CS 325 - Homework 5 - Solutions

1. **(7 points 1 pt each)** Let X and Y be two decision problems. Suppose we know that X reduces to Y in polynomial time. Which of the following can we infer? Explain

   a. If Y is NP-complete then so is X. False cannot be inferred

   b. If X is NP-complete then so is Y. False cannot be inferred

   c. If Y is NP-complete and X is in NP then X is NP-complete. False cannot be inferred

   d. If X is NP-complete and Y is in NP then Y is NP-complete. TRUE

   e. If X is in P, then Y is in P. False cannot be inferred

   f. If Y is in P, then X is in P. TRUE

   g. X and Y can't both be in NP. FALSE

2. **(3 points 1 pt each)** Two well-known NP-complete problems are 3-SAT and TSP, the Traveling Salesman Problem. The 2-SAT problem is a SAT variant in which each clause contains at most two literals. 2-SAT is known to have a polynomial-time algorithm. Are the following statements true or false? Justify your answer.

   a. 3-SAT $\leq_p$ TSP. TRUE – TSP in NP-complete so all problems in NP can be reduced to it in polynomial time

   b. If P $\neq$ NP, then 3-SAT $\leq_p$ 2-SAT. FALSE. 2-SAT is in P

   c. If TSP $\leq_p$ 2-SAT, then P = NP. TRUE. 2-SAT is in P

3. **(10 points)** A Hamiltonian path in a graph is a simple path that visits every vertex exactly once. Show that HAM-PATH = { (G, u, v): there is a Hamiltonian path from u to v in G} is NP-complete. You may use the fact that HAM-CYCLE is NP-complete

NP-Complete Proof - Prove that **HAM-PATH** is in NP-Complete

<u>Step 1:</u> Show that **HAM-PATH** $\in$ NI **3 points**

For an instance of HAM-PATH,(G,u,v), and a "candidate solution" Hamiltonian path P, we can verify the solution in polynomial time by the following steps:

a) Check the adjacency matrix (or list) of G to verify that there is an edge between every vertex in the path P. This takes at most $O(V^2)$.
b) Check that every vertex is listed in P exactly once. This takes at most $O(V^2)$.
c) Verify that P starts at u and ends at v. This takes O(1).

Therefore the candidate soltuion can be verified in polynomial time.

<u>Step 2:</u> Show that **HAM-CYCLE** $\leq_P$ **HAM-PATH** where **HAM-CYCLE** $\in$NP-Complete **7 points**

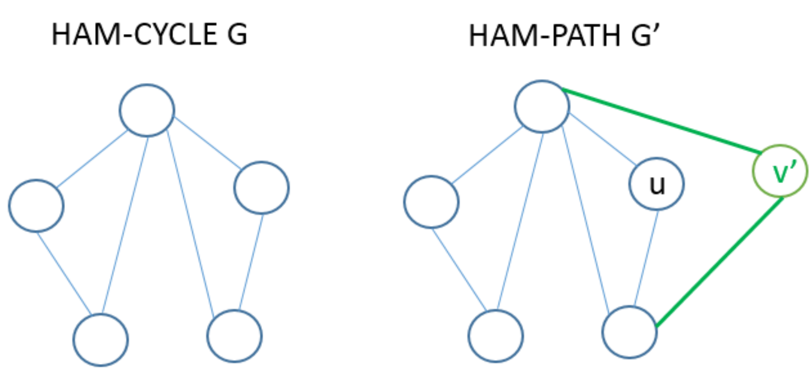
a) Show a polynomial algorithm to transform **HAM-CYCLE** into an instance of **HAM-PATH**.

   Given a graph G = (V,E) that is an instance of HAM-CYCLE create new graph G' = (V',E') such that G has a Hamiltonian cycle if and only if G' has a Hamiltonian path from (u,v).

   To create G'

   i. Duplicate all the edges and vertices in G so that G' = G with V=V' and E=E'. G and G' now have the same adjacency list/matrix. This takes $O(V^2)$.
   ii. Next select any vertex in V' to be u, create a new vertex v' and add v' to V'. Next for all edges (u,x) in E add the edge (v',x) to E'. Now v' will be adjacent to the same vertices as u. This can be completed in O(V) using either the adjacency list/matrix of G'.

   Below is an example of the transformation.



b) Show that the graph G has a Hamiltonian cycle if and only if graph G' has a Hamiltonian path.

$\Rightarrow$ If G = (V,E) has a Hamiltonian cycle then G' = (V', E') has a Hamiltonian Path from u to v'. Suppose that G has a Hamitonian cycle C. We can list the vertices in the cycle starting with u such that C = (u, v_i, …, v_k). Since C is a cycle then there is an edge from (u, v_k) in E and in E'. By the construction of G', the edge v' was added to G' such that it is adjacent to all edges adjacent to u. Therefore the edge (v_k, v') exists in E', and the path P = (u, v_i, …, v_k, v') will be a Hamiltonian path from u to v' in G'.

