Non-Propagating Solitary Waves

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1 Introduction

A fluid is a disorienting and continuous mess of many, many particles. The first, tightest knot we can tie around a mechanical description of this mess is to assume that we can obtain a complete description of it if equipped with five characteristic variables. These can be taken as the fluid pressure distribution, the velocity field, and the entropy per unit mass. To respect the messy nature of our fluid, the next knots we tie must be as "loose" as possible to accommodate for a generalized description. These are the statements of conservation of mass and momentum. One could pause here and stare at these equations, but this report is about what happens when we uninterruptedly *nag* the fluid by shaking it up and down in a gravitational field. The fluid in question is one that fills a rectangular channel mounted atop a loudspeaker, and is uniformly *irritated* using a signal generator. We suppose that the act of messing with our system in such a uniform manner has the effect of perturbing the descriptive variables of our fluid. Moreover, in the spirit of being as vague as possible, we suppose that these variables are composed from an assortment of functions of spatial and temporal variables that are organized in terms of the weight of their impact on the system as a whole.

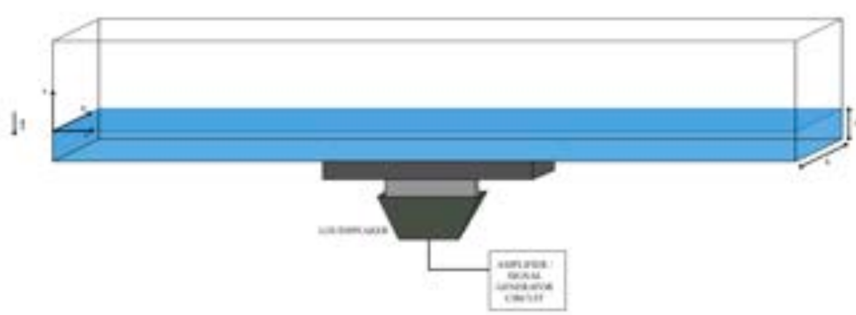


Figure 1: Experimental Setup

The exact problem we are investigating is depicted above in Figure 1. As we might expect, the linearized hydrodynamic equations for this problem yield standing surface waves set up across the width axis with the dispersion relation $\omega^2 = gk\tanh(kd)$, where $g = 9.81\text{ms}^{-2}$, d is the depth of the fluid at rest, b is the width of the channel, and $\{k = \frac{m\pi}{b} : m \in \mathbb{Z}^+\}$ is the spatial frequency of the widthwise standing waves. Here the channel is assumed to be long enough to be a horizontal waveguide such that there are no lengthwise standing waves, and if there are, they are teensy (since the denominator in the spatial frequency, i.e. the length of the channel is much greater than the width). So, in the linear approximation, i.e. the lowest resolution of our how we want to look at our system, we see that the system responds to a uniform disturbance with a uniform spread of waves that are localized in time.

Unfortunately, this isn't enough to describe what we observe. What we observe is a wave that is localized in *space*, such that there is no phase shift across the length of the channel. By including terms whose influence decreases asymptotically with higher orders of approximation, we find that the equations of motion do not retain the homogenous form of the linearized equations. In this system, the inhomogeneity in higher order equations is driven by the behavior of the system in lower orders of approximation. That is, the primary non-linear contribution is driven by the linear solution, and successive non-linear contributions are determined by solutions of preceding orders of approximation. By solving for these higher order contributions, we can better resolve what is happening within the fluid and see how the interplay between multiple processes can lead to non-uniform, localized behavior within a uniformly driven system. In this report, we will investigate the theory behind these solitary waves and determine the region of drive amplitude and frequency for which they arise upon parametric excitation by a tongue depressor.

2 Theory

2.1 Equations and Boundary Conditions

Conservation of Mass

The equation of conservation of mass requires that

$$\varrho_t + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0$$

For an incompressible fluid, the above evaluates to

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

The assumption of irrotational flow ensures that the velocity field can always be expressed as the gradient of a scalar potential, i.e. $\boldsymbol{v} = \boldsymbol{\nabla}\phi$. Substituting this into the reduced equation of conservation of mass, we obtain Laplace's equation for the velocity potential of an incompressible, irrotational fluid:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{1}$$

Since the channel is so very long, it can be considered as a horizontal waveguide, which allows us to treat the velocity potential as invariant over the characteristic length of the channel. Then, equation (1) can be thought of as constraint on the curvature of the velocity potential across the width of the channel. That is, the sum of principal curvatures of the surface described by $\phi(y, z, t)$ is zero, which implies that the velocity potential has zero mean curvature, and minimal (negative) gaussian curvature. If the term on the right is non-zero, as we will encounter in the third order equations, then the velocity potential can be thought of as having a source that perpetuates a "curvature" in the velocity potential over some small region of the characteristic length.

Conservation of momentum

Euler's equation for an incompressible fluid in a uniform gravitational field is

$$\boldsymbol{v}_t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} - \boldsymbol{\nabla}(gz)$$

For an irrotational fluid, the convective term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ can be expressed as $\boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{v}^2\right)$, so the above equation becomes

$$\boldsymbol{\nabla}\left(\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{p}{\varrho} + gz\right) = 0 \tag{2}$$

Conditions at the Free Surface

We define the depth of the fluid at rest in the channel to be d, and choose our origin so that the base of the channel is located at $z = -d$. In other words, the xy plane of our coordinate system lies on the surface of the fluid at rest. The next step is to suppose that the surface of the fluid open to the atmosphere can be described as a surface in $\mathbb{R}^3$ that evolves in time. The magnitude of ξ is the displacement of the fluid from $z = 0$ directed perpendicular to the xy plane.

