Fundamentally Sneaky: Applications of Vortex Dynamics

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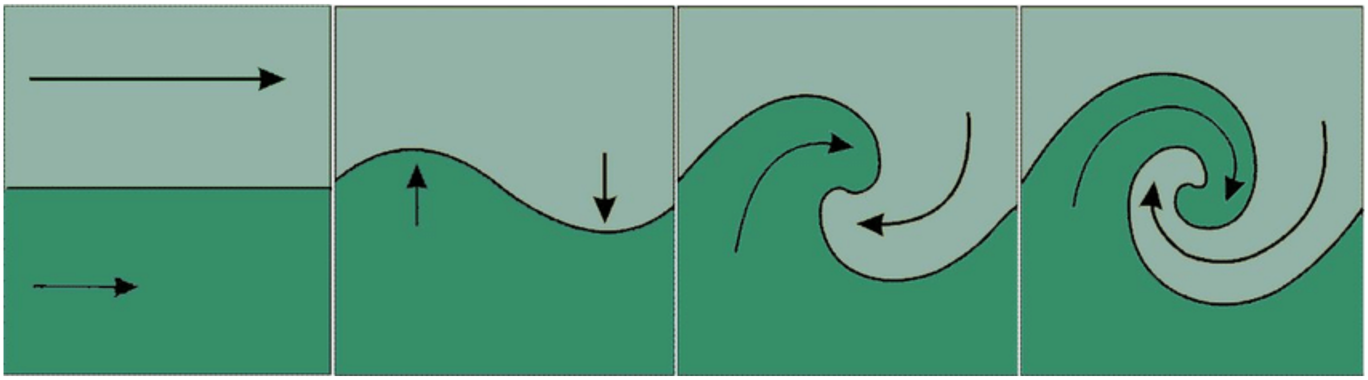
1 Kelvin-Helmholtz Instability

Perturbation Equations



We will be considering an instability at the interface between two horizontal, parallel streams of fluid with different densities. The denser fluid (B) is on the bottom, and the rarer on top (A). Suppose that in an unperturbed state, the surface of discontinuity (of zero thickness) is located at $y = 0$. Next, we assume that the fluid layers have an infinite depth.

Let $\boldsymbol{U}_A$, φ_A and ρ_A be the velocity, velocity potential and density of the top layer in the unperturbed state, and similarly, $\boldsymbol{U}_B$, φ_B and ρ_B for the bottom layer. The base flow is in the horizontal direction. Since we have incompressible flow,

$$\boxed{\nabla^2\varphi_A = 0 \quad , \quad \nabla^2\varphi_B = 0}$$

Moreover, the first boundary conditions we impose are

$$\boxed{\begin{aligned} \lim_{y\to\infty}(\boldsymbol{\nabla}\varphi_A) &= \boldsymbol{U}_A \\ \lim_{y\to-\infty}(\boldsymbol{\nabla}\varphi_B) &= \boldsymbol{U}_B \end{aligned}}$$

which simply means that the perturbation dies out far from the surface of separation of fluids. Suppose that the perturbed surface can be represented as $y = \xi(x,t)$. In the parametric form $f(x,y,z) = 0$, the surface can be represented as

$$f(x,y,t) = y - \xi(x,t)$$

Suppose the interface moves with a velocity $\boldsymbol{u}_{AB}$. The kinematic condition at the interface ($y = 0$) is $\dfrac{Df}{Dt} = 0$, or equivalently,

$$\begin{aligned} \frac{\partial f}{\partial t} + \boldsymbol{U}_{AB}\cdot\boldsymbol{\nabla}f &= 0 \\ \frac{\partial y}{\partial t} - \frac{\partial \xi}{\partial t} + \boldsymbol{U}_{AB}\cdot\boldsymbol{\nabla}y - \boldsymbol{U}_{AB}\cdot\boldsymbol{\nabla}\xi &= 0 \\ \frac{\partial \xi}{\partial t} + \boldsymbol{U}_{AB}\cdot\boldsymbol{\nabla}\xi &= \frac{\partial y}{\partial t} + \boldsymbol{U}_{AB}\cdot\boldsymbol{\nabla}y \\ \frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} &= v_\xi \end{aligned}$$

where u is the horizontal component of the velocity of the interface, and v_ξ is the vertical component of the fluid velocity at the interface. What this boundary condition ensures is that the interface moves up and down with a velocity that is equal to the vertical component of the fluid velocity.