$\Leftarrow$ If G' = (V', E') has a Hamiltonian Path from u to v' then G = (V,E) has a Hamiltonian cycle.

Suppose that P = (u, x, …,y, v') is a Hamiltonian path in G'. This implies thay there is an edge from y to v' in G'. By the construction of G' there must have been an edge from u to y in G. Since v' is the only vertex in G' but not in G, it will not be in any cycle in G, we can let C = (u, x, …,y). Since there exists an edge from y to u in G, C is a Hamiltonian cycle in G.

Therefore **HAM-CYCLE $\leq_P$ HAM-PATH.**

Since 1) and 2) are true. HAM-PATH is NP-Complete

4**. Graph-Coloring**. **(10 points)**

It has been proven that 3-COLOR is NP-complete by using a reduction from SAT. Use the fact that 3-COLOR is NP-complete to prove that 4-COLOR is NP-complete.

Step 1: **(3 points) Show that 4-COLOR is in NP. Give a polynomial time algorithm to verify solution.**

Given a Graph G=(V,E) and a 4-coloring certificate function c: V -> {1, 2, 3, 4} we can verify if c is a "legal" coloring function in polynomial time. To verify the solution, for each vertex u in V we must check the colors of the adjacent vertices. All colors of adjacent vertices must be different. If for any (u, w) $\in$ E, c(u) = c(w) then c is not a 4-COLORING of G. The verification of the 4-coloring is polynomial in n (the number of vertices) since 4 $\leq$ n and the time required to look at all edges in G is **O(n^2).**

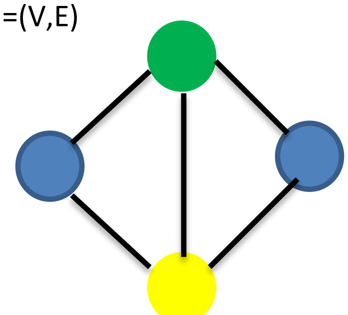
Step 2: (7 points) Show that there is a polynomial reduction from 3-COLOR to 4-COLOR.

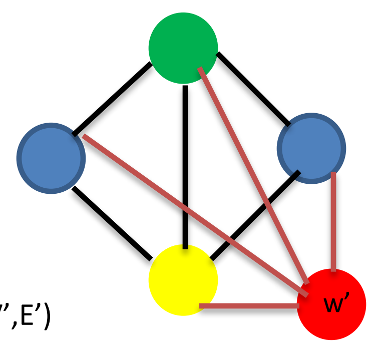
a) Reduce an instance G of 3-COLOR to an instance G' of 4-COLOR in polynomial time, and show that there is a 3-COLOR in G iff there is a 4-COLOR in G'. Let G=(V,E) be an instance of 3-COLOR transform G into G' by adding a new vertex w' that is connect t every other vertex. That is

G'=(V', E') where V'= V $\cup$\{w'\} and E' = E $\cup$ \{ (w', u) for all u $\in$ G\}

This reduction can be done in polynomial time since we are adding one vertex and at most n edges

G=(V,E)





G'=(V',E')

blue = 1, yellow = 2, green = 3, red = 4.

b) G has a 3-COLORing if and only if G' has a 4-COLORing

$\Rightarrow$ If G has a 3-COLORing then G' has a 4-COLORing. Assume G has a 3-COLORing then there exists a function c: V -> {1, 2, 3} such that for all u, w $\in$ V if (u,w) $\in$E then c(u) $\neq$c(w). Now define the 4-coloring function c' for G'

$$c'(u) = \begin{cases} c(u), & if\ u\ \in V \\ \quad 4, & if\ u\ \notin\ V\ \ (u = w') \end{cases}$$

Therefore, if there is a 3-COLORing in G then there is a 4-COLORing in G'

$\Leftarrow$ If G' has a 4-COLORing then G has a 3-COLORing. Assume G' has a 4-COLORing, since w' is adjacent to all other vertices in G' then w' must be a different color. Let c' be the coloring function for G', without loss of generality we can say that $c'(w')= 4$ and $c(u) \neq 4$ for all $u \in (V' - \{w\})$. However, $(V' - \{w\}) = (V \cup \{w'\} - \{w\}) = V$. So we have colored all of the original vertices in V using only colors 1, 2 and 3 proving that G is 3-COLORable.

Therefore **3-COLOR** $\leq_P$ **4-COLOR and** the 4-Color problem is NP-Hard

Since it was shown in Part 1 that 4-COLOR is in NP, and by Step 2 NP-Hard, 4-COLOR is NP-Complete.