Module 2

- Divide and conquer VS iterative algorithms
- Recursion
- Solving Recurrences
- Binary Search
- Merge Sort
- Towers of Hanoi

Recall from Week 1

- Asymptotic Analysis: O, Ω, Θ
- Used to compare functions that represent the running times of different algorithms that can be used to solve a problem.
- How did we get the functions

Iterative Algorithm Analysis

```
for (i=1; i<=n*n; i++)     Executed n*n times
  for (j=0; j<i; j++)       Executed <= n*n times
    sum++;                    O(1)
```

Exact # of times `sum++` is executed:

$$\sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2}$$

$$= \frac{n^4+n^2}{2}$$

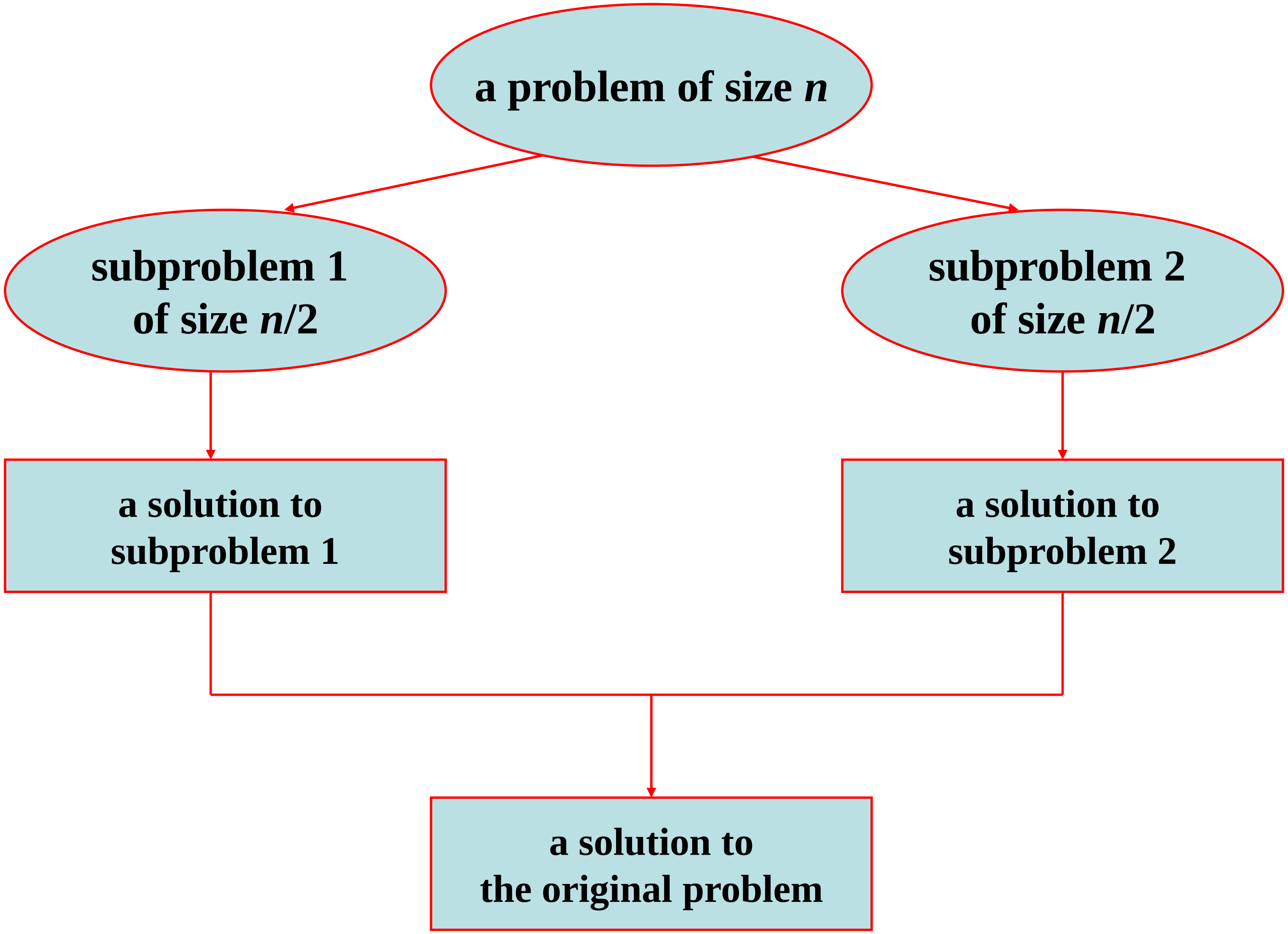
$$\in \Theta(n^4)$$

The Divide and Conquer Approach

The most well known algorithm design strategy:

1. **Divide** the problem into two or more smaller subproblems.
2. **Conquer** the subproblems by solving them recursively.
3. **Combine** the solutions to the subproblems into the solutions for the original problem.

A Typical Divide and Conquer Case



Merge-Sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining

Closed form: $T(n) = \Theta(n lg n)$

Recurrences and Running Time

- An equation or inequality that describes a function in terms of its value on smaller inputs.

$$T(n) = T\left(\frac{n}{4}\right) + 1$$

- Recurrences arise when an algorithm contains recursive calls to itself
- What is the actual running time of the algorithm?
- Need to solve the recurrence
  - Find an explicit formula of the expression
  - Bound the recurrence by an expression that involves n

Binary Search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

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Example: Find 9

3	5	7	8	9	12	15

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Find an element in a sorted array:

1. ***Divide:*** Check middle element.
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3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15

Binary Search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

Closed form: $T(n) = \Theta(lgn)$

Power of a Number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

Counting the number of operations which are multiplications

Example: $a^n = a*a*\ldots*a$

Example: $15^9 = 15*15*15*15*15*15*15*15*15$

Power of a Number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Divide-and-conquer algorithm:

Base cases $a^0 = 1$ and $a^1 = a$

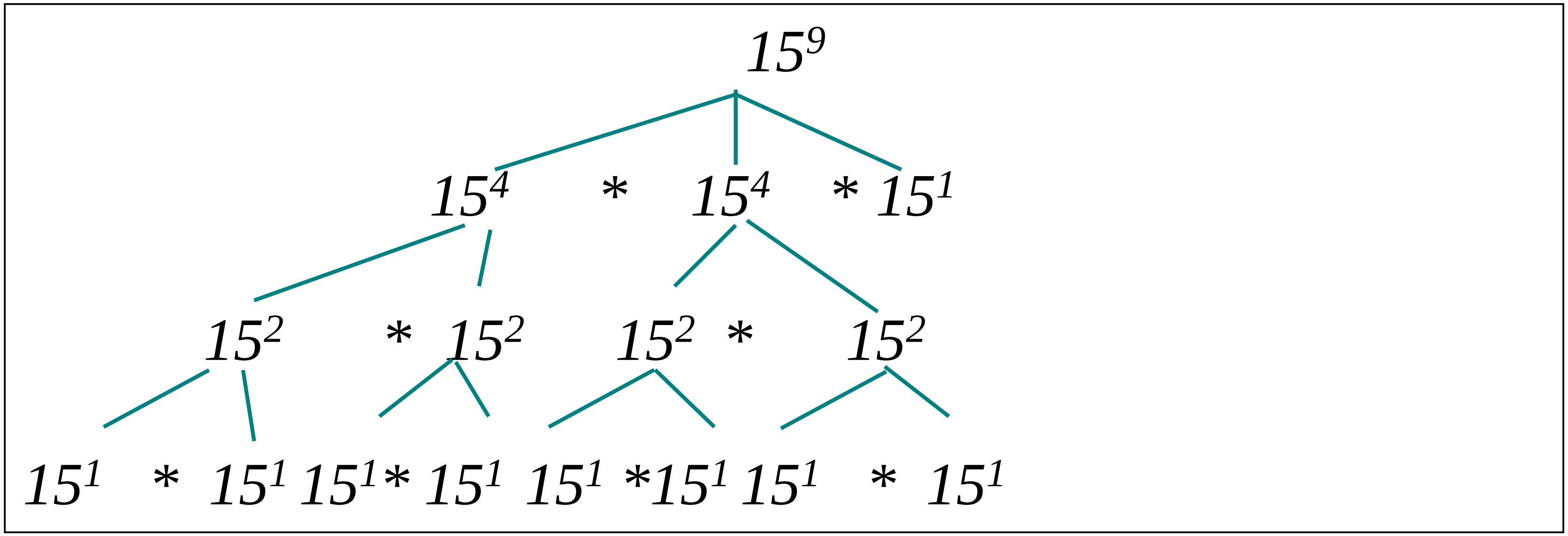
$$T(n) = T(n/2) + \Theta(1)$$

Problem: Compute a^n, where $n \in \mathbb{N}$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

Base cases $a^0 = 1$ and $a^1 = a$



Extra Recursion

```
long power (long x, long n) {
   if(n == 0)
     return 1;
   else if(n == 1)
        return x;
   else if ((n % 2) == 0)
    return power (x, n/2) * power (x, n/2);
   else
    return x * power (x,(n-1)/2) * power (x,(n-1)/2);
}
```

The recurrence relation is:

$$T(n) = 1 \qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) = 2T(n/2) + c \qquad \text{if } n > 2$$

Running time $\Theta(n)$

Problem: Compute a^n, where $n \in \mathbb{N}$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

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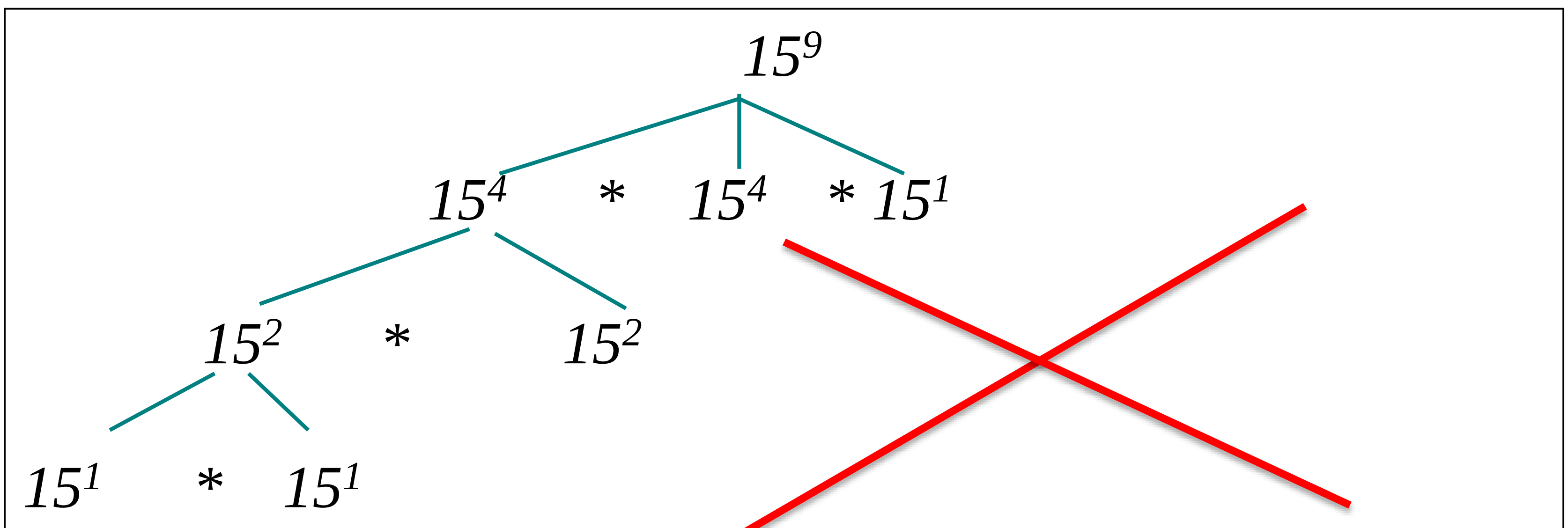