Just above the surface (evaluated at $y = \eta(x,t)$), the kinematic boundary condition requires that

$$\frac{\partial \xi}{\partial t} + (U_A + v_{Ax})\frac{\partial \xi}{\partial x} = v_{Ay}$$

where v_{Ax} and v_{Ay} are the horizontal and vertical components of the perturbed velocity of the top fluid. Since the flow is potential, we can rewrite this as (using the same argument for the bottom fluid):

$$\boxed{\begin{aligned} \left.\frac{\partial \varphi_A}{\partial y}\right|_{y=\eta(x,t)} &= \frac{\partial \xi}{\partial t} + (U_A + v_{Ax})\frac{\partial \xi}{\partial x} \\ \left.\frac{\partial \varphi_B}{\partial y}\right|_{y=\eta(x,t)} &= \frac{\partial \xi}{\partial t} + (U_B + v_{Bx})\frac{\partial \xi}{\partial x} \end{aligned}}$$

So, we have boundary conditions on φ_A and φ_B.

Since we are dealing with incompressible flow, we can apply the unsteady Bernoulli equation to both sides of the interface. This will allow us to derive a dynamic boundary condition.

$$\begin{aligned} \frac{\partial \varphi_A}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\varphi_A)^2 + \frac{p_A}{\rho_A} + gy = C_A(t) \\ \frac{\partial \varphi_B}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\varphi_B)^2 + \frac{p_B}{\rho_B} + gy = C_B(t) \end{aligned}$$

At the interface, we must have that the pressure is continuous across the interface (if we ignore the effects of surface tension). Hence, at the interface ($y = \eta(x,t)$), the condition is that $p_A = p_B$. Using the Bernoulli equation for each fluid, we can rewrite this as

$$\boxed{\rho_A\left(\frac{\partial \varphi_A}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\varphi_A)^2 - C_A\right) = \rho_B\left(\frac{\partial \varphi_B}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\varphi_B)^2 - C_B\right)}$$

The steady horizontal base flow (U_A, U_B) satisfies this problem with $\eta = 0$. Hence we can write the dynamic boundary condition as

$$\rho_A\left(\frac{1}{2}(U_A)^2 - C_A\right) = \rho_B\left(\frac{1}{2}(U_B)^2 - C_B\right)$$

Now, we *decompose* the flow into a base flow and perturbative flow. We can thereby express the potentials as

$$\begin{aligned} \varphi_A = U_A x + \varphi'_A \\ \varphi_B = U_B x + \varphi'_B \end{aligned}$$

Substituting this into the Laplace equations for the potentials, we find

$$\begin{aligned} \nabla^2(\varphi'_A) = 0 \\ \nabla^2(\varphi'_B) = 0 \end{aligned}$$

The first boundary condition (that the perturbations die at infinity) gives

$$\begin{aligned}\boldsymbol{\nabla}(\varphi'_A)|_{y\to\infty} &= 0\\ \boldsymbol{\nabla}(\varphi'_B)|_{y\to-\infty} &= 0\end{aligned}$$

Now, we will linearize the kinematic surface conditions by evaluating at the surface $y = 0$ instead of at $y = \xi(x,t)$, and omitting quadratic terms:

$$\boxed{\begin{aligned}\left.\frac{\partial\varphi'_A}{\partial y}\right|_{y=0} &= \frac{\partial\xi}{\partial t} + U_A\frac{\partial\xi}{\partial x}\\ \left.\frac{\partial\varphi'_B}{\partial y}\right|_{y=0} &= \frac{\partial\xi}{\partial t} + U_B\frac{\partial\xi}{\partial x}\end{aligned}}$$

Great. Now, we substitute the decomposed potentials into the dynamic boundary conditions (Bernoulli equations) evaluated at $y = \xi$.

