Mechanics of Wave Motion and Sound

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1 Ideal Fluids

An ideal fluid is one for which thermal conductivity and viscosity are unimportant. That is, we take no account of any process of energy dissipation due to internal friction and heat exchange. In this section, we are going to establish the basic equations governing the motion of ideal fluids, starting from first principles.

1.1 Conservation of Matter

Consider some volume V_0. The total mass of fluid in this volume is

$$\int \varrho \, dV$$

The velocity of the fluid at a given point (x, y, z) in space and at a particular time t is

$$\boldsymbol{v}(x, y, z, t)$$

This is a vector field and it refers to fixed points in space, not specific particles of fluid. In time, the particles will move around in space, with their velocities determined by their position in space.

The mass of fluid flowing in unit time through a surface element $d\boldsymbol{a}$ bounding the volume is

$$(\varrho \boldsymbol{v}) \cdot d\boldsymbol{a}$$

where $d\boldsymbol{a}$ points along the outward normal. Thus, the total mass flowing out of the entire volume V_0 in unit time is

$$\oint (\varrho \boldsymbol{v}) \cdot d\boldsymbol{a}$$

The decrease per unit time of the mass of the fluid in V_0 can be represented as

$$-\frac{\partial}{\partial t} \int \varrho \, dV$$

Conservation of matter requires that these time rates of change are equal, so, we equate them:

$$\begin{aligned}
\frac{\partial}{\partial t} \int \varrho \, dV &= -\oint (\varrho \boldsymbol{v}) \cdot d\boldsymbol{a} \\
\frac{\partial}{\partial t} \int \varrho \, dV &= -\int (\boldsymbol{\nabla} \cdot \varrho \boldsymbol{v}) \, dV \\
\int \frac{\partial \varrho}{\partial t} \, dV + \int (\boldsymbol{\nabla} \cdot \varrho \boldsymbol{v}) \, dV &= 0 \\
\int \left(\frac{\partial \varrho}{\partial t} + (\boldsymbol{\nabla} \cdot \varrho \boldsymbol{v}) \right) dV &= 0 \\
\boxed{\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0} \\
\frac{\partial \varrho}{\partial t} + \varrho (\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \varrho) &= 0
\end{aligned}$$

This is called the *equation of continuity.*

1.2 Euler's Equation

The total force acting on some volume can be expressed (by the definition of pressure, p) as

$$\boldsymbol{F}_{\text{tot}} = -\oint p \, d\boldsymbol{a}$$

To convert it to a volume integral, we start from the divergence theorem and then define $\boldsymbol{f} = -p\boldsymbol{n}$, where $\boldsymbol{n}$ is a constant.

$$\begin{aligned}
\oint (\boldsymbol{f} \cdot d\boldsymbol{a}) &= \int (\boldsymbol{\nabla} \cdot \boldsymbol{f}) \, dV \\
-\oint (p\boldsymbol{n} \cdot d\boldsymbol{a}) &= \int (\boldsymbol{\nabla} \cdot p\boldsymbol{n}) \, dV \\
-\boldsymbol{n} \cdot \oint p \, d\boldsymbol{a} &= -\int (p\boldsymbol{\nabla} \cdot \boldsymbol{n} + \boldsymbol{n} \cdot \boldsymbol{\nabla} p) \, dV \\
-\boldsymbol{n} \cdot \oint p \, d\boldsymbol{a} &= -\boldsymbol{n} \cdot \int \boldsymbol{\nabla} p \, dV \\
\therefore -\oint p \, d\boldsymbol{a} &= -\int (\boldsymbol{\nabla} p) \, dV = \boldsymbol{F}_{\text{tot}}
\end{aligned}$$

Thus, the *pressure gradient* ($-\boldsymbol{\nabla} p$) is the force per unit volume. But Newton's law tells us that the force per unit volume is

$$\varrho \frac{d\boldsymbol{v}}{dt}$$

where ϱ is the density (mass per unit volume) of the fluid. Equating these expressions, we arrive at

$$\varrho \frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla} p$$

But what is $\frac{d\boldsymbol{v}}{dt}$? It denotes the rate of change of velocity of a given fluid particle as it moves about in space. At different points in space, however, the fluid particle has different velocities, given by the field $\boldsymbol{v}$. So, the differential velocity is

$$\begin{aligned}
d\boldsymbol{v} &= \left[\frac{\partial \boldsymbol{v}}{\partial t}\right] dt + \left[\frac{\partial \boldsymbol{v}}{\partial x} \, dx + \frac{\partial \boldsymbol{v}}{\partial y} \, dy + \frac{\partial \boldsymbol{v}}{\partial z} \, dz\right] \\
d\boldsymbol{v} &= \left[\frac{\partial \boldsymbol{v}}{\partial t}\right] dt + [d\boldsymbol{r} \cdot \boldsymbol{\nabla}] \, \boldsymbol{v} \\
\frac{d\boldsymbol{v}}{dt} &= \left[\frac{\partial \boldsymbol{v}}{\partial t}\right] + \left[\frac{d\boldsymbol{r}}{dt} \cdot \boldsymbol{\nabla}\right] \boldsymbol{v} \\
\frac{D\boldsymbol{v}}{Dt} &= \left[\frac{\partial \boldsymbol{v}}{\partial t}\right] + [\boldsymbol{v} \cdot \boldsymbol{\nabla}] \, \boldsymbol{v}
\end{aligned}$$

$\frac{D\boldsymbol{v}}{Dt}$ is known as the *convective derivative*, and in general, it represents the time rate of change of any property of the fluid. So, Newton's law for a continuous fluid (or *Euler's equation*) is:

$$\frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p$$

$$\boxed{\varrho \left[\left(\frac{\partial \boldsymbol{v}}{\partial t}\right) + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \, \boldsymbol{v}\right] = -\boldsymbol{\nabla} p}$$

1.3 Entropy

The absence of heat exchange between different parts of the fluid means that the motion is *adiabatic*, so the motion of an ideal fluid is necessarily adiabatic. In adiabatic motion, the entropy of any particle of fluid remains constant as it roams through space. If the entropy per unit mass is s, the condition for adiabatic motion is

$$\frac{Ds}{Dt} = 0$$

which can be rewritten as

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} s) = 0$$

This equation can be combined with the equation for conservation of mass to as follows:

$$\begin{aligned} s\left(\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v})\right) + \varrho\left(\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} s)\right) &= 0 \\ s\frac{\partial \varrho}{\partial t} + \varrho\frac{\partial s}{\partial t} + s\boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) + \varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla} s) &= 0 \\ \boxed{\frac{\partial}{\partial t}(\varrho s) + (\boldsymbol{\nabla} \cdot \varrho s \boldsymbol{v}) = 0} \end{aligned}$$

This is the equation of continuity for entropy.

1.4 Boundary Conditions

For an ideal fluid, the boundary condition is that the fluid cannot penetrate a solid surface. That is, if $\boldsymbol{U}$ is the velocity of the boundary, then

$$(\boldsymbol{v} - \boldsymbol{U}) \cdot \hat{\boldsymbol{n}} = 0$$

So, if the surface is at rest, then $v_n = 0$.

At a boundary between two *immiscible fluids*, the boundary condition is that the pressure *and* the velocity component normal to the surface of separation must be the same for both fluids. Moreover, each of these velocity components must be equal to the corresponding component of the velocity of the surface.

The state of a moving fluid is determined by 5 quantities: the 3 components of $\boldsymbol{v}$, the pressure p and the density ϱ. So, we need a system of 5 equations to find $\boldsymbol{v}$, p, and ϱ. For an ideal fluid, these are:

(i) Euler's Equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho}$$

(ii) The Equation of Continuity

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0$$

(iii) The Adiabatic Equation

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} s) = 0$$

1.5 Hydrostatics

If the fluid is in a gravitational field, an additional force per unit volume of $\varrho \boldsymbol{g}$ is present. So, Euler's equation becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} + \boldsymbol{g}$$

For a fluid at rest in a uniform gravitational field, Euler's equation is

$$\begin{aligned} 0 &= -\frac{\boldsymbol{\nabla} p}{\varrho} + \boldsymbol{g} \\ \therefore \boldsymbol{\nabla} p &= \varrho \boldsymbol{g} \end{aligned}$$

If the density of the fluid is assumed to be constant throughout the volume, i.e. there is no significant compression of the fluid due to the external force (called an incompressible fluid), then we have

$$\begin{aligned} \frac{\partial p}{\partial x} = 0 \ , \ \frac{\partial p}{\partial y} = 0 \ , \ \frac{\partial p}{\partial z} &= -\varrho g \\ \therefore p &= \varrho g z + \text{const.} \end{aligned}$$

If the fluid at rest has a *free surface* to which the same external pressure p_0 is applied at every point, this surface must be the horizontal plane $z = h$. So,

$$\begin{aligned} p_0 &= \varrho g(h) + \text{const.} \\ \therefore \text{const.} &= p_0 - \varrho g h \\ \rightsquigarrow p &= p_0 + \varrho g(h - z) \end{aligned}$$

1.6 Circulation

Circulation, Γ, is defined as the integral:

$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{r}$$

taken around a closed contour. Consider a closed contour drawn into the fluid at any given instant. Keep in mind that this contour is a "fluid contour", that is, it is composed of the fluid particles that lie on it. Since we would like to see how this loop of fluid particles changes over time, we take the convective derivative of circulation:

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \oint \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r} + \oint \boldsymbol{v} \cdot \frac{D(d\boldsymbol{r})}{Dt} \\
&= \oint \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r} + \oint \boldsymbol{v} \cdot d\left(\frac{D\boldsymbol{r}}{Dt}\right) \\
&= \oint \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r} + \oint \boldsymbol{v} \cdot d\boldsymbol{v} \\
&= \oint \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r} + \oint d(\frac{1}{2}v^2) \\
&= \oint \frac{D\boldsymbol{v}}{Dt} \cdot d\boldsymbol{r}
\end{aligned}$$

From Newton's law, we know that

$$\frac{D\boldsymbol{v}}{dt} = -\frac{\boldsymbol{\nabla} p}{\varrho}$$

So,

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \oint \left(-\frac{\boldsymbol{\nabla} p}{\varrho}\right) \cdot d\boldsymbol{r} \\
&= \int \boldsymbol{\nabla} \times \left(-\frac{\boldsymbol{\nabla} p}{\varrho}\right) \cdot d\boldsymbol{a} \\
&= 0
\end{aligned}$$

because the curl of a gradient is zero. Thus,

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$

or $\Gamma = \text{const}$. This is the law of conservation of circulation, or Kelvin's Theorem.

If you apply Kelvin's Theorem to an infinitesimal closed contour ∂C:

$$\begin{aligned}
\oint_{\partial C} \boldsymbol{v} \cdot d\boldsymbol{r} &= \int (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{a} \\
&\approxeq (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{a} = \text{constant.}
\end{aligned}$$

This means that at any tiny packet of fluid, $\boldsymbol{\nabla} \times \boldsymbol{v}$ is a constant. So, $\boldsymbol{\nabla} \times \boldsymbol{v}$ – which we call the *vorticity* of a fluid at a given point– moves with the fluid.

1.7 Incompressible fluids

Flow in which $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$ in all space is called *potential flow*. By Kelvin's theorem, the velocity circulation around any closed contour must also be zero. Moreover, since the curl of the velocity field is zero, we can define a velocity potential such that $\boldsymbol{v} = \boldsymbol{\nabla}\phi$.

An incompressible fluid is one with a constant density. Let's see what characterizes *incompressible flow*.
Euler's equations give:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} + \boldsymbol{g}$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{\nabla} w$$

Using that $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{2}\boldsymbol{\nabla} v^2 - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v})$

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{\nabla}(w - \frac{1}{2}v^2)$$

Taking the curl of both sides,

$$\boxed{\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{\nabla} \times (\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}))}$$

The equation of continuity yields:

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0}$$

For potential flow, this becomes $\nabla^2\phi = 0$. Moreover, with potential flow, Euler's equations can be tweaked to get:

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{\nabla}(w - \frac{1}{2}v^2)$$
$$\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = \boldsymbol{\nabla}(w - \frac{1}{2}v^2)$$
$$\therefore \boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\varrho} = f(t)}$$

This is also known as Bernoulli's equation.

1.8 Energy

We would like to know how the energy of fluid contained in volume element changes over time. The fluid particles have kinetic and internal energy that is transported through the surface, and also there is work done by pressure on the fluid within the surface. In order to show this, consider the kinetic and internal energy per unit volume of the fluid: $\frac{1}{2}\varrho v^2 + \varrho\varepsilon$ where ε is the internal energy per unit mass. The time rate of change of this quantity is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) = \frac{1}{2}v^2\frac{\partial\varrho}{\partial t} + \varrho\boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial t} + \frac{\partial(\varrho\varepsilon)}{\partial t}$$

Using Euler's equation and the continuity equation,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) &= -\frac{1}{2}v^2(\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})) + \varrho\boldsymbol{v}\cdot\left(-\frac{\boldsymbol{\nabla}p}{\varrho} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) + \frac{\partial(\varrho\varepsilon)}{\partial t}\\ &= -\frac{1}{2}v^2(\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})) - \boldsymbol{v}\cdot\boldsymbol{\nabla}p - \varrho\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \frac{\partial(\varrho\varepsilon)}{\partial t}\\ &= -\frac{1}{2}v^2(\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})) - \boldsymbol{v}\cdot\boldsymbol{\nabla}p - \varrho\frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{\nabla}v^2) + \frac{\partial(\varrho\varepsilon)}{\partial t}\end{aligned}$$

Let w denote the heat function per unit mass of fluid (enthalpy). We have the thermodynamic relation that

$$\begin{aligned}dw &= T\,ds + V\,dp\\ \boldsymbol{\nabla}w &= T\,\boldsymbol{\nabla}s + \left(\frac{1}{\varrho}\right)\boldsymbol{\nabla}p\\ \boldsymbol{\nabla}p &= \varrho\boldsymbol{\nabla}w - \varrho T\,\boldsymbol{\nabla}s\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) = -\frac{1}{2}v^2(\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})) - \varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}(\frac{1}{2}v^2 + w) + \varrho T\,\boldsymbol{\nabla}s + \frac{\partial(\varrho\varepsilon)}{\partial t}$$

The differential internal energy per unit mass is

$$d\varepsilon = T\,ds - p\,dV = T\,ds + \left(\frac{p}{\varrho^2}\right)d\varrho$$

So,

$$\begin{aligned}d(\varrho\varepsilon) &= \varepsilon\,d\varrho + \varrho\,d\varepsilon\\ &= \varepsilon\,d\varrho + \varrho T\,ds + \left(\frac{p}{\varrho}\right)d\varrho\\ &= \left(\varepsilon + \frac{p}{\varrho}\right)d\varrho + \varrho T\,ds\\ d(\varrho\varepsilon) &= w\,d\varrho + \varrho T\,ds\\ \implies \frac{\partial(\varrho\varepsilon)}{\partial t} &= w\frac{\partial\varrho}{\partial t} + \varrho T\frac{\partial s}{\partial t}\end{aligned}$$

Using the continuity equation and considering adiabatic motion ($\frac{Ds}{Dt} = 0$), this is

$$\frac{\partial(\varrho\varepsilon)}{\partial t} = -w\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v}) - \varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s)$$

Therefore, the time rate of change of energy in the volume is

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) &= -\frac{1}{2}v^2(\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})) - \varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}(\frac{1}{2}v^2 + w) + \varrho T\,\boldsymbol{\nabla}s - w\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v}) - \varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s) \\
&= -\left(\frac{1}{2}v^2 + w\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v}) - \varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2 + w\right) \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) &= -\boldsymbol{\nabla}\cdot\left(\varrho\boldsymbol{v}\left[\frac{1}{2}v^2 + w\right]\right)
\end{aligned}$$

We are left with

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) + \boldsymbol{\nabla}\cdot\left(\varrho\boldsymbol{v}\left[\frac{1}{2}v^2 + w\right]\right) = 0$$

The quantity $\varrho\boldsymbol{v}\left[\frac{1}{2}v^2 + w\right]$ is known as the energy flux density vector.

2 Acoustics

2.1 Review of the basic equations

The basic assumption which gives rise to fluid mechanics is that 5 variables are a complete set of physical quantities which characterize the state of a simple fluid. Usually, these are:

$$\begin{aligned}
\varrho(\boldsymbol{r},t) &= \text{mass density } (\mathrm{g\,cm^{-3}}) \\
s(\boldsymbol{r},t) &= \text{specific entropy } (\mathrm{ergs\,g^{-1}\,K^{-1}}) \\
\boldsymbol{v}(\boldsymbol{r},t) &= \text{velocity } (\mathrm{cm\,s^{-1}})
\end{aligned}$$

To determine how these quantities are developed over time, we obtain equations which follow from macroscopic first principles. These are:

(i) *Conservation of Mass*: The net change of some mass enclosed in a volume is due to the flow of some flux across a surface bounding the enclosed volume (V_0).

$$\frac{\partial}{\partial t}\int\limits_{V_0} \varrho \, d\tau = -\int\limits_{\partial V_0} (\varrho \boldsymbol{v}) \boldsymbol{\cdot} d\boldsymbol{a} = -\int\limits_{V_0} (\boldsymbol{\nabla} \boldsymbol{\cdot} (\varrho \boldsymbol{v}))\, d\tau$$

$$\therefore \boxed{\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \boldsymbol{\cdot} (\varrho \boldsymbol{v}) = 0} \tag{1}$$

(ii) *Newton's Law for a Continuum*: From Newton's Law for a Continuum, we get the equation for $\boldsymbol{v}$. We write it in terms of the force per unit volume of the fluid. So,

$$\varrho \frac{D\boldsymbol{v}}{Dt} \equiv \varrho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right] = \text{Force per unit volume}$$

The total force acting on an enclosed surface is

$$-\oint p \, d\boldsymbol{a} = -\int (\boldsymbol{\nabla} p)\, dV$$

So, the force "density" is $-\boldsymbol{\nabla} p$. Hence, we arrive at

$$\boxed{\varrho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right] = -\boldsymbol{\nabla} p} \tag{2}$$

(Note that an external gravitational force per unit volume can be added on the RHS as $\varrho \boldsymbol{g}$)

(iii) *Momentum Conservation*: Multiply (1) by $\boldsymbol{v}$ and add (2):

$$\varrho\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\frac{\partial \varrho}{\partial t} + \varrho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{\nabla}\boldsymbol{\cdot}(\varrho\boldsymbol{v})) = -\boldsymbol{\nabla} p$$

$$\frac{\partial}{\partial t}(\varrho \boldsymbol{v}) + \boldsymbol{\nabla} p + [\, \varrho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{\nabla}\boldsymbol{\cdot}(\varrho\boldsymbol{v}))\,] = 0$$

We can rewrite this in a more convenient form, summing over repeated indices $j = 1, 2, 3$ for each component $i = 1, 2, 3$.

