	3
	3	4
	4	5
	5	15
	10	16

value = 16
Weight = 10

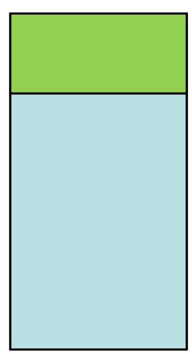


0-1 Knapsack problem:

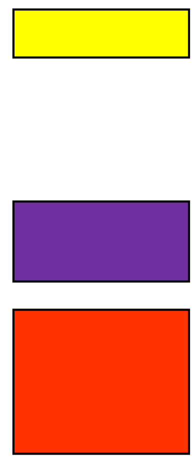
This is a knapsack
Max weight: W = 13

Is this maximum ?

value = 20
Weight = 13



Items



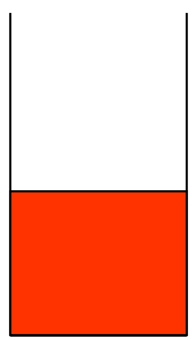
Weight w_i	value v_i
2	3
3	4
4	5
5	15
10	16

0-1 Knapsack problem:

This is a knapsack
Max weight: W = 13

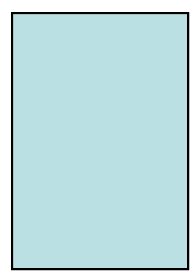
Items	Weight w_i	value vi
	2	3
	3	4
	4	5
	5	15
	10	16

value = 15
Weight = 5

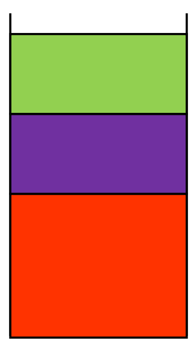


0-1 Knapsack problem:

This is a knapsack
Max weight: W = 13

Items	Weight w_i	value v_i
	2	3
	3	4
	4	5
	5	15
	10	16

value = 24
Weight = 12



0-1 Knapsack Problem

- Given a knapsack with maximum capacity W, and a set S consisting of n items
- Each item i has some weight w_i and value v_i (all w_i and W are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?

0-1 Knapsack problem

Let S be the set of items represented by the ordered pairs (w_i, v_i) and W be the capacity of the knapsack.

Find a T $\subseteq$ S such that

$$\max \sum_{i \in T} v_i \ subject\ to \sum_{i \in T} w_i \leq W$$

The problem is called a *"0-1"* problem, because each item must be entirely accepted or rejected.

0-1 Knapsack Brute-Force

Let's first solve this problem with a straightforward algorithm

- Since there are *n* items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with the most total value and with total weight less or equal to *W*
- Running time will be $O(n2^n)$

Can we do better?
Yes, with an algorithm based on dynamic programming
We need to carefully identify the subproblems

Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for

S_k = *{items labeled 1, 2, .. k}*

- This is a valid subproblem definition.
- The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?
- Unfortunately, we can't do that. ….Why???

Defining a Subproblem

$w_1 = 2$ $v_1 = 3$	$w_2 = 4$ $v_2 = 5$	$w_3 = 3$ $v_3 = 4$	$w_4 = 5$ $v_4 = 8$	**?**

Max weight: W = 20

For S_4: {1, 2, 3, 4}

Total weight: 14;

total value: 20

Item #	Weight w_i	Value v_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

S_4

S_5

$w_1 = 2$ $v_1 = 3$	$w_3 = 4$ $v_3 = 5$	$w_4 = 5$ $v_4 = 8$	$w_5 = 9$ $v_5 = 10$

For S_5: { 1, 3, 4, 5 }

Total weight: 20

total value: 26

Solution for S_4 is not part of the solution for S_5!!!

Defining a Subproblem

- As we have seen, the solution for S_4 is not part of the solution for S_5
- So our definition of a subproblem is flawed and we need another one!
- Let's add another parameter: *w*, which will represent the exact weight for each subset of items
- The subproblem then will be to compute v[k,w]

Recursive Formula

$$V[k,w] = \begin{cases} V[k-1,w], & if\ w_k > w \\ \max\{V[k-1,w], V[k-1,w-w_k]+v_k\}, & w_k \leq w \end{cases}$$

The best subset of S_k that has the total weight *w,* either contains item *k* or not.