$$\begin{aligned}\frac{\partial}{\partial t}(U_Ax+\varphi'_A) + \frac{1}{2}[\boldsymbol{\nabla}(U_Ax+\varphi'_A)]^2 + \frac{p_A}{\rho_A} + g\xi &= C_A\\ \frac{\partial\varphi'_A}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial}{\partial x}(U_Ax+\varphi'_A)\right)^2 + \left(\frac{\partial}{\partial y}(U_Ax+\varphi'_A)\right)^2\right] + \frac{p_A}{\rho_A} + g\xi &= C_A\\ \frac{\partial\varphi'_A}{\partial t} + \frac{1}{2}\left[\left(U_A + \frac{\partial\varphi'_A}{\partial x}\right)^2 + \left(\frac{\partial\varphi'_A}{\partial y}\right)^2\right] + \frac{p_A}{\rho_A} + g\xi &= C_A\end{aligned}$$

Now, we linearize this by getting rid of quadratic terms in the perturbed quantities:

$$\frac{\partial\varphi'_A}{\partial t} + \frac{1}{2}\left[U_A^2 + 2U_A\frac{\partial\varphi'_A}{\partial x}\right] + \frac{p_A}{\rho_A} + g\xi = C_A$$

Solving for pressure and doing the same thing for the bottom fluid, we have

$$\begin{aligned}-p_A &= \rho_A\left(\frac{\partial\varphi'_A}{\partial t} + \frac{1}{2}U_A^2 + U_A\frac{\partial\varphi'_A}{\partial x} + g\xi - C_A\right)\\ -p_B &= \rho_B\left(\frac{\partial\varphi'_B}{\partial t} + \frac{1}{2}U_B^2 + U_B\frac{\partial\varphi'_B}{\partial x} + g\xi - C_B\right)\end{aligned}$$

Since $p_A = p_B$ at $y = \xi$, we equate the above expressions

$$\rho_A\left(\frac{\partial\varphi'_A}{\partial t} + \frac{1}{2}U_A^2 + U_A\frac{\partial\varphi'_A}{\partial x} + g\xi - C_A\right) = \rho_B\left(\frac{\partial\varphi'_B}{\partial t} + \frac{1}{2}U_B^2 + U_B\frac{\partial\varphi'_B}{\partial x} + g\xi - C_B\right)$$

We saw earlier that the steady base flow satisfies that at $y = 0$,

$$\rho_A\left(\frac{1}{2}(U_A)^2 - C_A\right) = \rho_B\left(\frac{1}{2}(U_B)^2 - C_B\right)$$

Hence, we have, at $y = 0$:

$$\boxed{\rho_A\left(\frac{\partial\varphi'_A}{\partial t} + U_A\frac{\partial\varphi'_A}{\partial x} + g\xi\right) = \rho_B\left(\frac{\partial\varphi'_B}{\partial t} + U_B\frac{\partial\varphi'_B}{\partial x} + g\xi\right)}$$

What's left is to solve for ξ, φ'_A and φ'_B.

Attempt at a Solution

We assume that the perturbations can be represented as

$$\begin{aligned}\xi(x,t) &= \xi_0 e^{ik(x-ct)}\\ \varphi'_A(x,y,t) &= \phi'_A(y)e^{ik(x-ct)}\\ \varphi'_B(x,y,t) &= \phi'_B(y)e^{ik(x-ct)}\end{aligned}$$

where $k \in \mathbb{R}$ and $c \in \mathbb{C}$, so that $c = c_r + ic_i$. The coefficient ξ_0 is the initial amplitude of the displacement of the interface, and hence, it is a constant which specifies the magnitude of *all* perturbations. Note that when $c_i > 0$, the solutions grow exponentially, which is indicative of an unstable solution.

We substitute the last two into **Laplace's equations**:

$$\begin{aligned}\nabla^2(\varphi'_A) = \nabla^2\left(\phi'_A(y)e^{ik(x-ct)}\right) &= 0\\ \frac{\partial^2}{\partial x^2}\left(\phi'_A(y)e^{ik(x-ct)}\right) + \frac{\partial^2}{\partial y^2}\left(\phi'_A(y)e^{ik(x-ct)}\right) &= 0\\ \phi'_A\frac{\partial^2}{\partial x^2}\left(e^{ik(x-ct)}\right) + e^{ik(x-ct)}\frac{\partial^2\phi'_A}{\partial y^2} &= 0\\ e^{ik(x-ct)}\left(\frac{\partial^2\phi'_A}{\partial y^2} - k^2\phi'_A\right) &= 0\\ \therefore\quad \frac{\partial^2\phi'_A}{\partial y^2} &= k^2\phi'_A\\ \&\quad \frac{\partial^2\phi'_B}{\partial y^2} &= k^2\phi'_B\end{aligned}$$

where the last equation is obtained in the same way for the bottom fluid as for the top fluid. The solutions to the above equations are

$$\begin{aligned}\phi'_A &= Ae^{-ky} + Ce^{ky}\\ \phi'_B &= De^{-ky} + Be^{ky}\end{aligned}$$