Recurrence Relations from Code

```
long power (long x, long n) {
   if(n == 0)
        return 1;
   else if(n == 1)
        return x;
   else if ((n % 2) == 0){
        temp = power(x, n/2);
        return temp*temp;
   }
   else {
        temp = power(x,(n-1)/2)
        return x * temp* temp;
   }
}
```

The recurrence relation is:

$$T(n) = 1 \qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) = T(n/2) + c \qquad \text{if } n > 2$$

Solve the Recurrence

The recurrence relation is:

$$T(n) = 1 \qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) = T(n/2) + c \qquad \text{if } n > 2$$

$$\begin{aligned} T(n) &= T(n/2) + c \\ &= T(n/4) + c + c \\ &= T(n/8) + c + c + c \\ &\dots \\ &= T(n/2^k) + kc \end{aligned}$$

Stop when k = lgn

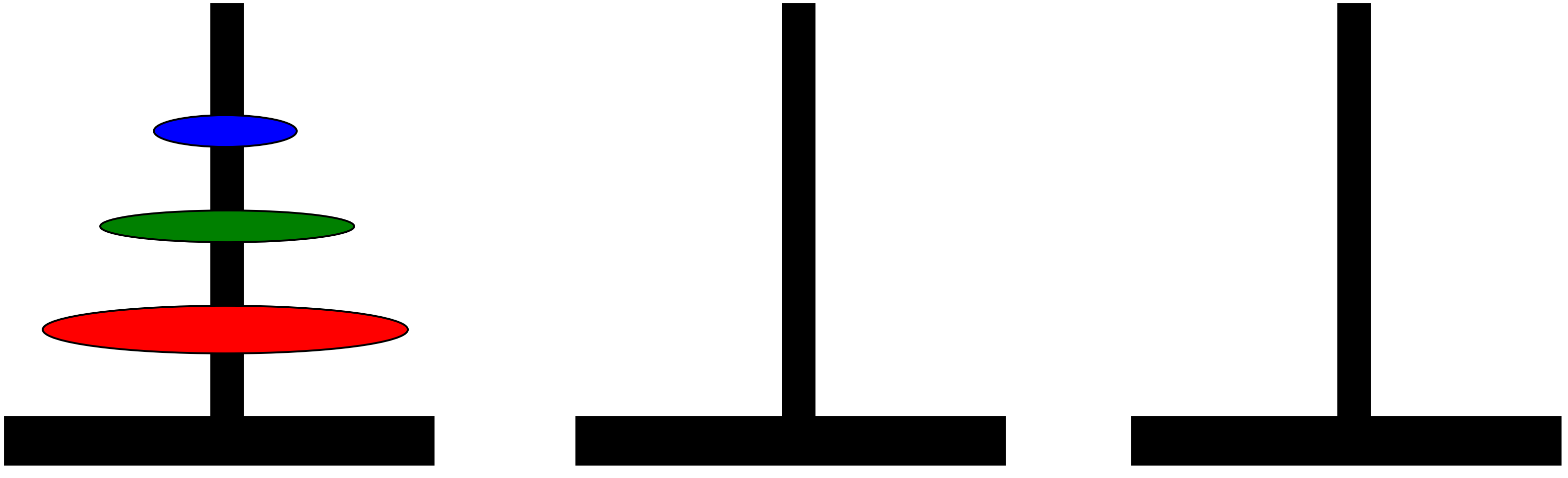
$$\begin{aligned} T(n) &= T(1) + c\lg n \\ &= 1 + c\lg n \end{aligned}$$

$$T(n) = \Theta(\lg n)$$

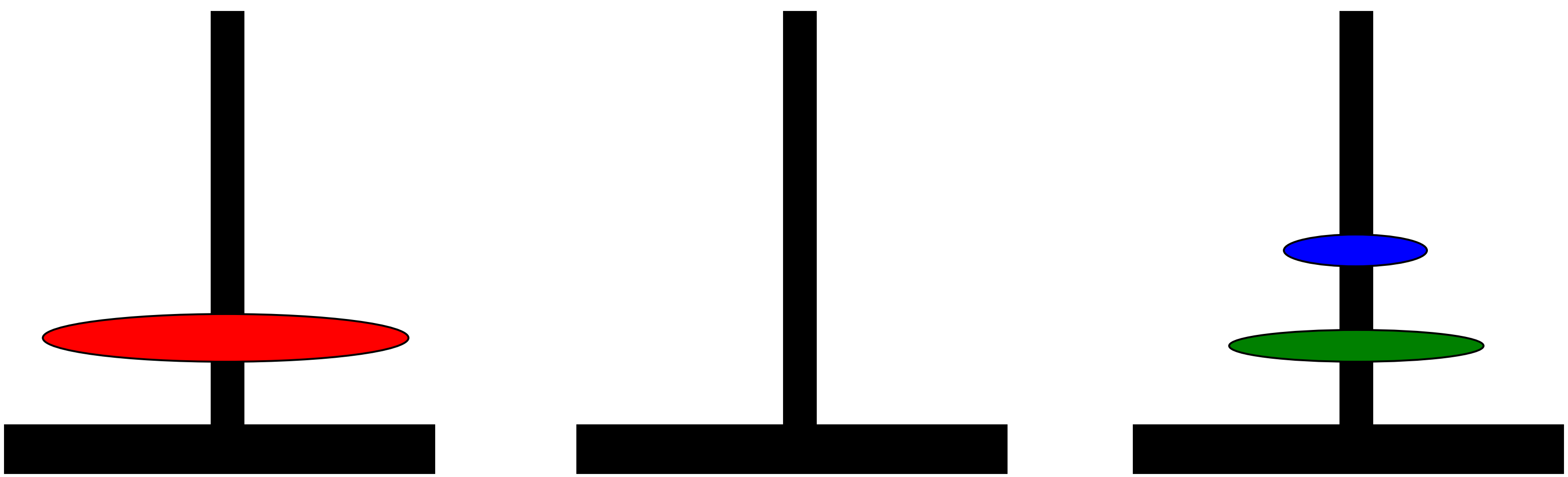
Tower of Hanoi

- There are three towers
- n gold disks, with decreasing sizes, placed on the first tower
- You need to move all of the disks from the first tower to the second tower
- Larger disks can not be placed on top of smaller disks
- The third tower can be used to temporarily hold disks

Recursive Solution



Recursive Solution



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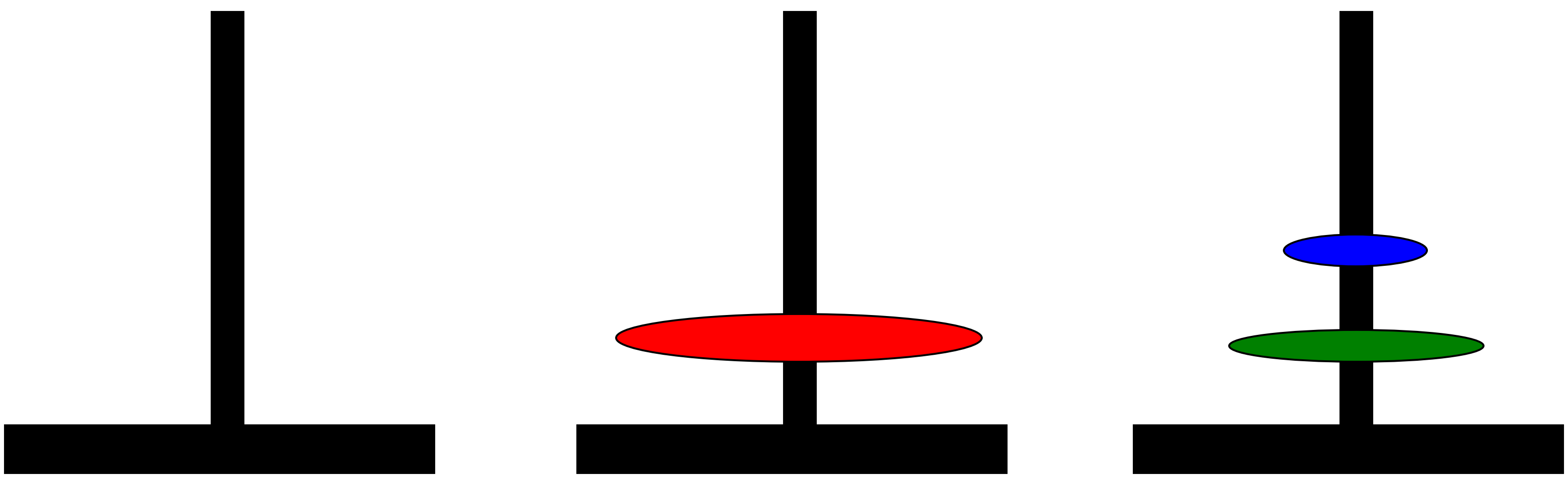