$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial p}{\partial r_i} + \left(\varrho v_j \frac{\partial v_i}{\partial r_j}\right) + \left(v_i \frac{\partial(\varrho v_j)}{\partial r_j}\right) = 0$$

$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial p}{\partial r_i} + \frac{\partial(\varrho v_i v_j)}{\partial r_j} = 0$$

$$\therefore \boxed{\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial r_j}(p\delta_{ij} + \varrho v_i v_j) = 0} \tag{3}$$

In the presence of a gravitational field, the RHS will include ϱg_i.

(iv) <u>1^{st} *and* 2^{nd} *Laws of Thermodynamics*</u>: The equation for entropy follows from the requirements of the second law and our assumption (for now) that we can neglect irreversible phenomena. That is, effects which lead to the production of entropy due to viscous heating and thermal diffusion. Under this approximation, entropy is conserved.

$$\boxed{\frac{D}{Dt}(s) = \frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})s = 0} \tag{4}$$

2.2 Linear Acoustics

In the limit of a small amplitude disturbance, the fluid equations possess a dispersion law in that ω is a function of k. To see this, we linearize around a constant reference state:

$$\begin{aligned} s &= s_0 + \delta s \\ \varrho &= \varrho_0 + \delta\varrho \\ \boldsymbol{v} &= \delta\boldsymbol{v} \end{aligned}$$

Small amplitude oscillations are isentropic, so $\delta s = 0$.
Around some constant density ϱ_0, the Taylor expansion of pressure is

$$p(\varrho) = p(\varrho_0) + \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} (\delta\varrho)$$

Evaluating Euler's equations and the continuity equation, to first order in $\delta\varrho$, we have the wave equation:

$$\frac{\partial^2(\delta\varrho)}{\partial t^2} - c_0^2 \nabla^2(\delta\varrho) - 0$$

If you look for a solution of the form

$$\delta\varrho = \varrho' e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

Then we find the dispersion law $\omega = c_0 k$.

If we linearize velocity around a constant steady flow $\boldsymbol{v} = \boldsymbol{v}_0 + \delta\boldsymbol{v}$, the continuity law gives

$$\begin{aligned} \frac{\partial \delta\varrho}{\partial t} + (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\delta\varrho &= -\varrho_0 \left(\boldsymbol{\nabla} \cdot \delta\boldsymbol{v}\right) \\ \rightsquigarrow \frac{D_0^2}{Dt^2}(\delta\varrho) - c_0^2 \nabla^2(\delta\varrho) &= 0 \end{aligned}$$

The traveling wave solution to this is

$$(\omega - \boldsymbol{k} \cdot \boldsymbol{v}_0)^2 = c_0^2 k^2$$

The relationship between velocity amplitude and density swing in a plane progressive sound wave is

$$\hat{\boldsymbol{k}} \cdot \delta\boldsymbol{v} = \pm \frac{c_0}{\varrho_0} \delta\varrho$$

2.3 1D Non-Linear Acoustics

An essential aspect of these equations is that they are nonlinear. When we are considering perturbations, we usually describe the fluid's characteristic variables as variations about some fixed reference. So, we have that $\varrho = \varrho_0 + \delta\varrho$.
Even if you take $s =$ constant so that $p = p(\varrho)$, nonlinearities come about from the presence of the convective derivative and deviations from Hooke's law. The Taylor expansion of $p(\varrho)$ about some constant ϱ_0 (and constant s) is:

$$\begin{aligned} p(\varrho) &= p(\varrho_0) + \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} (\varrho - \varrho_0) + \left(\frac{\partial^2 p}{\partial \varrho^2}\right)\bigg|_{\varrho_0} \frac{(\varrho-\varrho_0)^2}{2} + \dots \\ \therefore p(\varrho) &= p(\varrho_0) + \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} (\delta\varrho) + \left(\frac{\partial^2 p}{\partial \varrho^2}\right)\bigg|_{\varrho_0} \frac{(\delta\varrho)^2}{2} + \dots \end{aligned} \tag{5}$$

For now, we'll only be looking at second order terms in everything. In 1D, consider traveling waves. The equation of conservation of mass and momentum are:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v)}{\partial x}\right) &= \frac{\partial}{\partial t}(0) \\ \frac{\partial}{\partial x}\left(\frac{\partial(\varrho v)}{\partial t} + \frac{\partial}{\partial x}(p + \varrho v^2)\right) &= \frac{\partial}{\partial x}(0) \end{aligned}$$

Subtracting the second from the first leads to

$$\begin{aligned} \frac{\partial^2 \varrho}{\partial t^2} &= \frac{\partial^2}{\partial x^2}(p + \varrho v^2) \\ \frac{\partial^2(\delta\varrho)}{\partial t^2} &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2}{\partial x^2}(\varrho_0 v^2) \end{aligned} \tag{6}$$

From (5), we have that

$$\frac{\partial^2 p}{\partial x^2} = \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} \frac{\partial^2(\delta\varrho)}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \varrho^2}\right)\bigg|_{\varrho_0} \frac{\partial^2(\delta\varrho)^2}{\partial x^2}$$

We can set the coefficients as:

$$\begin{aligned} \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} &= c_0^2 \\ \left(\frac{\partial^2 p}{\partial \varrho^2}\right)\bigg|_{\varrho_0} &= \left(\frac{\partial}{\partial \varrho}\left(\frac{\partial p}{\partial \varrho}\right)\right)\bigg|_{\varrho_0} = \left(\frac{\partial}{\partial \varrho}(c^2)\right)\bigg|_{\varrho_0} = 2c_0\left(\frac{\partial c}{\partial \varrho}\right)\bigg|_{\varrho_0} \end{aligned}$$

The speed of sound changes! So, (6) becomes

$$\begin{aligned} \frac{\partial^2(\delta\varrho)}{\partial t^2} &= \left(\frac{\partial p}{\partial \varrho}\right)\bigg|_{\varrho_0} \frac{\partial^2(\delta\varrho)}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \varrho^2}\right)\bigg|_{\varrho_0} \frac{\partial^2(\delta\varrho)^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}(\varrho_0 v^2) \\ \frac{\partial^2(\delta\varrho)}{\partial t^2} - c_0^2\frac{\partial^2(\delta\varrho)}{\partial x^2} &= c_0\left(\frac{\partial c}{\partial \varrho}\right)\bigg|_{\varrho_0} \frac{\partial^2(\delta\varrho)^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}(\varrho_0 v^2) \\ \frac{\partial^2(\delta\varrho)}{\partial t^2} - c_0^2\frac{\partial^2(\delta\varrho)}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(\varrho_0 v^2 + c_0\left(\frac{\partial c}{\partial \varrho}\right)\bigg|_{\varrho_0} (\delta\varrho)^2\right) \end{aligned}$$

We know that $v = \frac{\delta\varrho}{\varrho_0}c$, so substituting this for v in (6), we get that

$$\frac{\partial^2(\delta\varrho)}{\partial t^2} - c_0^2\frac{\partial^2(\delta\varrho)}{\partial x^2} = \frac{c_0^2}{\varrho_0}\left(1 + \frac{\varrho_0}{c_0}\left(\frac{\partial c}{\partial\varrho}\right)\right)\frac{\partial^2(\delta\varrho)^2}{\partial x^2}$$

Setting $G = 1 + \frac{\varrho_0}{c_0}\left(\frac{\partial c}{\partial\varrho}\right)$,

$$\boxed{\frac{\partial^2(\delta\varrho)}{\partial t^2} - c_0^2\frac{\partial^2(\delta\varrho)}{\partial x^2} = \left(\frac{c_0^2}{\varrho_0}G\right)\frac{\partial^2(\delta\varrho)^2}{\partial x^2}} \tag{7}$$

2.4 Perturbation

Let us consider the case where:

$$\delta\varrho = \varrho_1(x - ct) + \varrho_2$$

so that ϱ_2 is of order $\mathcal{O}(\varrho_1^2)$. Evaluating the term $(\delta\varrho)^2$ in (7), we see that

$$(\delta\varrho)^2 = \underbrace{\varrho_1^2}_{\mathcal{O}(\varrho_1^2)} + \underbrace{\varrho_2^2}_{\mathcal{O}(\varrho_1^4)} + \underbrace{2\varrho_1\varrho_2}_{\mathcal{O}(\varrho_1^4)}$$

We neglect the terms of $\mathcal{O}(\varrho_1^4), \mathcal{O}(\varrho_1^3)$, so (7) becomes

$$\cancel{\left(\frac{\partial^2\varrho_1}{\partial t^2} - c_0^2\frac{\partial^2\varrho_1}{\partial x^2}\right)} + \boxed{\left(\frac{\partial^2\varrho_2}{\partial t^2} - c_0^2\frac{\partial^2\varrho_2}{\partial x^2}\right) = \left(\frac{c_0^2}{\varrho_0}G\right)\frac{\partial^2(\varrho_1)^2}{\partial x^2}} \tag{8}$$

This is the 2$^{\text{nd}}$ order equation for 1D traveling sound waves. The simple solution to (8) which becomes $\varrho_1(x)$ at $t = 0$ is:

$$\delta\varrho(x,t) = \varrho_1(x - ct) \underbrace{- \frac{c_0 t}{2\varrho_0}G\frac{\partial(\varrho_1)^2}{\partial x}}_{\varrho_2}$$

This solution checks out. You can substitute $\xi = x - ct$ and see that it solves (8).

Now, let's consider an example where we throw two waves of different frequency at each other.

$$\begin{aligned}
\varrho_1(x,t) &= A\cos(k_1x - \omega_1t) + B\cos(k_2x - \omega_2t) \\
{\varrho_1}^2 &= A^2\cos^2(k_1x - \omega_1t) + B^2\cos^2(k_2x - \omega_2t) + \underbrace{2AB\cos(k_1x - \omega_1t)\cos(k_2x - \omega_2t)}_{\text{We're only interested in this term for now}}
\end{aligned}$$

Differentiating that last term in ${\varrho_1}^2$, we get

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}(2AB\cos(k_1x - \omega_1t)\cos(k_2x - \omega_2t)) = &-AB(k_1 + k_2)^2\cos([k_1 + k_2]x - [\omega_1 + \omega_2]t) \\
&\underbrace{-AB(k_1 - k_2)^2\cos([k_1 - k_2]x - [\omega_1 - \omega_2]t)}_{\text{Let's only consider this term now lol}}
\end{aligned}$$

Our non-linear equation takes the form:

$$\begin{aligned}
\frac{\partial^2\varrho_2}{\partial t^2} - c_0^2\frac{\partial^2\varrho_2}{\partial x^2} &= -\left(\frac{c_0^2G}{\varrho_0}AB(k_1 - k_2)^2\right)\cos(\underbrace{[k_1 - k_2]}_{\kappa}x - \underbrace{[\omega_1 - \omega_2]}_{\Omega}t) \\
\frac{\partial^2\varrho_2}{\partial t^2} - c_0^2\frac{\partial^2\varrho_2}{\partial x^2} &= -\left(\frac{c_0^2G}{\varrho_0}AB\kappa^2\right)\cos(\kappa x - \Omega t)
\end{aligned}$$

We guess a solution to (8) in the form $\varrho_2 = Fx\sin(\kappa x - \Omega t)$. Evaluating the partial derivatives, we get

$$-Fx(-c_0^2\kappa^2 + \Omega^2)\sin(\kappa x - \Omega t) - 2Fc_0^2\kappa\cos(\kappa x - \Omega t) = -\left(\frac{c_0^2 G}{\varrho_0}AB\kappa^2\right)\cos(\kappa x - \Omega t)$$
$$\therefore \Omega^2 = c_0^2\kappa^2$$
$$F = \frac{\kappa G}{2\varrho_0}AB$$

So, ϱ_2 becomes:

$$\varrho_2(x,t) = \frac{(k_1 - k_2)G}{2\varrho_0}ABx\sin([k_1 - k_2]x - [\omega_1 - \omega_2]t)$$

Over time, this equation obviously blows up (the x term increases) and the perturbation approach becomes wrong (that is, unless all order terms are considered). So, our final solution becomes

$$\delta\varrho(x,t) = A\cos(k_1 x - \omega_1 t) + B\cos(k_2 x - \omega_2 t) + \frac{(k_1 - k_2)G}{2\varrho_0}ABx\sin([k_1 - k_2]x - [\omega_1 - \omega_2]t)$$

This solution has the property that it can at most be valid only for short periods of time. The origin of the blow-up with time is that the non-linear terms on the RHS of (7) are in resonance (i.e. in phase) with the wave equation; this yields a resonance just as for a mass on a spring driven by an external force at natural frequency.

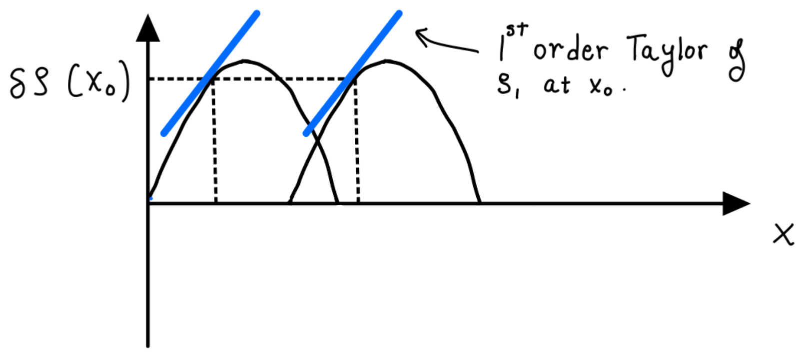
What this solution really means is that if you superimpose 2 waves of frequency ω_1 and ω_2, you will also end up with a wave of frequency $\omega_1 - \omega_2$. This is called an end-fire array.

2.5 Shock Waves

Using that

$$\delta\varrho(x,t) = \varrho_1(x-ct) - \frac{c_0 t}{2\varrho_0} G \frac{\partial(\varrho_1)^2}{\partial x}$$
$$= \varrho_1(x-ct) - \frac{c_0 t}{\varrho_0} G \varrho_1 \frac{\partial \varrho_1}{\partial x}$$

we can determine the velocity of a point on the profile of the wave (not a fluid particle) which has a fixed amplitude. Let's Taylor expand $\varrho_1(x-ct)$ about the point x_0. The variable t just represents a translation of the shape that the function ϱ_1 takes. This is just a visual aid:



We would like to see how a point of given amplitude on the initial "shape" of the wave moves. In a freeze-frame, the entire shape is determined only by x. So, the Taylor expansion is:

$$\varrho_1(x-ct) = \varrho_1(x_0) + \left.\frac{\partial \varrho_1}{\partial x}\right|_{x_0} ((x-ct)-x_0) + \dots$$

Now fix $\delta\varrho = \varrho_1(x_0)$. By taking a linear approximation of $\delta\varrho$ on the LHS and ϱ_1 on the RHS, we can write:

$$\varrho_1(x_0) = \varrho_1(x_0) + \left.\frac{\partial \varrho_1}{\partial x}\right|_{x_0} ((x-ct)-x_0) - \frac{c_0 t}{\varrho_0} G \varrho_1 \left.\frac{\partial \varrho_1}{\partial x}\right|_{x_0}$$
$$((x-c_0 t)-x_0) = \frac{c_0 t G \varrho_1}{\varrho_0}$$
$$\therefore \boxed{x = x_0 + \left(c_0 + c_0 G \frac{\varrho_1}{\varrho_0}\right) t}$$

This represents the position of a point of given amplitude on the profile of the wave. Hence, the speed of any such point on the wave profile is given by

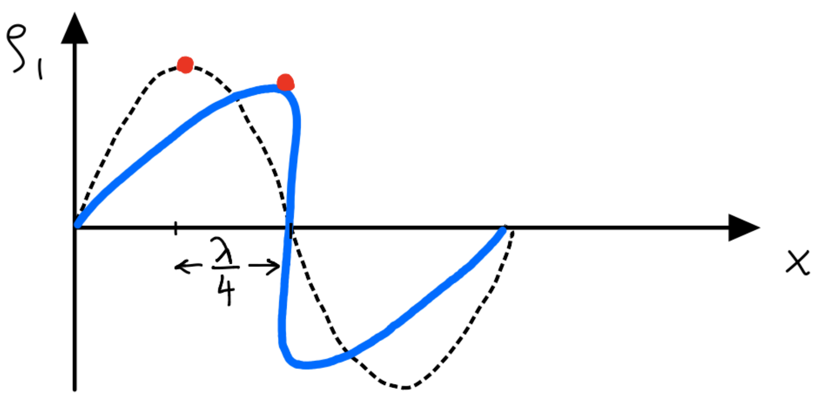
$$\dot{x} = c_0 \left(1 + G\frac{\varrho_1}{\varrho_0}\right)$$

Using that $G = 1 + \frac{\varrho}{c}\frac{\partial c}{\partial \varrho}$ and $\frac{v}{c} = \frac{\varrho}{\varrho_0}$, this can be rewritten as

$$\dot{x} = c_0 + v + \varrho_1 \frac{\partial c}{\partial \varrho}$$

where:
c_0 is the speed of sound in the medium,
v represents a self-doppler shift.
$\varrho_1 \frac{\partial c}{\partial \varrho}$ represents a self-change in the local speed of sound.