The vertical component of the fluid velocity evaluated at any point along this free surface must be equal to the velocity of the free surface at that point. This can be expressed as a kinematic boundary condition at the free surface:

$$\xi_t + \phi_x\xi_x + \phi_y\xi_y = \phi_z \quad \text{at } z = \xi(x,y,t) \tag{i}$$

Evaluating (2) at the free surface (where the pressure is equal to p_{atm}), we obtain a dynamic boundary condition for the free surface of the fluid:

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\xi = 0 \quad \text{at } z = \xi(x,y,t) \tag{ii}$$

Conditions at the Walls

Neglecting the effects of viscosity and surface tension, we require that the velocity of the fluid vanishes at the walls of the channel located at $y = 0$ and $y = b$ respectively:

$$\phi_y = 0 \quad \text{at } y = 0, b \tag{iii,iv}$$

2.2 Perturbations and Multiple Scales

The differential equation for the velocity potential we are trying to solve is:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{1}$$

with boundary conditions:

$$\xi_t + \phi_x\xi_x + \phi_y\xi_y - \phi_z = 0 \quad \text{at } z = \xi(x,y,t) \tag{i}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\xi = 0 \quad \text{at } z = \xi(x,y,t) \tag{ii}$$

$$\phi_y = 0 \quad \text{at } y = 0, b \tag{iii,iv}$$

In order to evaluate the boundary conditions (i) and (ii) at the free surface, we need to know what the free surface is. Unfortunately, we have no idea what it is. What we do know is that the free surface lives right around the plane $z = 0$, which is just the description of the free surface at rest. With this in mind, it may be a good idea to approximate the velocity potential as a Taylor series about $z = 0$. Omitting terms that are $\mathcal{O}(z^4)$ and higher, the third order approximation of ϕ is

$$\phi(x, y, z, t) \approxeq [\phi|_{z=0}] + [\phi_z|_{z=0}]\, z + \left[\frac{1}{2}\ \phi_{zz}|_{z=0}\right] z^2 + \left[\frac{1}{2}\ \phi_{zzz}|_{z=0}\right] z^3$$

In order to keep the notation simple, the coefficients in the above expansion will be assumed to be evaluated at $z = 0$. That is,

$$\phi(x, y, z, t) \approxeq \phi + \phi_z z + \frac{1}{2}\phi_{zz} z^2 + \frac{1}{6}\phi_{zzz} z^3 \tag{3}$$

We want to know what happens when we mess with the fluid in the channel by uniformly introducing energy into it. One way to tell this to our equations is by expressing ϕ and ξ as perturbative expansions that are asymptotic in the parameter $\epsilon \to 0$. That is,

$$\phi(x, y, z, t) = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \epsilon^3\phi^{(3)} + \ldots \tag{4}$$

$$\xi(x, y, z, t) = \epsilon\xi^{(1)} + \epsilon^2\xi^{(2)} + \epsilon^3\xi^{(3)} + \ldots \tag{5}$$

If we insert the perturbative expansion of ϕ into (1), we find

$$\epsilon\left[\phi^{(1)}_{xx} + \phi^{(1)}_{yy} + \phi^{(1)}_{zz}\right] + \epsilon^2\left[\phi^{(2)}_{xx} + \phi^{(2)}_{yy} + \phi^{(2)}_{zz}\right] + \epsilon^3\left[\phi^{(3)}_{xx} + \phi^{(3)}_{yy} + \phi^{(3)}_{zz}\right] = 0 \tag{1*}$$

Equipped with the Taylor expansion of the velocity potential about the unperturbed surface at $z = 0$, we can now evaluate a third order approximation of ϕ at $z = \xi$, by simply substituting the latter into the former. Evaluating the velocity potential (3) and its derivatives at the free surface, we can substitute these expressions into (i) and (ii) to obtain boundary conditions at $z = 0$. When the terms are organized with respect to degrees the order parameter ϵ, we obtain the kinematic condition:

$$\begin{aligned}
&\epsilon\left[\xi^{(1)}_t - \phi^{(1)}_z\right] \\
&+\epsilon^2\left[\xi^{(2)}_t + \phi^{(1)}_x\xi^{(1)}_x + \phi^{(1)}_y\xi^{(1)}_y - \phi^{(2)}_z - \phi^{(1)}_{zz}\xi^{(1)}\right] \\
&+\epsilon^3\left[\xi^{(3)}_t - \phi^{(3)}_z\right] \\
&+\epsilon^3\left[\phi^{(2)}_x\xi^{(1)}_x + \phi^{(1)}_x\xi^{(2)}_x + \phi^{(2)}_y\xi^{(1)}_y + \phi^{(1)}_y\xi^{(2)}_y - \phi^{(1)}_{zz}\xi^{(2)} - \phi^{(2)}_{zz}\xi^{(1)}\right] \\
&+\epsilon^3\left[\phi^{(1)}_{xz}\xi^{(1)}_x\xi^{(1)} + \phi^{(1)}_{yz}\xi^{(1)}_y\xi^{(1)} - \frac{1}{2}\phi^{(1)}_{zzz}\xi^{(1)}\xi^{(1)}\right] = 0 \qquad \text{at } z = 0
\end{aligned} \tag{i*}$$

and the dynamic condition:

$$\begin{aligned}
&\epsilon\left[\phi^{(1)}_t + g\xi^{(1)}\right] \\
&+\epsilon^2\left[\phi^{(2)}_t + g\xi^{(2)} + \phi^{(1)}_{tz}\xi^{(1)} + \frac{1}{2}\phi^{(1)}_x\phi^{(1)}_x + \frac{1}{2}\phi^{(1)}_y\phi^{(1)}_y + \frac{1}{2}\phi^{(1)}_z\phi^{(1)}_z\right] \\
&+\epsilon^3\left[\phi^{(3)}_t + g\xi^{(3)}\right] \\
&+\epsilon^3\left[\phi^{(1)}_{tz}\xi^{(2)} + \phi^{(2)}_{tz}\xi^{(1)} + \phi^{(1)}_x\phi^{(2)}_x + \phi^{(1)}_y\phi^{(2)}_y + \phi^{(1)}_z\phi^{(2)}_z\right] \\
&+\epsilon^3\left[\frac{1}{2}\phi^{(1)}_{tzz}\xi^{(1)}\xi^{(1)} + \phi^{(1)}_x\phi^{(1)}_{xz}\xi^{(1)} + \phi^{(1)}_y\phi^{(1)}_{yz}\xi^{(1)} + \phi^{(1)}_z\phi^{(1)}_{zz}\xi^{(1)}\right] = 0 \qquad \text{at } z = 0
\end{aligned} \tag{ii*}$$

In the set of equations above, any expression within parentheses must vanish identically, since ϵ is an order parameter that we have introduced, without specifying its value. Thus, we have a system of equations and boundary conditions associated with each order of approximation.

