EEE 598 ASSIGNMENT - 2

REPORT

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WRITTEN PROBLEMS

Prob 1. Homogeneous Coordinates, Points and Lines:

1) Given line $y = mx + b$

$l \Rightarrow y - mx - b = 0$

$\Rightarrow$ Decomposing into matrix form,

$$\begin{bmatrix} -m & 1 & -b \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = -mx + y - b = 0$$

$\Rightarrow$ This is of the form $l^T X = 0$, where

$$l = \begin{bmatrix} -m \\ 1 \\ -b \end{bmatrix} \quad \& \quad X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

These are the homogeneous parameters.

2) To convert a homogeneous coordinate system $[X\ Y\ Z]^T$ to a 2D coordinate, $[X'\ Y']^T$, we can use the fact,

$$X' = \frac{X}{Z} \quad , \quad Y' = \frac{Y}{Z}$$

Given point $\begin{bmatrix}3\\5\end{bmatrix}$, if $z<0$, say $z=-1$,

$X = X' \cdot Z = -3$, $Y = Y' \cdot Z = -5$

$\Rightarrow$ for $z<0$, $\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}-3\\-5\\-1\end{bmatrix}$

Similarly, if $z>0$, say $z=1$,

$X = X' \cdot Z = 3$, $Y = Y' \cdot Z = 5$

$\Rightarrow$ for $z>0$, $\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}3\\5\\1\end{bmatrix}$

3) Let $l_1 \Rightarrow ax+by+c=0$ be $x+2y+3=0$

and $l_2 \Rightarrow dx+ey+f=0$ be $2x+3y+5=0$

Their homogeneos representations are as follows:

$$l_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad l_2 = \begin{bmatrix}2\\3\\5\end{bmatrix}$$

$$l_1 \times l_2 = \det\begin{bmatrix}x & y & z\\1 & 2 & 3\\2 & 3 & 5\end{bmatrix} = x(10-9) - y(5-6) + z(3-4).$$

$$l_1 \times l_2 = x + y - z = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$\Rightarrow l_1 \times l_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$x = -1$, $y = -1$ satisfies both line equations.

Hence proved that $l_1 \times l_2$ gives the intersecting point of l_1 & l_2.

4) Given, 2 lines $x + y - 5 = 0$, $4x - 5y + 7 = 0$.

To compute point of intersection, take cross product of l_1 & l_2 after writing them in homogeneous coord.

$$\Rightarrow l_1 = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$$

$$\Rightarrow l_1 \times l_2 = \det\left(\begin{bmatrix} x & y & z \\ 1 & 1 & -5 \\ 4 & -5 & 7 \end{bmatrix}\right) = x(7-25) - y(7+20) + z(-5-4)$$

$$\Rightarrow l_1 \times l_2 = \begin{bmatrix} -18 \\ -27 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore$ point of intersection is $(2, 3)$.

5) $l_1 \Rightarrow ax + by + c = 0$

Parallel line $l_2 \Rightarrow kax + kby + d = 0$

$$l_1 \times l_2 = \det\left(\begin{bmatrix} x & y & z \\ a & b & c \\ ka & kb & d \end{bmatrix}\right) = x(bd - ckb) - y(ad - kba) + d(akb - akb) = 0$$

— general form.

$$\Rightarrow l_1 \times l_2 = \begin{bmatrix} bd - cKb \\ Kba - ad \\ 0 \end{bmatrix} = \begin{bmatrix} bd - cKb/0 \\ Kba - ad/0 \end{bmatrix} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

$\Rightarrow$ There is no intersection points, since in the 2d coordinate, such a point doesn't exist for parallel lines. It can be intersecting at infinity.

6) Let 2 homogeneous points

$$x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad x_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

The line that can go through both these homogeneos points is equivalent to their intersecting point in the 2D cartesian plane.

$\therefore$ $x_1 \times x_2$ gives the line equation which passes through both these points.

$$x_1 \times x_2 = \det\left(\begin{bmatrix} x & y & z \\ a & b & c \\ d & e & f \end{bmatrix}\right) = (bf - ec)x - y(af - cd) + z(ae - bd) \rightarrow l_1$$

To justify that l_1 goes through both x_1 & x_2, we can substitute x_1 & x_2 in l_1 and check if it equates to zero.