In order to satisfy the first boundary condition, that the perturbation dies out at infinity, we can see that $C = D = 0$. Hence,

$$\begin{aligned}\phi'_A &= Ae^{-ky}\\ \phi'_B &= Be^{ky}\end{aligned}$$

So, our perturbations are now

$$\begin{aligned}\xi(x,t) &= \xi_0 e^{ik(x-ct)}\\ \varphi'_A(x,y,t) &= Ae^{-ky}e^{ik(x-ct)}\\ \varphi'_B(x,y,t) &= Be^{ky}e^{ik(x-ct)}\end{aligned}$$

Now, we substitute all three of these equations into the **kinematic boundary conditions** we found:

$$\begin{aligned}
\left.\frac{\partial \varphi'_A}{\partial y}\right|_{y=0} &= \frac{\partial \xi}{\partial t} + U_A \frac{\partial \xi}{\partial x} \\
\left.(-kAe^{ik(x-ct)}e^{-ky})\right|_{y=0} &= -ikc\xi_0 e^{ik(x-ct)} + ikU_A\xi_0 e^{ik(x-ct)} \\
-kA &= -ikc\xi_0 + ikU_A\xi_0 \\
\therefore\ \ A &= -i(U_A - c)\xi_0
\end{aligned}$$

Similarly, for the bottom fluid:

$$\begin{aligned}
\left.\frac{\partial \varphi'_B}{\partial y}\right|_{y=0} &= \frac{\partial \xi}{\partial t} + U_B \frac{\partial \xi}{\partial x} \\
\left.(kBe^{ik(x-ct)}e^{ky})\right|_{y=0} &= -ikc\xi_0 e^{ik(x-ct)} + ikU_B\xi_0 e^{ik(x-ct)} \\
kB &= -ikc\xi_0 + ikU_B\xi_0 \\
\therefore\ \ B &= i(U_B - c)\xi_0
\end{aligned}$$

Now, we have to make use of our **dynamic boundary condition** evaluated at $y = 0$. This will allow us to find the dispersion relation. Once again, we substitute our perturbations:

$$\begin{aligned}
\rho_A \left.\left(\frac{\partial \varphi'_A}{\partial t} + U_A \frac{\partial \varphi'_A}{\partial x} + g\xi\right)\right|_{y=0} &= \rho_B \left.\left(\frac{\partial \varphi'_B}{\partial t} + U_B \frac{\partial \varphi'_B}{\partial x} + g\xi\right)\right|_{y=0} \\
\rho_A e^{ik(x-ct)} \left.\left(-ikcAe^{-ky} + ikU_A Ae^{-ky} + g\xi_0\right)\right|_{y=0} &= \rho_B e^{ik(x-ct)} \left(-ikcBe^{ky} + ikU_B Be^{ky} + g\xi\right. \\
\rho_A \left(ikA(U_A - c) + g\xi_0\right) &= \rho_B \left(ikB(U_B - c) + g\xi_0\right) \\
\rho_A \left(k(U_A - c)^2\xi_0 + g\xi_0\right) &= \rho_B \left(-k(U_B - c)^2\xi_0 + g\xi_0\right) \\
\rho_A k(U_A - c)^2 + \rho_A g &= -\rho_B k(U_B - c)^2 + \rho_B g \\
\rho_A(U_A - c)^2 + \rho_B(U_B - c)^2 &= \frac{g}{k}(\rho_B - \rho_A) \\
(\rho_A + \rho_B)c^2 - (2\rho_A U_A + 2\rho_B U_B)c + \left(\rho_A U_A^2 + \rho_B U_B^2 - \frac{g}{k}(\rho_B - \rho_A)\right) &= 0
\end{aligned}$$