One thing we must acknowledge at this point is that we are trying to arrive at a description of a fluid that is driven uniformly across the entire characteristic length, but tends to form localized disturbances with no dispersion. This suggests that we introduce variables of multiple scales in x and t, so that $x_j = \epsilon^j x$ and $t_j = \epsilon^j t$. From the chain rule, it follows that

$$\begin{aligned}\partial_x &= \partial_{x_0} + \epsilon\partial_{x_1} + \epsilon^2\partial_{x_2} + \dots \\ \partial_{xx} &= \partial_{x_0x_0} + \epsilon(2\partial_{x_0x_1}) + \epsilon^2(2\partial_{x_0x_2} + \partial_{x_1x_1}) + \dots\end{aligned}$$

and

$$\begin{aligned}\partial_t &= \partial_{t_0} + \epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \dots \\ \partial_{tt} &= \partial_{t_0t_0} + \epsilon(2\partial_{t_0t_1}) + \epsilon^2(2\partial_{t_0t_2} + \partial_{t_1t_1}) + \dots\end{aligned}$$

Now, we make a change of variables from x and t to x_j and t_j, which characterize the different length and time scales of our system. Substituting this into the kinematic and dynamic free surface conditions, we obtain

$$\begin{aligned}&\epsilon\left[\xi^{(1)}_{t_0} - \phi^{(1)}_z\right] \\ &+\epsilon^2\left[\xi^{(2)}_{t_0} + \phi^{(1)}_{x_0}\xi^{(1)}_{x_0} + \phi^{(1)}_y\xi^{(1)}_y - \phi^{(2)}_z - \phi^{(1)}_{zz}\xi^{(1)} + \xi^{(1)}_{t_1}\right] \\ &+\epsilon^3\left[\xi^{(3)}_{t_0} - \phi^{(3)}_z + \xi^{(1)}_{t_2} + \xi^{(2)}_{t_1}\right] \\ &+\epsilon^3\left[\phi^{(1)}_{x_0}(\xi^{(1)}_{x_1} + \xi^{(2)}_{x_0}) + (\phi^{(1)}_{x_1} + \phi^{(2)}_{x_0})\xi^{(1)}_{x_0} + \phi^{(2)}_y\xi^{(1)}_y + \phi^{(1)}_y\xi^{(2)}_y - \phi^{(1)}_{zz}\xi^{(2)} - \phi^{(2)}_{zz}\xi^{(1)}\right] \\ &+\epsilon^3\left[\phi^{(1)}_{x_0z}\xi^{(1)}_{x_0}\xi^{(1)} + \phi^{(1)}_{yz}\xi^{(1)}_y\xi^{(1)} - \frac{1}{2}\phi^{(1)}_{zzz}\xi^{(1)}\xi^{(1)}\right] = 0 \qquad \text{at } z = 0 \qquad \text{(a)}\end{aligned}$$

and

$$\begin{aligned}&\epsilon\left[\phi^{(1)}_{t_0} + g\xi^{(1)}\right] \\ &+\epsilon^2\left[\phi^{(1)}_{t_1} + \phi^{(2)}_{t_0} + g\xi^{(2)} + \phi^{(1)}_{t_0z}\xi^{(1)} + \frac{1}{2}\phi^{(1)}_{x_0}\phi^{(1)}_{x_0} + \frac{1}{2}\phi^{(1)}_y\phi^{(1)}_y + \frac{1}{2}\phi^{(1)}_z\phi^{(1)}_z\right] \\ &+\epsilon^3\left[\phi^{(1)}_{t_2} + \phi^{(2)}_{t_1} + \phi^{(3)}_{t_0} + g\xi^{(3)}\right] \\ &+\epsilon^3\left[\phi^{(1)}_{t_1z}\xi^{(1)} + \phi^{(1)}_{t_0z}\xi^{(2)} + \phi^{(2)}_{t_0z}\xi^{(1)} + \phi^{(1)}_{x_0}\phi^{(2)}_{x_0} + \phi^{(1)}_y\phi^{(2)}_y + \phi^{(1)}_z\phi^{(2)}_z + \phi^{(1)}_{x_0}\phi^{(1)}_{x_1}\right] \\ &+\epsilon^3\left[\frac{1}{2}\phi^{(1)}_{t_0zz}\xi^{(1)}\xi^{(1)} + \phi^{(1)}_{x_0}\phi^{(1)}_{x_0z}\xi^{(1)} + \phi^{(1)}_y\phi^{(1)}_{yz}\xi^{(1)} + \phi^{(1)}_z\phi^{(1)}_{zz}\xi^{(1)}\right] = 0 \qquad \text{at } z = 0 \qquad \text{(b)}\end{aligned}$$