The third term is proportional to the amplitude on the profile of the wave. So, points of greater amplitude are shifted faster at any given time.

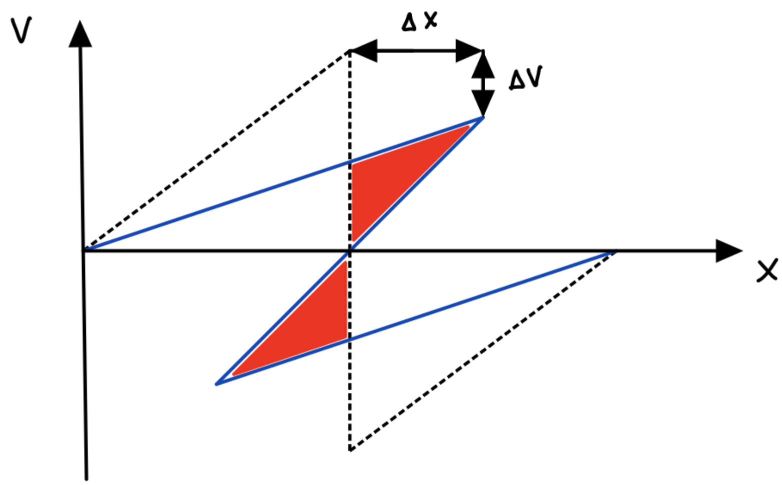


For the sine wave above, the point of greatest amplitude is displaced by $\lambda/4$ when the shock takes place. Using the expression $\dot{x} = c_0(1 + \frac{G\varrho_1}{\varrho_0})$, we can describe the shock formation time as

$$\left(\frac{G\varrho_1 c_0}{\varrho_0}\right)\tau_{\text{shock}} = \frac{\lambda}{4}$$

$$\text{or,} \qquad \omega\tau_{\text{shock}} = \frac{8\pi\varrho_0}{\varrho_1 G}$$

The first result we got here was that $x = x_0 + \left(c_0 + c_0 G \frac{\varrho_1}{\varrho_0}\right) t$. Consider the time after which the shock extends beyond the point of discontinuity:



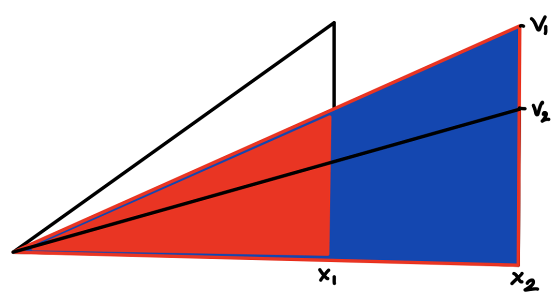
We see that

$$\Delta x = c_0 G \frac{\varrho_1}{\varrho_0} \Delta t$$

and since $\frac{v}{c} = \frac{\varrho_1}{\varrho_0}$,

$$\Delta x = G v \, \Delta t$$

By similar triangles,



$$x_1 v_2 = x_2 v_1$$
$$x_1 v_2 + (-x_1 v_1) = x_2 v_1 + (-x_1 v_1)$$
$$x_1 \Delta v = v_1 \Delta x$$
$$\therefore \frac{\Delta v}{v_1} = \frac{\Delta x}{x_1}$$

Hence,

$$
\begin{aligned}
\frac{\Delta v}{v} &= \frac{\Delta x}{\frac{\lambda}{2}} \\
\frac{\Delta v}{v} &= \frac{2Gv\Delta t}{\lambda} \\
v &= \frac{\Delta v}{\left(\frac{2Gv\Delta t}{\lambda}\right)} \\
v\left(\frac{2Gvt}{\lambda} - 1\right) &= -v(0) \\
\therefore \boxed{v(t) = \frac{v(0)}{1 - \frac{2Gvt}{\lambda}}}
\end{aligned}
$$

3 Surface Waves

3.1 Equations and Boundary Conditions

Incompressible potential flow ($\varrho =$ constant, $\boldsymbol{v} = \boldsymbol{\nabla}\phi$, $s = s_0 + \delta s$, $\boldsymbol{v} = \delta \boldsymbol{v}$)

Conservation of mass gives:

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) &= 0 \\ \rightsquigarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \end{aligned}$$

$$\therefore \boxed{\nabla^2 \phi = 0}$$

Euler's equations give:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \frac{-\boldsymbol{\nabla} p}{\varrho} - \boldsymbol{\nabla}(gz\hat{\boldsymbol{z}})$$

$$\rightsquigarrow \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t}\right) + \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) - \boldsymbol{v} \times (\cancel{\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi}) = -\boldsymbol{\nabla}\left(\frac{p}{\varrho}\right) - \boldsymbol{\nabla}(gz\hat{\boldsymbol{z}})$$

$$\therefore \boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\varrho} + gz = \frac{p_0}{\varrho}}$$

The boundary conditions are:

(i) Let us consider the free surface, where $z = 0$, and there is a small perpendicular perturbative displacement at this surface. So, we can say that $z = 0 \pm \eta$. The boundary condition then must ensure that the vertical speed of this displacement at the surface must be that of the fluid, because the fluid is that which is displaced.

$$\boxed{\left.\frac{\partial \eta}{\partial t}\right|_{z=0} = v_z\big|_{z=0} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0}}$$

(ii) At the bottom of the rectangular trough, there is a floor. At all points on the floor, the velocity of the fluid cannot be into or out of the floor, so it must be zero.

$$\boxed{v_z\big|_{z=-d} = \left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0}$$

(iii) For a free surface,

$$\boxed{p\big|_{z=0} = p_{\text{atm}}}$$

Evaluated at the free surface, Euler's equation in the form above gives:

$$\begin{aligned}
\frac{\partial \phi}{\partial t} + \frac{1}{2}(v_x^2 + v_z^2) + \frac{p_{\text{atm}}}{\varrho} + g\eta &= \frac{p_{\text{atm}}}{\varrho} \\
\rightsquigarrow \frac{\partial \phi}{\partial t} + g\eta &= -\frac{1}{2}(\delta v_x^2 + \delta v_z^2) \\
\rightsquigarrow \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + g\eta\right) &= 0 \\
\therefore \boxed{\frac{\partial^2 \phi}{\partial t^2} + g\left(\left.\frac{\partial \phi}{\partial z}\right|_{z=0}\right) = 0}
\end{aligned}$$

Laplace's equation can be written (in this case) as:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

3.2 Solution

We will try to approach this using separation of variables. Suppose that $\phi(x,z,t) = \Psi(x)\Gamma(z)\Lambda(t)$. Then,

$$\frac{\partial \phi}{\partial z} = \Psi\Lambda\frac{\partial \Gamma}{\partial z}\ , \quad \begin{aligned} \frac{\partial^2 \phi}{\partial x^2} &= \Gamma\Lambda\frac{\partial^2 \Psi}{\partial x^2} \\ \frac{\partial^2 \phi}{\partial z^2} &= \Psi\Lambda\frac{\partial^2 \Gamma}{\partial z^2} \\ \frac{\partial^2 \phi}{\partial t^2} &= \Psi\Gamma\frac{\partial^2 \Lambda}{\partial t^2} \end{aligned}$$

So, Laplace's equation becomes

$$\begin{aligned} \Gamma\Lambda\frac{\partial^2 \Psi}{\partial x^2} &= -\Psi\Lambda\frac{\partial^2 \Gamma}{\partial z^2} \\ \rightsquigarrow \frac{1}{\Psi}\frac{\partial^2 \Psi}{\partial x^2} &= -\frac{1}{\Gamma}\frac{\partial^2 \Gamma}{\partial z^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x^2} &= -k^2\Psi \\ \rightsquigarrow \Psi(x) &= a_1 e^{ikx} + b_1 e^{-ikx} \\ \frac{\partial^2 \Gamma}{\partial x^2} &= k^2\Gamma \\ \rightsquigarrow \Gamma(z) &= a_2 e^{kz} + b_2 e^{-kz} = c_1 \sinh(kz) + c_2 \cosh(kz) \end{aligned}$$

where a_1, a_2 are complex (with equal phase), and c_1, c_2 are real.

Great. Now, let's evaluate the second boundary condition that $\frac{\partial \phi}{\partial z}\big|_{z=-d} = 0$. We know that $\frac{\partial \phi}{\partial z} = \Psi\Lambda\frac{\partial \Gamma}{\partial z}$. Therefore, the boundary condition be restated as $\frac{\partial \Gamma}{\partial z}\big|_{z=-d} = 0$. So,

$$\begin{aligned} \frac{\partial \Gamma}{\partial z} &= kc_1 \cosh(kz) + kc_2 \sinh(kz) \\ \therefore 0 &= kc_1 \cosh(-kd) + kc_2 \sinh(-kd) \\ \rightsquigarrow c_1 \cosh(kd) &= c_2 \sinh(kd) \\ \therefore c_2 &= \frac{c_1}{\tanh(kd)} \end{aligned}$$

So, we can rewrite $\Gamma(z)$ as

$$\begin{aligned} \Gamma(z) &= c_1\left(\frac{\sinh(kz)\sinh(kd) + \cosh(kz)\cosh(kd)}{\sinh(kd)}\right) \\ \therefore \Gamma(z) &= c_1\left(\frac{\cosh(k[z+d])}{\sinh(kd)}\right) \end{aligned}$$

And we can rewrite $\frac{\partial \phi}{\partial z}$ as

$$\begin{aligned}\frac{\partial \phi}{\partial z} &= \Psi\Lambda\frac{\partial \Gamma}{\partial z}\\ &= c_1 k\Psi\Lambda\left(\frac{\sinh\left(k[z+d]\right)}{\sinh\left(kd\right)}\right)\\ \therefore \left.\frac{\partial \phi}{\partial z}\right|_{z=0} &= c_1 k\Psi\Lambda\end{aligned}$$

We would now like to find $\Lambda(t)$, which suspiciously crept away by being cancelled on both sides of the laplacian (because it involved no time derivative). So, we know that

$$\begin{aligned}\frac{\partial^2 \phi}{\partial t^2} &= \Psi\Gamma\frac{\partial^2 \Lambda}{\partial t^2}\\ &= c_1\Psi\left(\frac{\cosh\left(k[z+d]\right)}{\sinh\left(kd\right)}\right)\frac{\partial^2 \Lambda}{\partial t^2}\\ \therefore \left.\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0} &= c_1\Psi\frac{1}{\tanh\left(kd\right)}\frac{\partial^2 \Lambda}{\partial t^2}\end{aligned}$$

Now that we have $\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0}$ and $\left.\frac{\partial \phi}{\partial z}\right|_{z=0}$, we can use Euler's equation evaluated at the free surface:

$$\begin{aligned}\frac{\partial^2 \phi}{\partial t^2} + g\left(\left.\frac{\partial \phi}{\partial z}\right|_{z=0}\right) &= 0\\ c_1\Psi\frac{1}{\tanh\left(kd\right)}\frac{\partial^2 \Lambda}{\partial t^2} &= -gc_1 k\Psi\Lambda\\ \frac{\partial^2 \Lambda}{\partial t^2} &= -\left[gk\tanh\left(kd\right)\right]\Lambda\end{aligned}$$

Now, we can set $\omega^2 = gk\tanh\left(kd\right)$, and this brings us to the solution for Λ:

$$\Lambda(t) = |d_1|e^{i(\omega t+\theta)} + |f_1|e^{i(-\omega t+\theta)}$$

So, we've arrived expressions for the three equations that characterize ϕ:

$$\begin{aligned}\Psi(x) &= |a_1|e^{i(kx+\vartheta)} + |b_1|e^{i(-kx+\vartheta)}\\ \Gamma(z) &= |c_1|\left(\frac{\cosh\left(k[z+d]\right)}{\sinh\left(kd\right)}\right)\\ \Lambda(t) &= |d_1|e^{i(\omega t+\theta)} + |f_1|e^{i(-\omega t+\theta)}\end{aligned}$$

So, by making a huge mess of all the constants after multiplying these three equations, we finally have an expression for ϕ:

$$\phi(x,z,t) = \Psi\Gamma\Lambda = |c_1|\left(\frac{\cosh\left(k[z+d]\right)}{\sinh\left(kd\right)}\right) * \left(|a_1|e^{i(kx+\vartheta)} + |b_1|e^{i(-kx+\vartheta)}\right) * \left(|d_1|e^{i(\omega t+\theta)} + |f_1|e^{i(-\omega t+\theta)}\right)$$

I'm not going to bother multiplying that out.

Taking $\mathrm{Re}(\phi)$, we arrive at:

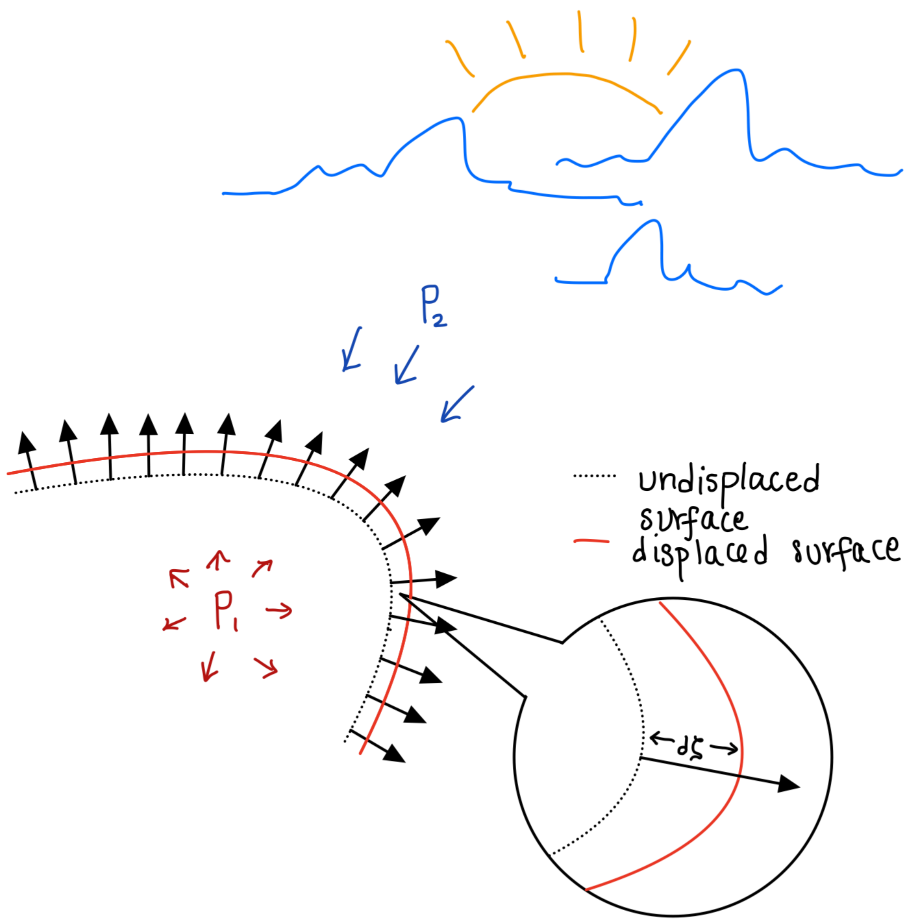
$$\boxed{\phi(x,z,t) = A(e^{k[z+d]} + e^{-k[z+d]})\cos\left(kx-\omega t+\Theta\right)}$$

where $\Theta = \theta + \vartheta$,
and $\omega^2 = gk\tanh\left(kd\right)$.

3.3 Laplace's Formula

3.3.1 Derivation

We're going to consider phenomena which occur near the surface separating two continuous media. In reality, the media are separated by a narrow transitional later, but it is so thin that it can be considered a surface.



Suppose that the surface separating two media undergoes an infinitesimal displacement. At each point on the un-displaced surface, there are normal vectors pointing into one medium. The length of the segment of the normal between the points where it intersects the displaced and un-displaced surfaces is $d\zeta$. An element of volume between these two surfaces would then be $d\zeta da$ where da is the surface element. Between the two surfaces lies a thinfinitesimal change in volume.

Let the displacement of the surface be towards medium 2 (as depicted in the figure). The work necessary to bring about the above change in volume (simply due to the difference in pressure on either side of the surface) is:

$$
\begin{aligned}
&-\int \boldsymbol{F}\cdot d\boldsymbol{l}\\
&= -\int\left(\int (\boldsymbol{\nabla} p \cdot d\boldsymbol{l})\right) dV\\
&= -\int (p_1 - p_2)\, d\zeta\, da
\end{aligned}
$$

There is also work connected with the change in the area of the surface. This is proportional to the total change in area of the surface (δa). So, the total work δW done in displacing the surface is:

$$\delta W = \alpha \cdot \delta a - \int (p_1 - p_2)\, d\zeta\, da$$

where α is the *surface tension coefficient*.