The solution to this quadratic equation for the wave speed is

$$c = \left[\frac{\rho_A U_A + \rho_B U_B}{\rho_A + \rho_B}\right] \pm \left[\left(\frac{g}{k}\frac{\rho_A - \rho_B}{\rho_A + \rho_B}\right) - \rho_A\rho_B \left(\frac{U_A - U_B}{\rho_A + \rho_B}\right)^2\right]^{\frac{1}{2}}$$

We noted earlier that if the complex portion of c is positive, then the system becomes exponentially unstable. So, in order to set $c_i = 0$, we take the quantity in the square root to be positive, and this condition ensures stable solutions. Hence, if

$$\left(\frac{g}{k}\frac{\rho_A - \rho_B}{\rho_A + \rho_B}\right) - \rho_A\rho_B \left(\frac{U_A - U_B}{\rho_A + \rho_B}\right)^2 \geq 0$$

then we find stable waves. This can be rewritten as a condition on k, so if

$$k \leq \frac{g}{\rho_A\rho_B}\frac{\rho_A^2 - \rho_B^2}{(U_A - U_B)^2}$$

then the system are stable. On the other hand, if

$$k > \frac{g}{\rho_A \rho_B} \frac{\rho_A^2 - \rho_B^2}{(U_A - U_B)^2}$$

then the flow is unstable.

We can now write a complete solution:

$$\begin{aligned} \xi(x,t) &= \xi_0 e^{ik(x-ct)} \\ \varphi_A'(x,y,t) &= (U_A - c)\xi_0 e^{-ky} e^{ik(x-ct-\pi/2)} \\ \varphi_B'(x,y,t) &= (U_B - c)\xi_0 e^{ky} e^{ik(x-ct+\pi/2)} \end{aligned}$$

where $c = \left[\frac{\rho_A U_A + \rho_B U_B}{\rho_A + \rho_B}\right] \pm \left[\left(\frac{g}{k}\frac{\rho_A - \rho_B}{\rho_A + \rho_B}\right) - \rho_A \rho_B \left(\frac{U_A - U_B}{\rho_A + \rho_B}\right)^2\right]^{\frac{1}{2}}$.

Implications

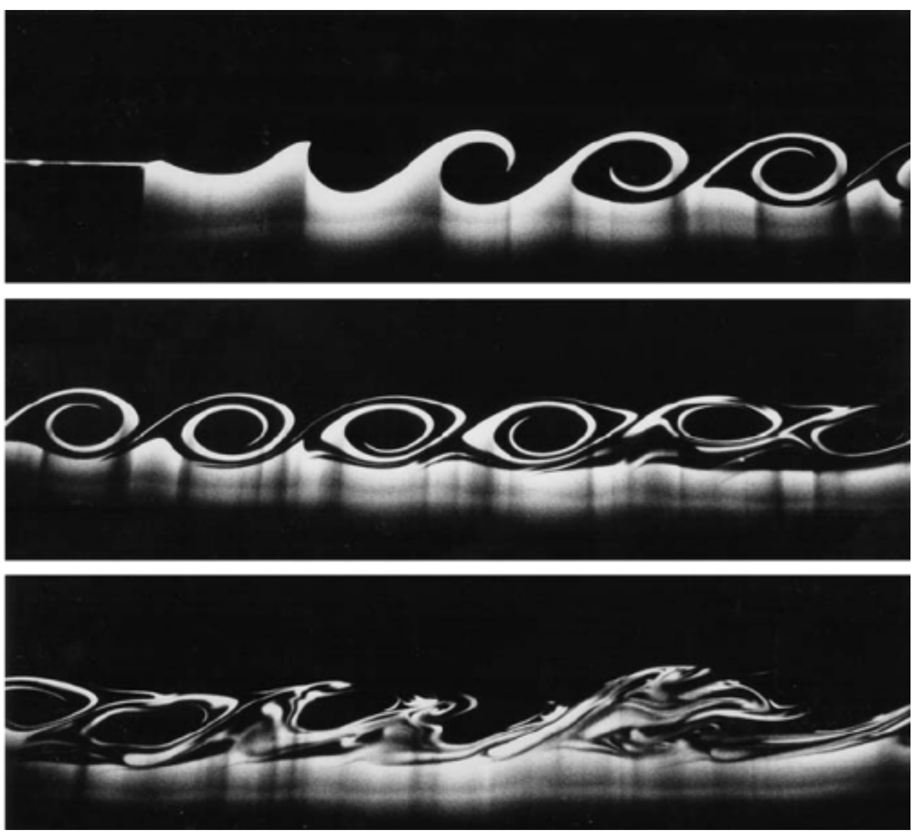
There's so much you can do with this! First, let's consider waves at the interface between two different immiscible liquids ($\rho_A \neq \rho_B$) that are initially stationary ($U_A = U_B = 0$). Substituting these conditions into the expression for c, we find