Doing the same for Laplace's equation, we obtain

$$\begin{aligned}&\epsilon\left[\phi^{(1)}_{x_0x_0} + \phi^{(1)}_{yy} + \phi^{(1)}_{zz}\right] \\ &+\epsilon^2\left[\phi^{(2)}_{x_0x_0} + \phi^{(2)}_{yy} + \phi^{(2)}_{zz} + 2\phi^{(1)}_{x_0x_1}\right] \\ &+\epsilon^3\left[\phi^{(3)}_{x_0x_0} + \phi^{(3)}_{yy} + \phi^{(3)}_{zz} + 2\phi^{(2)}_{x_0x_1} + 2\phi^{(1)}_{x_0x_2} + \phi^{(1)}_{x_1x_1}\right] = 0 \qquad \text{(I)}\end{aligned}$$

The boundary conditions (iii,iv) remain the same for all orders of approximation, since the fluid velocity *must* vanish at the walls if viscosity and surface tension are neglected. Equation (I) and the boundary conditions (a), (b),(iii,iv) form a set of three systems of Laplace equations and boundary conditions corresponding to the first, second, and third order solutions for the velocity potential. This is because the terms in parentheses associated with ϵ, ϵ^2, and ϵ^3 must vanish identically, as the order parameter ϵ is a common factor that arises from the definition of the velocity potential as a perturbative series. In the spirit of the first order-solution, we will group terms in the second and third order equations so that they may be represented as *augmented* versions of the first order equations. That is, the second and third order equations can be represented as linear equations with non-linear forcing terms. This allows us to write the 15 governing equations in a concise manner. That is,

$$
\begin{aligned}
\phi^{(n)}_{x_0x_0} + \phi^{(n)}_{yy} + \phi^{(n)}_{zz} &= \Delta^{(n)} && \text{for } -d \le z \le \xi && \text{(I.n)}\\
\xi^{(n)}_{t_0} - \phi^{(n)}_{z} &= \Lambda^{(n)} && \text{at } z = 0 && \text{(a.n)}\\
g\xi^{(n)} + \phi^{(n)}_{t_0} &= \Pi^{(n)} && \text{at } z = 0 && \text{(b.n)}\\
\phi^{(n)}_{y} &= 0 && \text{at } y = 0, b && \text{(iii.n, iv.n)}\\
\phi^{(n)}_{z} &= 0 && \text{at } z = -d && \text{(v.n)}
\end{aligned}
$$

where $n = 1, 2, 3$ correspond to the first, second and third order equations respectively, with the forcing terms defined for each order of approximation as

1st order

$$
\begin{aligned}
\Delta^{(1)} &\equiv 0\\
\Lambda^{(1)} &\equiv 0\\
\Pi^{(1)} &\equiv 0
\end{aligned}
$$

2st order

$$
\begin{aligned}
\Delta^{(2)} &\equiv -2\phi^{(1)}_{x_0x_1}\\
\Lambda^{(2)} &\equiv \phi^{(1)}_{zz}\xi^{(1)} - \left(\xi^{(1)}_{t_1} + \phi^{(1)}_{x_0}\xi^{(1)}_{x_0} + \phi^{(1)}_{y}\xi^{(1)}_{y}\right)\\
\Pi^{(2)} &\equiv -\left(\phi^{(1)}_{t_1} + \phi^{(1)}_{t_0z}\xi^{(1)} + \frac{1}{2}(\phi^{(1)}_{x_0}\phi^{(1)}_{x_0} + \phi^{(1)}_{y}\phi^{(1)}_{y} + \phi^{(1)}_{z}\phi^{(1)}_{z})\right)
\end{aligned}
$$

3st order

$$
\begin{aligned}
\Delta^{(3)} \equiv & - \left(2\phi^{(2)}_{x_0x_1} + 2\phi^{(1)}_{x_0x_2} + \phi^{(1)}_{x_1x_1}\right) \\
\Lambda^{(3)} \equiv & - \left(\xi^{(1)}_{t_2} + \xi^{(2)}_{t_1}\right) \\
& - \left(\phi^{(1)}_{x_0}(\xi^{(1)}_{x_1} + \xi^{(2)}_{x_0}) + (\phi^{(1)}_{x_1} + \phi^{(2)}_{x_0})\xi^{(1)}_{x_0}\right) \\
& - \left(\phi^{(2)}_{y}\xi^{(1)}_{y} + \phi^{(1)}_{y}\xi^{(2)}_{y} - \phi^{(1)}_{zz}\xi^{(2)} - \phi^{(2)}_{zz}\xi^{(1)}\right) \\
& - \left(\phi^{(1)}_{x_0z}\xi^{(1)}_{x_0}\xi^{(1)} + \phi^{(1)}_{yz}\xi^{(1)}_{y}\xi^{(1)} - \frac{1}{2}\phi^{(1)}_{zzz}\xi^{(1)}\xi^{(1)}\right) \\
\Pi^{(3)} \equiv & - \left(\phi^{(1)}_{t_2} + \phi^{(2)}_{t_1}\right) \\
& - \left(\phi^{(1)}_{t_1z}\xi^{(1)} + \phi^{(1)}_{t_0z}\xi^{(2)} + \phi^{(2)}_{t_0z}\xi^{(1)}\right) \\
& - \left(\phi^{(1)}_{x_0}\phi^{(2)}_{x_0} + \phi^{(1)}_{y}\phi^{(2)}_{y} + \phi^{(1)}_{z}\phi^{(2)}_{z} + \phi^{(1)}_{x_0}\phi^{(1)}_{x_1}\right) \\
& - \left(\frac{1}{2}\phi^{(1)}_{t_0zz}\xi^{(1)}\xi^{(1)} + \phi^{(1)}_{x_0}\phi^{(1)}_{x_0z}\xi^{(1)} + \phi^{(1)}_{y}\phi^{(1)}_{yz}\xi^{(1)} + \phi^{(1)}_{z}\phi^{(1)}_{zz}\xi^{(1)}\right)
\end{aligned}
$$