Let R_1 and R_2 be the principal radii of curvature at a given point on the surface. Assume that they are drawn into medium 1 so that R_1 and R_2 are positive. Elements of length dl_1 and dl_2 on the un-displaced surface can be thought of as elements of the circumferences of the principal radii of curvature. Consider dl_1 and dl_2 sweeping out an infinitesimal angular change. The arc lengths spanned by the new radii of curvature are primed.

$$
\begin{aligned}
\frac{dl_1'}{dl_1} &= \frac{(R_1 + d\zeta)d\theta}{R_1 d\theta}\\
\therefore dl_1' - dl_1 = \left(\frac{R_1 + d\zeta}{R_1} - 1\right) dl_1 &= \frac{d\zeta}{R_1}\, dl_1
\end{aligned}
$$

So, dl_1 and dl_2 are incremented by $\frac{d\zeta}{R_1}\, dl_1$ and $\frac{d\zeta}{R_2}\, dl_2$ respectively, when the surface undergoes an infinitesimal displacement. The surface is element da is related to these elements of length by $da = dl_1\, dl_2$ because they are orthogonal. Hence, the new surface element after displacement is:

$$
\begin{aligned}
da' = dl_1'\, dl_2' &= (1 + \frac{d\zeta}{R_1})(1 + \frac{d\zeta}{R_2})\, dl_1\, dl_2\\
&\approxeq \left(1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\zeta\right) dl_1\, dl_2
\end{aligned}
$$

So the infinitesimal change in the area element is

$$da' - da = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\zeta\, dl_1\, dl_2 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\zeta\, da$$

Hence the total change in area, δa is

$$\delta a = \int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\zeta\, da$$

The condition of thermodynamic equilibrium is that $\delta W = 0$. So, this gives us:

$$\int d\zeta \left\{(p_1 - p_2) - \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\} da = 0$$

Since this must be true for every displacement on the surface $d\zeta$, the expression in braces must be identically equal to zero. So, we arrive at *Laplace's formula*:

$$\boxed{p_1 - p_2 = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

3.3.2 Minimal Surface

Let the equation of a surface be given by $z = \zeta(x, y)$. Suppose that ζ only deviates slightly from the plane $z = 0$. The area of the surface is given by

$$a = \iint \left(1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right)^{\frac{1}{2}} dx\, dy$$

Using that the Taylor expansion of $\sqrt{1+x} = 1 + \frac{x}{2}$, we can approximate this area as

$$a \approx \iint \left(1 + \frac{1}{2}\left(\frac{\partial \zeta}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \zeta}{\partial y}\right)^2\right) dx\, dy$$

Suppose that $z = \zeta(x, y)$ is the minimal surface of a boundary C. Let $z^\epsilon = \zeta(x, y) + \epsilon g(x, y)$. Here, g is a function on the domain of ζ which has the effect, when multiplied by a small ϵ and added to ζ, of moving points on the minimal a tiny little bit whilst C stays fixed. We retrieve the minimal surface by setting $\epsilon = 0$.

The surface area of the "wrong" surface is:

$$a^\epsilon = \iint \left(1 + \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + \epsilon\frac{\partial g}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \zeta}{\partial y} + \epsilon\frac{\partial g}{\partial y}\right)^2\right) dx\, dy$$

$$\text{Differentiating w.r.t } \epsilon, \quad \frac{\partial a^\epsilon}{\partial \epsilon} = \iint \left(\frac{\partial \zeta}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial \zeta}{\partial y}\frac{\partial g}{\partial y}\right) dx\, dy = \delta a$$

Now, set $P = \frac{\partial \zeta}{\partial x} g$ and $Q = -\frac{\partial \zeta}{\partial y} g$. Computing $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial y}$ and applying Green's Theorem gives us:

$$\oint_C \underbrace{\left(\frac{\partial \zeta}{\partial x} g\right) dy}_{P\,dy} + (\underbrace{-\left(\frac{\partial \zeta}{\partial y} g\right) dx}_{Q\,dx}) = \iint \left(\underbrace{\left(\frac{\partial^2 \zeta}{\partial x^2} g + \frac{\partial \zeta}{\partial x}\frac{\partial g}{\partial x}\right)}_{\frac{\partial P}{\partial x}} - \underbrace{\left(-\frac{\partial^2 \zeta}{\partial y^2} g - \frac{\partial \zeta}{\partial y}\frac{\partial g}{\partial y}\right)}_{\frac{\partial Q}{\partial x}}\right) dx\, dy$$

The LHS is zero, as $g = 0$ on C. Noting that $g = \delta\zeta$ (the variation of the surface), the above equation becomes:

$$\iint \left(\frac{\partial \zeta}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial \zeta}{\partial y}\frac{\partial g}{\partial y}\right) dx\, dy = -\iint \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) g\ dx\, dy$$

$$\therefore \delta a = -\iint \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) \delta\zeta\ dx\, dy$$

Comparing this to the earlier result for δa, we have:

$$\boxed{\frac{1}{R_1} + \frac{1}{R_2} = -\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)}$$

So, we have successfully related the principal curvatures at a point to the second partial derivatives of the surfaces.

3.4 Gravity Waves

Back to our equations for surface waves. We had

$$\phi(x,z,t) = A(e^{k[z+d]} + e^{-k[z+d]}) \cos(kx - \omega t + \Theta)$$

where $\Theta = \theta + \vartheta$, and $\omega^2 = gk \tanh(kd)$. If we take surface tension into consideration, the boundary condition on pressure is modified by the curvature of the surface. Hence,

$$p - p_0 = \alpha \left[\frac{1}{R_1}\right] = -\alpha \frac{\partial^2 \zeta}{\partial x^2}$$

Thus, for a plane wave, the Euler equations evaluated at the free surface become:

$$\begin{aligned} \frac{\partial \phi}{\partial t} + g\eta - \frac{\alpha}{\varrho}\frac{\partial^2 \eta}{\partial x^2} &= 0 \\ \rightsquigarrow \left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} - \frac{\alpha}{\varrho}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi}{\partial z}\right)\right]_{z=0} &= 0 \\ \rightsquigarrow \omega^2 = [gk + \frac{\alpha k^3}{\varrho}] \tanh(kd) \end{aligned}$$

The limiting cases are:

- $kd \ll 1$: Long gravity waves.

$$\omega^2 = gdk + \frac{\alpha d k^4}{\varrho}$$

- $kd \gg 1$: Short gravity waves modified by ripples.

$$\omega^2 = gk + \frac{\alpha k^3}{\varrho}$$

4 Sonoluminescence

4.1 Spherical Waves

A wave that is spherically symmetric is called a *spherical wave*. That is, the distribution of density, velocity, etc... depends only on the distance from some point. In spherical coordinates, the wave equation for velocity potential is:

$$\left(\frac{1}{c^2}\right)\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi$$
$$\rightsquigarrow \frac{\partial^2\phi}{\partial t^2} = c^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)$$

We guess a solution of the form $\phi = \frac{f(r,t)}{r}$. Plugging this into the wave equation leads us to:

$$\phi = \underbrace{\frac{f_1(ct-r)}{r}}_{\text{outward}} + \underbrace{\frac{f_2(ct+r)}{r}}_{\text{inward}}$$

So, the amplitude of the spherical wave decreases inversely as the distance from the centre. The intensity of the wave is given by the square of the amplitude, and hence it falls off inversely as the square of the distance.

If there is no source of sound at the origin, then the potential must be finite at $r = 0$. Hence,

$$f_1(ct) = -f_2(ct)$$

which is to say that the wave going into the origin is equal and opposite to the wave going out of the origin–there's no source! This is a stationary spherical wave, and the potential will take the form:

$$\phi = \frac{f(ct-r) - f(ct+r)}{r}$$

A stationary monochromatic spherical wave has the form:

$$\phi = Ae^{-i\omega t}\frac{\sin(kr)}{r}$$

where $k = \omega/c$.

On the other hand, if there is a source at the origin, the potential of the outgoing wave is

$$\phi = \frac{f(ct-r)}{r}$$

Note that this function can be constant if and only if f is zero (because it is a function of $(ct-r)$). An outgoing monochromatic spherical wave has the form:

$$\phi = \frac{A}{r}e^{i(kr-\omega t)}$$

Before the arrival of a wave, at every point in space we have $\phi = 0$ (that is, if the medium is at rest). After the arrival of an outgoing wave, the motion must eventually die out. Hence the potential must be zero before and after the passage of the wave.
Since $p = -\varrho\frac{\partial\phi}{\partial t}$, integrating pressure over all time we get

$$\int_{-\infty}^{\infty} p \; dt = 0$$

because the potential must be zero before and after the wave. Apparently this means that as the spherical wave passes through a point, both condensations ($p > 0$) and rarefaction ($p < 0$) will be observed at that point. This is unlike a plane wave, where only condensations or rarefactions.

4.2 Rayleigh-Plesset Equation

Here's a nice long list of all the assumptions we're going to make in this section:

- The bubble exists in an infinite medium.
- The bubble stays spherical at all times during the pulsation.
- Spatially uniform conditions exist within the bubble.
- The bubble radius is much smaller than the wavelength of the driving sound field.
- There are no body forces acting (e.g. gravity).
- Bulk viscous effects can be ignored.
- The density of the surrounding fluid is much greater than that of the internal gas.
- The gas content is constant.

The total velocity potential is the sum of two parts: that inside the bubble, and that outside. Thus, we have

$$\phi = \phi_{\text{cavity}} + \phi_{\text{acoustic}}$$

The radius of the bubble (R) is very small compared to the acoustic wavelength (λ). So, we can assume that there is not much variation of the potential in the volume enclosed by the bubble. So, we can write:

$$\phi_{\text{cavity}} = \frac{1}{r} f(t - \frac{r}{c}) \approxeq \frac{1}{r} f(t)$$

We are going to assume incompressibility ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) around the bubble as well, so around $r = R$, we have $\nabla^2 \phi = 0$. Hence,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right)\right]_{r=R} = 0$$

$$\left.\frac{\partial^2 f(t - \frac{r}{c})}{\partial r^2}\right|_{r=R} = 0$$

Since $R \ll \lambda$,

$$f = f(t)$$

which identically comes from the small wavelength assumption.

Since we want things to be nice and smooth, at the boundary between the cavity and the medium, we must have that

$$\left.\frac{\partial \phi_{\text{cavity}}}{\partial r}\right|_{r=R} = \dot{R}$$

$$\left.\frac{\partial \phi_{\text{acoustic}}}{\partial r}\right|_{r=R} = 0$$

where $R = R(t)$ is the radius of the bubble and r is the radial distance measured from the center of the bubble. Evaluating this boundary condition, we get:

$$\left.\frac{\partial \phi_{\text{cavity}}}{\partial r}\right|_{r=R} = -\frac{f(t)}{R^2} + \frac{1}{R}\cancel{\frac{\partial f(t)}{\partial r}}$$

$$\therefore f(t) = -\dot{R}R^2$$
$$\rightsquigarrow \phi_{\text{cavity}} = \frac{-\dot{R}R^2}{r}$$

Interestingly, this says that $f(t) = -\dot{R}R^2$ is constant in space. It has units of energy per density.

Evaluating the time derivative of the potential:

$$\begin{aligned}\left.\frac{\partial \phi}{\partial t}\right|_{r=R} &= \left.\frac{\partial \phi_{\text{cavity}}}{\partial t}\right|_{r=R} + \left.\frac{\partial \phi_{\text{acoustic}}}{\partial t}\right|_{r=R} \\ &= (-R\ddot{R} - 2\dot{R}^2) + \left(\frac{1}{\varrho}\; p_{\text{acoustic}}|_{r=R}\right)\end{aligned}$$

Since the flow is incompressible, we can use Bernoulli's equation:

$$\begin{aligned}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\frac{\partial \phi}{\partial r})^2 + \frac{p}{\varrho}\right]_{r=R} &= \frac{p_0}{\varrho} \\ (-R\ddot{R} - 2\dot{R}^2) + \frac{1}{2}\dot{R}^2 + \frac{p_{\text{gas}}}{\varrho} &= -\left(\frac{1}{\varrho}\, p_{\text{acoustic}}(R,t)\right) + \frac{p_0}{\varrho} \\ -R\ddot{R} - \frac{3}{2}\dot{R}^2 + \frac{p_{\text{gas}}}{\varrho} &= \frac{p_0 - p_{\text{acoustic}}(R,t)}{\varrho} \\ \therefore \boxed{R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_{\text{gas}}}{\varrho} = \frac{p_{\text{acoustic}}(R,t) - p_0}{\varrho}}\end{aligned}$$

This is the Rayleigh-Plesset equation.

4.3 Collapsing Bubbles

We just derived the Rayleigh-Plesset equation, which is a differential equation for the radius of the bubble. We saw that it was

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_{\text{gas}}}{\varrho} = \frac{p_{\text{acoustic}}(R,t) - p_0}{\varrho}$$

Suppose that there is so very little gas inside that, for a very rapid collapse, we can neglect p_{gas} and $p_{\text{acoustic}}(R,t)$. This gives:

$$\varrho\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) + p_0 = 0$$

Multiply by $R^2\dot{R}$ on both sides,

$$\rightarrow \quad \varrho\left(R^3\dot{R}\ddot{R} + \frac{3}{2}R^2\dot{R}^3\right) + p_0(R^2\dot{R}) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\varrho R^3\dot{R}^2 + \frac{p_0R^3}{3}\right) = 0$$

What the heck. Conveniently, this has units of energy, so we say:

$$\frac{1}{2}\varrho R^3\dot{R}^2 + \frac{p_0R^3}{3} = E \tag{9}$$

After a short time, the R^3 term becomes very small (as R is decreasing). The term on the left remains constant, as the bubble collapses faster (increasing $\dot{R}^2$) whilst the radius decreases (decreasing R^3). This keeps the first term close to E whilst the second term goes to zero pretty quickly.
Initially, the cavity starts at $R(0) = R_{\text{max}}$ and $\dot{R}(0) = 0$. Hence,

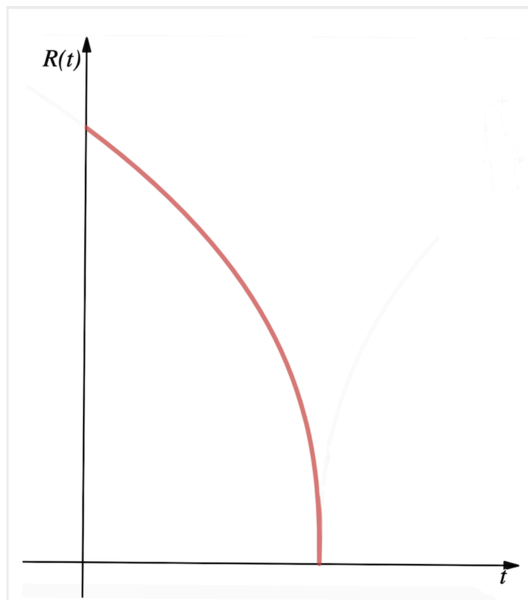
$$E = \frac{p_0R_{\text{max}}^3}{3}$$

Under these approximations and initial conditions, (9) becomes:

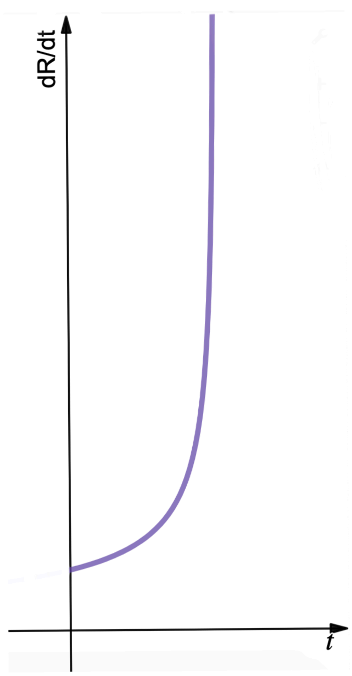
$$\begin{aligned}
\frac{1}{2}\varrho R^3\dot{R}^2 &= \frac{p_0R_{\text{max}}^3}{3} \\
\frac{dR}{dt} &= -\sqrt{\frac{2p_0}{3\varrho}\left(\frac{R_{\text{max}}}{R}\right)^3} \\
(R)^{\frac{3}{2}}\,dR &= -\sqrt{\frac{2p_0}{3\varrho}}\,(R_{\text{max}})^3\,dt \\
\int (R)^{\frac{3}{2}}\,dR &= -\int\sqrt{\frac{2p_0}{3\varrho}}\,(R_{\text{max}})^3\,dt \\
[R(t)]^{\frac{5}{2}} - [R(0)]^{\frac{5}{2}} &= -\frac{5}{2}\left(\sqrt{\frac{2p_0}{3\varrho}}\,(R_{\text{max}})^{\frac{3}{2}}\right)t \\
\therefore\ &\boxed{R(t) = \left[[R(0)]^{\frac{5}{2}} - 5\sqrt{\frac{p_0}{6\varrho}}\,(R_{\text{max}})^{\frac{3}{2}}\,t\right]^{\frac{2}{5}}}
\end{aligned}$$

Simply, $R(t)$ takes the form:

$$R(t) \sim [A - \alpha t]^{\frac{2}{5}}$$



and $\dot{R}(t) = \frac{2}{5}\alpha[A - \alpha t]^{-\frac{3}{5}}$. This means there is a finite time singularity in the velocity of the boundary.



If the surrounding fluid is water, then water is slightly compressible (weak hydrogen bonds). In that case, we get:

$$R(t) = [A_w - \alpha_w t]^{\frac{5}{9}}$$

compared to the incompressible case where $R(t) = [A - \alpha t]^{\frac{4}{10}}$. The speed of the disturbance is $c + v$, and at the cavity, $v = \dot{R}$. When the fluid is slightly compressible like water, energy is released in the form of light. If the fluid is incompressible, then the disturbance produces shock waves. These shock waves then implode, causing another singularity. What in the world...

5 Geometrical Acoustics

5.1 Hamilton's Canonical Equations

The Lagrangian is $\mathcal{L} = T - U$. The generalized momentum is

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

and the Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= (\sum_j p_j \dot{q}_j) - \mathcal{L} \\ &= (2T) - (T - U) \\ \therefore \mathcal{H} &= T + U = \text{constant} \end{aligned}$$

The Canonical Equations of Motion are

$$\begin{aligned} \dot{q}_k &= \frac{\partial \mathcal{H}}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial \mathcal{H}}{\partial q_k} \end{aligned}$$

5.2 An Analogy

In a plane wave, the direction of propagation, and amplitude of the wave are the same in all space. In a small region of space, then, any wave that is not a plane wave may be regarded as a plane wave. In order to do this, we must have that the amplitude and direction of the wave vary very little over distances of the order of λ. This allows us to introduce *rays* tangent to the direction of propagation and say that sound is propagated along those rays. We're going to see what happens as $\lambda \to 0$. Over small regions of space (and during short intervals of time), the phase ψ can be expanded, and up to first order we have

$$\psi = \psi_0 + \boldsymbol{r} \cdot \boldsymbol{\nabla} \psi + t \frac{\partial \psi}{\partial t}$$

From this, we can define (at each point):

$$\begin{aligned} \frac{\partial \psi}{\partial \boldsymbol{r}} &= \boldsymbol{k} \equiv \boldsymbol{\nabla} \psi \\ -\frac{\partial \psi}{\partial t} &= \omega \end{aligned}$$

Recall that in a sound wave:

$$\begin{aligned} \left(\frac{\omega}{c_0}\right)^2 &= k^2 = k_x^2 + k_y^2 + k_z^2 \\ \frac{1}{c^2}\left(\frac{\partial \psi}{\partial t}\right)^2 &= \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \end{aligned}$$

This is the basic equation of geometrical acoustics. Since the fluid is not homogeneous, $\frac{1}{c^2}$ will be a function of the coordinates.