$$c = \sqrt{\frac{g}{k}\frac{\rho_A - \rho_B}{\rho_A + \rho_B}}$$
$$kc = \omega = \sqrt{gk\frac{\rho_A - \rho_B}{\rho_A + \rho_B}}$$

Now, let's consider the flow of one homogenous fluid ($\rho_A = \rho_B$) with a velocity discontinuity at $y = 0$. This gives us

$$c = \frac{1}{2}(U_A + U_B) \pm i\frac{1}{2}|U_A - U_B|$$

In this case we can see that we must necessarily have $c_i > 0$, so the discontinuity is unstable. This is what a vortex sheet is, and it is unstable for all k. The real wave speed $c_r = \frac{1}{2}(U_A + U_B)$ is equal to the average velocity of the base flow. That is, the perturbations move with a speed equal to the average of both base flow velocities.



2 Scattering of Acoustic Waves by a Vortex

We'll be looking at the scattering of a plane acoustic wave by a toroidal (axisymmetric) vortex in 2D. The wavelength of the acoustic waves is assumed to be much greater than the dimensions of the vortex. This allows us to divide the problem into two asymptotic regions:

1. The inner vortical region

2. The outer wave region

The problem, stated simply, is to understand the scattered sound field when a plane sound wave is incident on an axisymmetric vortex in 2D. There are two limits to this problem, namely:

1. The WKB limit: The acoustic waves have small wavelength compared to the scale of the vortex.

   As the acoustic waves propagate through the vortex, their ray paths are deflected by the vortical flow. When the Mach number of vortex flow is small, this deflection causes a caustic to form along a straight line that extends from the vortex in the direction opposite to the plane wave source. This direction is called the *forward scatter direction*

2. The Born limit: The acoustic waves have long wavelength compared to the scale of the vortex.

   The large acoustic wave introduces pressure and velocity perturbations in the vortex, which in turn contribute to the scattering of the acoustic field.

In this section, we will be looking at the scattering of sound by a perturbed vortex in the Born limit. Moreover, the acoustic plane wave amplitude is small, so that terms quadratic in it can be ignored. The Mach number of the flow within the vortex is small. In the inner, vortical region, the flow will consist of the initial vortex and the perturbations induced by the acoustic wave. The outer, wave region will be composed of long-range azimuthal velocity of the vortex combined with the incident and scattered waves.

The governing equations are

$$\frac{\partial \varrho_a}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\varrho_a + \varrho_a \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$$\varrho_a(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla}))\boldsymbol{u} = -\boldsymbol{\nabla} p$$

$$\frac{p_a}{p_0} = \left(\frac{\varrho_a}{\varrho_0}\right)^{\gamma}$$

where $\gamma = \frac{c_p}{c_v}$ is the ratio of specific heats of the fluid. p_0 and ϱ_0 are the pressure and density respectively when the fluid is at rest. The Mach number of the vortex is defined as

$$M \equiv \frac{U}{c_0} \ll 1$$

where U is the "typical" magnitude of azimuthal velocity due to the vortex, and

$$c_0 \equiv \left(\frac{\gamma p_0}{\varrho_0}\right)^{\frac{1}{2}}$$

is the speed of linear sound in the medium. Assume that the incident acoustic wave is a monochromatic plane wave (ω, $k = \dfrac{\omega}{c_0}$, $\lambda = \frac{2\pi}{k}$). If the length scale of the vortical region is characterized by L, then the limit we are considering is $\lambda \gg L$. So, for flow in the vortical region, the time scale is $\tau_v \equiv \frac{L}{U}$, and for the period of the wave, the other time scale, we have $\tau_w = \dfrac{2\pi}{\omega}$.