2.3 First-Order Approximation

The system of first-order equations is:

$$\phi^{(1)}_{x_0x_0} + \phi^{(1)}_{yy} + \phi^{(1)}_{zz} = 0 \qquad \text{for } -d \le z \le \xi \tag{I.1}$$
$$\xi^{(1)}_{t_0} - \phi^{(1)}_{z} = 0 \qquad \text{at } z = 0 \tag{a.1}$$
$$g\xi^{(1)} + \phi^{(1)}_{t_0} = 0 \qquad \text{at } z = 0 \tag{b.1}$$
$$\phi^{(1)}_{y} = 0 \qquad \text{at } y = 0, b \tag{iii.1, iv.1}$$
$$\phi^{(1)}_{z} = 0 \qquad \text{at } z = -d \tag{v.1}$$

This system of equations can be solved using the method of separation of variables. We suppose that

$$\phi^{(1)} = \alpha(x)\beta(y)\gamma(z)\zeta(t)$$

and find, for the spatial contributions:

$$\begin{aligned} \alpha_{x_0x_0} &= -k_0^2\alpha \\ \beta_{yy} &= -k_y^2\beta \\ \gamma_{zz} &= (k_0^2 + k_y^2)\gamma = k^2\gamma \end{aligned}$$

which have the general solutions

$$\begin{aligned} \alpha(x) &= A_1 e^{-ik_0x_0} + A_2 e^{ik_0x_0} \\ \beta(y) &= \frac{B_1}{2} e^{-ik_y y} + B_2 e^{ik_y y} \\ \gamma(z) &= C_1 \sinh(kz) + C_2 \cosh(kz) \end{aligned}$$

Utilizing boundary conditions (iii.1, iv.1), we find that $B_1 = B_2$, and $k_y = \dfrac{m\pi}{b}$, where $m = 1, 2, 3, \ldots$. Boundary condition (v.1) implies that $C_2 = \tanh(kd)C_1$ If we differentiate (b.1) with respect to t_0, we find that $\xi^{(1)}_t = -\dfrac{1}{g}\phi^{(1)}_{t_0t_0}$, which, when substituted into (a.1), produces the relation

$$\phi^{(1)}_{z} = -\frac{1}{g}\phi^{(1)}_{t_0t_0} \qquad \text{at } z = 0 \tag{c.1}$$

Substituting the separable solution for $\phi^{(1)}$ into the above relation yields a differential equation for $\zeta(t)$, namely

$$\zeta_{t_0t_0} = -gk\tanh(kd)\,\zeta = -\omega^2\zeta$$

where we have defined $\omega = \sqrt{gk\tanh(kd)}$, which is the usual dispersion relation for water waves.

Hence, the separable solutions for the first order velocity potential, $\phi^{(1)}$ reduce to

$$\begin{aligned} \alpha(x) &= A_1 e^{-ik_0x_0} + A_2 e^{ik_0x_0} \\ \beta(y) &= B_1 \cos\left(\frac{m\pi}{b} y\right) \\ \gamma(z) &= C_1 \frac{\cosh(k[z+d])}{\cosh(kd)} \\ \zeta(t) &= D_1 e^{-i\omega t_0} + D_2 e^{i\omega t_0} \end{aligned}$$

Since the length of the channel is much greater than the width, any standing waves produced across the x axis will have a small spatial frequency. That is, $k_0 \ll k_y$. Assuming $k_0 \to 0$, the exponential terms in α will limit to unity, leaving behind a sum of complex constants. The constants $A_1, A_2 \in \mathbb{C}$ and $D_1, D_2 \in \mathbb{C}$ are undetermined functions that depend on the variables $\{x_j : j > 0\}$ and $\{t_j : j > 0\}$ respectively. We have $\alpha(x) \approxeq A_1 + A_2 = A \in \mathbb{C}$, which we can absorb into ζ. All other coefficients, namely $B_1,\ C_1 \in \mathbb{C}$ are complex constants.

Thus, we have a solution for the first order velocity potential

$$\phi^{(1)} = \left(\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \frac{\cosh(k[z+d])}{\cosh(kd)} \tag{A.1}$$

where $\{\psi = \psi(x_j, t_j) : j > 0 \in \mathbb{Z}^+\}$ and $k = \dfrac{m\pi}{b}$, with the dispersion relation $\omega = \sqrt{gk\tanh(kd)}$.

The first order free surface displacement can then be found using (b.1):

$$\xi^{(1)} = -\left[\frac{i\omega}{g}\right] \left(\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \tag{B.1}$$

The free surface displacement is plotted below for

2.4 Second-Order Approximation

The system of second-order equations is:

$$\phi^{(2)}_{x_0x_0} + \phi^{(2)}_{yy} + \phi^{(2)}_{zz} = \Delta^{(2)} \qquad \text{for } -d \le z \le \xi \tag{I.2}$$

$$\xi^{(2)}_{t_0} - \phi^{(2)}_{z} = \Lambda^{(2)} \qquad \text{at } z = 0 \tag{a.2}$$

$$g\xi^{(2)} + \phi^{(2)}_{t_0} = \Pi^{(2)} \qquad \text{at } z = 0 \tag{b.2}$$

$$\phi^{(2)}_{y} = 0 \qquad \text{at } y = 0, b \tag{iii.2, iv.2}$$

$$\phi^{(2)}_{z} = 0 \qquad \text{at } z = -d \tag{v.2}$$

After evaluating the forcing terms $\Delta^{(2)}$, $\Lambda^{(2)}$, and $\Pi^{(2)}$ using the first order solutions (A.1) and (B.1), the second order Laplace equation and free surface boundary conditions assume the form:

$$\phi^{(2)}_{x_0x_0} + \phi^{(2)}_{yy} + \phi^{(2)}_{zz} = 0 \tag{I.2}$$

$$\begin{aligned}\xi^{(2)}_{t_0} - \phi^{(2)}_{z} &= \left[\frac{i\omega}{g}\right]\left(\psi_{t_1}e^{i\omega t_0} + \text{c.c.}\right)\cos(ky) \\ &\quad - \left[\frac{i\omega k^2}{g}\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky) \qquad \text{at } z = 0\end{aligned} \tag{a.2}$$

$$\begin{aligned}g\xi^{(2)} + \phi^{(2)}_{t_0} &= -\left(\psi_{t_1}e^{i\omega t_0} + \text{c.c.}\right)\cos(ky) \\ &\quad + \left[\frac{k^2}{4}(1-3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky) \\ &\quad - \left[\frac{k^2}{4}(1+3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right) \\ &\quad + \left[\frac{k^2}{2}(1+T^2)\right]|\psi|^2\cos(2ky) \\ &\quad - \left[\frac{k^2}{2}(1-T^2)\right]|\psi|^2 \qquad \text{at } z = 0\end{aligned} \tag{b.2}$$