Substituting x_1 in l_1,

$(bf - ec).a - (af - cd)b + (ae - bd)c$

$= \cancel{abf} - \cancel{aec} - \cancel{abf} + \cancel{bcd} + \cancel{aec} - \cancel{bcd}$

$= 0 \implies l_1$ passes through x_1

Substituting x_2 in l_1,

$(bf - ec)d - (af - cd)e + (ae - bd)f$

$= \cancel{bfd} - \cancel{dec} - \cancel{aef} + \cancel{cde} + \cancel{aef} - \cancel{bdf} = 0 \Rightarrow l_1$ passes through x_2

Problem ②: 2D TRANSFORMATIONS.

1. Rotate by angle θ around (a,b)

To do this, first the subject should be brought to the (a,b) origin, perform rotation and translate it back to its original point.

$$\Rightarrow \underset{\text{(Transformed point)}}{x'} = T_x \cdot \underset{\text{Rotation by } \theta.}{R} \cdot \underset{\text{negative translation by } a,b}{(T_x')} \cdot \underset{\text{(Point to be transformed)}}{X}$$

$$T_x = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T_x' = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Transformation Matrix } T_M = T_x \cdot R \cdot T_x'$$

$$T_M = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_M = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{T_M = \begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{bmatrix}}$$

2) $P_1 = (1,1)$ $P_2 = (2,1)$ $P_3 = (2,2)$ $P_4 = (1,2)$.

To construct the transformation matrix, $(a,b) = (2,1)$, $\theta = 45^\circ$

From previous problem,

$$x' = x\cos\theta - y\sin\theta - a\cos\theta + b\sin\theta + a, \quad y' = x\sin\theta + y\cos\theta - a\sin\theta - b\cos\theta + b$$

plugging in (a,b) & $\theta = 45^\circ$, we get,

$$x' = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 2, \quad y' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1$$

Using these equations to find each vertex after rotation around P_2,

For $P_1 = (1,1)$

$$x'_{P_1} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 2 = 2 - \frac{1}{\sqrt{2}} = 2 - 0.707 = 1.292$$

$$y'_{P_1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 = 1 - \frac{1}{\sqrt{2}} = 0.292$$

$\Rightarrow P_1' = (1.292, 0.292)$

For $P_2 = (2,1)$

Since P_2 is the point of origin around which the rotation is done, it doesn't need to be transformed.

$\Rightarrow P_2' = (2,1)$

For $P_3 = (2,2)$

$$x'_{P_3} = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 2 = 2 - \frac{1}{\sqrt{2}} = 1.292$$

$$y'_{P_3} = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 = 1 + \frac{1}{\sqrt{2}} = 1.707$$

$\Rightarrow P_3' = (1.292, 1.707)$

For $P_4 = (1,2)$

$$x'_{P_4} = \cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{2}{\sqrt{2}}} - \frac{2}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} + 2 = 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2(0.292) = 0.584$$

$$y'_{P_4} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 = 1 \quad \Rightarrow P_4' = (0.584, 1)$$

c) Reflection around a line $ax+by+c=0$

$$\Rightarrow y = \frac{-ax}{b} - \frac{c}{b}$$

$$\Rightarrow y = mx + c_1 \quad \left[m = \frac{-a}{b},\ c_1 = c/b\right]$$

To achieve the reflection we must do the following

→ 1st Translate the point by $(0,-c_1)$ since the line intersects y-axis at c_1.

→ Rotate with an angle $-\theta \approx \tan^{-1}(m)$, to align it with x-axis as θ is the angle the line makes with x-axis.