- **First case**: $w_k > w$. Item *k* can’t be part of the solution, since if it was, the total weight would be > *w*. So we select the “optimal” using items 1,.., k-1
- **Second case**: $w_k \leq w$. Then the item *k* <u>can</u> be in the solution, and we choose the case with greater value

Recursive Formula for subproblems

$$V[k,w] = \begin{cases} \boxed{V[k-1,w],} & if\ w_k > w \\ \max\{\boxed{V[k-1,w]}, \boxed{V[k-1,w-w_k]+v_k}\}, & w_k \le w \end{cases}$$

It means, that the best subset of S_k that has total weight w is one of the two:

Item k is too big to fit in the knapsack with capacity w

Do not use item k: the best subset of S_{k-1} that has total weight w, **or**

Use item k: the best subset of S_{k-1} that has total weight w-w_k plus the item k *with value* v_k

Recursive Code

```
Knapsack(W, n)
{
        // Base Case
        if (n = 0 or W = 0)
            return 0;

        if (w_n > W)
            return Knapsack(W, n-1);
        else
            return max(v_n + Knapsack(W-w_n, n-1),  Knapsack(W, n-1) );
}
```

Knapsack(W-w_n, n-1), Knapsack(W, n-1))

KS(5, 4)

KS(4, 3) KS(5, 3)

KS(3, 2) KS(4, 2) KS(4, 2) KS(5, 2)

KS(2, 1) KS(3, 1) KS(3, 1) KS(4, 1) KS(3, 1) KS(4, 1) KS(5, 1)

KS(1, 0) KS(2, 0) KS(2, 0)

KS(2, 0) KS(3, 0) KS(3, 0)

Etc …….

0-1 DP Knapsack Algorithm

```
for w = 0 to W
    V[0,w] = 0      // 0 item's
for i = 1 to n
    V[i,0] = 0        // 0 weight
    for w = 1 to W
        if wi <= w // item i can be part of the solution
            if vi + V[i-1,w-wi] > V[i-1,w]
                V[i,w] = vi + V[i-1,w- wi]
            else
                V[i,w] = V[i-1,w]
        else V[i,w] = V[i-1,w]  // wi > w item i is too big
```

Running time

```
for w = 0 to W
   V[0,w] = 0
for i = 0 to n
   V[i,0] = 0
   for w = 0 to W
         < the rest of the code >
```

$\Theta(W)$

Repeat *n* times

$\Theta(W)$

What is the running time of this algorithm?

$\Theta(nW)$ pseudo-polynomial

Remember that the brute-force algorithm takes $\Theta(n2^n)$.

Better than Brute force if $W << 2^n$

Example

Let's run our algorithm on the following data:

n = 4 (# of elements)

W = 5 (max weight)

Elements (weight, value):

S = {(2,3), (3,4), (4,5), (5,6) }

w \ i	0	1	2	3	4
0	0				
1	0				
2	0				
3	0				
4	0				
5	0				

for w = 0 to W

    V[0,w] = 0

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				
5	0				

for i = 0 to n

  V[i,0] = 0

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0				
3	0				
4	0				
5	0				

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=1$

$v_i=3$

$w_i=2$

$w=1$

$w-w_i=-1$

if $w_i <= w$ // item i can be part of the solution
    if $v_i + V[i-1,w-w_i] > V[i-1,w]$
        $V[i,w] = v_i + V[i-1,w- w_i]$
    else
        $V[i,w] = V[i-1,w]$
else **V[i,w] = V[i-1,w]** // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	**3**			
3	0				
4	0				
5	0				

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=1$

$v_i=3$

$w_i=2$

$w=2$

$w-w_i=0$

```
if w_i <= w // item i can be part of the solution
    if v_i + V[i-1,w-w_i] > V[i-1,w]
        V[i,w] = v_i + V[i-1,w- w_i]
    else
        V[i,w] = V[i-1,w]
else V[i,w] = V[i-1,w]  // w_i > w
```