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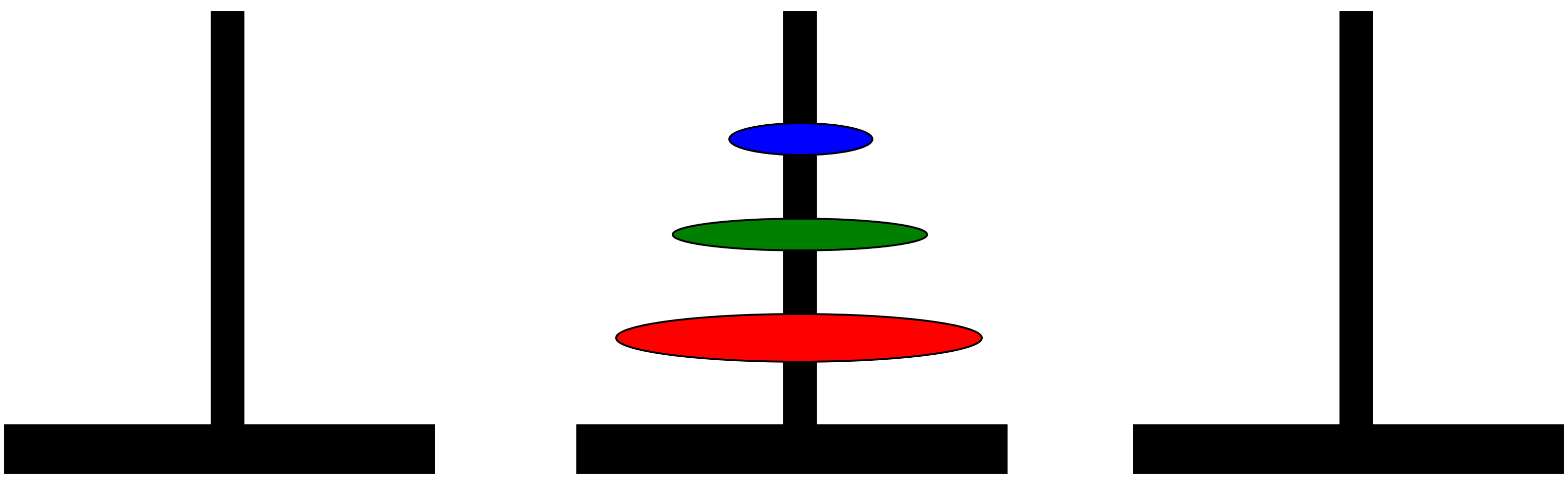
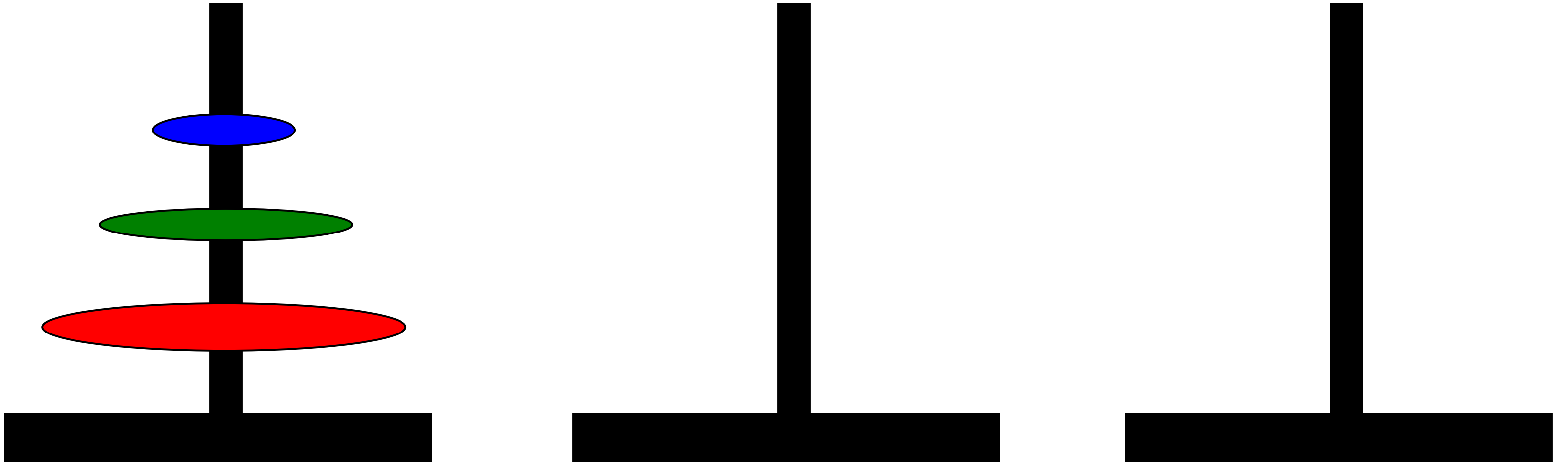


Tower of Hanoi



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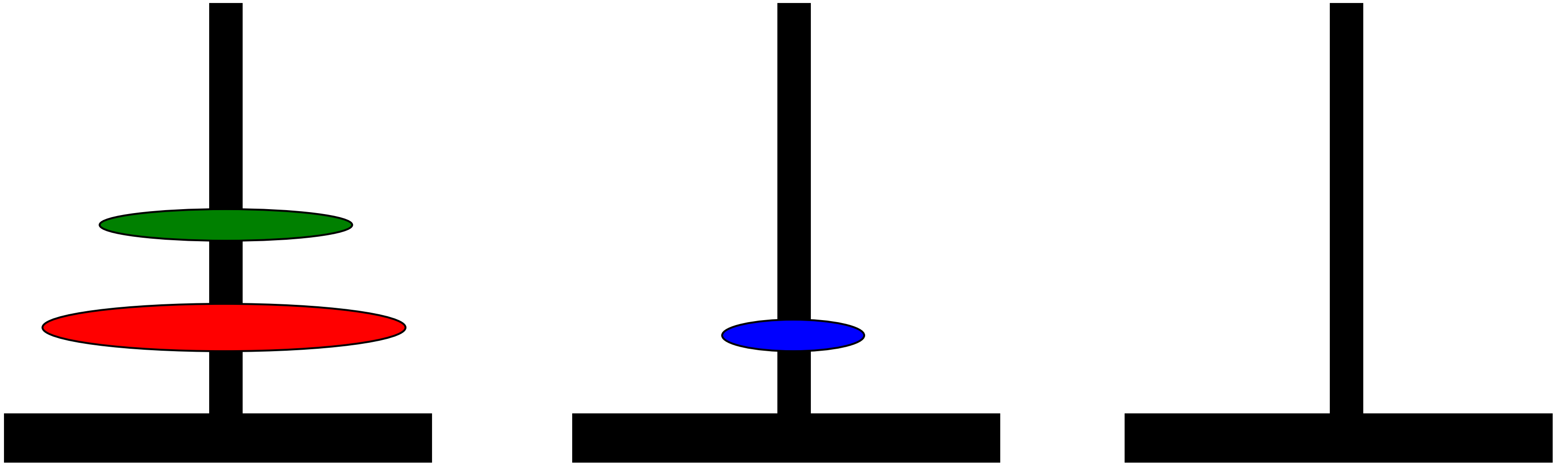

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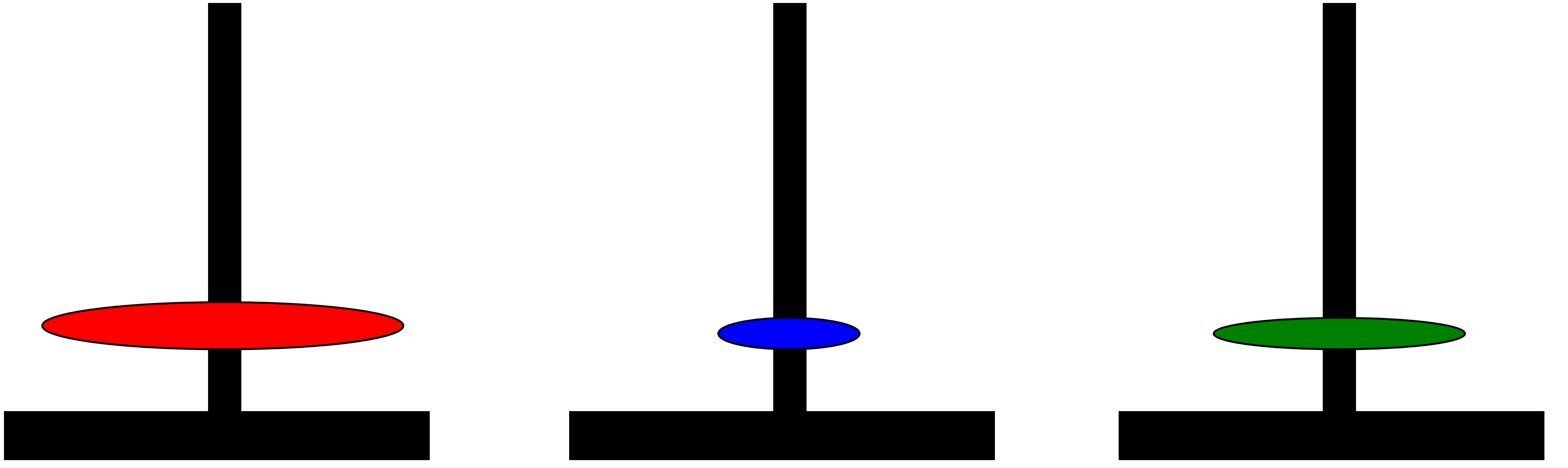

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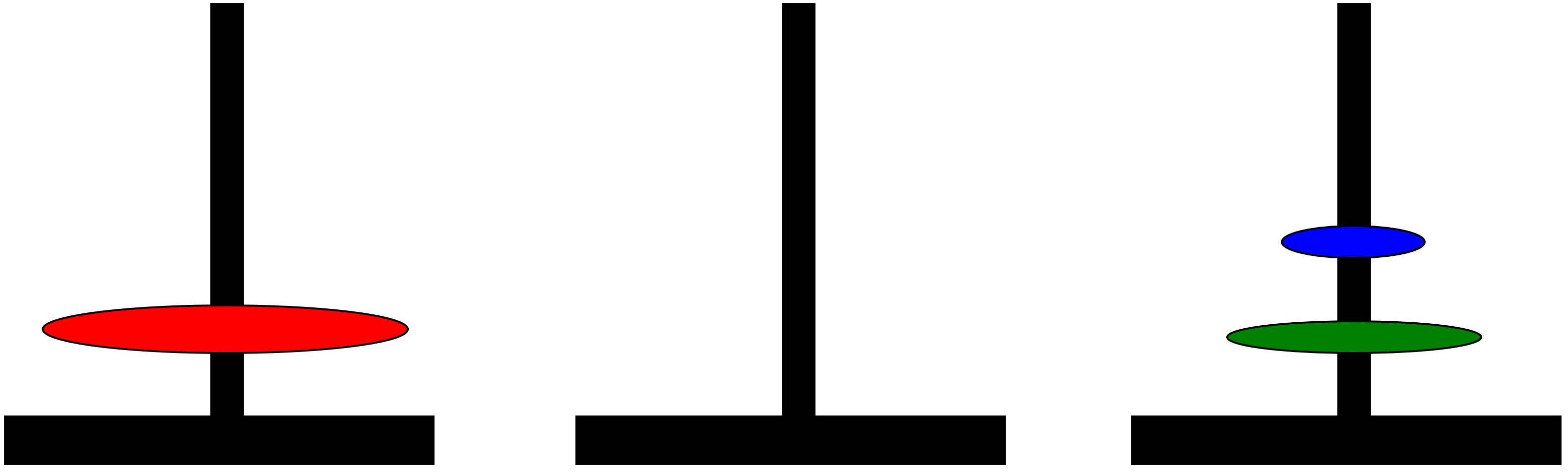

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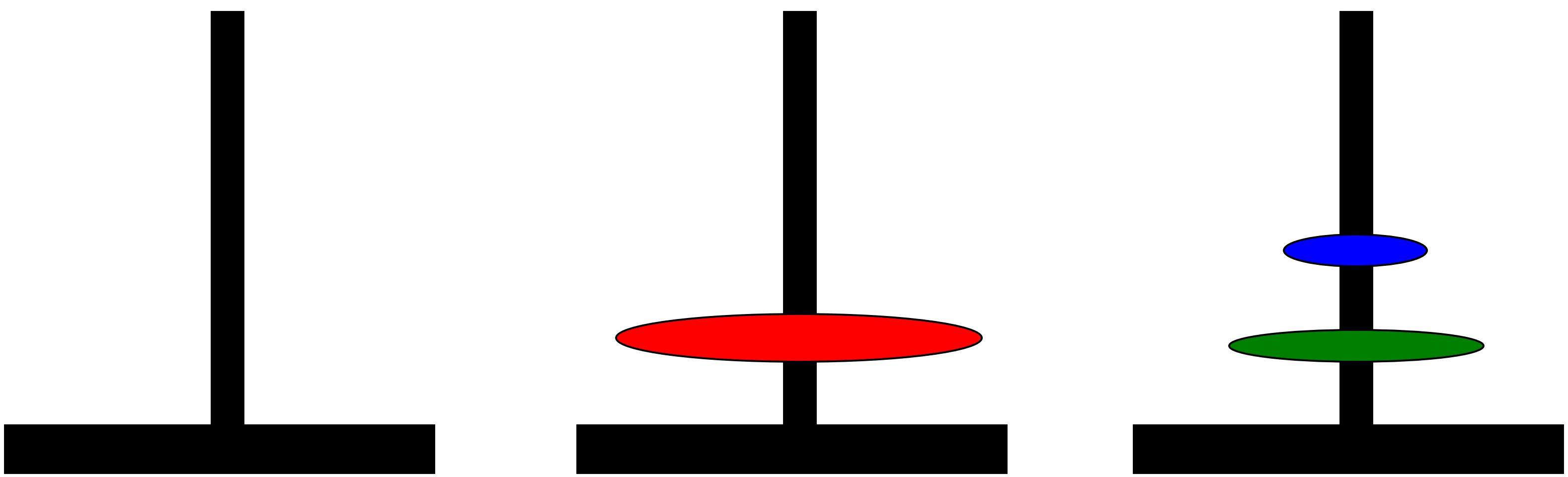