The Strouhal number is defined as

$$St = \frac{\omega L}{U}$$

and we define the ratio

$$\frac{\tau_v}{\tau_w} = \frac{St}{2\pi}$$

The incoming wave has frequency ω, wavenumber $k = \frac{\omega}{c_0}$, and pressure amplitude p_i.

3 Vorticity Dynamics

The Navier-Stokes equation is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} + \frac{\eta}{\varrho}\boldsymbol{\nabla}^2\boldsymbol{v} + (\frac{\zeta}{\varrho} + \frac{1}{3}\frac{\eta}{\varrho})\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})$$

where $\nu = \eta/\varrho$. Taking the curl on both sides and using that $\boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{\omega}$, we obtain:

$$\begin{aligned}
\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times\left(\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{v}) - \boldsymbol{v}\times\boldsymbol{\omega}\right) &= \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}\\
\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) &= \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}\\
\frac{\partial \boldsymbol{\omega}}{\partial t} - (\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}) &= \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}\\
\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} + \boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) &= (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}\\
\frac{D\boldsymbol{\omega}}{Dt} &= (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}
\end{aligned}$$

In the case of incompressible flow, this reduces to

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}$$

This first term on the right encapsulates the *enhancement* of vorticity by velocity gradients that are parallel to the direction of $\boldsymbol{\omega}$. That is, it represents vortex *stretching*. The second term governs the diffusion of vorticity throughout the medium over time. In two dimensional flow, this equation further reduces to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}$$

which is just the diffusion equation. When $\nu \to 0$, the vorticity remains constant and is just advected with the flow.

3.1 Decay of a Line Vortex

The velocity field of a 2D line vortex at $t = 0$ (in polar coordinates) is

$$\boldsymbol{v}(r, t = 0) = v(r, 0)\boldsymbol{e}_\theta = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta$$

The vorticity diffusion equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}$$

but we don't know what $\boldsymbol{v}(r, t)$ is, so we should probably find that first. The azimuthal component of Navier Stokes is

$$\varrho\frac{\partial v}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$

However, we assume that $\dfrac{\partial p}{\partial \theta} = 0$, so the equation reduces to

$$\frac{\partial v}{\partial t} = \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \right)$$

We look for a solution of the form $v(r,t) = \dfrac{\Gamma}{2\pi r} f\left(s = \dfrac{r}{\sqrt{t}}\right)$. We did this because we want f to vanish at $t = 0$, which means that no term in f can be independent of r. We evaluate the following:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= -\frac{\Gamma}{4\pi}\frac{1}{t\sqrt{t}}\frac{df}{ds} \\
\frac{\partial v}{\partial r} &= \frac{\Gamma}{2\pi r^2}\left(s\frac{df}{ds} - f\right) \\
\frac{\partial^2 v}{\partial r^2} &= \frac{\Gamma}{2\pi r^3}\left(s^2\frac{d^2 f}{ds^2} - 2s\frac{df}{ds} + 2f\right)
\end{aligned}$$

Substituting this into our Navier Stokes equation, we get

$$-\frac{df}{ds} = 2\nu\frac{d}{ds}\left(\frac{1}{s}\frac{df}{ds}\right)$$

Integrating both sides,

$$\begin{aligned}
-f &= \frac{2\nu}{s}\frac{df}{ds} + A \\
\frac{df}{ds} + \frac{s}{2\nu}f &= -A\frac{2}{2\nu} \\
f(\nu) &= -A + Be^{-\frac{s^2}{4\nu}}
\end{aligned}$$

Hence,

$$v(r,t) = \frac{\Gamma}{2\pi r}(-A + Be^{-\frac{r^2}{4\nu t}})$$

From the initial condition and the fact that the term in parentheses must be zero at $r = 0$, we get $A = B = -1$. Hence,

$$v(r,t) = \frac{\Gamma}{2\pi r}(1 - e^{-\frac{r^2}{4\nu t}})$$

If we taylor expand the exponential term, we get

$$v(r,t) = \frac{\Gamma}{8\pi\nu t}r$$

which is just solid body rotation at an angular velocity of $\dfrac{\Gamma}{8\pi\nu t}$

4 Linearized Kelvin Wake