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	**3**			
4	0				
5	0				

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=1$
$v_i=3$
$w_i=2$
$w=3$
$w-w_i=1$

if $w_i <= w$ // item i can be part of the solution
    if $v_i + V[i-1,w-w_i] > V[i-1,w]$
        $V[i,w] = v_i + V[i-1,w-w_i]$
    else
        $V[i,w] = V[i-1,w]$
else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	3			
4	0	**3**			
5	0				

$i=1$

$v_i=3$

$w_i=2$

$w=4$

$w-w_i=2$

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if $w_i <= w$ // item i can be part of the solution

  if $v_i + V[i-1,w-w_i] > V[i-1,w]$

    $V[i,w] = v_i + V[i-1,w- w_i]$

  else

    $V[i,w] = V[i-1,w]$

else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	3			
3	0	3			
4	0	3			
5	0	**3**			

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=1$
$v_i=3$
$w_i=2$
$w=5$
$w-w_i=2$

if $w_i <= w$ // item i can be part of the solution
    if $v_i + V[i-1,w-w_i] > V[i-1,w]$
        $\mathbf{V[i,w] = v_i + V[i-1,w- w_i]}$
    else
        $V[i,w] = V[i-1,w]$
else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	→ **0**		
2	0	3			
3	0	3			
4	0	3			
5	0	3			

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=2$

$v_i=4$

$w_i=3$

$w=1$

$w-w_i=-2$

if w_i <= w // item i can be part of the solution
    if $v_i + V[i-1,w-w_i] > V[i-1,w]$
        $V[i,w] = v_i + V[i-1,w- w_i]$
    else
        V[i,w] = V[i-1,w]
else **V[i,w] = V[i-1,w]** // $w_i > w$

Item : (w, v)

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	**3**		
3	0	3			
4	0	3			
5	0	3			

$i=2$

$v_i=4$

$w_i=3$

$w=2$

$w-w_i=-1$

if w_i <= w // item i can be part of the solution
    if v_i + V[i-1,w-w_i] > V[i-1,w]
        V[i,w] = v_i + V[i-1,w- w_i]
    else
        V[i,w] = V[i-1,w]
else **V[i,w] = V[i-1,w]** // w_i > w

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	**4**		
4	0	3			
5	0	3			

Item : (w, v)

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=2$

$v_i=4$

$w_i=3$

$w=3$

$w-w_i=0$

if $w_i <= w$ // item i can be part of the solution

    if $v_i + V[i-1,w-w_i] > V[i-1,w]$

        $\mathbf{V[i,w] = v_i + V[i-1,w-w_i]}$

    else

        $V[i,w] = V[i-1,w]$

else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	4		
4	0	3	**4**		
5	0	3			

i=2

v_i=4

w_i=3

w=4

$w-w_i$=1

1: (2,3)

2: (3,4)

3: (4,5)

4: (5,6)

if w_i <= w // item i can be part of the solution

    if $v_i + V[i-1,w-w_i] > V[i-1,w]$

        $V[i,w] = v_i + V[i-1,w- w_i]$

    else

        V[i,w] = V[i-1,w]

else V[i,w] = V[i-1,w] // w_i > w

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0		
2	0	3	3		
3	0	3	4		
4	0	3	4		
5	0	3	**7**		

$i=2$

$v_i=4$

$w_i=3$

$w=5$

$w-w_i=2$

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if w_i <= w // item i can be part of the solution
    if v_i + V[i-1,w-w_i] > V[i-1,w]
        V[i,w] = v_i + V[i-1,w- w_i]
    else
        V[i,w] = V[i-1,w]
else V[i,w] = V[i-1,w] // w_i > w

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0 →	**0**	
2	0	3	3 →	**3**	
3	0	3	4 →	**4**	
4	0	3	4		
5	0	3	7		

Item : (w, v)

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=3$
$v_i=5$
$w_i=4$
$w=1..3$

```
if w_i <= w // item i can be part of the solution
    if v_i + V[i-1,w-w_i] > V[i-1,w]
        V[i,w] = v_i + V[i-1,w- w_i]
    else
        V[i,w] = B[i-1,w]
else V[i,w] = V[i-1,w]  // w_i > w
```