→ Apply a reflection in the x-axis, which is done with the following transformation matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Rotate back about the origin by θ, and then translate it by $(0, c_1)$

The concatenation of the above transformations is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & \frac{2c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -\frac{2c}{b}\cos^2\theta \\ 0 & 0 & 1 \end{bmatrix} \quad \text{— } (1)$$

Now, since $\tan\theta = -a/b$,

$$\Rightarrow \cos^2\theta = 1/1+\tan^2\theta = \frac{b^2}{(a^2+b^2)}$$

$$\Rightarrow \sin^2\theta = 1-\cos^2\theta = \frac{a^2}{(a^2+b^2)}$$

$$\Rightarrow \sin\theta\cos\theta = \tan\theta\cos^2\theta = \frac{-ab}{(a^2+b^2)}$$

Substituting the parameters in ①, we get,

$$\begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix}$$

Now, scaling it by (a^2+b^2) since homogeneous coordinates do not get affected by a multiplication of a factor. This way we can simplify the transformation to remove the denominators.

$$\Rightarrow \underset{\text{around } ax+by+c=0}{\text{Reflection Transformation}} = \underset{(a,b,c)}{Ref} = \begin{bmatrix} b^2-a^2 & -2ab & -2ac \\ -2ab & a^2-b^2 & -2bc \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Problem 3: (Affine + Second order Warp)

Given $(x,y) \xrightarrow{maps} (x',y')$

s.t $x' = ax+by+t_x+\alpha x^2+\beta y^2 \Rightarrow ax+by+t_x+\alpha x^2+\beta y^2-x' = 0$

& $y' = cx+dy+t_y+\gamma x^2+\theta y^2 \Rightarrow cx+dy+t_y+\gamma x^2+\theta y^2-y' = 0$

This can be rewritten as $AX = 0$, here X is all unknown parameters.

$$\begin{bmatrix} x & y & x^2 & y^2 & 1 & 0 & 0 & 0 & 0 & 0 & x' \\ 0 & 0 & 0 & 0 & 0 & x & y & x^2 & y^2 & 1 & y' \\ & & & & \vdots & & & & & & \end{bmatrix} * \begin{bmatrix} a \\ b \\ \alpha \\ \beta \\ t_x \\ c \\ d \\ \gamma \\ \theta \\ t_y \\ 1 \end{bmatrix} = 0$$

This is of the form $AX = 0$, which can be solved using the SVD decomposition of A where,

if $SVD(A) = U \cdot S \cdot V^T$, then the V vector corresponding to the smallest singular value, is a solution for X, given there are equal number of Known variables in A as the number of parameters (degrees of freedom)

Currently in X, there are 10 degrees of freedom.

From each $(x,y)-(x',y')$ correspondences, we have 6 known variables namely

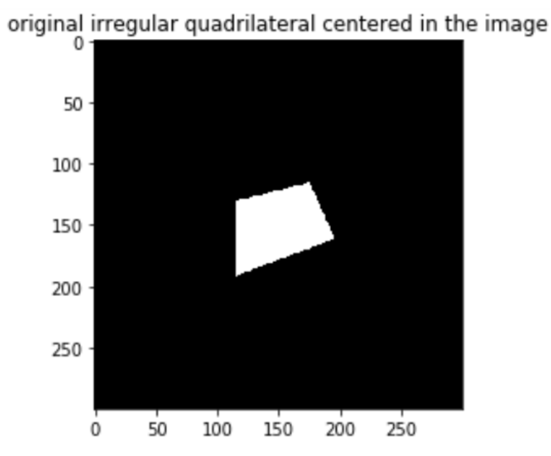
4 $\rightarrow x, y, x^2, y^2$

2 $\rightarrow x', y'$

$\therefore$ With a minimum of $\underline{\underline{2}}$ $(x,y)-(x',y')$ pairs, we can construct the A matrix & solve for X & get the parameters. (2 points minimum)

