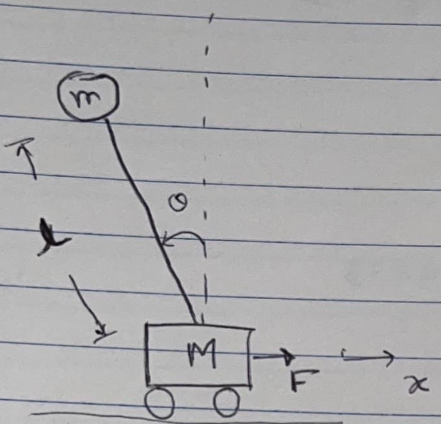
Cart-pole system

Note: 'm' is the total mass of pole assumed at the Centre of mass of Pole which is at 'l' distance from cart.



Let,

m - mass of bob / pendulum / pole

M - mass of cart

l - length of pendulum / pole

x - position of cart

$\dot{x}$ - velocity of cart

Position & velocity of pole can be written as,

$$x_{pole} = -l \sin\theta$$

$$\dot{x}_{pole} = -l \cos\theta . \dot{\theta} + \dot{x}$$

... (since cart's motion will add up)

$$y_{pole} = l \cos\theta$$

$$\dot{y}_{pole} = -\dot{l} \sin\theta . \dot{\theta}$$

Let,

T = Total kinetic energy of the system:

V = Total potential energy — " —

$$\therefore T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_{pole}^2 + \dot{y}_{pole}^2)$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m [(l^2 \cos^2\theta . \dot{\theta}^2 - 2l \cos\theta . \dot{\theta} . \dot{x} + \dot{x}^2) + (l^2 \sin^2\theta . \dot{\theta}^2)]$$

$$= \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} m (l^2 \cos^2\theta \dot{\theta}^2 + l^2 \sin^2\theta . \dot{\theta}^2) - ml \cos\theta . \dot{\theta} . \dot{x}$$

$$= \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 - ml \cos\theta . \dot{\theta} . \dot{x}$$

Also, $V = mg\,y_{pole}$

$$V = mgL\cos\theta \quad \text{—— (2)}$$

We know, Lagrangian of the system can given as,

$$L = T - V$$

from (1) & (2),

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}mL^2\dot{\theta}^2 - mL\cos\theta.\dot{\theta}.\dot{x} - mgl\cos\theta$$

The generalized coordinates can be selected as

$$X = \begin{bmatrix} x \\ \theta \end{bmatrix}$$

∴ Lagrangian equations are ⇒

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F \quad \text{—— (3)} \quad \& \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \text{—— (4)}$$

Now, $\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - ml\cos\theta.\dot{\theta}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - ml\cos\theta.\ddot{\theta} + ml\sin\theta.(\dot{\theta})^2 \quad \text{—— (5)}$$

$$\frac{\partial L}{\partial x} = 0 \quad \text{—— (6)}$$

Also $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\cos\theta\,\dot{x}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - ml\cos\theta.\ddot{x} + ml\sin\theta.\dot{\theta}.\dot{x} \quad \text{—— (7)}$$

$$\frac{\partial L}{\partial \theta} = ml \sin\theta . \dot{\theta} . \dot{x} + m g l \sin\theta \quad \text{--- (8)}$$

From eq^{ns} (3), (5), (6) & (4), (7), (8) we get following eqn^s,

$$(M+m)\ddot{x} + ml \sin\theta . \dot{\theta}^2 - ml \cos\theta . \ddot{\theta} = F \quad \text{--- (9)}$$

$$ml^2\ddot{\theta} - ml\cos\theta . \ddot{x} - mgl\sin\theta = 0 \quad \text{--- (10)}$$

These are the dynamic equations of motion of a cart-pole system.