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	
2	0	3	3	3	
3	0	3	4	4	
4	0	3	4	**5**	
5	0	3	7		

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=3$
$v_i=5$
$w_i=4$
$w=4$
$w- w_i=0$

if $w_i <= w$ // item i can be part of the solution
  if $v_i + V[i-1,w-w_i] > V[i-1,w]$
    $V[i,w] = v_i + V[i-1,w- w_i]$
  else
    $V[i,w] = V[i-1,w]$
else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	
2	0	3	3	3	
3	0	3	4	4	
4	0	3	4	5	
5	0	3	7 →	7	

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

$i=3$
$v_i=5$
$w_i=4$
$w=5$
$w- w_i=1$

if $w_i <= w$ // item i can be part of the solution
  if $v_i + V[i-1,w-w_i] > V[i-1,w]$
    $V[i,w] = v_i + V[i-1,w- w_i]$
  else
    V[i,w] = V[i-1,w]
else $V[i,w] = V[i-1,w]$ // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0 →	**0**
2	0	3	3	3 →	**3**
3	0	3	4	4 →	**4**
4	0	3	4	5 →	**5**
5	0	3	7	7	

$i=4$

$v_i=6$

$w_i=5$

$w=1..4$

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if w_i <= w // item i can be part of the solution
    if $v_i + V[i-1,w-w_i] > V[i-1,w]$
        $V[i,w] = v_i + V[i-1,w- w_i]$
    else
        $V[i,w] = V[i-1,w]$
else **V[i,w] = V[i-1,w]** // $w_i > w$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	3	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	3	7	7 →	7

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i=4

v_i=6

w_i=5

w=1..4

if w_i <= w // item i can be part of the solution

 if v_i + V[i-1,w-w_i] > V[i-1,w]

  V[i,w] = v_i + V[i-1,w- w_i]

 else

  V[i,w] = V[i-1,w]

else V[i,w] = V[i-1,w] // w_i > w

Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
- To know the items that make this maximum value, an addition to this algorithm is necessary
- See LCS algorithm for the example how to extract this data from the table we built using “parent pointers”.
- Or use the information already in the table.

How to find actual Knapsack Items

Using current table

- V[n,W] is the maximal value of items that can be placed in the Knapsack.
- Let i=n and k=W

if $V[i,k] \neq V[i-1,k]$ then

mark the i^{th} item as in the knapsack

$i = i-1$, $k = k-w_i$

else

$i = i-1$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	3	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	3	7	**7**	**7**

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i=4

k=5

if V[i,k] ≠ V[i–1,k] then
    mark the i^{th} item as in the knapsack
    i = i–1, k = k-w_i
else
    i = i–1

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	3	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	3	7	**7**	**7**

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if V[i,k] ≠ V[i–1,k] then
  mark the i^{th} item as in the knapsack
  i = i–1, k = k-w_i
else
  i = i–1

i=3

k=5

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	3	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	**3**	7	**7**	**7**

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if $V[i,k] \neq V[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
$i = i-1$

$i=2$

$k=5$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	**0**	**3**	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	**3**	7	**7**	**7**

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if $V[i,k] \neq V[i-1,k]$ then
    mark the i^{th} item as in the knapsack
    $i = i-1$, $k = k-w_i$
else
    $i = i-1$

$i=1$

$k=2$

Item : (w, v)

w \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	**0**	**3**	3	3	3
3	0	3	4	4	4
4	0	3	4	5	5
5	0	**3**	7	**7**	**7**

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

if $V[i,k] \neq V[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
$i = i-1$

i=0

k=0

Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary
- Running time (Dynamic Programming algorithm vs. brute force algorithm):
  - 0-1 Knapsack problem: **$O(Wn)$** vs. **$O(n2^n)$**

    Pseudo-polynomial

  - LCS: **$O(mn)$** vs. **$O(n\ 2^m)$**

Longest Common Subsequence

- Given two sequences x[1..m] and y[1..n]

  $X = \langle x_1, x_2, \ldots, x_m \rangle$

  $Y = \langle y_1, y_2, \ldots, y_n \rangle$

  find a maximum length common subsequence (LCS) of X and Y

- *E.g.:*

  $X = \langle A, B, C, B, D, A, B \rangle$

- Subsequences of X:
  - A subset of elements in the sequence taken in order

  $\langle A, B, D \rangle$, $\langle B, C, D, B \rangle$, etc.