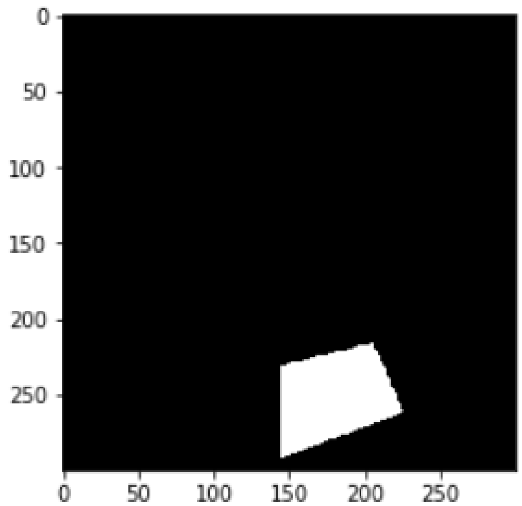
CODING ASSIGNMENTS

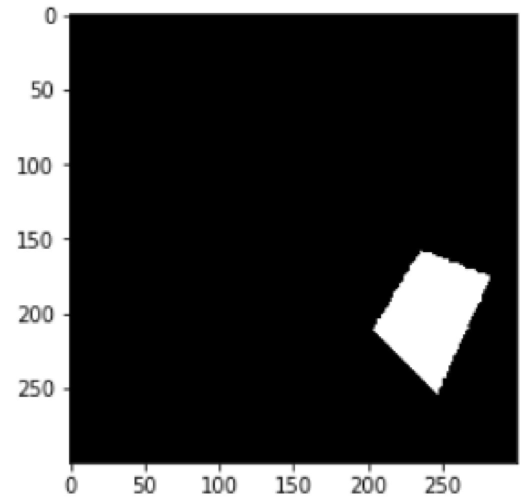
Problem 4.



Rotation and Translation is not commutative. Hence to get a rotated quadrilateral at a different location, we must first perform translation by (30,100) and then rotation of the polygon around the center.

- Translation and rotation done

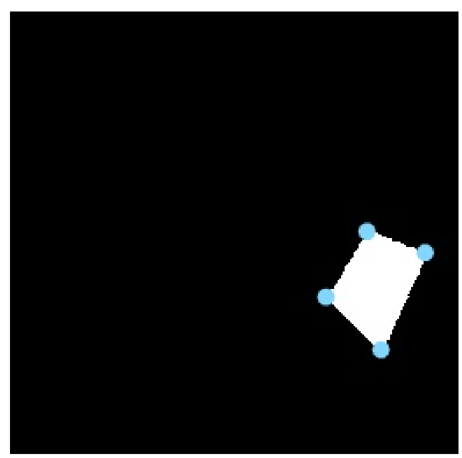




4.2) I implemented the Harris Corner detection from scratch. Check the jupyter notebook for the code. The most challenging part proved to be thresholding since there were many corners detected due to the scale of the image.

- The corners detected for each image were:

```
original image corners  [[175, 116], [115, 131], [195, 161], [115, 191]]
Transformed image corners  [[235, 158], [281, 175], [204, 211], [246, 253]]
```

For feature descriptor, I implemented angles at each corner feature. I made sure that the polygons had unique angles while constructing to avoid any false matches.
To implement this, I constructed vectors with corners and added the angles from 1 corner to the other 3. I used the angles between vectors formula as shown below

$$\theta = \cos^{-1}\frac{A.B}{|A||B|}$$

These were the angles found for both the images and then using some error threshold, corners were matched as shown below.

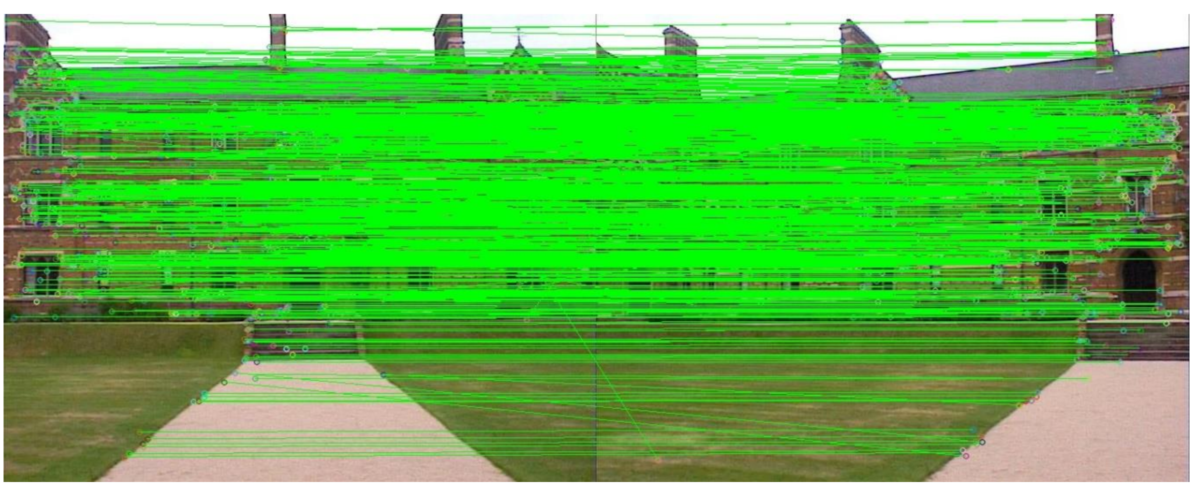
```
[99.92624550665171, 104.03624346792648, 86.59355624500529, 69.44395478041653]
[98.9529845932068, 86.64330788265305, 103.98611593958493, 70.41759158455523]
[[180, 60], [120, 75], [200, 105], [120, 135]]
[[199, 115], [168, 168], [245, 133], [210, 209]]
```