We saw the canonical equations of motion to be

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}$$
$$-\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} = \dot{\boldsymbol{p}}$$

We assumed that $\boldsymbol{k} = \boldsymbol{\nabla}\psi$ and $\omega = -\frac{\partial \psi}{\partial t}$, so we can manipulate this to get:

$$\frac{\partial \boldsymbol{k}}{\partial t} = \frac{\partial}{\partial t}\boldsymbol{\nabla}\psi \quad , \quad -\boldsymbol{\nabla}\omega = \boldsymbol{\nabla}\frac{\partial \psi}{\partial t}$$
$$\therefore \dot{\boldsymbol{k}} = -\frac{\partial \omega}{\partial \boldsymbol{r}}$$

Since the frequency is determined by the source, we can draw an analog to the canonical equations by considering ω to be conserved (like $\mathcal{H}$). We can analogize $\boldsymbol{k}$ with the momentum $\boldsymbol{p}$. Of course, $\dot{\boldsymbol{r}}$ would be the speed of the ray. Hence, the equations we arrive at are:

$$\dot{\boldsymbol{r}} = \frac{\partial \omega}{\partial \boldsymbol{k}}$$
$$\dot{\boldsymbol{k}} = -\boldsymbol{\nabla}\omega$$

For a homogenous isotropic medium, we have $\omega = ck$ with constant c, which gives $\dot{\boldsymbol{k}} = 0$ and $\dot{\boldsymbol{r}} = c\hat{\boldsymbol{n}}$ and the rays are propagated in straight lines.

5.3 Wave Packets

We guessed, by analogizing with Hamilton's equations, that

$$\dot{\boldsymbol{r}} = \frac{\partial \omega}{\partial \boldsymbol{k}}$$

In order to understand where this comes from, or what it is saying, let's consider wave packets. We assume that the spectral composition includes monochromatic components whose frequencies and wave vectors lie only in a small range. Let $\boldsymbol{k}_0$ be some mean wave vector. We can expand ω as

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_0) + \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} (\boldsymbol{k} - \boldsymbol{k}_0)$$

We can then rephrase the eikonal as

$$\begin{aligned}
\psi &= \psi_0 + \boldsymbol{k}\cdot\boldsymbol{r} - \omega t \\
&= \psi_0 + \boldsymbol{k}\cdot\boldsymbol{r} - \left(\omega(\boldsymbol{k}_0) + \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} (\boldsymbol{k} - \boldsymbol{k}_0)\right) t + \{\boldsymbol{k}_0\cdot\boldsymbol{r} - \boldsymbol{k}_0\cdot\boldsymbol{r}\} \\
&= \psi_0 + \boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k}_0)t - \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} (\boldsymbol{k} - \boldsymbol{k}_0)t + \{\boldsymbol{k}_0\cdot\boldsymbol{r} - \boldsymbol{k}_0\cdot\boldsymbol{r}\} \\
&= (\boldsymbol{k}_0\cdot\boldsymbol{r} - \omega(\boldsymbol{k}_0)t) + (\boldsymbol{k} - \boldsymbol{k_0})\cdot(\boldsymbol{r} - \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} t)
\end{aligned}$$

For the perturbative case that $\delta\varrho(\boldsymbol{r}, t) = \varrho_1 e^{i\psi}$,

$$\delta\varrho(\boldsymbol{r}, t) = \varrho_1 e^{i(\boldsymbol{k}-\boldsymbol{k_0})\cdot(\boldsymbol{r} - \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} t)} e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r} - \omega(\boldsymbol{k}_0)t)}$$

The first exponential therefore describes the phase of the amplitude distribution, that is, the wave packet. If we move along $\boldsymbol{r}$ so that the wave packet is stationary, then the phase of the first exponential must be constant. Thus, we have the requirement that

$$d\left((\boldsymbol{k} - \boldsymbol{k_0})\cdot(\boldsymbol{r} - \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} t)\right) = 0$$

$$(\boldsymbol{k} - \boldsymbol{k_0})\cdot d\boldsymbol{r} - (\boldsymbol{k} - \boldsymbol{k_0})\cdot \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0} dt = 0$$

$$\therefore \boxed{\frac{d\boldsymbol{r}}{dt} = \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0}}$$

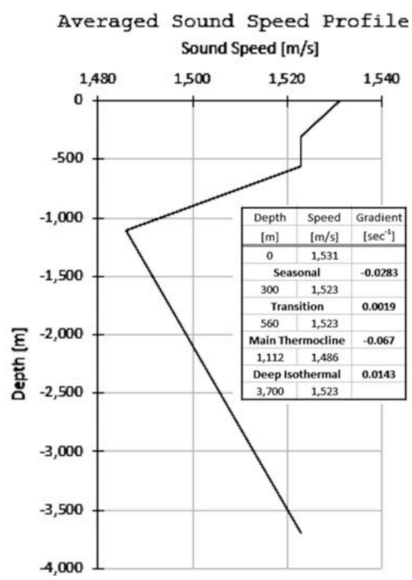
The speed at which we would move for the wave packet to be stationary, $\frac{d\boldsymbol{r}}{dt}$, is thus the speed of the wave packet, and is equal to $\left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0}$.

5.4 The Ocean

How does a sound wave propagate in a non-constant medium? That is, a medium in which:

$$\varrho = \varrho_0(\boldsymbol{r}, t) + \delta\varrho(\boldsymbol{r}, t)$$

As an example, check this out:



Deep-ocean sound speed profiles were taken every 2 weeks over a 9-year period at a location 15 miles SE of Bermuda. At a depth of approximately 3700 m, the sound speed is the same as it is at a depth of approximately 560 m. Between those two depths, sound can be trapped in the deep sound channel in the same way that light is trapped in an optical fiber, except the core of an optical fiber is about 10 m and the sound channel is about 1 km, a scale ratio of about 108. Damn.

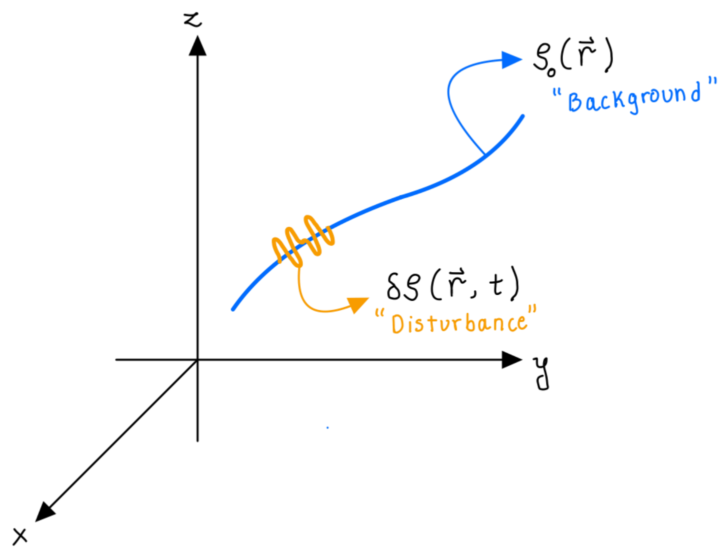
So, our goal is to solve for the motion of rays of sound. We are going to make the assumption that the wavelength is small as hell compared to the ambient density gradient. That is, like in the ocean, the density of water varies only over great distances. So,

$$\lambda \ll \boldsymbol{\nabla}\varrho_0 \ll m \cdot \varrho_0$$

For a first-order perturbation, we can write:

$$\boxed{\varrho = \varrho_0(\boldsymbol{r}, t) + \varrho_1(\boldsymbol{r}, t)e^{i\psi(\boldsymbol{r}, t)}}$$
$$\boxed{\delta\boldsymbol{v} = \boldsymbol{v}_1(\boldsymbol{r}, t)e^{i\psi(\boldsymbol{r}, t)}}$$

Let's look at a short-wavelength wave in a slowly varying medium. The disturbance "rides" the background density as the total fluid density is just the sum of the two.



Ignoring non-linear terms, the continuity equation gives:

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v}) = 0$$

$$\Rightarrow \frac{\partial(\delta\varrho)}{\partial t} + \nabla \cdot (\varrho_0 \, \delta \boldsymbol{v}) = 0$$

and Euler's equation reduces to:

$$\varrho \frac{D\boldsymbol{v}}{Dt} = -\nabla p(\varrho)$$

$$\Rightarrow \varrho \frac{\partial(\delta \boldsymbol{v})}{\partial t} = -\nabla p(\varrho)$$

Since the pressure can be expanded as

$$p(\varrho) = p(\varrho_0) + c_0^2 \cdot \delta\varrho + \dots$$

We can write

$$\frac{\partial(\delta \boldsymbol{v})}{\partial t} = -\nabla \left[\frac{c_0^2(\boldsymbol{r}, t)}{\varrho_0(\boldsymbol{r}, t)} \delta\varrho \right]$$

We know the disturbance is characterized by $\delta \boldsymbol{v} = \boldsymbol{v}_1 e^{i\psi}$ and $\delta\varrho = \varrho_1 e^{i\psi}$. By substituting these into the two equations (Euler and Continuity), we obtain the relations:

$$\omega \boldsymbol{v}_1 = \frac{c_0^2 \varrho_1}{\varrho_0} \boldsymbol{k}$$

$$\omega \varrho_1 = \varrho_0 (\boldsymbol{v}_1 \cdot \boldsymbol{k})$$

which boil down to the dispersion law:

$$\omega^2 = c_0^2 \, k^2$$

$$\frac{1}{c_0^2} \left(\frac{\partial \psi}{\partial t} \right)^2 = \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2$$

Earlier, we saw that

$$\dot{\boldsymbol{r}} = \frac{\partial \omega}{\partial \boldsymbol{k}}$$
$$\dot{\boldsymbol{k}} = -\boldsymbol{\nabla}\omega$$

In a medium in which the speed of sound is slowly varying, we can write $c_0 = c_0(\boldsymbol{r})$. So, we get:

$$\dot{\boldsymbol{k}} = -\boldsymbol{\nabla}\omega = -(k\boldsymbol{\nabla}c_0 + \cancel{c_0\boldsymbol{\nabla}k})$$
$$\therefore \boxed{\dot{\boldsymbol{k}} = -k\boldsymbol{\nabla}c_0}$$

Moreover,

$$\begin{aligned}\dot{\boldsymbol{r}} = \frac{\partial \omega}{\partial \boldsymbol{k}} &= k\cancel{\frac{\partial c_0}{\partial \boldsymbol{k}}} + c_0\frac{\partial k}{\partial \boldsymbol{k}} \\ &= c_0\langle \frac{\partial k}{\partial k_x}, \frac{\partial k}{\partial k_y}, \frac{\partial k}{\partial k_z}\rangle \\ &= c_0\langle \frac{k_x}{k}, \frac{k_y}{k}, \frac{k_z}{k}\rangle \\ \therefore &\boxed{\dot{\boldsymbol{r}} = c_0\hat{\boldsymbol{k}}}\end{aligned}$$

Noting that $\boldsymbol{k} = \frac{\omega}{c_0}\hat{\boldsymbol{k}}$. Differentiating with respect to time,

$$\begin{aligned}\frac{d\boldsymbol{k}}{dt} &= \frac{\omega}{c_0}\frac{d\hat{\boldsymbol{k}}}{dt} - \frac{\omega}{c_0^2}\hat{\boldsymbol{k}}(\frac{dc_0}{dt}) \\ &= \frac{\omega}{c_0}\frac{d\hat{\boldsymbol{k}}}{dt} - \frac{\omega}{c_0^2}\hat{\boldsymbol{k}}(\dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}c_0) = -k\boldsymbol{\nabla}c_0\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\omega}{c_0}\frac{d\hat{\boldsymbol{k}}}{dt} - \frac{\omega}{c_0^2}\hat{\boldsymbol{k}}(\dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}c_0) &= -\frac{\omega}{c_0}\boldsymbol{\nabla}c_0 \\ \frac{d\hat{\boldsymbol{k}}}{dt} &= -\boldsymbol{\nabla}c_0 + \frac{1}{c_0}\hat{\boldsymbol{k}}(\dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}c_0) \\ \frac{d\hat{\boldsymbol{k}}}{dt} &= -\boldsymbol{\nabla}c_0 + \frac{1}{c_0}\hat{\boldsymbol{k}}(c_0\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}c_0) \\ \therefore \frac{d\hat{\boldsymbol{k}}}{dt} &= -\boldsymbol{\nabla}c_0 + \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}c_0)\end{aligned}$$

If the ray moves with speed c_0, we can introduce the length element along the ray as $dl = c_0\,dt$ and rewrite the above expression as:

$$\boxed{\frac{d\hat{\boldsymbol{k}}}{dl} = -\frac{1}{c_0}\boldsymbol{\nabla}c_0 + \frac{1}{c_0}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}c_0)}$$

Theorem. Let I be an interval, and let $\mathcal{F}$ be a vector valued function on I such that $|\mathcal{F}(t)| = \alpha$ for all $t \in I$. Then, $\mathcal{F} \cdot \mathcal{F}' = 0$ on I. That is, $\mathcal{F}'(t)$ is perpendicular to $\mathcal{F}(t)$ for each $t \in I$.

Proof. Let $g(t) = |\mathcal{F}(t)|^2 = \mathcal{F}(t) \cdot \mathcal{F}(t)$. By assumption, g is constant on I, and there fore $g' = 0$ on I. We have that

$$g' = 2\mathcal{F}(t) \cdot \mathcal{F}'(t) = 0$$

Therefore, $\mathcal{F}(t) \cdot \mathcal{F}'(t) = 0$. □

The unit tangent vector $\hat{\boldsymbol{k}}$ has a constant magnitude, 1. By the theorem above, $\frac{d\hat{\boldsymbol{k}}}{dl}$ must be perpendicular to $\hat{\boldsymbol{k}}$. In view of this, we can define a principal normal vector $\hat{\boldsymbol{N}}$ to the curve at any point such that $\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{k}} = 0$

$$\begin{aligned}
\hat{\boldsymbol{N}} &= \frac{\frac{d\hat{\boldsymbol{k}}}{dl}}{\left|\frac{d\hat{\boldsymbol{k}}}{dl}\right|} = \frac{\frac{d\hat{\boldsymbol{k}}}{dl}}{(\frac{1}{R})} \\
\frac{1}{R}\hat{\boldsymbol{N}} &= \frac{d\hat{\boldsymbol{k}}}{dl} \\
\therefore \hat{\boldsymbol{N}} \cdot \frac{d\hat{\boldsymbol{k}}}{dl} &= \frac{1}{R} \\
\rightarrow \hat{\boldsymbol{N}} \cdot \left(-\frac{1}{c_0}\boldsymbol{\nabla} c_0 + \frac{1}{c_0}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} c_0)\right) &= \frac{1}{R} \\
-\frac{1}{c_0}\hat{\boldsymbol{N}} \cdot \boldsymbol{\nabla} c_0 + \frac{1}{c_0}\cancel{\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{k}}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} c_0) &= \frac{1}{R}
\end{aligned}$$

This leaves us with

$$-\frac{1}{c_0}\hat{\boldsymbol{N}} \cdot \boldsymbol{\nabla} c_0 = \frac{1}{R}$$

$$\Rightarrow \boxed{-\frac{1}{c_0}\hat{\boldsymbol{N}} \cdot \boldsymbol{\nabla} c_0 = \frac{\frac{d^2y}{dx^2}}{(1 + (\frac{dy}{dx})^2)^{\frac{3}{2}}}}$$

We can conduct experiments to approximate c_0 as a function of position, and accordingly find $\hat{\boldsymbol{N}}$. This gives us a differential equation for the path $y(x)$ traced out by the ray as it moves through a slowly changing medium.

6 Navier Stokes

6.1 The Real, Difficult World

Continuity of mass:

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0$$

Continuity of entropy:

$$\frac{\partial}{\partial t}(\varrho s) + \boldsymbol{\nabla} \cdot (\varrho s \boldsymbol{v}) = 0$$

Conservation of momentum:

$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial r_j}(p\delta_{ij} + \varrho v_i v_j) = 0$$

In order to incorporate the effects of heat diffusion and friction within the fluid, we are going to introduce three parameters to our equations, and in a special way. The parameters are:

The viscous stress tensor $\rightarrow \tau_{ij}$

Heat flow due to thermal conduction $\rightarrow \boldsymbol{q}$

Entropy production $\rightarrow \Sigma$

We manipulate our equations <u>without affecting the conservation laws</u>:

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0$$
$$\frac{\partial}{\partial t}(\varrho s) + \boldsymbol{\nabla} \cdot \left(\varrho s \boldsymbol{v} + \frac{\boldsymbol{q}}{T}\right) = \Sigma > 0$$
$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial r_j}(p\delta_{ij} + \varrho v_i v_j + \tau_{ij}) = 0$$

and energy conservation is

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{Q}_0 + \boldsymbol{Q}') = 0$$

where $\boldsymbol{Q}'$ is the off-equilibrium energy flow. Don't worry, the spooky-scary will go away after you read the rest of this section.