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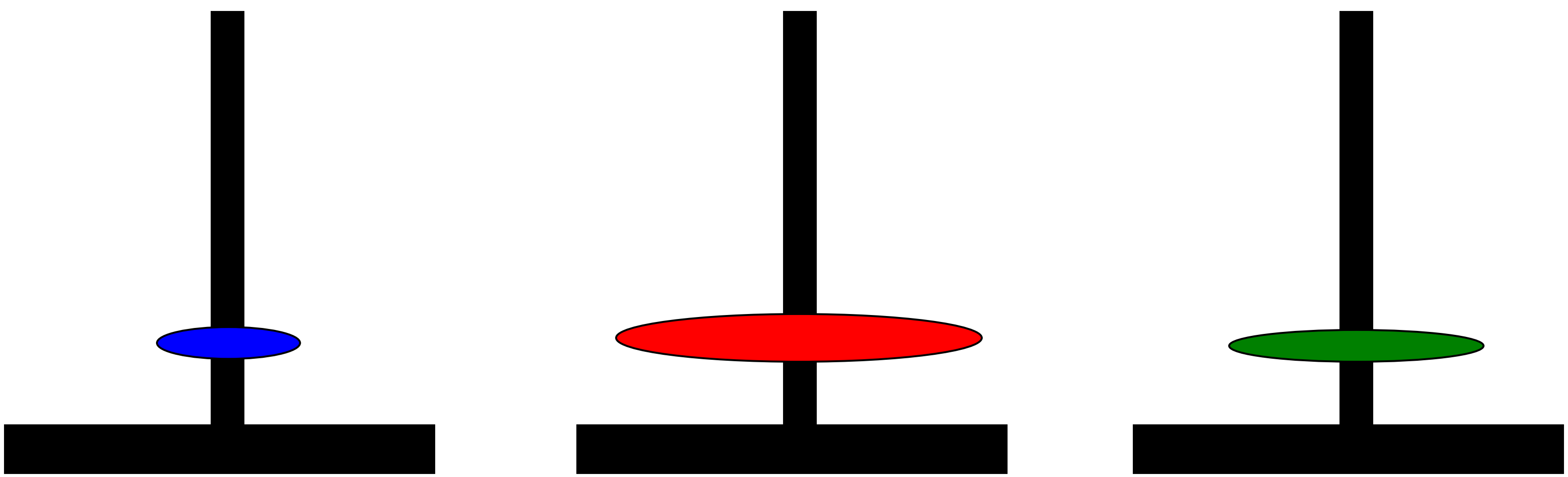

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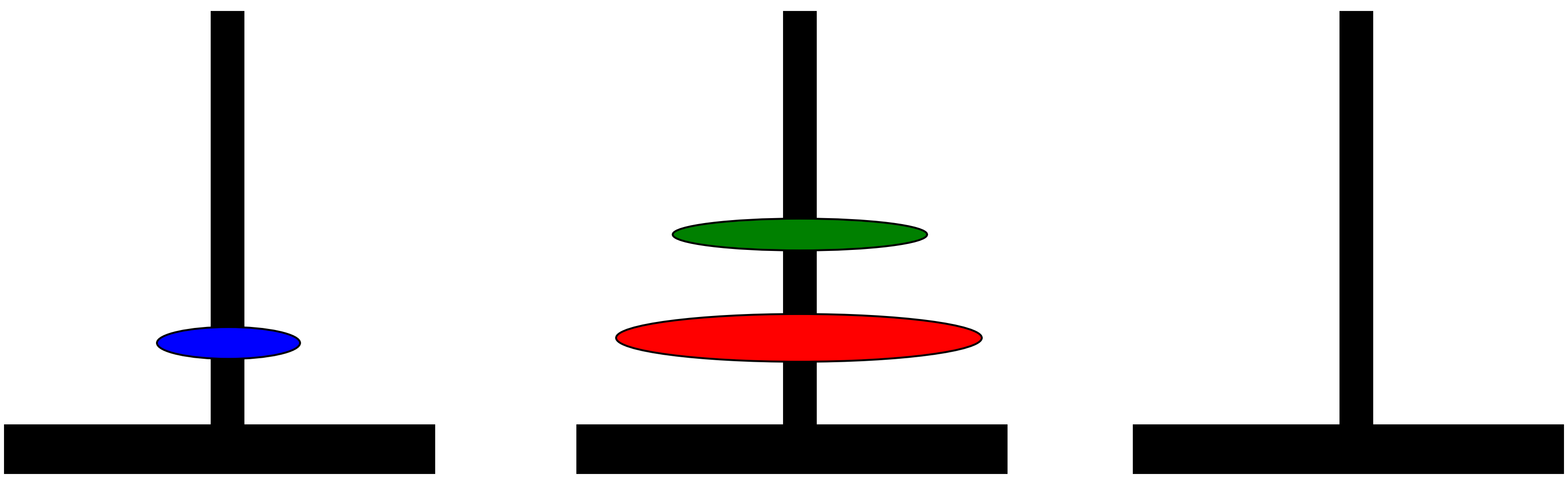

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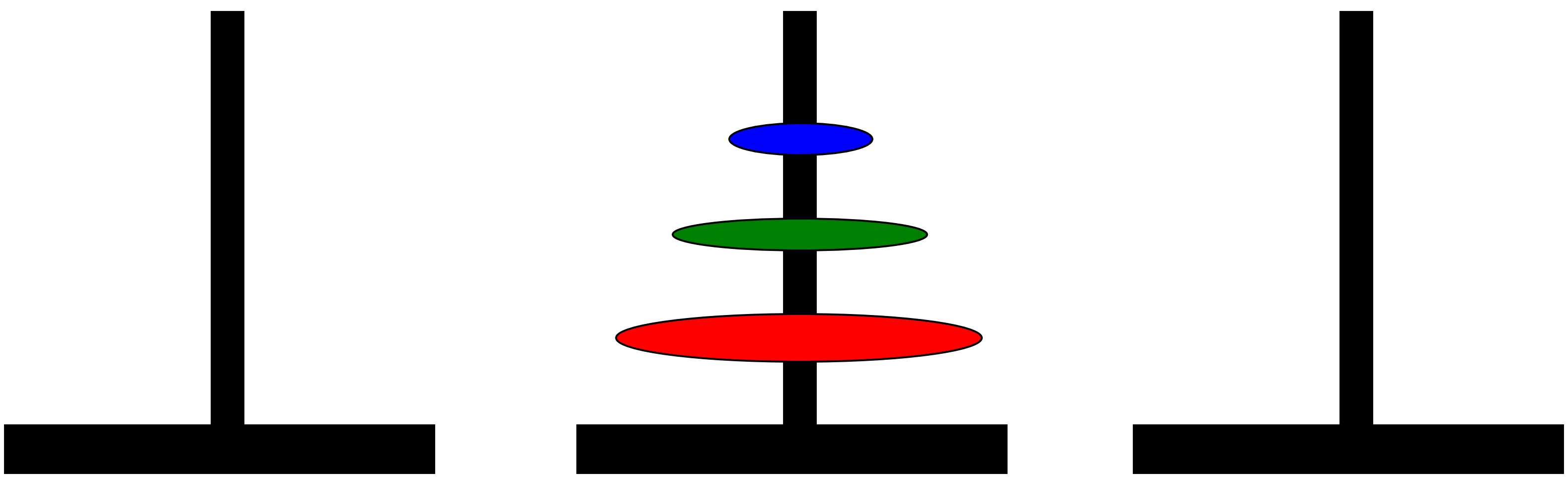


Tower of Hanoi

- The disks must be moved within one week. Assume one disk can be moved in 1 second. Is this possible?

- To create an algorithm to solve this problem, it is convenient to generalize the problem to the “n-disk” problem, where in our case n = 64.

Towers of Hanoi

```
Hanoi(n, from, to, temp){
   if (n == 1)
      Move(from, to);
   else{
      Hanoi(n - 1, from, temp, to);
      Move(from, to);
      Hanoi(n - 1, temp, to, from);
   }
}
```

The recurrence relation for the running time of the method **Hanoi** is:

$$T(1) = 1$$

$$T(n) = 2T(n - 1) + 1 \qquad \text{if } n > 1$$

$$T(n) = \Theta(2^n)$$

Guess and Prove

- Calculate T(n) for small n and look for a pattern.
- Guess the result and prove your guess correct using induction.

$$T(n) = 2T(n-1) + 1$$

n	T(n)
1	1
2	3
3	7
4	15
5	31

$$T(n) = 2^n - 1$$

Iteration Method

Unwind recurrence, by repeatedly replacing T(n) by the r.h.s. of the recurrence until the base case is encountered.

$$
\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2*[2*T(n-2)+1] + 1 \\
&= 2^2 * T(n-2) + 1+2 \\
&= 2^2 * [2*T(n-3)+1] + 1 + 2 \\
&= 2^3 * T(n-3) + 1+2 + 2^2
\end{aligned}
$$

….

$$T(n) = 2^k * T(n-k) + 1+2 + 2^2 + \ldots + 2^{k-1}$$

Common Summations

- Arithmetic series: $$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$
- Geometric series: $$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \left(x \neq 1\right)$$
  - Special case: $|x| < 1$: $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$
- Harmonic series: $$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \ln n$$
- Other important formulas: $$\sum_{k=1}^{n} \lg k \approx n \lg n$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \ldots + n^p \approx \frac{1}{p+1} n^{p+1}$$

Geometric Series

After k steps

$T(n) \;\; = 2^{k} * T(n\text{-}k) + 1+2 + 2^{2} + \ldots + 2^{k\text{-}1}$

$T(n) \;\; = 2^{n\text{-}1} * T(n\text{-}(n\text{-}1)) + 1+2 + 2^{2} + \ldots + 2^{n\text{-}2}$

$T(n) \;\; = 2^{n\text{-}1} * T(1) + 1+2 + 2^{2} + \ldots + 2^{n\text{-}2}$

$$= 1 + 2 + \ldots + 2^{n\text{-}1} = \sum_{i=0}^{n-1} 2^{i}$$

$\Theta(2^{n})$

If n=64 the 2^{64} seconds about 1.84×10^{19} seconds or 584+billion years

Forming Recurrence Relations

```
public void f (int n) {
    if (n > 0) {
        System.out.println(n);
        f(n-1);
    }else
        return;
}
```

