DP Hotel Problem

You are going on a long trip. You start on the road at mile post 0. Along the way there are n hotels, at mile posts a_1, < a_2 < … < a_n, where each a_i is measured from the starting point. The only places you are allowed to stop are at these hotels, but you can choose which of the hotels you stop at. You must stop at the final hotel (at distance an), which is your destination.

You'd ideally like to travel 200 miles a day, but this may not be possible (depending on the spacing of the hotels). If you travel x miles during a day, the penalty for that day is $(200-x)^2$. You want to plan your trip so as to minimize the total penalty – that is the sum, over all travel days, of daily penalties. Give an efficient algorithm that determines the minimum penalty for the optimal sequence of hotels at which to stop.

Let S[j] be the minimum total penalty when you stop at hotel j.

```
Let S[0] = 0
for  j = 1,  j <= n
        S[j] = infinity
        for i = 0,  i < j
            S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
Return  S[n]
```

If you want to retrieve a list of hotels in the optimal solution you will need a “back pointer” array, bp, such that bp[j] = i iff you stop at hotel i immediately before hotel j.

Initialize bp[0] = -1;   // no hotels (mile posts) before mile post 0

Let bp[j] = i ; // if I is associated with the minimum **S[i] + (200- (a_j – a_i))^2**

For $j = 1$, $j <= n$
    $S[j] = inf$;
    For $i = 0$, $i < j$
        $S[j] = \min \{ S[i] + (200 - (a_j - a_i))^2 , S[j]\}$

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] **= 0**
S[1] = 0 + (200-150)^2 **= 2500**

```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2 , S[j]}
```

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0** bp[0] = -1

S[1] = **2500** bp[1] = 0

S[2] = 0 + (200-200)^2 = **0**

2500 + (200-(200-150))^2 = 25000

```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
```

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0**	bp[0] = -1
S[1] = **2500**	bp[1] = 0
S[2] = **0**	bp[2] = 0

S[2] = 0 + (200-200)^2 = **0**

2500 + (200-(200-150))^2 = 25000

For $j = 1, j <= n$
    $S[j] = inf;$
    For $i = 0, i < j$
        $S[j] = \min \{ S[i] + (200 - (a_j - a_i))^2 , S[j]\}$

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0**
S[1] = **2500**
S[2] = **0**

S[3] = 0 + (295-200)^2 = 9025
= 2500 + (200-(295-150)^2) = **5525**
= 0 + (200-(295-200))^2 = 11025

For $j = 1, j <= n$
    $S[j] = inf;$
    For $i = 0, i < j$
        $S[j] = \min\{ S[i] + (200 - (a_j - a_i))^2 , S[j]\}$

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0** bp[0] = -1
S[1] = **2500** bp[1] = 0
S[2] = **0** bp[2] = 0
S[3] = **5525** bp[3] = 1

S[3] = 0 + (295-200)^2 = 9025
= 2500 + (200-(295-150)^2) = **5525**
= 0 + (200-(295-200))^2 = 11025

```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
```

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0**
S[1] = **2500**
S[2] = **0**
S[3] = **5525**

S[4] = 0 + (375-200)^2 = 30625
= 2500 + (200-(375-150)^2) = **3125**
= 0 + (200-(375-200))^2 = 625
= 5525 + (200-(375-295)^2 = 19925

For $j = 1$, $j <= n$

$S[j] = inf;$

For $i = 0$, $i < j$

$S[j] = \min \{ S[i] + (200- (a_j - a_i))^2 , S[j]\}$

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[0] = **0**
S[1] = **2500**
S[2] = **0**
S[3] = **5525**
S[4] = **625**

bp[0] = -1
bp[1] = 0
bp[2] = 0
bp[3] = 1
bp[4] = 2

S[4] = 0 + (375-200)^2 = 30625
= 2500 + (200-(375-150)^2) = **3125**
= 0 + (200-(375-200))^2 = 625
= 5525 + (200-(375-295)^2 = 19925

For j = 1, j <= n
    S[j] = inf;
    For i = 0, i < j
        $S[j] = \min \{ S[i] + (200 - (a_j - a_i))^2 , S[j]\}$

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[5] = 0 + (480-200)^2 = 78,400
= 2500 + (200-(480-150)^2) =19,400
= 0 + (200-(480-200))^2 =6,400
= 5525 + (200-(480-295)^2 = **5750**
= 625 + (200 – (480-375))^2 = 9650

S[0] = **0**	bp[0] = -1
S[1] = **2500**	bp[1] = 0
S[2] = **0**	bp[2] = 0
S[3] = **5525**	bp[3] = 1
S[4] = **625**	bp[4] = 2
S[5] = **5750**	bp[5] = 3

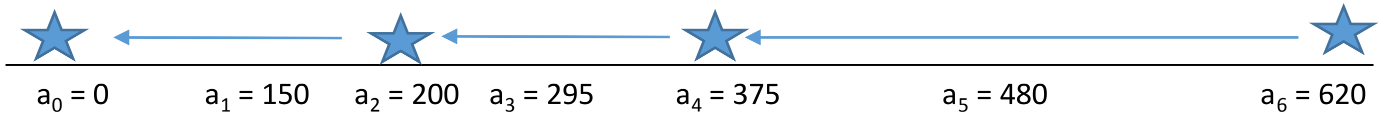
```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
```

$a_0 = 0$ $a_1 = 150$ $a_2 = 200$ $a_3 = 295$ $a_4 = 375$ $a_5 = 480$ $a_6 = 620$

S[6] = 0 + (620-200)^2 = 176,400
= 2500 + (200-(620-150)^2) =75,400
= 0 + (200-(620-200))^2 =48,400
= 5525 + (200-(620-295)^2 = 21,150
= 625 + (200 – (620-375))^2 = 2,650
= 5750 + (200 – (620-480))^2 = 9350

S[0] = **0**	bp[0] = -1
S[1] = **2500**	bp[1] = 0
S[2] = **0**	bp[2] = 0
S[3] = **5525**	bp[3] = 1
S[4] = **625**	bp[4] = 2
S[5] = **5750**	bp[5] = 3
S[6] = **2650**	bp[6] = 4

```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
```



S[6] = 0 + (620-200)^2 = 176,400
= 2500 + (200-(620-150)^2) =75,400
= 0 + (200-(620-200))^2 =48,400
= 5525 + (200-(620-295)^2 = 21,150
= 625 + (200 – (620-375))^2 = 2,650
= 5750 + (200 – (620-480))^2 = 9350

S[0] = **0**	bp[0] = -1
S[1] = **2500**	bp[1] = 0
S[2] = **0**	bp[2] = 0
S[3] = **5525**	bp[3] = 1
S[4] = **625**	bp[4] = 2
S[5] = **5750**	bp[5] = 3
S[6] = **2650**	bp[6] = 4

The minimum cost is $2650
Stop at hotels a2, a4 and a6.

Counter Example that greedy does not work

Greedy solution

DP Solution

Running time

- Nested for loops $\Theta(n^2)$

```
For  j = 1,  j <= n
        S[j] = inf;
        For i = 0,  i < j
                S[j] = min { S[i] + (200- (a_j – a_i))^2  , S[j]}
```