6.2 The Heat Diffusion Equation

Conservation of energy requires that the energy flowing into a small volume must be compensated by either:

- Energy flowing out of the volume
- A change in total energy
- A combination of both

Let's only consider thermal energy in a small rectangular volume element $dx\,dy\,dz$. The total quantity of heat (in J) flowing *into* a control volume through the face $dy\,dz$ is

$$h_{\text{in}} = (q_x)\,dy\,dz\,dt$$

where q_x is the heat flux ($\mathrm{J\,m^{-2}\,s^{-1}}$). If we Taylor expand q_x from the entrance of the control volume, we get, to first order:

$$(q_x)_{\text{out}} = q_x + \frac{\partial q_x}{\partial x}\,dx$$

So, we can express the quantity of heat flowing *out* of the control volume through the face $dy\,dz$ is

$$h_{\text{out}} = \left(q_x + \frac{\partial q_x}{\partial x}\,dx\right) dy\,dz\,dt$$

The net flow of heat is therefore

$$\begin{aligned} N_x &= h_{\text{out}} - h_{\text{in}} \\ &= \left(q_x + \frac{\partial q_x}{\partial x}\,dx\right) dy\,dz\,dt - (q_x)\,dy\,dz\,dt \\ &= \left(\frac{\partial q_x}{\partial x}\right) dx\,dy\,dz\,dt \end{aligned}$$

So, we find the net flows of heat across all faces of our small volume to be:

$$\begin{aligned} N_x &= \left(\frac{\partial q_x}{\partial x}\right) dx\,dy\,dz\,dt \\ N_y &= \left(\frac{\partial q_y}{\partial y}\right) dx\,dy\,dz\,dt \\ N_z &= \left(\frac{\partial q_z}{\partial z}\right) dx\,dy\,dz\,dt \end{aligned}$$

The sum of these ($N = N_x + N_y + N_z$) is the **total net flow of of heat** out of the control volume. Thus,

$$N = \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dx\,dy\,dz\,dt$$

With our sign convention, if $N > 0$, then heat is lost from the control volume. By energy conservation, this must be compensated by an equivalent decrease of energy within the volume. Since the control volume $dx\,dy\,dz$ is constant over dt, the only way to decrease the energy is to either:

- Decrease the temperature (T)

- Decrease the specific heat (c)

- Decrease the density (ϱ)

or some combination of the three. At a time t, the quantity of heat $h(t)$ within the volume element is

$$h(t) = (\varrho c T)\, dx\, dy\, dz$$

Taylor expanding this with time as the independent variable, the quantity of heat within the volume after a small time dt is

$$h(t + dt) = \left(\varrho c T + \frac{\partial(\varrho c T)}{\partial t}\, dt\right) dx\, dy\, dz$$

. Thus, the change in energy during dt is:

$$\begin{aligned} dh &= h(t + dt) - h(t) \\ &= \left(\varrho c T + \frac{\partial(\varrho c T)}{\partial t}\, dt\right) dx\, dy\, dz - (\varrho c T)\, dx\, dy\, dz \\ \therefore dh &= \left(\frac{\partial(\varrho c T)}{\partial t}\right) dt\, dx\, dy\, dz \end{aligned}$$

By conservation of energy, $N + dh = 0$, so:

$$\begin{aligned} \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + \frac{\partial(\varrho c T)}{\partial t}\right) dt\, dx\, dy\, dz = 0 \\ \therefore \frac{\partial(\varrho c T)}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \end{aligned}$$

Assuming that the density and specific heat do not change over time,

$$\begin{aligned} \frac{\partial T}{\partial t} &= -\frac{1}{\varrho c}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \\ \therefore \frac{\partial T}{\partial t} &= -\frac{1}{\varrho c}\boldsymbol{\nabla} \cdot \boldsymbol{q} \end{aligned}$$

Great, but what to do about the heat flux? Well, Fourier's law is a constitutive relation (one specific to materials, derived from experiments) that says:

$$\boldsymbol{q} = -k\boldsymbol{\nabla} T$$

Thus, if we assume that k

$$\begin{aligned} \frac{\partial T}{\partial t} &= \boldsymbol{\nabla} \cdot \left(\frac{k}{\varrho c}\boldsymbol{\nabla} T\right) \\ &= \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T) \end{aligned}$$

where k is the *thermal conductivity* (in $\mathrm{J\,m^{-1}}$), and $\kappa = \frac{k}{\varrho c}$ is the *thermal diffusivity* (in $\mathrm{m^2 s^{-1}}$). If the thermal diffusivity does not vary in space, this simplifies to

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

6.3 Modifying Momentum Conservation

We obtained the equation for conservation of momentum by multiplying the mass continuity equation by $\boldsymbol{v}$ and adding Euler's equation:

$$\frac{\partial}{\partial t}(\varrho \boldsymbol{v}) + \boldsymbol{\nabla} p + [\, \varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}))\,] = 0$$

and we found that it could more conveniently be expressed as

$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial r_j}(p\delta_{ij} + \varrho v_i v_j) = 0$$

We denote the quantity on the right as

$$\Pi_{ij} = p\delta_{ij} + \varrho v_i v_j$$

So, we have that $\dfrac{\partial}{\partial t}(\varrho v_i) + \dfrac{\partial \Pi_{ij}}{\partial r_j} = 0$. Integrating over a volume,

$$\begin{aligned} \frac{\partial}{\partial t}\int \varrho v_i \, dV &= -\int \frac{\partial \Pi_{ij}}{\partial r_j}\, dV \\ \frac{\partial}{\partial t}\int \varrho v_i \, dV &= -\oint \Pi_{ij}\, df_j \end{aligned}$$

The LHS is the time rate of change of the i^{th} component of momentum contained in the volume. The RHS is the amount of momentum flowing out of the bounding surface in unit time. Thus, Π_{ij} is called the momentum flux density tensor. So, we can say that Π_{ij} represents the transfer of momentum due to

- The *mechanical transport* of different fluid particles.
- The *pressure forces* acting in the fluid.

For a more complete theory, we must try to incorporate the effects of energy dissipation due to the thermodynamic irreversibility of motion due to internal friction, i.e. viscosity and thermal conduction.

The viscosity introduces an irreversible transfer of momentum from points where the velocity is large to where it is small. So, we can add a term to Π_{ij} which represents an irreversible, *viscous* transfer of momentum within the fluid. Thus, denoting this extra term as σ'_{ij}, we can write

$$\Pi_{ij} = -\underbrace{(\sigma'_{ij} - p\delta_{ij})}_{\sigma_{ij}} + \varrho v_i v_j$$

We call $\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$ the *stress tensor*, and σ'_{ij} the *viscous stress tensor*. By making this distinction, we can see that σ_{ij} is the part of momentum flux that is *not* due to the mechanical transport of fluid particles.

In order to figure out the general form of the viscous stress tensor σ'_{ij}, we must lay down the constraints on it.

- By assumption, we have that processes of internal friction occur only when different fluid particles move with different velocities. Thus, viscous momentum transfer must depend on the space derivatives of the fluid velocity.

- For small velocity gradients, we may suppose that this dependence only entails the first space derivatives of velocity (first order terms only).

- We know that if $\boldsymbol{v} = \text{const}$, then ${\sigma'}_{ij}$ must vanish. So, there can be no terms that are independent of $\dfrac{\partial v_i}{\partial r_j}$.

- We know that if the fluid is in uniform rotation, such that $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$, then once again ${\sigma'}_{ij}$ must vanish. So, $\boldsymbol{v}$ must be of the form $\begin{bmatrix} \omega_2 r_3 - \omega_3 r_2 \\ \omega_3 r_1 - \omega_1 r_3 \\ \omega_1 r_2 - \omega_2 r_1 \end{bmatrix}$. If we compute the set of possible space derivatives of velocity, we have

$$\begin{array}{ccc} \frac{\partial v_1}{\partial r_1} = 0 & \frac{\partial v_1}{\partial r_2} = -\omega_3 & \frac{\partial v_1}{\partial r_3} = \omega_2 \\ \\ \frac{\partial v_2}{\partial r_1} = \omega_3 & \frac{\partial v_2}{\partial r_2} = 0 & \frac{\partial v_2}{\partial r_3} = -\omega_1 \\ \\ \frac{\partial v_1}{\partial r_1} = -\omega_2 & \frac{\partial v_1}{\partial r_2} = \omega_1 & \frac{\partial v_1}{\partial r_3} = 0 \end{array}$$

  So, we see that sums of the form $\dfrac{\partial v_i}{\partial r_j} + \dfrac{\partial v_j}{\partial r_i}$ vanish when $\boldsymbol{v} = \Omega \times \boldsymbol{r}$. Moreover, we have that $\boldsymbol{\nabla} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) = \dfrac{\partial v_l}{\partial r_l} = 0$ by observing the components of $\boldsymbol{v}$, so we may also include linear combinations of the divergence of $\boldsymbol{v}$ to the viscous stress tensor.

With all of this in mind, we can write down the most general form of σ'_{ij} as

$$\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial r_l} \right) + \zeta\delta_{ij}\frac{\partial v_l}{\partial r_l}$$

This looks like

$$\begin{bmatrix} 2\eta\frac{\partial v_1}{\partial r_1} + (\zeta - \frac{2}{3}\eta)\boldsymbol{\nabla}\cdot\boldsymbol{v} & \eta\left(\frac{\partial v_1}{\partial r_2} + \frac{\partial v_2}{\partial r_1}\right) & \eta\left(\frac{\partial v_1}{\partial r_3} + \frac{\partial v_3}{\partial r_1}\right) \\ \eta\left(\frac{\partial v_2}{\partial r_1} + \frac{\partial v_1}{\partial r_2}\right) & 2\eta\frac{\partial v_2}{\partial r_2} + (\zeta - \frac{2}{3}\eta)\boldsymbol{\nabla}\cdot\boldsymbol{v} & \eta\left(\frac{\partial v_2}{\partial r_3} + \frac{\partial v_3}{\partial r_2}\right) \\ \eta\left(\frac{\partial v_3}{\partial r_1} + \frac{\partial v_1}{\partial r_3}\right) & \eta\left(\frac{\partial v_3}{\partial r_2} + \frac{\partial v_2}{\partial r_3}\right) & 2\eta\frac{\partial v_3}{\partial r_3} + (\zeta - \frac{2}{3}\eta)\boldsymbol{\nabla}\cdot\boldsymbol{v} \end{bmatrix}$$

and as you can see, it is symmetric. The scalars η and ζ are called the coefficients of viscosity. ζ is called the second viscosity, and depends on the compressibility of the fluid.

Hence, Euler's equation can be written as

$$\begin{aligned}
\varrho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial r_j}\right) &= -\frac{\partial p}{\partial r_i}+\frac{\partial}{\partial r_j}\left(\eta\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial r_l}\right)+\zeta\delta_{ij}\frac{\partial v_l}{\partial r_l}\right)\\
&= -\frac{\partial p}{\partial r_i}+\frac{\partial}{\partial r_j}\left(\eta\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial r_l}\right)\right)+\frac{\partial}{\partial r_j}\left(\zeta\delta_{ij}\frac{\partial v_l}{\partial r_l}\right)\\
&= -\frac{\partial p}{\partial r_i}+\eta\left(\frac{\partial^2 v_i}{\partial r_j^2}+\frac{\partial^2 v_j}{\partial r_j\partial r_i}-\frac{2}{3}\frac{\partial}{\partial r_i}\left(\frac{\partial v_l}{\partial r_l}\right)\right)+\zeta\frac{\partial}{\partial r_i}\left(\frac{\partial v_l}{\partial r_l}\right)\\
&= -\frac{\partial p}{\partial r_i}+\eta\left(\frac{\partial^2 v_i}{\partial r_j^2}\right)+(\zeta-\frac{2}{3}\eta)\frac{\partial}{\partial r_i}\left(\frac{\partial v_l}{\partial r_l}\right)+\eta\left(\frac{\partial^2 v_j}{\partial r_i\partial r_j}\right)\\
\therefore\varrho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial r_j}\right) &= -\frac{\partial p}{\partial r_i}+\eta\left(\frac{\partial^2 v_i}{\partial r_j^2}\right)+(\zeta+\frac{1}{3}\eta)\frac{\partial}{\partial r_i}\left(\frac{\partial v_l}{\partial r_l}\right)
\end{aligned}$$

We have regarded the viscosity coefficients to remain constant throughout the fluid, so they may be taken outside the gradient operators. In general, however, η and ζ are functions of pressure and temperature. Since these are not constant, the coefficients of viscosity are not constant, but screw that.

The above result may be written in vector notation as

$$\boxed{\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right)=-\boldsymbol{\nabla}p+\eta\,\nabla^2\boldsymbol{v}+(\zeta+\frac{1}{3}\eta)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})}$$

This is the *Navier-Stokes Equation.*

6.4 Incompressible Limit

When we have that $\varrho = \text{const}$, viscous stress tensor becomes

$$\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right)$$

and the Navier-Stokes equation reduces to

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} + \frac{\eta}{\varrho}\nabla^2\boldsymbol{v}$$

The coefficient of viscosity η is called the *dynamic viscosity*, and the quantity $\nu = \dfrac{\eta}{\varrho}$ is called the *kinematic viscosity*. The dynamic viscosity (η) of a gas at a fixed temperature is independent of pressure. The kinematic viscosity (ν), however, is inversely proportional to the pressure.

6.4.1 Energy Dissipation in an Incompressible Fluid

The time derivative of kinetic energy within some fixed volume of fluid is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho\,\boldsymbol{v}\cdot\boldsymbol{v}\right) = \varrho\,\boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial t} = \varrho\, v_i\frac{\partial v_i}{\partial t}$$

From Navier-Stokes, we know that

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial r_j} = -\frac{1}{\varrho}\frac{\partial p}{\partial r_i} + \frac{1}{\varrho}\frac{\partial \sigma'_{ij}}{\partial r_j}$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho\, v^2\right) &= \varrho v_i\left(-v_j\frac{\partial v_i}{\partial r_j} - \frac{1}{\varrho}\frac{\partial p}{\partial r_i} + \frac{1}{\varrho}\frac{\partial \sigma'_{ij}}{\partial r_j}\right)\\
&= \varrho v_i\left(-v_j\frac{\partial v_i}{\partial r_j} - \frac{1}{\varrho}\frac{\partial p}{\partial r_i}\right) + v_i\frac{\partial \sigma'_{ij}}{\partial r_j}\\
&= -\varrho v_j v_i\frac{\partial v_i}{\partial r_j} - \varrho v_i\frac{1}{\varrho}\frac{\partial p}{\partial r_i} + \frac{\partial}{\partial r_j}(v_i\sigma'_{ij}) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\
&= -\varrho v_j\frac{\partial}{\partial r_j}\left(\frac{1}{2}v^2\right) - \varrho v_i\frac{1}{\varrho}\frac{\partial p}{\partial r_i} + \frac{\partial}{\partial r_j}(v_i\sigma'_{ij}) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\
&= -\varrho v_j\frac{\partial}{\partial r_j}\left(\frac{1}{2}v^2 + \frac{p}{\varrho}\right) + \frac{\partial}{\partial r_j}(v_i\sigma'_{ij}) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}
\end{aligned}$$

Since $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ in the incompressible limit, we can express the first term as a divergence,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho\, v^2\right) &= -\frac{\partial}{\partial r_j}\left(v_j(\frac{1}{2}v^2 + \frac{p}{\varrho})\right) + \frac{\partial}{\partial r_j}(v_i\sigma'_{ij}) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\
&= -\frac{\partial}{\partial r_j}\left(v_j(\frac{1}{2}v^2 + \frac{p}{\varrho}) - v_i\sigma'_{ij}\right) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\
\therefore \frac{\partial}{\partial t}\left(\frac{1}{2}\varrho\, v^2\right) &= -\boldsymbol{\nabla}\cdot\left[\varrho\boldsymbol{v}\left(\frac{1}{2}v^2 + \frac{p}{\varrho}\right) - \boldsymbol{v}\cdot\boldsymbol{\sigma}'\right] - \sigma'_{ij}\frac{\partial v_i}{\partial r_j}
\end{aligned}$$

The term $\varrho\boldsymbol{v}\left(\frac{1}{2}v^2+\frac{p}{\varrho}\right)$ is the energy flux due to the actual transfer of fluid mass. It is equal to the energy flux of an ideal fluid, as we have seen before. The term $\boldsymbol{v}\cdot\boldsymbol{\sigma}'$ is the energy flux due to internal friction. We saw that the presence of viscosity in the fluid introduces a momentum flux σ'_{ij}. A transfer of momentum always involves a transfer of energy.

If we integrate this over a volume, we get an expression for the rate of change of kinetic energy in that volume:

$$\frac{\partial}{\partial t}\int\frac{1}{2}\varrho\,v^2\;dV=-\int\boldsymbol{\nabla}\cdot\left[\varrho\boldsymbol{v}\left(\frac{1}{2}v^2+\frac{p}{\varrho}\right)-\boldsymbol{v}\cdot\boldsymbol{\sigma}'\right]dV-\int\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\;dV$$

$$\frac{\partial}{\partial t}\int\frac{1}{2}\varrho\,v^2\;dV=-\oint\left[\varrho\boldsymbol{v}\left(\frac{1}{2}v^2+\frac{p}{\varrho}\right)-\boldsymbol{v}\cdot\boldsymbol{\sigma}'\right]\cdot d\boldsymbol{a}-\int\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\;dV$$

The surface integral gives the rate of change of kinetic energy of the fluid in the volume due to the energy flux through the surface bounding the volume. The volume integral on the right denotes the decrease in the kinetic energy as a result of dissipation (per unit time). If we integrate over the entire fluid, such that the velocity vanishes at all boundaries (i.e. a fluid in a container, or an infinite volume), then the surface integral vanishes. Thus,

$$\dot{E}_{\text{kinetic}}=-\int\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\;dV=-\frac{1}{2}\int\sigma'_{ij}\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}\right)dV$$

since σ'_{ij} is symmetric. In the incompressible limit, we know that $\sigma'_{ij}=\eta\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}\right)$. Hence,

$$\dot{E}_{\text{kinetic}}=-\frac{1}{2}\eta\int\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}\right)^2 dV$$

Since the dissipation always leads to a decrease in the mechanical energy of the fluid, it must be that $\dot{E}_{\text{kinetic}}<0$. So, we have shown that for an incompressible fluid, since the integral is always positive, it must be true that viscosity (η) is positive.