The recurrence relation is:

$$T(0) = 1$$

$$T(n) = T(n-1) + c \qquad \text{if } n > 0$$

$$
\begin{aligned}
T(n) &= T(n-1) + c \\
&= T(n-2) + c + c \\
&= T(n-3) + c + c + c \\
&\ldots \\
&= T(n-k) + kc
\end{aligned}
$$

Stop when $k = n$

$T(n) = T(0) + cn$

$T(n) = b + cn = \Theta(n)$

$$\mathbf{T(n) = \Theta(n)}$$

Write a recurrence for the running time T(n) of Algo1(n).

```
Algo1(n) {
  total = 0
  if n <= 1 return 2
  else {
        total = Algo1(n/4) + Algo1(n/4)
        for i = 1 to n do
            for j = 1 to n do
                    total = i + j
      return total
  }
}
```

a) $T(n) = T(2n/4) + cn$

b) $T(n) = 2T(n/4) + cn^2$

c) $T(n) = 2T(n/4) + c$

d) $T(n) = T(2n/4) + cn^2$

e) $T(n) = T(n/4) + cn^2$

Write a recurrence for the running time T(n) of Algo2(n).

```
Algo2(n) {
   total = 0
      if n <= 1 return 2
   else {
         Algo2(n/2)
          print total
         Algo2(n/2)
         for j = 1 to n do
                 total = n+j
         print total
         return
      }
}
```

a) $T(n) = T(n/2) + cn$

b) $T(n) = 2T(n/2) + cn^2$

c) $T(n) = 2T(n/2) + cn$

d) $T(n) = T(n/4) + cn^2$

e) $T(n) = T(n/4) + cn^2$

Recurrences Solutions

- $T(n) = T(n-1) + cn$ $\Theta(n^2)$
  - Recursive algorithm that loops through the input to eliminate one item
- $T(n) = T(n/2) + c$ $\Theta(\lg n)$
  - Recursive algorithm that halves the input in one step
- $T(n) = T(n/2) + cn$ $\Theta(n)$
  - Recursive algorithm that halves the input but must examine every item in the input
- $T(n) = 2T(n/2) + c$ $\Theta(n)$
  - Recursive algorithm that splits the input into 2 halves and does a constant amount of other work

Methods for Solving Recurrences

- Iteration method
- Substitution method
- Recursion tree method
- Master method

The Iteration Method

- Convert the recurrence into a summation and try to bound it using a known series
  - Iterate the recurrence until the initial condition is reached.
  - Use back-substitution to express the recurrence in terms of n and the initial (boundary) condition.

Iteration Method – Binary Search

$$\mathbf{T(n) = c + T(n/2)}$$

$T(n) = c + T(n/2)$

$\quad = c + c + T(n/4)$

$\quad = c + c + c + T(n/8)$

$T(n/2) = c + T(n/4)$

$T(n/4) = c + T(n/8)$

Stop when $n/2^i = 1 \quad \Rightarrow i = \lg n$

$$T(n) = \underbrace{c + c + \ldots + c}_{\text{n times}} + T(1)$$

$\quad = c\lg n + T(1)$

$\quad = \Theta(\lg n)$

Iteration - Mergesort

$$\mathbf{T(n) = n + 2T(n/2)}$$

$T(n) = n + 2T(n/2)$ $\qquad$ $T(n/2) = n/2 + 2T(n/4)$

$= n + 2(n/2 + 2T(n/4))$

$= n + n + 4T(n/4)$

$= n + n + 4(n/4 + 2T(n/8))$

$= n + n + n + 8T(n/8)$

… $= in + 2^iT(n/2^i)$ stop at $i = \lg n$

$= n\lg n + 2^{\lg n}T(1)$

$= n\lg n + nT(1)$

$= \Theta(n\lg n)$

Substitution Method

- Guess a solution

  $T(n) = O(g(n))$

  Induction goal: apply the definition of the asymptotic notation

  $T(n) \leq c\, g(n)$, for some $c > 0$ and $n \geq n_0$

- Induction hypothesis: $T(k) \leq c\, g(k)$ for all $k < n$

- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

Substitution: $T(n) = T(n-1)+T(n-2)$

Guess: $T(n) = O(\phi^n)$

Induction goal: $T(n) \le c\phi^n$, for some c and $n \ge n_0$

– Induction hypothesis: $T(k) \le c\phi^k$ for $k < n$

– Proof of induction goal:

$$
\begin{aligned}
T(n) &= T(n-1) + T(n-2) \\
&\le c\phi^{n-1} + c\phi^{n-2} \\
&\le c\phi^{n-2} (\phi + 1) \\
&\le c\phi^{n-2} (\phi^2)
\end{aligned}
$$

$$T(n) \le c\,\phi^n$$

$$T(n) = O(\phi^n)$$

Properties

$$\boldsymbol{\Phi = \frac{1+\sqrt{5}}{2}}$$

$$\boldsymbol{\Phi^2 = \frac{3+\sqrt{5}}{2}}$$

$$\boldsymbol{\Phi + 1 = \Phi^2}$$

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- Convert the recurrence into a tree:
  - Each node represents the cost incurred at various levels of recursion
  - Sum up the costs of all levels
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- Usually involves geometric series

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$T(n)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$T(n/4)$ $\qquad\qquad$ $T(n/2)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

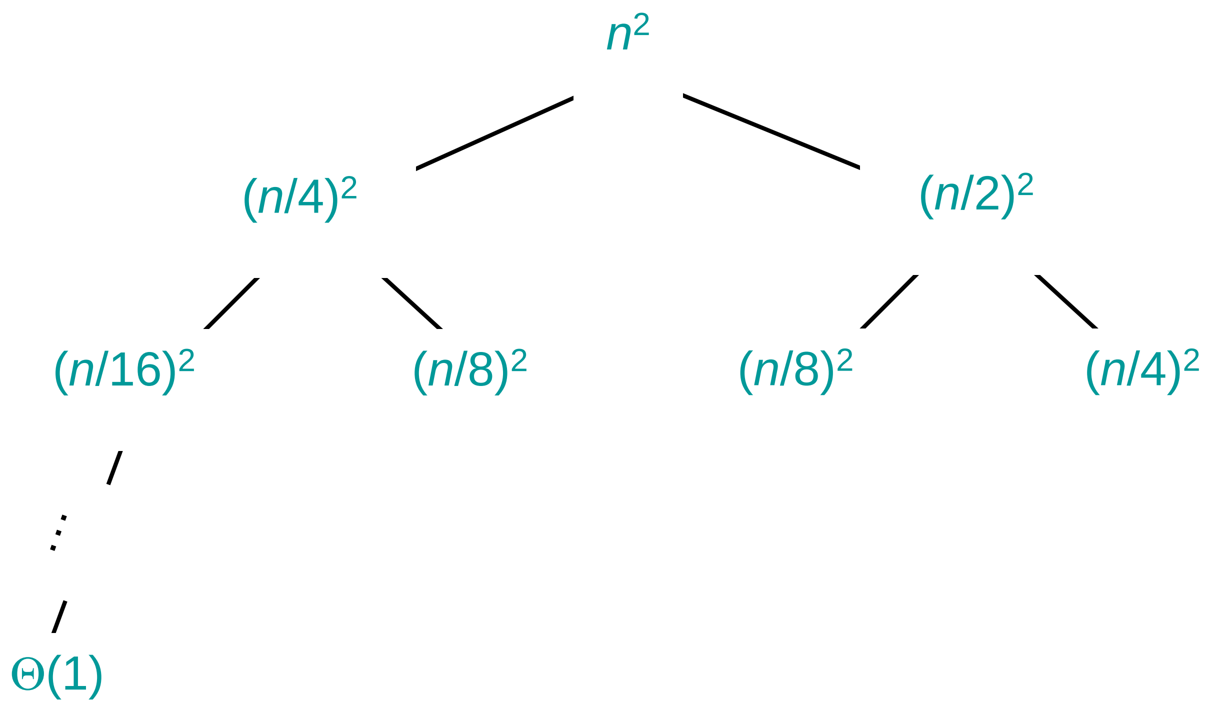
n^2

$(n/4)^2$ $(n/2)^2$

$T(n/16)$ $T(n/8)$ $T(n/8)$ $T(n/4)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

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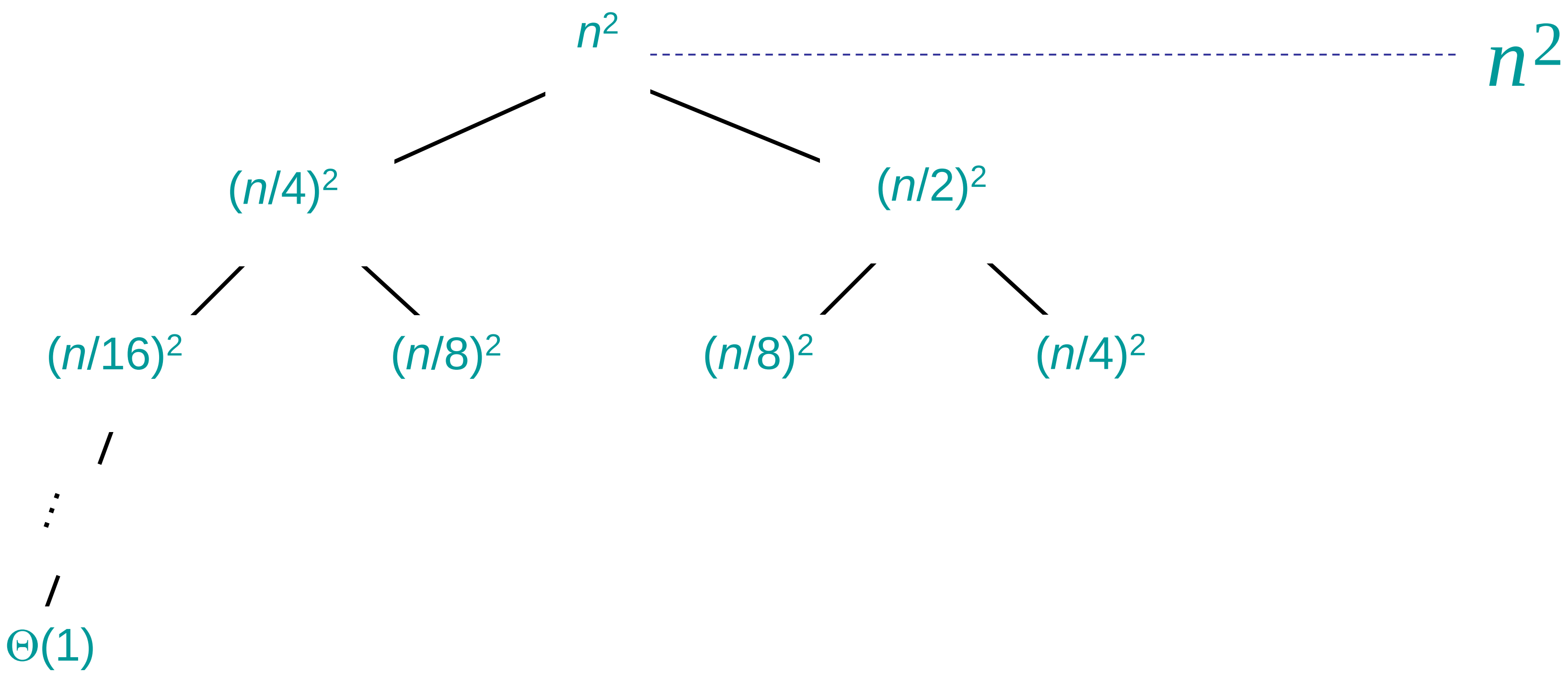