The term $-\psi_{t_1}\cos(ky)e^{i\omega t_0}$ on the right side of the free surface conditions is a secular term, since the spatial and temporal frequencies with which ψ_{t_1} oscillates are governed by the first-order solution. Thus, the condition to avoid secular terms in the second order solution is $\psi_{t_1} = 0$. Utilizing this fact and differentiating the dynamic condition with respect to t_0, we find

$$g\xi^{(2)}_{t_0} + \phi^{(2)}_{t_0t_0} = \left[\frac{i\omega k^2}{2}(1-3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky) - \left[\frac{i\omega k^2}{2}(1+3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right) \tag{b.2.1}$$

Moreover, we know from the kinematic condition that

$$-g\xi^{(2)}_{t_0} + g\phi^{(2)}_{z} = \left(i\omega k^2\right)\psi^2\cos(2ky)e^{2i\omega t_0} + \text{c.c.} \tag{a.2}$$

Adding equations (b.2.1) and (a.2), and dividing the entire expression by $(2\omega)^2 = 4gkT$, we find

$$\begin{aligned}\left[\frac{1}{4gkT}\right]\phi^{(2)}_{t_0t_0} + \left[\frac{1}{4kT}\right]\phi^{(2)}_{z} &= \left[\frac{3ik^2}{8\omega}(1-T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky) \\ &\quad - \left[\frac{ik^2}{8\omega}(1+3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right) \qquad \text{at } z = 0\end{aligned} \tag{c.2}$$

Since $\Delta^{(2)} = 0$, the most general solution to equation (I.2) is a linear combination of separable functions. The second term on the right hand side of (c.2) suggests that one of these functions is independent of y, and probably z. So, we propose that the solution is of the form:

$$\phi^{(2)}(x,y,z,t) = \cos(\kappa y)\frac{\cosh(\kappa[z+d])}{\cosh(\kappa d)}\left(\varphi_1 e^{i\Omega t_0} + \text{c.c.}\right) + \left(\varphi_2 e^{i\Omega t_0} + \text{c.c.}\right)$$

where φ_1 and φ_2 are complex functions of the slowly varying variables. The goal is to figure out what they are. If we compute $\phi^{(2)}_{t_0 t_0}$ and $\phi^{(2)}_z$ using the above proposal, and substitute it into (c.2), we find that

$$\left[\frac{\kappa\tanh(\kappa d)}{4kT} - \left(\frac{\Omega}{2\omega}\right)^2\right]\left(\varphi_1 e^{i\Omega t_0} + \text{c.c.}\right)\cos(\kappa y) - \left[\left(\frac{\Omega}{2\omega}\right)^2\right]\left(\varphi_2 e^{i\Omega t_0} + \text{c.c.}\right)$$
$$= \left[\frac{3ik^2}{8\omega}(1-T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky) - \left[\frac{ik^2}{8\omega}(1+3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)$$

By inspecting the arguments of the cosine and exponential terms on the right, we set $\kappa = 2k$ and $\Omega = 2\omega$.
Then, using the identity $\tanh(2kd) = \dfrac{2T}{1+T^2}$, equation (c.2) boils down to

$$-\left[\left(\frac{1}{1+T^{-2}}\right)\varphi_1\right]\cos(2ky)e^{2i\omega t_0} - [\varphi_2]\, e^{2i\omega t_0} + \text{c.c.}$$
$$= \left[\frac{3ik^2}{8\omega}(1-T^2)\psi^2\right]\cos(2ky)e^{2i\omega t_0} - \left[\frac{ik^2}{8\omega}(1+3T^2)\psi^2\right]e^{2i\omega t_0} + \text{c.c.}$$

By coefficients of the y-dependent and independent terms, it follows that

$$\varphi_1 = -\frac{3ik^2}{8\omega}(T^{-2}-T^2)\psi^2$$
$$\varphi_2 = \frac{ik^2}{8\omega}(1+3T^2)\psi^2$$

Thus, the second order velocity potential is

$$\phi^{(2)} = -\left[\frac{3ik^2}{8\omega}\left(T^{-2}-T^2\right)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky)\frac{\cosh(2k[z+d])}{\cosh(2kd)}$$
$$+ \left[\frac{ik^2}{8\omega}(1+3T^2)\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right) \qquad \text{(A.2)}$$

and the free surface displacement is obtained by substituting (A.2) into (b.2) and excluding the secular term. We obtain:

$$\xi^{(2)} = \left[\frac{k^2}{4g}(1-3T^{-2})\right]\left(\psi^2 e^{2i\omega t_0} + \text{c.c.}\right)\cos(2ky)$$
$$+ \left[\frac{k^2}{2g}(1+T^2)\right]|\psi|^2\cos(2ky)$$
$$- \left[\frac{k^2}{2g}(1-T^2)\right]|\psi|^2 \qquad \text{(B.2)}$$