Example

$X = \langle A, B, C, B, D, A, B \rangle$ $X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$ $Y = \langle B, D, C, A, B, A \rangle$

- $\langle B, C, B, A \rangle$ and $\langle B, D, A, B \rangle$ are longest common subsequences of X and Y (length = 4)

- $\langle B, C, A \rangle$, however is not a LCS of X and Y

Longest Common Subsequence (LCS)

Application: comparison of two DNA strings

Ex: X= ⟨A, B, C, B, D, A, B ⟩, Y= ⟨B, D, C, A, B, A⟩

Longest Common Subsequence:

X = A **B** **C** **B** D **A** B

Y = **B** D **C** A **B** **A**

Brute force algorithm would compare each subsequence of X with the symbols in Y

Brute-Force Solution

- For every subsequence of X, check whether it's a subsequence of Y
- There are 2^m subsequences of X to check
- Each subsequence takes $\Theta(n)$ time to check
  - scan Y for first letter, from there scan for second, and so on
- Running time: $\Theta(n2^m)$

Steps in Dynamic Programming

1. Characterize structure of an optimal solution.
2. Define value of optimal solution recursively.
3. Compute optimal solution values bottom-up in a table.
4. Construct an optimal solution from computed values.

We'll study these with the help of examples.

Making the choice

$X = \langle A, B, D, E \rangle$

$Y = \langle Z, B, E \rangle$

- Choice: include one element into the common sequence (E) and solve the resulting subproblem

$X = \langle A, B, D, G \rangle$

$Y = \langle Z, B, D \rangle$

- Choice: exclude an element from a string and solve the resulting subproblem

Notations

- Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$ we define the i-th prefix of X, for i = 0, 1, 2, …, m

  $X_i = \langle x_1, x_2, \ldots, x_i \rangle$ or x[1,..,i]

  $Y_j = \langle y_1, y_2, \ldots, y_j \rangle$ or y[1,..,j]

- c[i, j] = the length of a LCS of the sequences $X_i = \langle x_1, x_2, \ldots, x_i \rangle$ and $Y_j = \langle y_1, y_2, \ldots, y_j \rangle$

A Recursive Solution

Case 1: $x_i = y_j$

e.g.: $X_i = \langle A, B, D, E \rangle$

$Y_j = \langle Z, B, E \rangle$

$$c[i, j] = c[i-1, j-1] + 1$$

- Append $x_i = y_j$ to the LCS of X_{i-1} and Y_{j-1}
- Must find a LCS of X_{i-1} and Y_{j-1} $\Rightarrow$ optimal solution to a problem includes optimal solutions to subproblems

A Recursive Solution

Case 2: $x_i \neq y_j$

e.g.: $X_i = \langle A, B, D, G \rangle$

$Y_j = \langle Z, B, D \rangle$

$$c[i, j] = \max \{ c[i-1, j], c[i, j-1] \}$$

- Must solve two problems
  - find a LCS of X_{i-1} and Y_j: $X_{i-1} = \langle A, B, D \rangle$ and $Y_j = \langle Z, B, D \rangle$
  - find a LCS of X_i and Y_{j-1}: $X_i = \langle A, B, D, G \rangle$ and $Y_j = \langle Z, B \rangle$

- Optimal solution to a problem includes optimal solutions to subproblems

Overlapping Subproblems

- To find a LCS of X and Y
  - we may need to find the LCS between X and Y_{n-1} and that of X_{m-1} and Y
  - Both the above subproblems has the subproblem of finding the LCS of X_{m-1} and Y_{n-1}
- Subproblems share subsubproblems

Finding LCS Length

Define c[i,j] to be the length of the LCS of x[1..,i] and y[1.., j]

Theorem:

$$c[i,j] = \begin{cases} 0 & i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

LCS Algorithm

- First we'll find the length of LCS. Later we'll modify the algorithm to find LCS itself.
- Define X_i, Y_j to be the prefixes of X and Y of length *i* and *j* respectively
- Define *c[i,j]* to be the length of LCS of X_i and Y_j
- Then the length of LCS of X and Y will be *c[m,n]*

$$c[i,j] = \begin{cases} 0 & i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

LCS recursive solution

$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- We start with $i = j = 0$ (empty substrings of x and y)
- Since X_0 and Y_0 are empty strings, their LCS is always empty (i.e. $c[0,0] = 0$)
- LCS of empty string and any other string is empty, so for every i and j: $c[0, j] = c[i,0] = 0$

LCS recursive solution

$$c[i, j] = \begin{cases} c[i-1, j-1]+1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

- When we calculate *c[i,j]*, we consider two cases:
- **First case:** *x[i]=y[j]*: one more symbol in strings X and Y matches, so the length of LCS X_i and Y_j equals to the length of LCS of smaller strings X_{i-1} and Y_{i-1} , plus 1

LCS recursive solution

$$c[i,j] = \begin{cases} c[i-1, j-1]+1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

- **Second case:** *x[i] != y[j]*
- As symbols don't match, our solution is not improved, and the length of LCS(X_i , Y_j) is the same as before (i.e. maximum of LCS(X_i, Y_{j-1}) and LCS(X_{i-1},Y_j)

LCS Length Algorithm

```
LCS-Length(X, Y)
1. m = length(X)  // get the # of symbols in X
2. n  = length(Y) // get the # of symbols in Y
3. for i = 1 to m      c[i,0] = 0      // special case: Y0
4. for j = 1 to n      c[0,j] = 0      // special case: X0
5. for i = 1 to m                      // for all Xi
6.       for j = 1 to n                      // for all Yj
7.             if ( Xi == Yj )
8.                   c[i,j] = c[i-1,j-1] + 1
9.             else c[i,j] = max( c[i-1,j], c[i,j-1] )
10. return c
```

LCS Example

We'll see how LCS algorithm works on the following example:

- X = ABCB
- Y = BDCAB

LCS Example (0)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi						
1	**A**						
2	**B**						
3	**C**						
4	**B**						

X = ABCB; m = |X| = 4
Y = BDCAB; n = |Y| = 5
Allocate array c[4,5]

LCS Example (1)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**					
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

for i = 1 to m   c[i,0] = 0
for j = 1 to n   c[0,j] = 0

LCS Example (2)

ABCB
BDCAB

		j	0	1	2	3	4	5
i			Yj	**B**	**D**	**C**	**A**	**B**
0	Xi		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**				
2	**B**		**0**					
3	**C**		**0**					
4	**B**		**0**					

if ($X_i == Y_j$)
$c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (3)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**		
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (4)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

if (X_i == Y_j)
    c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (5)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (6)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**				
3	**C**	**0**					
4	**B**	**0**					

if ($X_i == Y_j$)
    $c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (7)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	
3	**C**	**0**					
4	**B**	**0**					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (8)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**					
4	**B**	**0**					

if ($X_i == Y_j$)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (10)

ABCB
BDCAB

		j	0	1	2	3	4	5
i			Yj	**B**	**D**	**C**	**A**	**B**
0	Xi		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**	**0**	**0**	**1**	**1**
2	**B**		**0**	**1**	**1**	**1**	**1**	**2**
3	**C**		**0**	**1**	**1**			
4	**B**		**0**					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (11)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**		
4	**B**	**0**					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (12)

ABCB
BDCAB

		j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**	
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**	
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**	
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**	
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**	
4	**B**	**0**						

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (13)

ABCB
BDCAB

i		j	0	1	2	3	4	5
		Yj		B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	2
3	C		0	1	1	2	2	2
4	B		0	1				

if ($X_i == Y_j$)
$c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

ABCB

BDCAB

LCS Example (14)

i	Xi	j	0	1	2	3	4	5
		Yj		B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	2
3	C		0	1	1	2	2	2
4	B		0	1	1	2	2	

if ($X_i == Y_j$)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (15)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**B**	**0**	**1**	**1**	**2**	**2**	**3**

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Algorithm Running Time

- LCS algorithm calculates the values of each entry of the array c[m,n]
- So what is the running time?

O(mn)

since each c[i,j] is calculated in constant time, and there are m*n elements in the array

Finding LCS

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**B**	**0**	**1**	**1**	**2**	**2**	**3**

Finding LCS (2)

i		j	0	1	2	3	4	5
		Yj		B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	2
3	C		0	1	1	2	2	2
4	B		0	1	1	2	2	3

LCS (reversed order): **B C B**

LCS (straight order): **B C B**

Additional Information

$$c[i, j] = \begin{cases} 0 & \text{if } i,j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

b & c:

		0	1	2	3		n
		y_j:	A	C	D		F
0	x_i	0	0	0	0	0	0
1	A	0					
2	B	0			c[i-1,j]		
3	C	0		↖ c[i,j-1]	↑		
		0					
m	D	0					

i

j

A matrix b[i, j]:

- For a subproblem [i, j] it tells us what choice was made to obtain the optimal value
- If $x_i = y_j$
  b[i, j] = " ↖ "
- Else, if
  c[i - 1, j] ≥ c[i, j-1]
  b[i, j] = " ↑ "
  else
  b[i, j] = " ← "

Example

$X = \langle A, B, C, B, D, A \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

If $x_i = y_j$

$b[i, j] =$ " $\nwarrow$ "

Else if

$c[i - 1, j] \geq c[i, j-1]$

$b[i, j] =$ " $\uparrow$ "

else

$b[i, j] =$ " $\leftarrow$ "

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	$\uparrow$ 0	$\uparrow$ 0	$\uparrow$ 0	$\nwarrow$ 1	$\leftarrow$ 1	$\nwarrow$ 1
2	B	0	$\nwarrow$ 1	$\leftarrow$ 1	$\leftarrow$ 1	$\uparrow$ 1	$\nwarrow$ 2	$\leftarrow$ 2
3	C	0	$\uparrow$ 1	$\uparrow$ 1	$\nwarrow$ 2	$\leftarrow$ 2	$\uparrow$ 2	$\uparrow$ 2
4	B	0	$\nwarrow$ 1	$\uparrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\nwarrow$ 3	$\leftarrow$ 3
5	D	0	$\uparrow$ 1	$\nwarrow$ 2	$\uparrow$ 2	$\uparrow$ 2	$\uparrow$ 3	$\uparrow$ 3
6	A	0	$\uparrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\nwarrow$ 3	$\uparrow$ 3	$\nwarrow$ 4
7	B	0	$\nwarrow$ 1	$\uparrow$ 2	$\uparrow$ 2	$\uparrow$ 3	$\nwarrow$ 4	$\uparrow$ 4

Constructing a LCS

- Start at b[m, n] and follow the arrows
- When we encounter a "↖" in b[i, j] ⇒ $x_i = y_j$ is an element of the LCS

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

PRINT-LCS(b, X, i, j)

Running time: $\Theta(m + n)$

```
if i = 0 or j = 0
  then return
if b[i, j] = "↖"
   then PRINT-LCS(b, X, i - 1, j - 1)
        print x_i
elseif b[i, j] = "↑"
        then PRINT-LCS(b, X, i - 1, j)
        else PRINT-LCS(b, X, i, j - 1)
```

Initial call: PRINT-LCS(b, X, length[X], length[Y])

Improving the Code

- What can we say about how each entry $c[i, j]$ is computed?
  - It depends only on $c[i-1, j-1]$, $c[i-1, j]$, and $c[i, j-1]$
  - Eliminate table b and compute in $O(1)$ which of the three values was used to compute $c[i, j]$
  - We save $\Theta(mn)$ space from table b
  - However, we do not asymptotically decrease the auxiliary space requirements: still need table c

Improving the Code

- If we only need the length of the LCS
  - LCS-LENGTH works only on two rows of c at a time
    - The row being computed and the previous row
  - We can reduce the asymptotic space requirements by storing only these two rows