6.5 Boundary Conditions

Between a viscous fluid and a solid surface, there will be forces of molecular attraction. The result is that the layer of fluid immediately adjacent to the surface is brought to a complete rest, and *adheres* to the solid surface.

Therefore, at fixed solid surfaces, the normal *and* tangential components of fluid velocity must vanish:

$$\boldsymbol{v} = 0$$

In an ideal fluid, as we've discussed before, only the normal component of the fluid velocity vanishes at the boundary.

For a moving surface, the velocity of the fluid at the boundary must be equal to the velocity of the boundary, so

$$\boldsymbol{v} = \boldsymbol{U}$$

At a solid surface, since $\boldsymbol{v} = 0$, the momentum flux through the surface element $d\boldsymbol{f}$ is

$$\begin{aligned}
\Pi_{ij}\, df_j &= (\cancel{\varrho v_i v_j} - \sigma'_{ij} + p\delta_{ij})\, df_j \\
&= (-\sigma'_{ij} + p\delta_{ij}) n_j\, df \\
&= ((-\sigma'_{ij}) n_j + (p\delta_{ij}) n_j)\, df \\
&= (-\sigma'_{ij} n_j + p n_i)\, df
\end{aligned}$$

Therefore, the force acting on unit surface area is $P_i = pn_i - \sigma'_{ij} n_j$. The first term, pn_i is the usual pressure of the fluid. The second term, $-\sigma'_{ij} n_j$ is the force of friction due to viscosity that acts on the surface. Note that $\boldsymbol{n}$ is along the inward normal to the boundary.

At a surface of separation between two immiscible fluids, the velocities of the fluids at the surface of separation must be equal. Moreover, the forces they exert on each other must be equal and opposite:

$$\sigma_{1,ij}\, n_{1,j} + \sigma_{2,ij}\, n_{2,j} = 0$$

So, at a free surface, it must be that

$$\begin{aligned}
\sigma_{ij}\, n_j &= 0 \\
{\sigma'}_{ij}\, n_j - pn_i &= 0
\end{aligned}$$

6.6 Poiseuille Flow

Let's look at the case of steady, incompressible, viscous flow through a pipe of constant, arbitrary cross sectional area. The fluid velocity is along the z axis at any point. Evidently, the fluid velocity is only a function of the x and y coordinates. For a pipe with circular cross section, we write $\boldsymbol{v} = v(r)\hat{\boldsymbol{z}}$. Navier-Stokes equation is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\varrho} + \frac{\eta}{\varrho}\nabla^2 \boldsymbol{v}$$

We are looking for steady flow, so $\dfrac{\partial \boldsymbol{v}}{\partial t} = 0$. Moreover, our constraint on the form of the fluid velocity results in:

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = v(r)\frac{\partial v(r)}{\partial z}\hat{\boldsymbol{z}} = 0$$

Therefore, Navier Stokes is reduced to

$$\begin{aligned}
\boldsymbol{\nabla} p &= \eta \nabla^2 \boldsymbol{v} \\
\frac{dp}{dz} &= \eta \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{\Delta p}{\Delta z} = \text{const} \\
\implies \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) &= \left(\frac{\Delta p}{\eta L}\right) \\
\frac{d}{dr}\left(r\frac{dv}{dr}\right) &= \left(\frac{\Delta p}{\eta L}\right) r \\
r\frac{dv}{dr} &= \left(\frac{\Delta p}{2\eta L}\right) r^2 + c_1 \\
\frac{dv}{dr} &= \left(\frac{\Delta p}{2\eta L}\right) r + \frac{c_1}{r} \\
\therefore v &= \left(\frac{\Delta p}{4\eta L}\right) r^2 + c_1 \ln r + c_2
\end{aligned}$$

Since the velocity at $r = 0$ must be finite, we must set $c_1 = 0$.
At $r = R$, we have that $v = 0$, so

$$c_2 = -\left(\frac{\Delta p}{4\eta L}\right) R^2$$

Hence, our solution is

$$v = \left(\frac{\Delta p}{4\eta L}\right)(r^2 - R^2)$$

6.7 Couette Flow

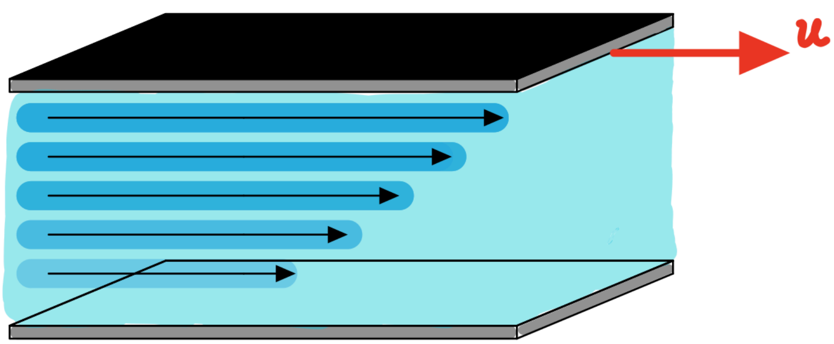
6.7.1 Between Two Planes

Suppose there is an incompressible, viscous fluid enclosed between two parallel plates. The top plate is moving to the right at a velocity $\boldsymbol{u} = u\hat{\boldsymbol{x}}$ relative to the bottom plate. The fluid "sticks" to the top plate, so the boundary condition there is that $\boldsymbol{v} = 0$. We are going to suppose that the fluid velocity depends only on the depth (y), and is in the x direction everywhere. Hence, $\boldsymbol{v} = v(y)\hat{\boldsymbol{x}}$, and $p = p(y)$. This implies that $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0$. Moreover, we are considering steady flow, so $\dfrac{\partial \boldsymbol{v}}{\partial t} = 0$. Navier Stokes becomes

$$\begin{aligned}\boldsymbol{\nabla} p &= \eta \nabla^2 \boldsymbol{v} \\ \langle 0, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \rangle &= \eta \langle \frac{\partial^2 v}{\partial y^2}, 0, 0 \rangle \\ &\implies p = \text{const} \\ &\implies v = ay + b\end{aligned}$$

We know that $v(y = 0) = 0$ and $v(y = h) = u$. So, $b = 0$ and $a = \dfrac{u}{h}$. Therefore,

$$v = \left(\frac{u}{h}\right) y$$



Now suppose that there is a pressure gradient, i.e. $\dfrac{\partial p}{\partial x} \neq 0$. This time, we have

$$\frac{1}{\eta}\frac{\partial p}{\partial x} = \frac{\partial^2 v}{\partial y^2}$$

Both sides must be constant. Thus,

$$v = \frac{1}{2\eta}\frac{dp}{dx} y^2 + cy + d$$

With the same boundary conditions as before we can obtain the constants c and d. So, we get

$$\boldsymbol{v} = v(y)\hat{\boldsymbol{x}} = -\left(\frac{1}{2\eta}\frac{dp}{dx}\right) y(y - h)\hat{\boldsymbol{x}}$$

6.7.2 Between Two Cylinders

Consider the steady, viscous flow of a fluid between two infinite coaxial cylinders with radii R_1, R_2, rotating about their axis with angular velocities Ω_1, Ω_2. In cylindrical polar coordinates, we have the following restrictions on our variables:

$$\begin{aligned} v_r = v_z &= 0 \\ \boldsymbol{v} &= v(r)\hat{\boldsymbol{\phi}} \\ p &= p(r) \end{aligned}$$

Moreover, $\dfrac{\partial \boldsymbol{v}}{\partial t} = 0$, and $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0$. In polar coordinates, Navier Stokes for this flow is of the form

$$\begin{aligned} \boldsymbol{\nabla} p &= \eta \, \nabla^2 \boldsymbol{v} \\ \frac{dp}{dr}\hat{\boldsymbol{r}} + 0\hat{\boldsymbol{\phi}} + 0\hat{\boldsymbol{z}} &= 0\hat{\boldsymbol{r}} + \eta\left(\nabla^2 v - \frac{v}{r^2}\right)\hat{\boldsymbol{\phi}} + 0\hat{\boldsymbol{z}} \\ \implies p &= p_0 \\ \implies \frac{d^2 v}{dr^2} &+ r^{-1}\frac{dv}{dr} + r^{-2} v = 0 \end{aligned}$$

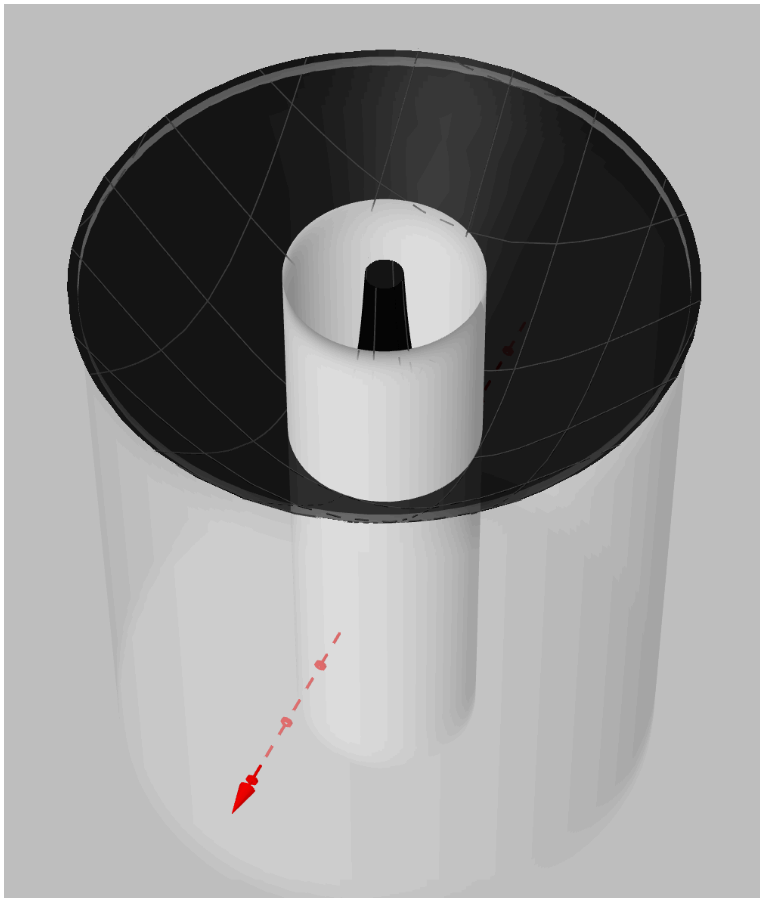
Hence, $v(r) = Ar + Br^{-1}$ and the boundary conditions give

$$\begin{aligned} v(r = R_1) \implies A + \frac{B}{R_1^2} &= R_1\Omega_1 \\ v(r = R_2) \implies A + \frac{B}{R_2^2} &= R_2\Omega_2 \\ \implies A &= \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \\ B &= \frac{(\Omega_1 - \Omega_2)(R_1 R_2)^2}{R_2^2 - R_1^2} \end{aligned}$$

Therefore,

$$v(r) = \left(\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\right) r + \left(\frac{(\Omega_1 - \Omega_2)(R_1 R_2)^2}{R_2^2 - R_1^2}\right)\frac{1}{r}$$

.



The azimuthal velocity between the cylinders is shown above in a scalar plot. In the limit that $\Omega_2 = 0$, $R_2 \to \infty$, notice that $v(r) = \dfrac{\Omega_1 R_1^2}{r}$.

6.8 Shear Waves

Consider an incompressible, viscous fluid about an infinite plate oscillating with a velocity $u_0 e^{-i\omega t}\hat{\boldsymbol{x}}$. The boundary condition at the plate is that the fluid velocity will vanish, so that $\boldsymbol{v}(z=0,t)=0$. That is, fluid sticks to the plate. This will generate shear waves driven by viscosity. We will look for a solution $\boldsymbol{v}=v(z,t)\hat{\boldsymbol{x}}$. Moreover, there is no pressure gradient in x direction, and $p=p_0$, so Navier-Stokes becomes

$$\frac{\partial v}{\partial t}=\frac{\eta}{\varrho}\frac{\partial^2 v}{\partial x^2}$$

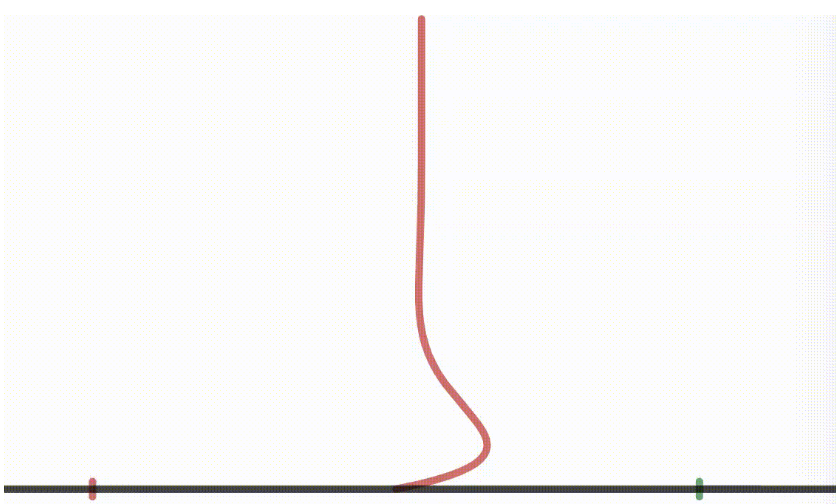
We look for a solution of the form $v(z,t)=v'e^{-i\omega t+ikz}$

$$\begin{aligned}
i\omega &= \frac{\eta k^2}{\varrho}\\
k &= \sqrt{i\frac{\omega}{\eta/\varrho}}=(1+i)\sqrt{\frac{\omega}{2\eta/\varrho}}\\
k &= (1+i)\frac{1}{\sqrt{\frac{2\eta/\varrho}{\omega}}}\\
\implies k &= (1+i)\frac{1}{\delta}
\end{aligned}$$

where $\delta=\sqrt{\dfrac{2\eta/\varrho}{\omega}}$ is called the viscous penetration depth. It is the region over which you feel the effect of the boundary, and is sometimes called the boundary layer. Using the boundary condition at the plate, we find that

$$\begin{aligned}
v(z,t) &= u_0 e^{-i\omega t+ikz}\\
&= u_0 e^{-i\omega t}e^{ikz}\\
\therefore v(z,t) &= u_0 e^{i(-\omega t+\frac{1}{\delta}z)}e^{-\frac{1}{\delta}z}
\end{aligned}$$

We can see that the effect of vibrating the plate diminishes as $e^{-\frac{1}{\delta}z}$ as you move out into the liquid.



6.9 The Wine Cellar Problem

Setting up the equations

We ought to write down a statement for conservation of energy when there is viscosity and thermal conduction in the fluid. Consider the kinetic and internal energy per unit volume of the fluid, just as we did in section 2.8. The time rate of change is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) = \frac{1}{2}v^2\frac{\partial\varrho}{\partial t} + \varrho\boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial t} + \varrho\frac{\partial\varepsilon}{\partial t} + \varepsilon\frac{\partial\varrho}{\partial t}$$

We substitute expressions for $\dfrac{\partial\varrho}{\partial t}$ and $\dfrac{\partial\boldsymbol{v}}{\partial t}$ using the continuity equation and Navier-Stokes respectively:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) =& -\frac{1}{2}v^2\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\\
&+\varrho\boldsymbol{v}\cdot\left(-\frac{\boldsymbol{\nabla}p}{\varrho} + \frac{1}{\varrho}\frac{\partial\sigma'_{ij}}{\partial r_j} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right)\\
&+\varrho\frac{\partial\varepsilon}{\partial t}\\
&-\varepsilon\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\\
\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) =& -\frac{1}{2}v^2\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\\
&-\boldsymbol{v}\cdot\boldsymbol{\nabla}p + \boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j} + \varrho\boldsymbol{v}\cdot\left(-\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) + \cancel{\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v})}^{\text{fu@k this!}}\right)\\
&+\varrho\frac{\partial\varepsilon}{\partial t}\\
&-\varepsilon\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\\
\therefore\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) =& -\frac{1}{2}v^2\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v}) - \boldsymbol{v}\cdot\boldsymbol{\nabla}p + \boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j} - \varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) - \varepsilon\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\\
&+\varrho\frac{\partial\varepsilon}{\partial t}
\end{aligned}$$

Expression for $\dfrac{\partial\varepsilon}{\partial t}$, coming right up. We have the thermodynamic relation:

$$\begin{aligned}
d\varepsilon &= T\,ds - p\,dV\\
&= T\,ds + \frac{p}{\varrho^2}\,d\varrho
\end{aligned}$$

$$\begin{aligned}
\implies\quad \frac{\partial\varepsilon}{\partial t} &= T\frac{\partial s}{\partial t} + \frac{p}{\varrho^2}\frac{\partial\varrho}{\partial t}\\
\therefore\frac{\partial\varepsilon}{\partial t} &= T\frac{\partial s}{\partial t} - \frac{p}{\varrho^2}\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})
\end{aligned}$$

Substitute this into the expression we were working on:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)=-\frac{1}{2}v^2\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})-\varrho\boldsymbol{v}\cdot\frac{\boldsymbol{\nabla}p}{\varrho}+\boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j}-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)-\varepsilon\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})$$
$$+\varrho T\frac{\partial s}{\partial t}-\frac{p}{\varrho}\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})$$
$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)=-\left(\frac{1}{2}v^2+\frac{p}{\varrho}+\varepsilon\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)-\boldsymbol{v}\cdot\boldsymbol{\nabla}p+\varrho T\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j}$$