2.5 Third-Order Approximation

The system of third-order equations is:

$$\phi^{(3)}_{x_0x_0} + \phi^{(3)}_{yy} + \phi^{(3)}_{zz} = \Delta^{(3)} \qquad \text{for } -d \le z \le \xi \tag{I.3}$$

$$\xi^{(3)}_{t_0} - \phi^{(3)}_{z} = \Lambda^{(3)} \qquad \text{at } z = 0 \tag{a.3}$$

$$\phi^{(3)}_{t_0} + g\xi^{(3)} = \Pi^{(3)} \qquad \text{at } z = 0 \tag{b.3}$$

$$\phi^{(3)}_{y} = 0 \qquad \text{at } y = 0, b \tag{iii.3, iv.3}$$

$$\phi^{(3)}_{z} = 0 \qquad \text{at } z = -d \tag{v.3}$$

In order to compute the forcing terms $\Delta^{(3)}$, $\Lambda^{(3)}$, and $\Pi^{(3)}$, we require the first and second order solutions for the velocity potential and free surface displacement. Now equipped with these, we find that equation (I.3) is

$$\phi^{(3)}_{x_0x_0} + \phi^{(3)}_{yy} + \phi^{(3)}_{zz} = -(\psi_{x_1x_1} e^{i\omega t_0} + \text{c.c.}) \cos(ky) \frac{\cosh(k[z+d])}{\cosh(kd)} \tag{I.3}$$

The third order kinematic surface condition is:

$$\begin{aligned}
\xi^{(3)}_{t_0} - \phi^{(3)}_{z} = & \left[\frac{i\omega}{g}\right] \left(\psi_{t_2} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8g}(6T^{-2} - 7)\right] \left(|\psi|^2 \psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8g}(2T^2 - 6T^{-2})\right] \left(|\psi|^2 \psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8g}(2T^2 - 1)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8g}(-18T^{-2} - 2T^2 + 1)\right] \left(|\psi|^2 \psi e^{i\omega t_0} + \text{c.c.}\right) \cos(3ky) \\
& + \left[\frac{k^4}{8g}(-18T^{-2} + 6T^2 + 3)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(3ky)
\end{aligned} \tag{a.3}$$

and the third order dynamic condition is

$$\begin{aligned}
\phi^{(3)}_{t_0} + g\xi^{(3)} = & -\left(\psi_{t_2} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8(i\omega)}(3T^{-2} - 55T^2 + 6T^4 + 15)\right] \left(|\psi|^2 \psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8(i\omega)}(3T^{-2} - 31T^2 + 15)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
& + \left[\frac{k^4}{8(i\omega)}(-3T^{-2} - 24T^2 + 2T^4 + 15)\right] \left(|\psi|^2 \psi e^{i\omega t_0} + \text{c.c.}\right) \cos(3ky) \\
& + \left[\frac{k^4}{8(i\omega)}(-3T^{-2} - 18T^2 + 15)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(3ky)
\end{aligned} \tag{b.3}$$

Just using equation (I.3) and boundary conditions (iii.3, iv.3) and (v.3), we can propose a possible solution for the third order velocity potential. Before we do that, we will differentiate (b.3) with respect to t_0 in order to substitute it into the kinematic condition. Differentiating, we find

$$\begin{aligned}
\phi^{(3)}_{t_0t_0} + g\xi^{(3)}_{t_0} = &- [i\omega] \left(\psi_{t_2} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \left[\frac{k^4}{8}(3T^{-2} - 55T^2 + 6T^4 + 15)\right] \left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \left[\frac{k^4}{8}(9T^{-2} - 93T^2 + 45)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \left[\frac{k^4}{8}(-3T^{-2} - 24T^2 + 2T^4 + 15)\right] \left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(3ky) \\
&+ \left[\frac{k^4}{8}(-9T^{-2} - 54T^2 + 45)\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(3ky)
\end{aligned}$$

Adding (a.3) and the above equation to eliminate $\xi^{(3)}_{t_0}$ we find:

$$\begin{aligned}
\phi^{(3)}_{t_0t_0} + g\phi^{(3)}_{z} = &\, 2i\omega \left(\psi_{t_2} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \frac{k^4}{8}\left[3T^{-2} - 57T^2 + 6T^4 + 22\right] \left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \frac{k^4}{8}\left[9T^{-2} - 95T^2 + 46\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(ky) \\
&+ \frac{k^4}{8}\left[15T^{-2} - 22T^2 + 2T^4 + 14\right] \left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right) \cos(3ky) \\
&+ \frac{k^4}{8}\left[9T^{-2} - 60T^2 + 42\right] \left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right) \cos(3ky)
\end{aligned} \tag{c.3}$$

Equation (I.3) was

$$\phi^{(3)}_{x_0x_0} + \phi^{(3)}_{yy} + \phi^{(3)}_{zz} = -(\psi_{x_1x_1} e^{i\omega t_0} + \text{c.c.}) \cos(ky) \frac{\cosh(k[z+d])}{\cosh(kd)} \tag{I.3}$$

with boundary conditions

$$\phi^{(3)}_{y} = 0 \qquad \text{at } y = 0, b \tag{iii.3, iv.3}$$

$$\phi^{(3)}_{z} = 0 \qquad \text{at } z = -d \tag{v.3}$$

This system of equations has the solution

$$\phi^{(3)} = -\frac{1}{2k}\left(\psi_{x_1x_1} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \left[(z+d)\frac{\sinh(k[z+d])}{\cosh(kd)} - (d)\frac{\cosh(k[z+d])}{\cosh(kd)}\right]$$

Now, we substitute this result into (c.3). The left hand side is

$$\begin{aligned}
\phi^{(3)}_{t_0t_0} + g\phi^{(3)}_{z} = &\, \frac{gT}{2}\left(\psi_{x_1x_1} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \left[(z+d)\frac{\sinh(k[z+d])}{\cosh(kd)} - (d)\frac{\cosh(k[z+d])}{\cosh(kd)}\right] \\
&- \frac{g}{2k}\left(\psi_{x_1x_1} e^{i\omega t_0} + \text{c.c.}\right) \cos(ky) \left[(1-kd)\frac{\sinh(k[z+d])}{\cosh(kd)} + (k[z+d])\frac{\cosh(k[z+d])}{\cosh(kd)}\right]
\end{aligned}$$