Example of recursion tree

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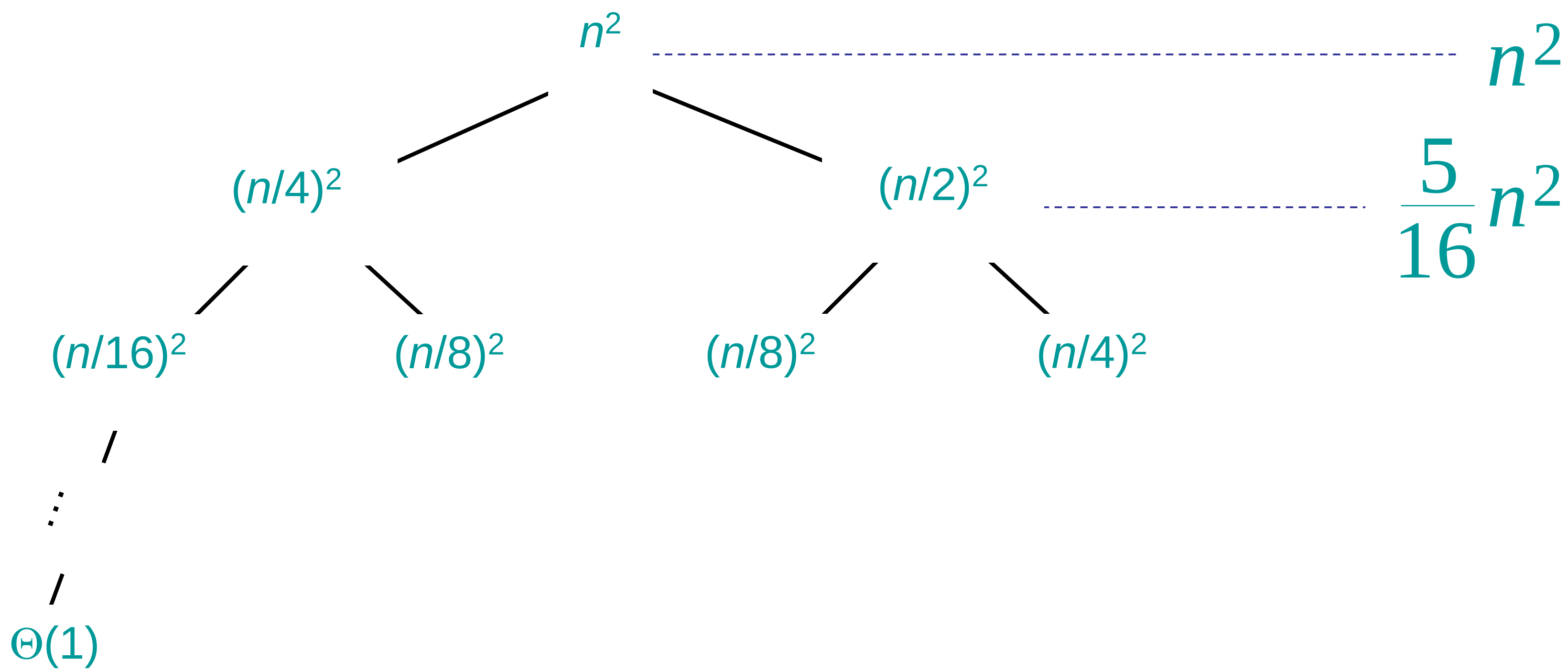
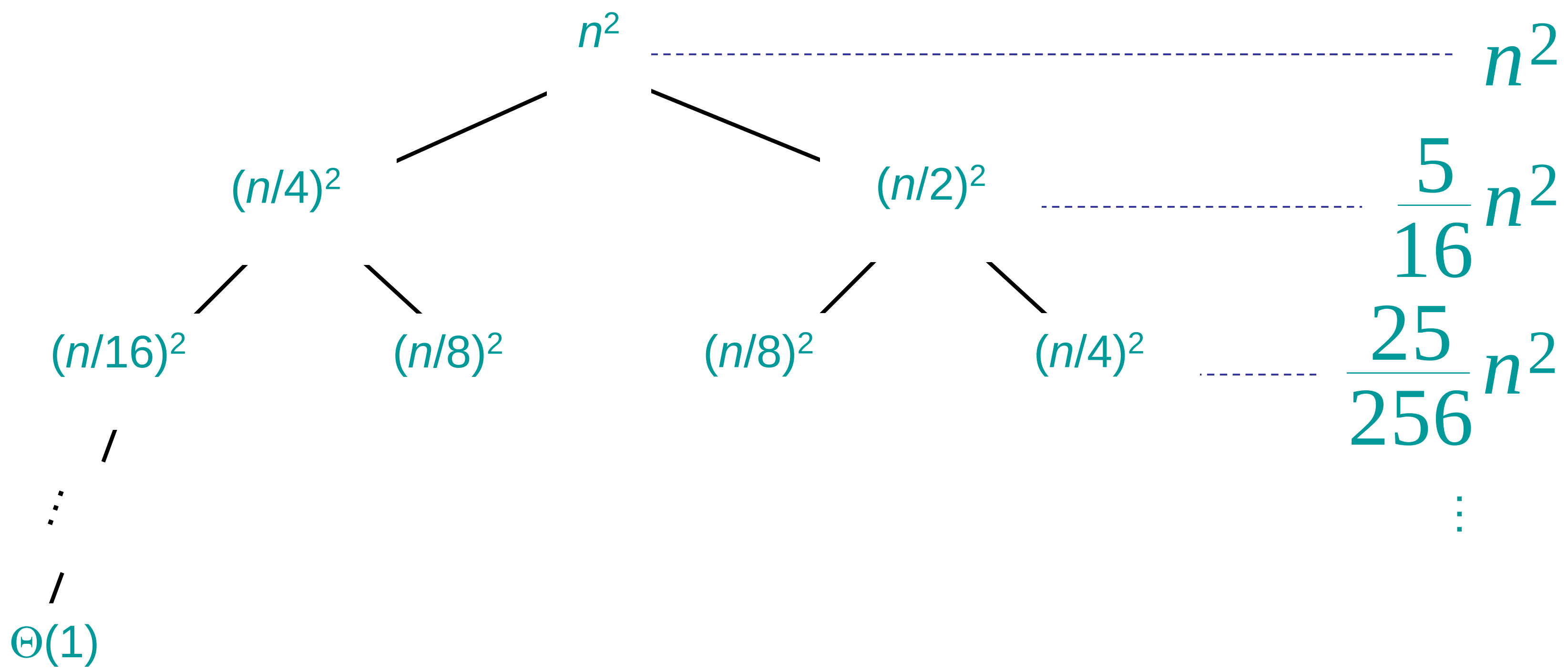


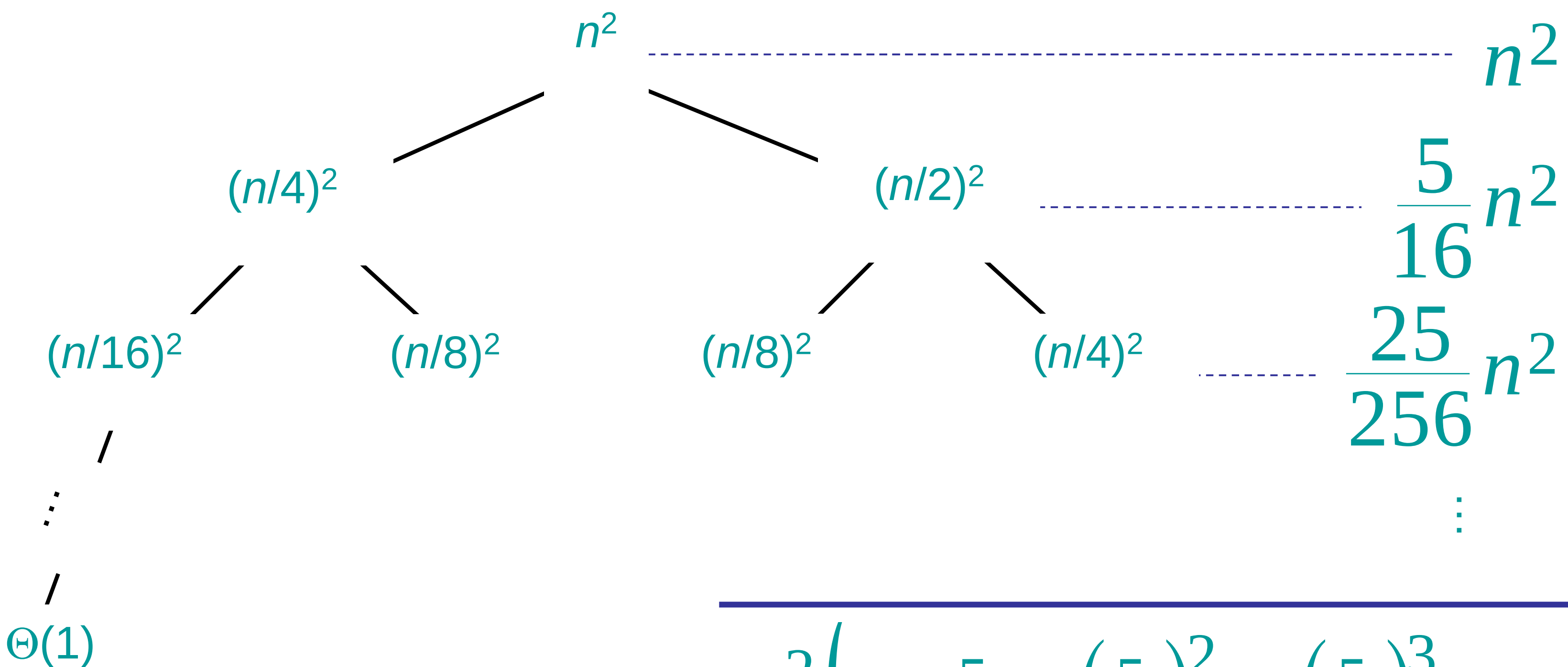
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} \leq n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