Also, the heat function $w=\varepsilon+\dfrac{p}{\varrho}$ can be used to clean this up a tiny bit:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)=-\left(\frac{1}{2}v^2+w\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)-\boldsymbol{v}\cdot\boldsymbol{\nabla}p+\varrho T\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j}$$

Moreover, since $dw=T\,ds+\dfrac{1}{\varrho}\,dp$, we have that $\boldsymbol{\nabla}p=\varrho\boldsymbol{\nabla}w-\varrho T\boldsymbol{\nabla}s$. And so we continue,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)=-\left(\frac{1}{2}v^2+w\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}w+\varrho\boldsymbol{v}\cdot T\boldsymbol{\nabla}s+\varrho T\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j}$$
$$=-\left(\frac{1}{2}v^2+w\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)-\varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}w+\varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s+\frac{\partial s}{\partial t})+\boldsymbol{v}\cdot\frac{\partial\sigma'_{ij}}{\partial r_j}$$

The last term, $v_i\dfrac{\partial\sigma'_{ij}}{\partial r_j}$, can be rewritten (using the chain rule) as

$$\begin{aligned}v_i\frac{\partial\sigma'_{ij}}{\partial r_j}&=\frac{\partial(v_i\sigma'_{ij})}{\partial r_j}-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\&=\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\sigma}')-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\end{aligned}$$

So,

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)\\&=-\left[\left(\frac{1}{2}v^2+w\right)\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})+(\varrho\boldsymbol{v})\cdot\left(\boldsymbol{\nabla}\left(\frac{1}{2}v^2\right)+\boldsymbol{\nabla}w\right)\right]+\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\sigma}')+\varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s+\frac{\partial s}{\partial t})-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\&=-\boldsymbol{\nabla}\cdot\left[(\varrho\boldsymbol{v})\left(\frac{1}{2}v^2+w\right)\right]+\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\sigma}')+\varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s+\frac{\partial s}{\partial t})-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\\&=-\boldsymbol{\nabla}\cdot\left[(\varrho\boldsymbol{v})\left(\frac{1}{2}v^2+w\right)-(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\right]+\varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s+\frac{\partial s}{\partial t})-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}\end{aligned}$$

Adding and subtracting $\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$, we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2+\varrho\varepsilon\right)=-\boldsymbol{\nabla}\cdot\left[(\varrho\boldsymbol{v})\left(\frac{1}{2}v^2+w\right)-(\boldsymbol{v}\cdot\boldsymbol{\sigma}')-\kappa\boldsymbol{\nabla}T\right]+\varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s+\frac{\partial s}{\partial t})-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}-\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$$

Therefore,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right) + \boldsymbol{\nabla}\cdot\left[(\varrho\boldsymbol{v})\left(\frac{1}{2}v^2 + w\right) - (\boldsymbol{v}\cdot\boldsymbol{\sigma}') - \kappa\boldsymbol{\nabla}T\right] = \varrho T(\boldsymbol{v}\cdot\boldsymbol{\nabla}s + \frac{\partial s}{\partial t}) - \sigma'_{ij}\frac{\partial v_i}{\partial r_j} - \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$$

The total energy flux in a fluid when there is viscosity and thermal conductivity is given by the divergence term on the left. If energy is to be conserved, the stuff on the right must be zero. So,

$$\boxed{\varrho T(\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}s) = \sigma'_{ij}\frac{\partial v_i}{\partial r_j} + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)}$$

This is called the *general equation of heat transfer.* $\frac{Ds}{Dt}$ denotes the rate of change of entropy of unit mass of the fluid. Therefore, $\varrho T\frac{Ds}{Dt}$ denotes the quantity of heat gained per unit volume! So, $\sigma'_{ij}\frac{\partial v_i}{\partial r_j}$ represents the energy dissipated into heat by viscosity, and $\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$ represents the heat conducted *into* the volume.

Notice that if there is no temperature gradient ($\boldsymbol{\nabla}T = 0$), the heat flux density due to conduction ($\boldsymbol{q} = \kappa\boldsymbol{\nabla}T$) is zero. If there is no viscosity, then $\sigma'_{ij} = 0$. With this in mind, this equation becomes the equation for conservation of entropy of an ideal fluid ($\frac{Ds}{Dt} = 0$).

Using math that I am too tired to type out, the heat transfer equation can be written in terms of coefficients of viscosities

$$\boxed{\varrho T\left(\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}s\right) = \frac{1}{2}\eta\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\frac{\partial v_k}{\partial r_k}\right)^2 + \zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)}$$

The chain rule blessed us with

$$\varrho\frac{\partial s}{\partial t} = \frac{\partial(\varrho s)}{\partial t} - s\frac{\partial\varrho}{\partial t}$$
$$(\varrho\boldsymbol{v})\cdot\boldsymbol{\nabla}s = \boldsymbol{\nabla}\cdot(\varrho s\boldsymbol{v}) - s\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})$$

and the equation of continuity tells us what to do with that blessing: more algebra.

$$\varrho\frac{\partial s}{\partial t} + \varrho\boldsymbol{v}\cdot\boldsymbol{\nabla}s = \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\frac{\partial v_k}{\partial r_k}\right)^2 + \frac{\zeta}{T}(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{1}{T}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$$
$$\Longrightarrow \frac{\partial(\varrho s)}{\partial t} - s\frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho s\boldsymbol{v}) - s\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v}) = \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\frac{\partial v_k}{\partial r_k}\right)^2 + \frac{\zeta}{T}(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{1}{T}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$$
$$\Longrightarrow \frac{\partial(\varrho s)}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho s\boldsymbol{v}) - s\cancel{\left(\frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho\boldsymbol{v})\right)} = \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\frac{\partial v_k}{\partial r_k}\right)^2 + \frac{\zeta}{T}(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{1}{T}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)$$

Thus, we have an equation of discontinuity for entropy (not sure if it's that's what it's called, but it is what it is):

$$\boxed{T\left(\frac{\partial(\varrho s)}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho s\boldsymbol{v})\right) = \frac{\eta}{2}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\frac{\partial v_k}{\partial r_k}\right)^2 + \zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)}$$

Wine Cellar Problem

For an incompressible fluid, assuming constant pressure, we have

$$\frac{\partial s}{\partial t} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\partial T}{\partial t} = \left(\frac{c_p}{T}\right)\frac{\partial T}{\partial t}$$
$$\implies T\frac{\partial s}{\partial t} = c_p\frac{\partial T}{\partial t}$$

$$\boldsymbol{\nabla} s = \left(\frac{\partial s}{\partial T}\right)_p \boldsymbol{\nabla} T$$
$$\implies T\boldsymbol{\nabla} s = c_p\boldsymbol{\nabla} T$$

The general equation of heat transfer is then

$$\varrho T(\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} s) = \sigma'_{ij}\frac{\partial v_i}{\partial r_j} + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla} T)$$
$$T\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot(T\,\boldsymbol{\nabla} s) = \frac{1}{\varrho}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla} T) + \frac{1}{\varrho}\sigma'_{ij}\frac{\partial v_i}{\partial r_j}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} T = \frac{1}{c_p\varrho}\left(\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla} T) + \sigma'_{ij}\frac{\partial v_i}{\partial r_j}\right)$$

If we assume the Earth is an incompressible fluid with $\boldsymbol{v} = 0$ and uniform thermal conductivity, this becomes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_p\varrho}\nabla^2 T$$
$$\boxed{\frac{\partial T}{\partial t} = \chi\nabla^2 T}$$

which is the diffusion equation!
Suppose the temperature at the surface of the earth is

$$T(z=0) = T_0 + \Delta T e^{-i\omega t}$$

We can guess a solution of the form $T(z,t) = T_0 + \Delta T e^{i(kz-\omega t)}$, since it satisfies the condition at the surface. The dispersion law is

$$\frac{\partial}{\partial t}\big(T_0 + \Delta T e^{i(kz-\omega t)}\big) = \chi\nabla^2\big(T_0 + \Delta T e^{i(kz-\omega t)}\big)$$
$$\implies -i\omega = \chi k^2$$
$$\implies k = \sqrt{\frac{\omega}{2\chi}}(1-i)$$

As we had a viscous penetration depth with shear waves, we now have a thermal penetration depth, $\delta_T = \sqrt{\frac{2\chi}{\omega}}$,which tells us around where we want our wine too keep the sun from messin' with it too much. The solution to the diffusion equation is

$$T(z,t) = T_0 + \Delta T e^{i\frac{z}{\delta_T}}e^{-i\omega t}e^{-\frac{z}{\delta_T}}$$

6.10 Attenuation of Sound

In one dimension, and in the linear approximation ($v = \delta v$, $\varrho = \varrho_0 + \delta\varrho$), the continuity equation is expressed as

$$\begin{aligned} \frac{\partial \varrho}{\partial t} &= -\frac{\partial(\varrho v)}{\partial x} \\ \implies \frac{\partial(\delta\varrho)}{\partial t} &= -\varrho_0 \frac{\partial(\delta v)}{\partial x} \end{aligned}$$

and Navier-Stokes is

$$\begin{aligned} \varrho \frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v}{\partial x^2} \\ \implies \varrho_0 \frac{\partial(\delta v)}{\partial t} &= -c^2 \frac{\partial(\delta\varrho)}{\partial x} + \eta \frac{\partial^2(\delta v)}{\partial x^2} \end{aligned}$$

where, as usual, we expand the perturbation in pressure as $p(\varrho) = p_0 + c^2\delta\varrho$. Hooray, now we can look for a dispersion law assuming solutions of the form

$$\begin{aligned} \delta v &= v' e^{i(kx - \omega t)} \\ \delta\varrho &= \varrho' e^{i(kx - \omega t)} \end{aligned}$$

Our simplified equations (upon substitution of the above) yield a system of equations for ω and k:

$$\begin{aligned} \omega\varrho' = \varrho_0 k v' \quad &\implies \quad \frac{v'}{\varrho'} = \frac{\omega}{\varrho_0 k} \\ i(\omega\varrho_0 v') = i(kc^2\varrho') + k^2\eta v' \quad &\implies \quad i(\omega\varrho_0 \frac{v'}{\varrho'}) = i(kc^2) + k^2\eta\frac{v'}{\varrho'} \\ &\implies \quad i(\frac{\omega^2}{k}) = i(c^2 k) + \frac{\eta\omega}{\varrho_0} k \\ &\implies \quad \left(\frac{\omega^2}{c^2}\right) = \left(1 - i\frac{\eta\omega}{\varrho_0 c^2}\right) k^2 \\ &\implies \quad k = \left(\frac{\omega}{c}\right) \cdot \frac{1}{\sqrt{1 - i\frac{\eta\omega}{\varrho_0 c^2}}} \end{aligned}$$

Since we have that $\dfrac{1}{\sqrt{1-x}} = 1 + \dfrac{x}{2} + \ldots$ the dispersion law becomes

$$k = \frac{\omega}{c}\left(1 + i\frac{\eta\omega}{2\varrho_0 c^2}\right)$$

$$\implies \boxed{k = \frac{\omega}{c} + i\frac{\eta\omega^2}{2\varrho_0 c^3}}$$

That is, the dispersion law is of the form $k = \dfrac{\omega}{c} + i\gamma$, where $\gamma = \dfrac{\eta\omega^2}{2\varrho_0 c^3}$ is called the attenuation coefficient.

7 Problems

The following problems are from the weekly problem sets provided by Dr. Seth Putterman for the Physics 114 class.

Problem 1

Consider a sealed 1D pipe of length L. At $t = 0$, $v = 0$ everywhere, but the pressure is $p(x, 0) = p_0 + \delta p$, where $\delta p = \frac{\bar{p}x}{L}$. Find $\varrho(x, t)$. This is a linear sound wave problem. The initial state will set of oscillations at various frequencies. (Note that p_0, $\bar{p}$ are constants).

Problem 2

Consider a tube which is sealed at $x = 0$ and fitted with a moveable piston at $x = L$. If the piston moves with a velocity

$$U_{\text{pist}} = v' e^{-i\omega t}$$

find $v(x,t)$ for undamped motion. This is a small amplitude problem. Let the speed of sound be c and the ambient density ϱ_0.

Problem 3

Consider a fluid moving with the velocity field of a cylindrical vortex. In this case

$$v(r,t) = \left(\frac{\kappa}{r}\right)\hat{\boldsymbol{\theta}}$$

which is written in cylindrical coordinates so that gravity is in the z direction. Find the pressure everywhere inside the fluid as a function of r, θ, z. Take the density as constant. Find the equation of a surface of constant pressure.

Problem 4

Show that $\delta p = \frac{A}{r} f(t - \frac{r}{c})$ is a solution to the 3D wave equation with a point source at the origin. Evaluate the velocity and interpret the point source. Small amplitude, linear problem.

Problem 5

Consider fluid which fills a rectangular trough to a depth d. The trough has a width w, and it extends from $z = 0$ to $z \to \infty$. Find the traveling sound wave solutions for this system. The top surface of the water is horizontal and open to the atmosphere. Take g=0.

Problem 6

Consider a pipe that is open to the atmosphere at one end and sealed at the other end. It is 3.4m long and is oscillating in its lowest resonant frequency with a mach # of 10^{-4}. The pipe has a diameter of 0.1m. Air data: $\varrho_0 = 1.2\text{kgm}^{-3}$, $c_0 = 340\text{ms}^{-1}$, $\frac{c_p}{c_v} = 1.4$, $T = 300\text{K}$.

(a) What is k (the wave-number)?

(b) What is the frequency in cycles per second?

(c) What is the energy of the acoustic motion?

(d) What is the temperature swing?

(e) What is $\frac{\delta\varrho}{\varrho_0}$?

Problem 7

Consider 2 vortices separated by a distance d. The circulation around each vortex is the same magnitude but has a different sign. At $t = 0$, they are located at $(0, \frac{d}{2})$ and $(0, -\frac{d}{2})$. What is the pressure as a function of time?

Problem 8

Consider a sphere of radius R and mass density ϱ. This fluid interacts with itself via gravitational interaction, i.e. G. It also has a speed of sound $\frac{\partial p}{\partial \varrho} = c^2$. Find the lowest mode of this system that includes c and G.

Problem 9

Consider a 3D rectangular box which has a square cross-section with edge length A, and a length L. Orient the length along the z axis with one square side at $z = 0$ and the other at $z = L$. All sides are solid walls except for the square side at $z = L$, which is open to the atmosphere. The speed of sound is c and the ambient density is ϱ.

(a) Write down the complete set of normal modes of sound in the box (neglect g).

(b) Write down the mode which has:

- One velocity node inside the box along the x direction.
- No nodes inside the box along the y direction.
- One velocity node inside the box along the z direction.

Let the amplitude of this mode be ϱ'.

(c) Now, take $c = 340\text{ms}^{-1}$ and $\varrho = 0.0012\text{gcm}^{-3}$ for air. Moreover, let $L = 1.7\text{m}$ and $A = 0.85\text{m}$. The Mach # is 10^{-3}.

(i) What is the frequency in Hz?

(ii) What is the energy of the acoustic motion?

(iii) What is $\dfrac{\delta\varrho}{\varrho_0}$?

(iv) If one places a dB meter inside the box near the bottom corner, what would the approximate reading be?

Problem 10

Consider a sound wave in air with frequency 1kHz, and a Mach # of 10^{-5}. It is incident on an expansion at $x = 0$. The area expands by a factor of 10 from $A_i = 25\text{cm}^2$.

(a) Calculate the energy density of the sound wave, and its loudness in dB on the right of the expansion. Calculate the transmitted power.

(b) What is the amplitude of the second harmonic 1m after the expansion? Assume that the expanded pipe is > 10m long.

To set this problem up, you will need to have incident and reflected waves on the left and a transmitted wave on the right.
At the interface of the 2 regions, make pressure continuous, and also make the mass flow continuous. (The mass flow is $\varrho v A$).

Problem 11

(a) Consider a Cylinder of radius R moving perpendicular to its axis with constant velocity U (in the x direction) through a fluid with density ϱ. The fluid is otherwise at rest.

(i) Find the velocity field everywhere for this 2D problem.

(ii) Find the force on the cylinder.

(b) Let there be a circulation κ around the cylinder. Think of this as a vortex trapped to the cylinder. Now, find the velocity field and the force on the cylinder.

Problem 12

If the speed of sound in water is 1500ms^{-1} at the surface, and increases linearly with depth at a rate of 0.017s^{-1}, find the range at which a ray emitted horizontally from a source at 100m depth will reach the surface.

Problem 13

Consider a tube which is open to the atmosphere at $x = 0$, and fitted with a moveable piston at $x = L$. The speed of sound is c and the ambient density is ϱ_0. If the piston moves with a velocity

$$v(L,t) = v' e^{-i\omega t}$$

solve for $p(x,t)$, and $v(x,t)$ in each of the following cases:

(a) Undamped motion.

(b) Damped motion where α is the attenuation coefficient of sound so that (for example) a wave moving to the right decays as $e^{-\alpha x}$. You may take $\alpha L \ll 1$ (but not zero).

Problem 14

Use the diffusion equation to calculate $T(x, t)$ if at $t = 0$ the temperature is $T(x, 0)$. The thermal diffusivity is χ. Use Fourier integrals.

Problem 15

Consider a U-tube of radius R. It is partially filled with water which occupies a length L in the tube. Initially the water is higher in one end by a level difference h as shown in the figure. This all sits in a g field open to the atmosphere. When allowed to oscillate freely, what is the frequency and the damping? This is incompressible flow with viscosity η and fluid density ϱ. Different levels of approximation are possible. Do this for the limit where the damping is small compared to the frequency, or the other way around. The picture shows a large displacement but you should set this up for a small level difference.

Problem 16

Liquid Helium II has 2 velocity fields. In addition, there are the equations of mass and entropy conservation. Set up the linear 1D equations using equations of state for the various parameters as a function of p, T. And write down the dispersion law. Solve the dispersion law in the limit where the expansion coefficient is zero.

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