At $z = 0$, this evaluates to

$$\phi^{(3)}_{t_0 t_0} + g\phi^{(3)}_z = -\frac{g}{2k}\left[T + kd(1-T^2)\right]\left(\psi_{x_1 x_1} e^{i\omega t_0} + \text{c.c.}\right)\cos(ky)$$

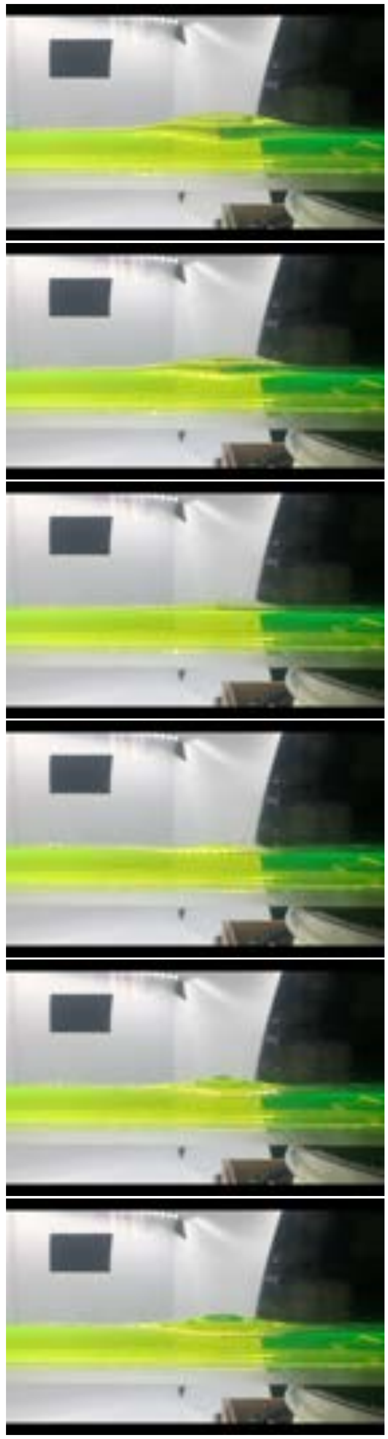
Hence, we have

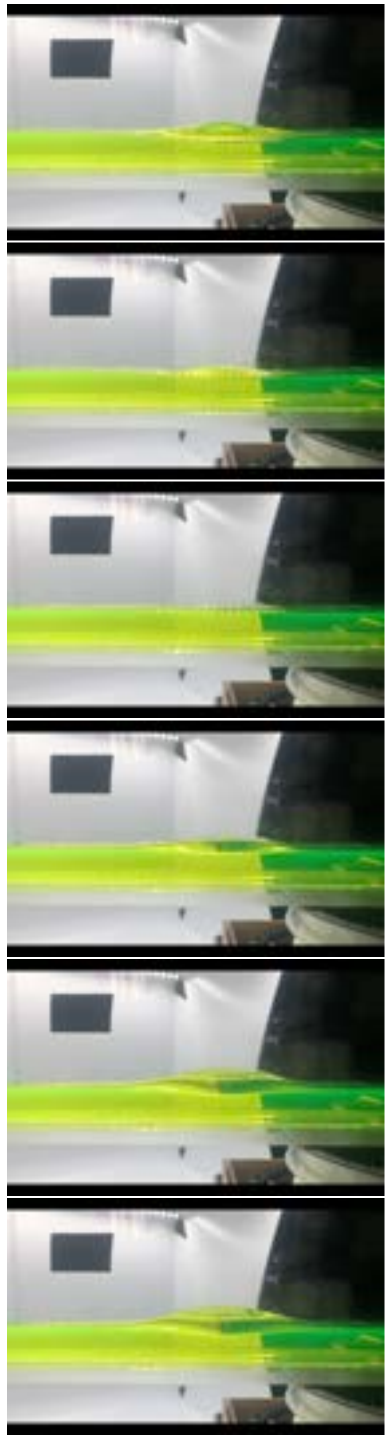
$$\begin{aligned}
-\frac{g}{2k}\left[T + kd(1-T^2)\right]&\left(\psi_{x_1 x_1} e^{i\omega t_0} + \text{c.c.}\right)\cos(ky) \\
&= 2i\omega\left(\psi_{t_2} e^{i\omega t_0} + \text{c.c.}\right)\cos(ky) \\
&\quad + \frac{k^4}{8}\left[3T^{-2} - 57T^2 + 6T^4 + 22\right]\left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right)\cos(ky) \\
&\quad + \frac{k^4}{8}\left[9T^{-2} - 95T^2 + 46\right]\left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right)\cos(ky) \\
&\quad + \frac{k^4}{8}\left[15T^{-2} - 22T^2 + 2T^4 + 14\right]\left(|\psi|^2\psi e^{i\omega t_0} + \text{c.c.}\right)\cos(3ky) \\
&\quad + \frac{k^4}{8}\left[9T^{-2} - 60T^2 + 42\right]\left(\psi^3 e^{3i\omega t_0} + \text{c.c.}\right)\cos(3ky)
\end{aligned}$$

The equation above gives rise to a secular condition which evaluates to a non-linear Schrodinger equation for the surface displacement. This equation has an exact hyperbolic secant soliton solution observed in the experiments.

3 Pictures of Experiment

Shown below are lengthwise images (taken at 60 frames per second) of a single non-propagating soliton through one cycle. This is a hyperbolic secant that is localized at a single point where a tongue depressor was used to uniformly introduce energy into the channel.





Across the width, the surface displacement is characterized by $\cos\left(\frac{m\pi}{b}y\right)$. This is shown in the images below.