Linearization of the cart-pole system around upright pole position $\Rightarrow$

While controlling around upright position,

θ & $\dot{\theta}$ → very small

$\therefore \sin\theta \approx \theta$

$\cos\theta \approx 1$

$\dot{\theta}^2 \approx 0$

Substituting approximate values in eq^n's (9) & (10) we get,

$$(M+m)\ddot{x} - ml\ddot{\theta} = F \quad \text{--- (11)}$$

$$ml^2\ddot{\theta} - ml\ddot{x} - mgl\theta = 0 \Rightarrow l^2\ddot{\theta} - l\ddot{x} - gl\theta = 0 \quad \text{--- (12)}$$

From (12),

$$\ddot{\theta} = \frac{l\ddot{x} + gl\theta}{l^2} = \frac{\ddot{x} + g\theta}{l} \quad \text{--- (13)}$$

Put value of $\ddot{\theta}$ from (13) to eqn (11)

$$(M+m)\ddot{x} - ml\left(\frac{\ddot{x} + g\theta}{l}\right) = F$$

$$\cancel{\therefore \ddot{x} = \frac{F + m(\ddot{x} + g\theta)}{M+m}}$$

$$\cancel{\ddot{x} = m}$$

$$(M+m)\ddot{x} - m(\ddot{x} + g\theta) = F$$

$$(M+m)\ddot{x} - m\ddot{x} - mg\theta = F$$

$$M\ddot{x} = F + mg\theta$$

$$\ddot{x} = \frac{m}{M}g\theta + \frac{1}{M}F \quad \text{—— (14)}$$

Put $\ddot{x}$ from eqn (14) in eqn (13) to get $\ddot{\theta}$

$$\therefore \ddot{\theta} = \frac{\left(\frac{m}{M}g\theta + \frac{1}{M}F\right) + g\theta}{l}$$

$$\ddot{\theta} = \frac{m}{M}\frac{g}{l}\theta + \frac{1}{Ml}F + \frac{g}{l}\theta$$

$$\ddot{\theta} = \left(1 + \frac{m}{M}\right)\frac{g}{l}\theta + \frac{1}{Ml}F \quad \text{—— (15)}$$

equation (14) & (15) are decoupled linearized dynamic equation of motion for cart-pole system.

For part d, we need to represent these equation in state space form

Let, $X = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$ be the state vector of our system

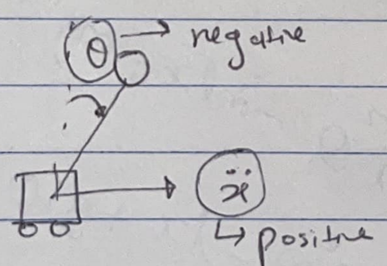
& $u = F$

State space form for linear time invariant system:

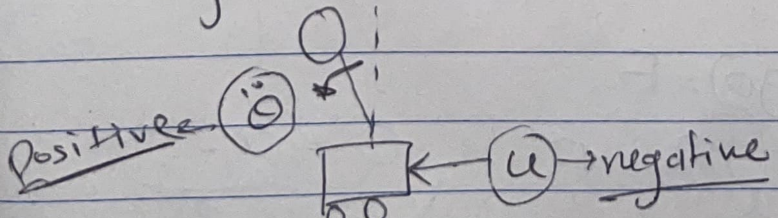
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \qquad D = 0$$

Before converting eq^{ns} (14) & (15) into state-space form, we need to take the sign consideration since it'll have particular directions of $\ddot{x}, \ddot{\theta}, \theta$ & u while controlling.

i.e. when $\ddot{x}$ is positive then θ must be negative while controlling. It is explained below.



Also, when $\ddot{\theta}$ is positive then u must be negative while controlling. It is explained in the following diagram



$\therefore$ State space form can be given as,

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{m}{M} g(-\theta) + \frac{1}{M} u \\ \dot{\theta} \\ \left(1+\frac{m}{M}\right)\frac{g}{l}\theta + \frac{1}{Ml}(-u) \end{bmatrix}$$

$$\therefore \dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \left(1+\frac{m}{M}\right)\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u$$

$$y = \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X$$

We will use this form in LQR control.