geometric series

$$= \mathrm{O}(n^2)$$

Geometric series

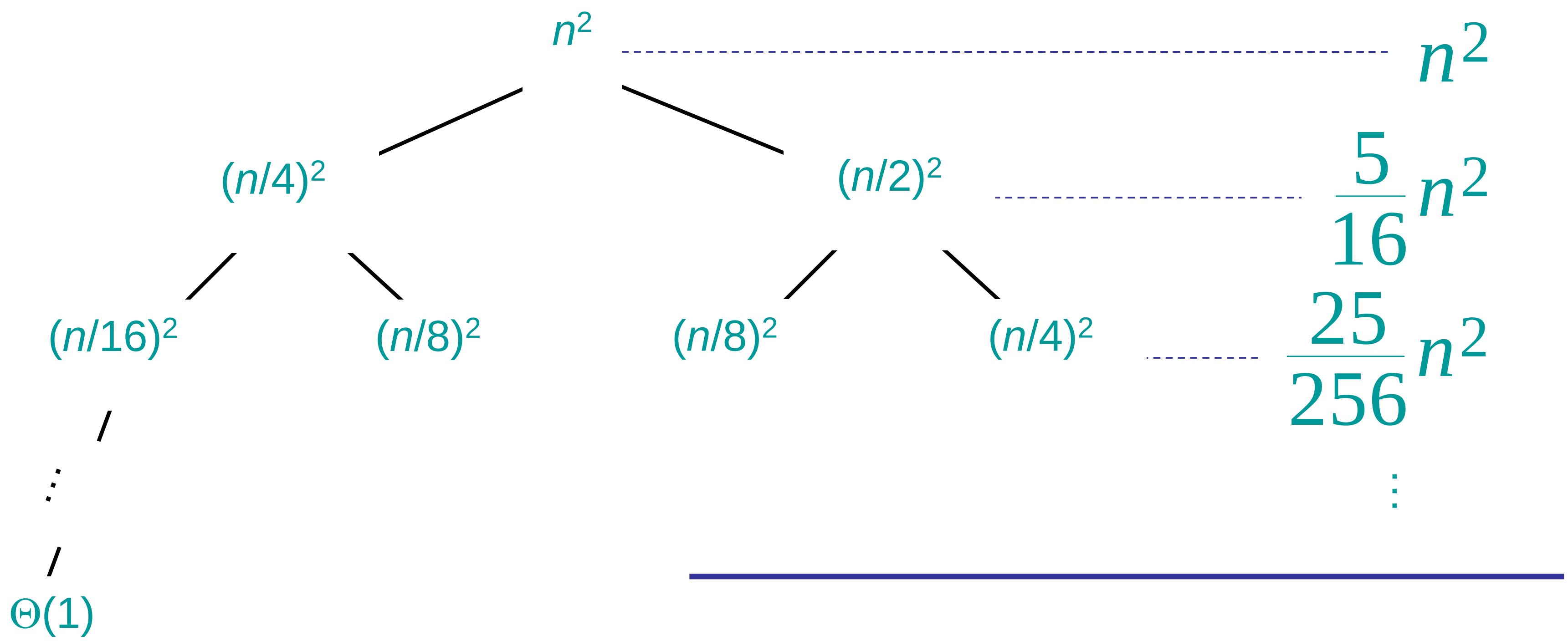
$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$ for $x \neq 1$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x}$$ for $|x| < 1$

$$n^2\left(1 + \tfrac{5}{16} + \left(\tfrac{5}{16}\right)^2 + \left(\tfrac{5}{16}\right)^3 + \cdots\right) = n^2\left(\frac{1}{1 - \frac{5}{16}}\right) = \tfrac{16}{11} n^2$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Total $\geq n^2$ $= \Omega(n^2)$

Therefore $T(n) = \Theta(n^2)$

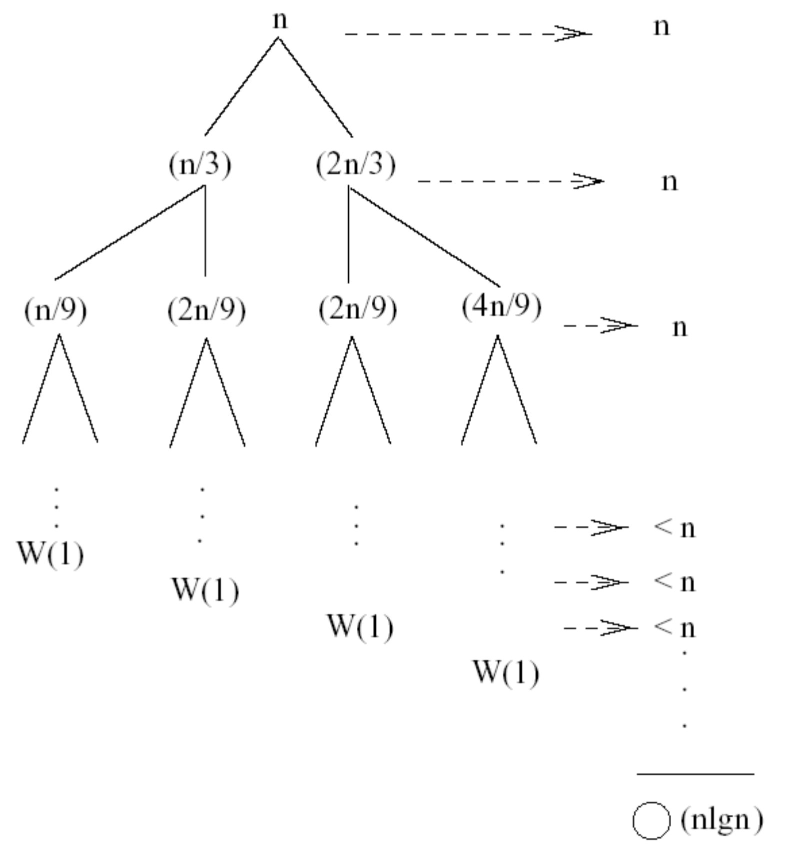
Recursion Tree – Example 2

$T(n) = T(n/3) + T(2n/3) + n$

- The longest path from the root to a leaf is:

  $n \rightarrow (2/3)n \rightarrow (2/3)^2\, n \rightarrow \dots \rightarrow 1$
- Subproblem size hits 1 when

  $1 = (2/3)^i n \Leftrightarrow i = \log_{3/2} n$
- cost of the problem at level i = n
- Total cost:

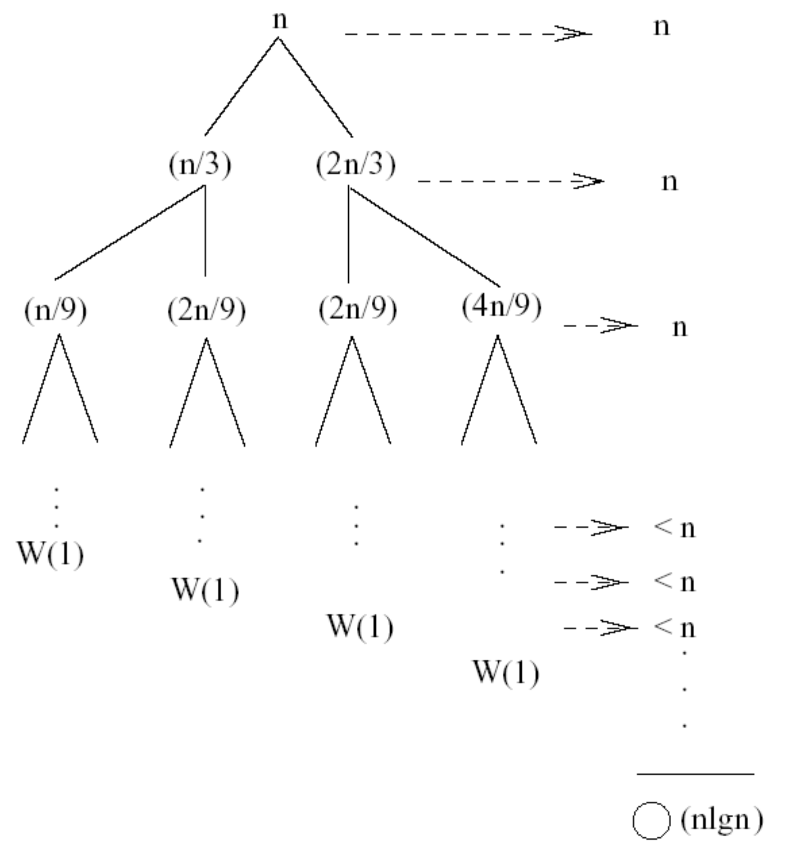


$$T(n) < n + n + \ldots = n(\log_{3/2} n) = n\frac{\lg n}{\lg \frac{3}{2}} = O(n \lg n)$$

$\Rightarrow T(n) = O(n\lg n)$

Recursion Tree – Example 3

$T(n) = T(n/3) + T(2n/3) + n$



$T(n) = \Omega(n)$

$T(n) = O(n\lg n)$

The Master Method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Master Method

"Formula" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$, and $f(n) > 0$

case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

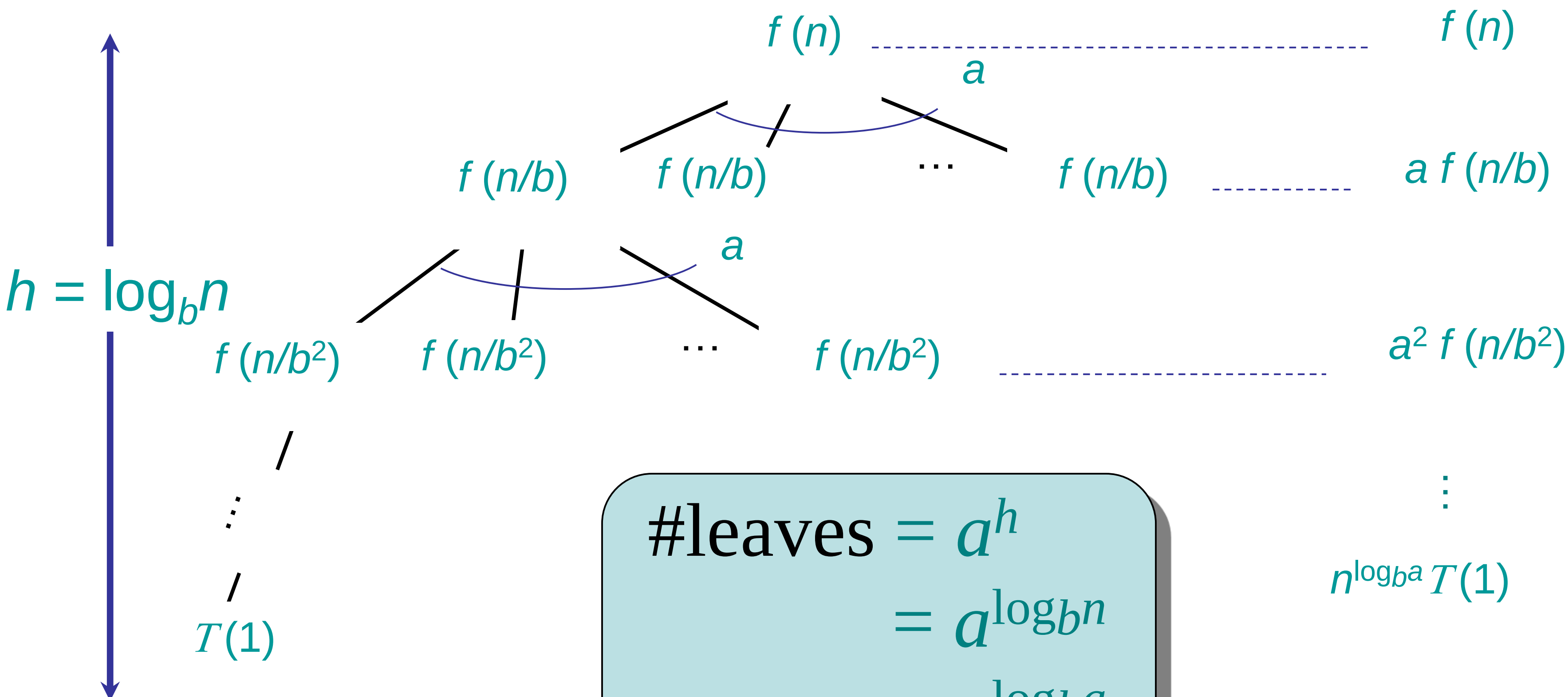
$af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then:

$$T(n) = \Theta(f(n))$$

regularity

Idea of Master Method

Recursion tree:

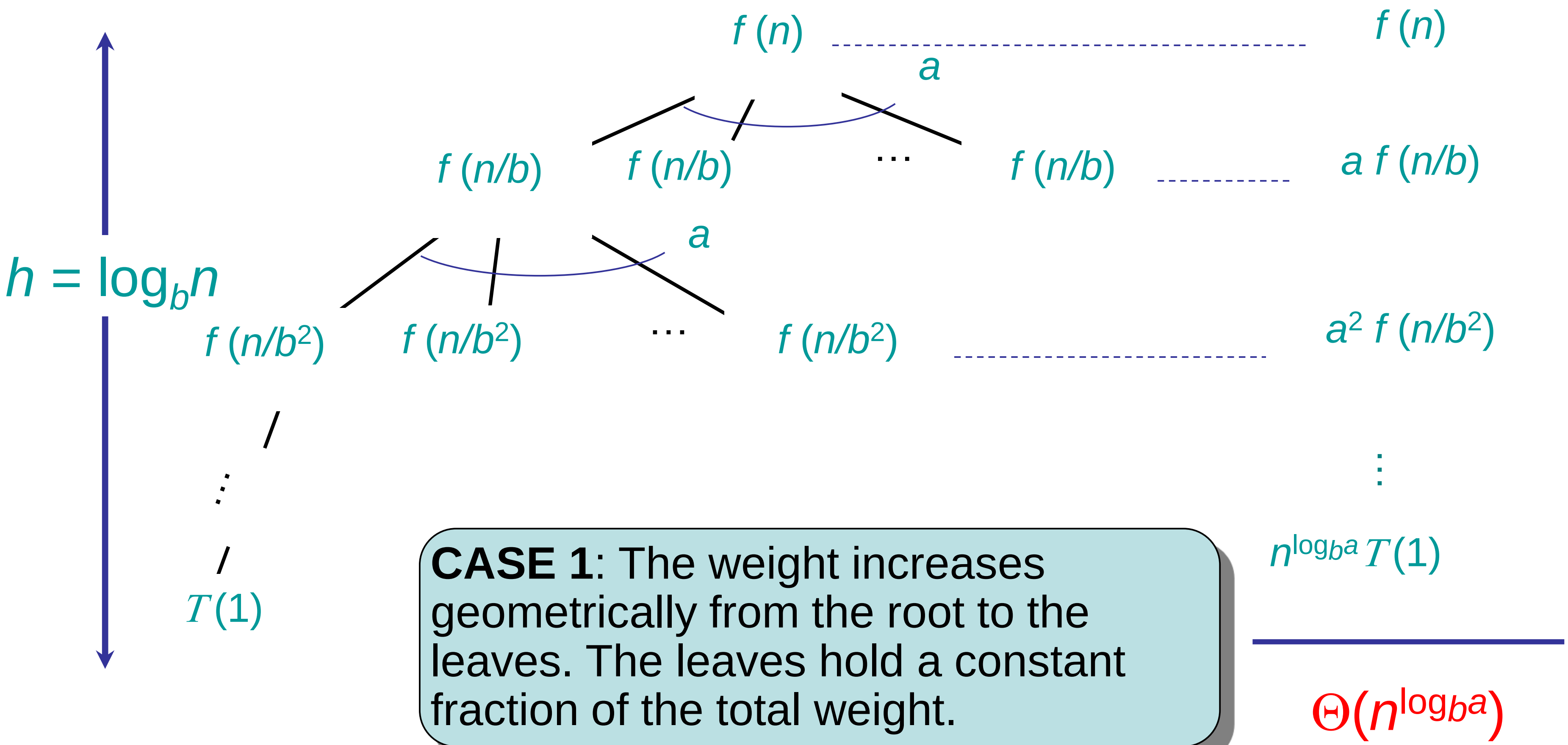


$$
\begin{aligned}
\#\text{leaves} &= a^h \\
&= a^{\log_b n} \\
&= n^{\log_b a}
\end{aligned}
$$

Idea of Master Method

$$f(n) = O(n^{\log_b a - \varepsilon})$$

Recursion tree:



CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

   $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

Case 1

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

Idea of Master Method

$$f(n) = \Theta(n^{\log_b a})$$

Recursion tree:

$f(n)$ .... $f(n)$

a

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ .... $a\,f(n/b)$

a

$h = \log_b n$

$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ .... $a^2 f(n/b^2)$

$\vdots$ $\vdots$

$T(1)$ $n^{\log_b a}\, T(1)$

CASE 2: The weight is approximately the same on each of the $\log_b n$ levels.

$\Theta(n^{\log_b a} \lg n)$

Case 2

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

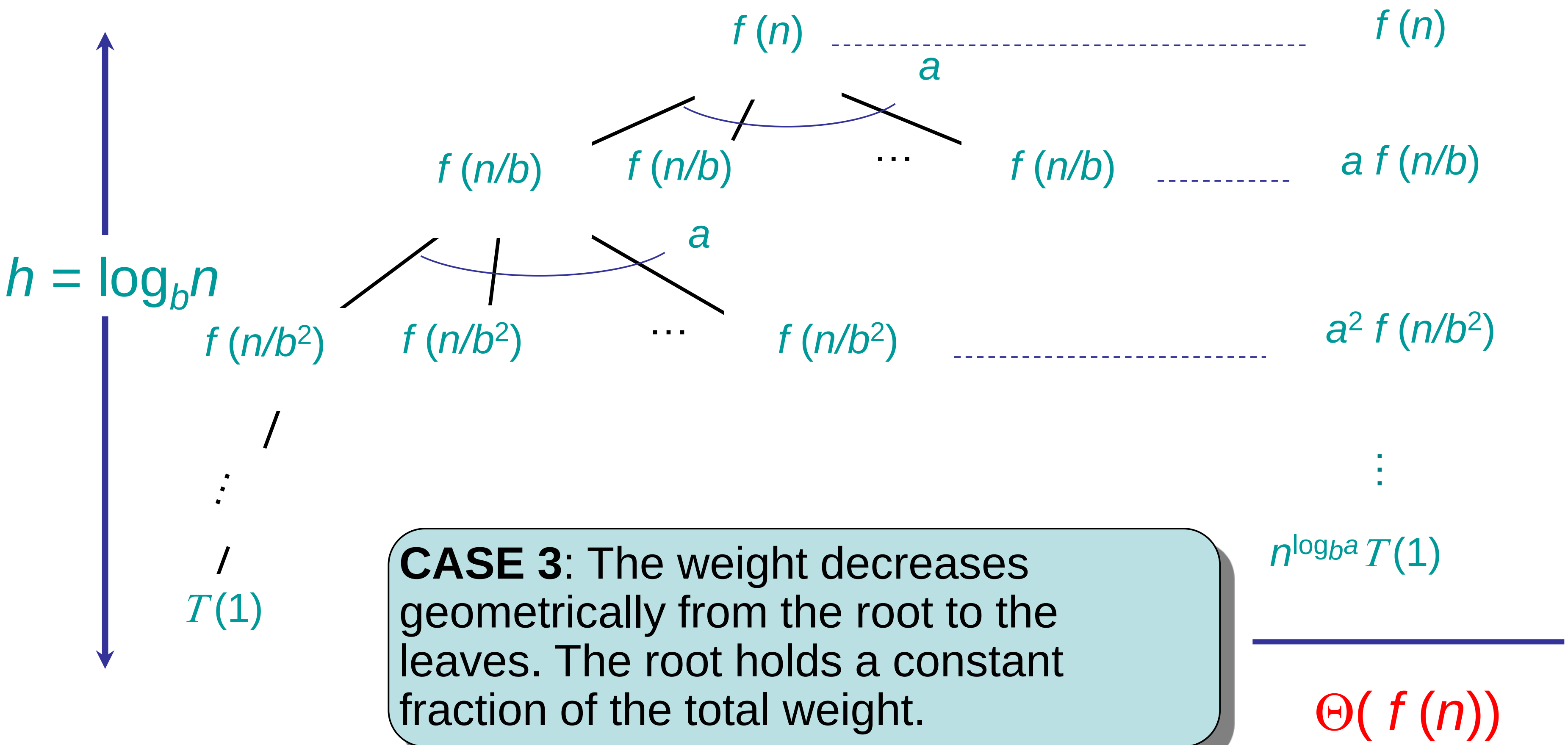
CASE 2: $f(n) = \Theta(n^2)$

$\therefore\ T(n) = \Theta(n^2 \lg n).$

Idea of master theorem

$$f(n) = \Omega(n^{\log_b a + \varepsilon})$$

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Case 3

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ ***and***

$4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3).$

Master Method

"Formula" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$, and $f(n) > 0$

case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then:

$T(n) = \Theta(f(n))$

regularity

Master Method – Binary Search

$T(n) = T(n/2) + c$ $a = 1, b = 2, \log_2 1 = 0$

compare $n^{\log_2 1} = n^0 = 1$ with $f(n) = c$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

$f(n) = \Theta(1) \Rightarrow$ case 2

$\Rightarrow T(n) = \Theta(\lg n)$

Master Method – Example 1

$T(n) = 2T(n/2) + n^2$ $\quad\quad$ $a = 2$, $b = 2$, $\log_2 2 = 1$

compare n with $f(n) = n^2$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$

$\Rightarrow f(n) = \Omega(n^{1+\varepsilon})$ case 3 $\Rightarrow$ verify regularity cond.

$a\, f(n/b) \leq c\, f(n)$

$\Leftrightarrow 2\,(n/2)^2 \leq c\, n^2$

$\Leftrightarrow 2\, n^2/4 \leq c\, n^2 \Rightarrow c = ½$ is a solution ($c<1$)

$\Rightarrow T(n) = \Theta(n^2)$

Master Method – Example 2

$T(n) = 2T(n/2) + \sqrt{n}$ $\quad$ $a = 2$, $b = 2$, $\log_2 2 = 1$

compare n with $f(n) = n^{1/2}$

$\Rightarrow f(n) = O(n^{1-\varepsilon})$ $\quad$ case 1

$\Rightarrow T(n) = \Theta(n)$

Master Method - Example 3

$T(n) = 3T(n/4) + n\lg n$ $a = 3, b = 4, \log_4 3 = 0.793$

compare $n^{0.793}$ with $f(n) = n\lg n$

$f(n) = \Omega(n^{\log_4 3+\varepsilon})$ case 3

check regularity condition:

$a\, f(n/b) \le c\, f(n)$

$3 * (n/4)\lg(n/4) \le 3/4 n\lg n \Rightarrow c = 3/4$

$T(n) = \Theta(n\lg n)$

Master Method: Merge-Sort

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + kn$$

where, $a = 2$, $b = 2$, and $f(n) = nn$

$n^{\log_b a} = n^{\log_2 2} = n$

case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$$T(n) = \Theta(n\lg n)$$