4.3) I implemented the linear least squares to get back the rotation and translation from transformed image w.r.t original image. It can be noted that, LLS optimization can give the rotation and translation relative to the correspondences. Therefore, recovering the actual transformation performed would be possible only if we did rotation first and then translation. However, the algorithm could still retrieve the angle well, but not the translation.

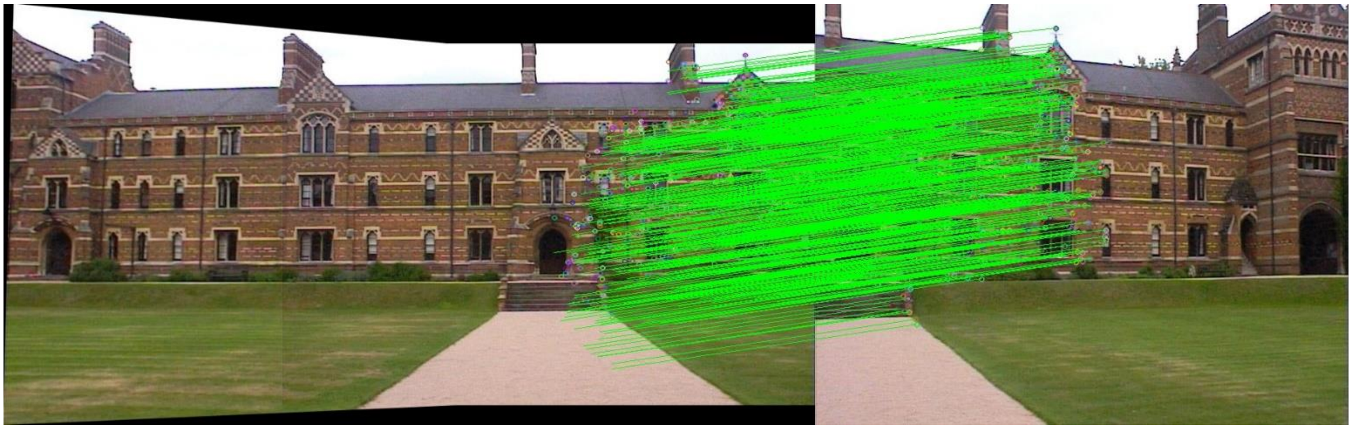
```
The rotation angle dervicd from Least Square optimization is  45.51210834563913
translation retained is [[410.47440267]
 [ 56.64011386]]
```

5) **Image Stitching.**

To derive the good matches or tentative matches, I first implemented KNN from scratch on the feature vector after performing SIFT detection operation. Then I used Lowe's ratio to eliminate matches that had the 2NN too close to the 1NN. This is done to make sure only the matches that has only 1 true correspondence is considered.



Then I implemented RANSAC using these good Matches to estimate Homography with the maximum inliers. By keeping the geometric distance threshold of 5 increasing the iterations to 10000, I could get a good 88% inlier accuracy for left and center and 68% when matching the below case. The inliers for the warped left and center image with respect to the right image is illustrated below.



Final Stitching: