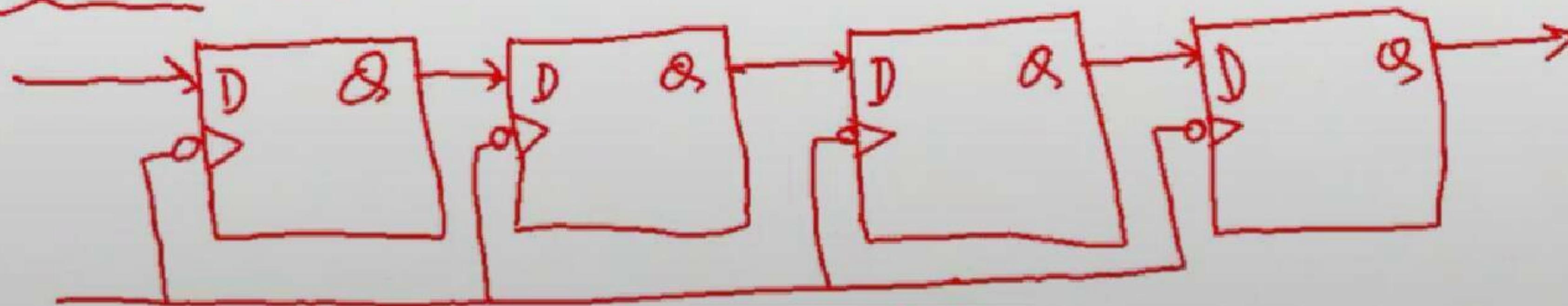


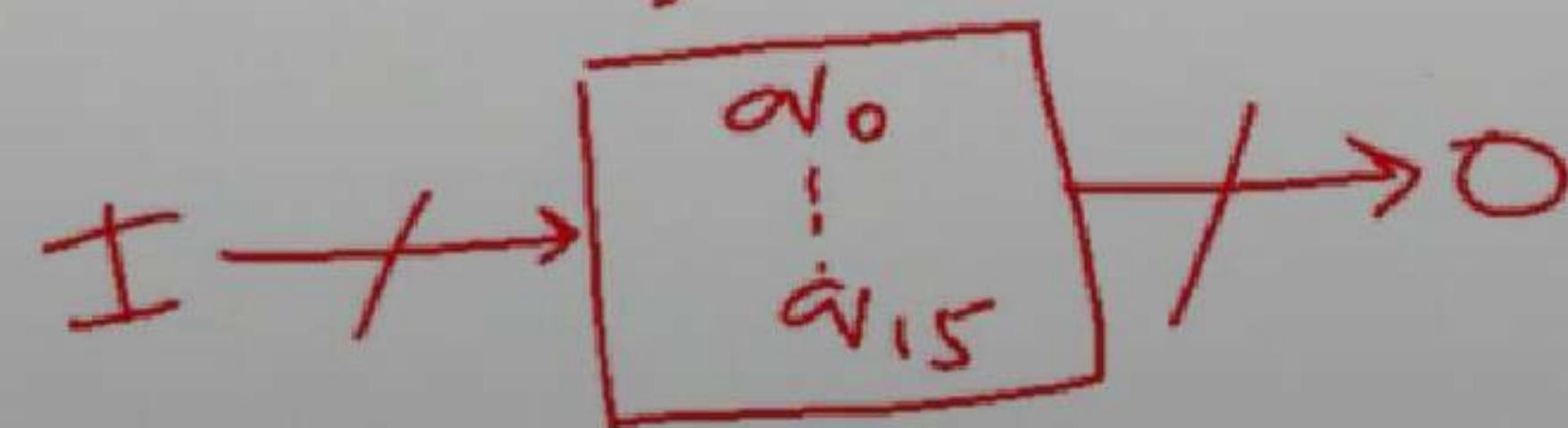
- (i) Output: O_1, O_2, \dots, O_n are the outputs which can take fixed values from the set O .
- (ii) States: At any instant the automaton can be in one of the states a_1, a_2, \dots, a_n .
- (iii) State relation: The next state is determined by the present state & present input.
- (iv) Output relation: The output is related to either state only or both the input & the state.

Example: Consider a shift register



There are $2^4 = 16$ states
($0000 \rightarrow 1111$)

$$\Sigma = \{0, 1\}, O = \{0, 1\}$$



Definition of Finite Automata

An automata can be defined as five tuples

$\delta \rightarrow$ finite set of states

$\Sigma \rightarrow$ finite set of input symbols called alphabet

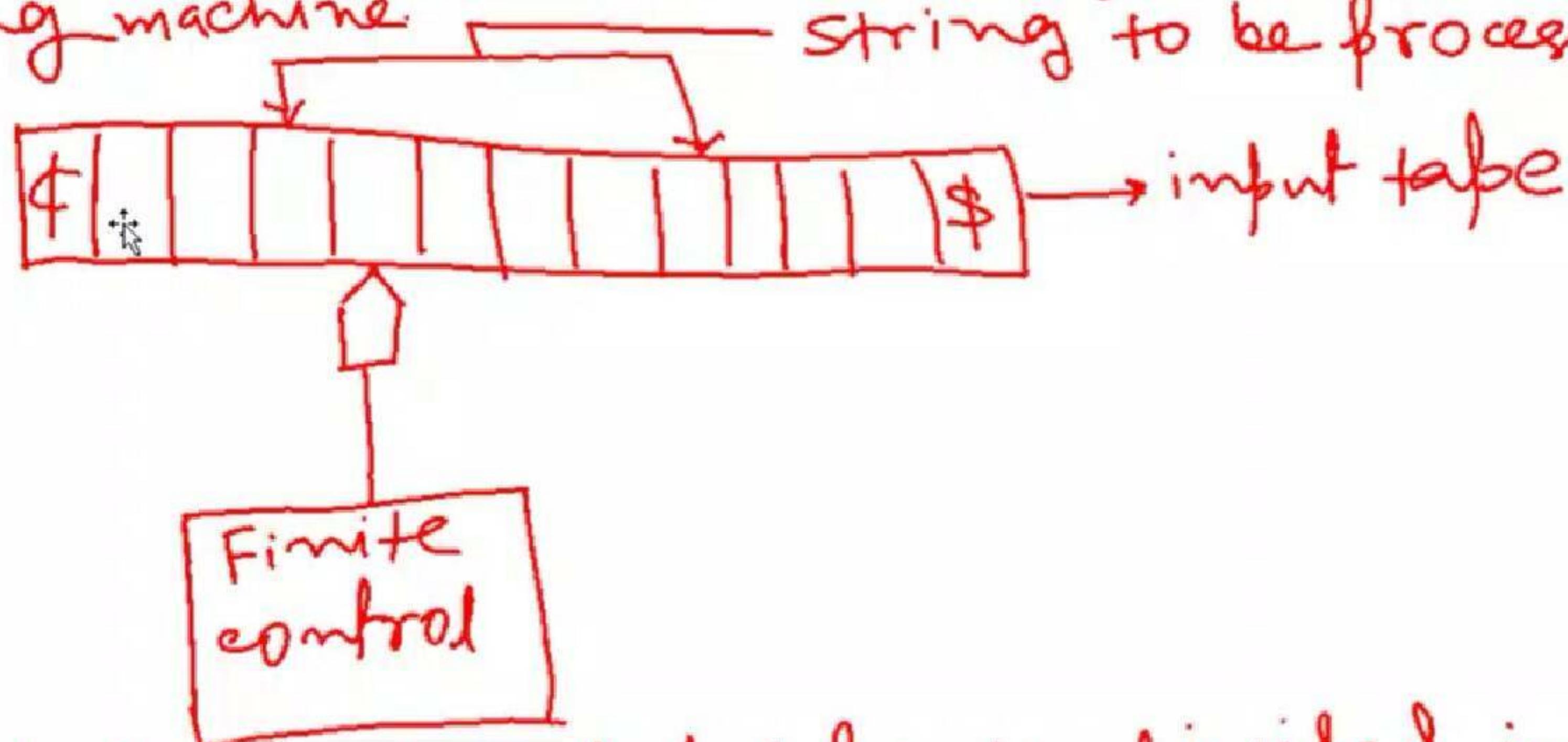
$\gamma \rightarrow$ is called transition function

$$\gamma: \delta \times \Sigma \longrightarrow \delta$$

$a_0 \rightarrow$ initial state, where $a_0 \in \delta$

$F \subseteq \delta$ is the set of final states.

The automaton can be visualized as the following machine.



Input tape: Input tape is divided into squares, each square containing a symbol from the alphabet Σ .
from the alphabet Σ .
 $\$ \rightarrow$ start marker, $\$ \rightarrow$ End marker

Reading head: reading head examines one symbol at a time.

(i.e. one square)

It can move either right or left. (We restrict the movement to the right for now).

Finite control: input to the finite control is a symbol under head and the present state of the machine.

- output of the finite control

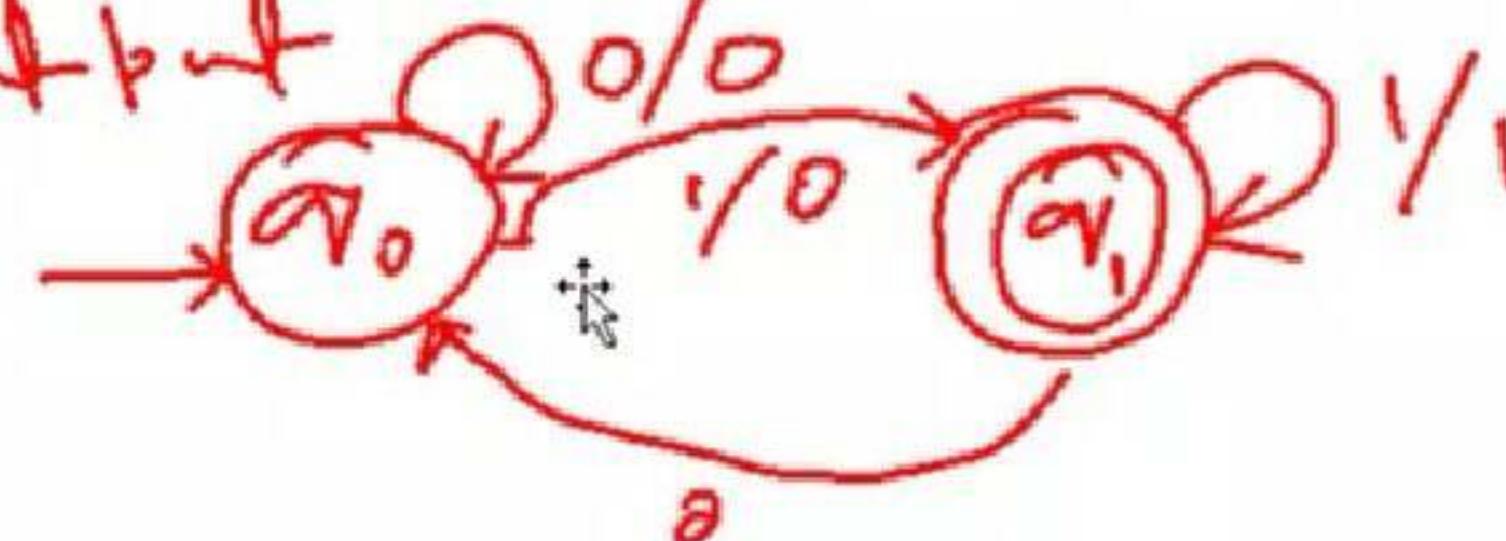
(a) A motion of the R-head to the next square or a null move.

(b) The next state is given by $S(a, a)$.

Transition Systems

A transition graph or transition system is a visual representation of an automaton.

- A transition system consists of states and possible transitions shown by directed edges labelled by input/output



- States are represented as a single circle and final states are represented as double circle.

- The δ is represented as

$$\emptyset \times \Sigma^* \times \emptyset$$

i.e. $(\alpha_1, w, \alpha_2) \rightarrow$ means automation at state α_1 , after consuming the string w moves to α_2 .

Acceptance of a string

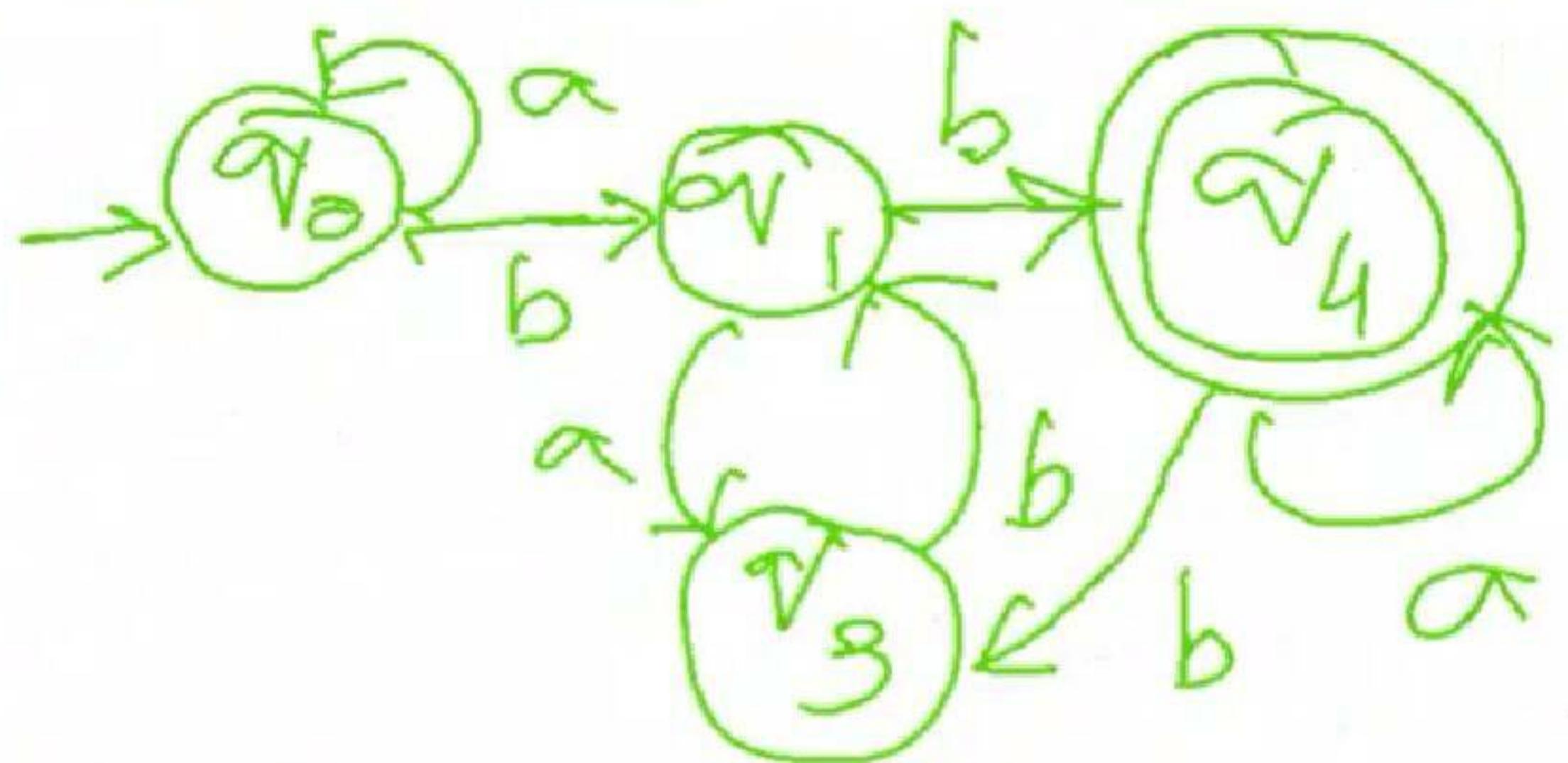
A transition system accepts a string w in Σ^* if

- (a) there exists a path from initial state to some final state labelled with symbols from w .
- (b) The path value obtained by concatenation of all edge labels of the path is equal to w .

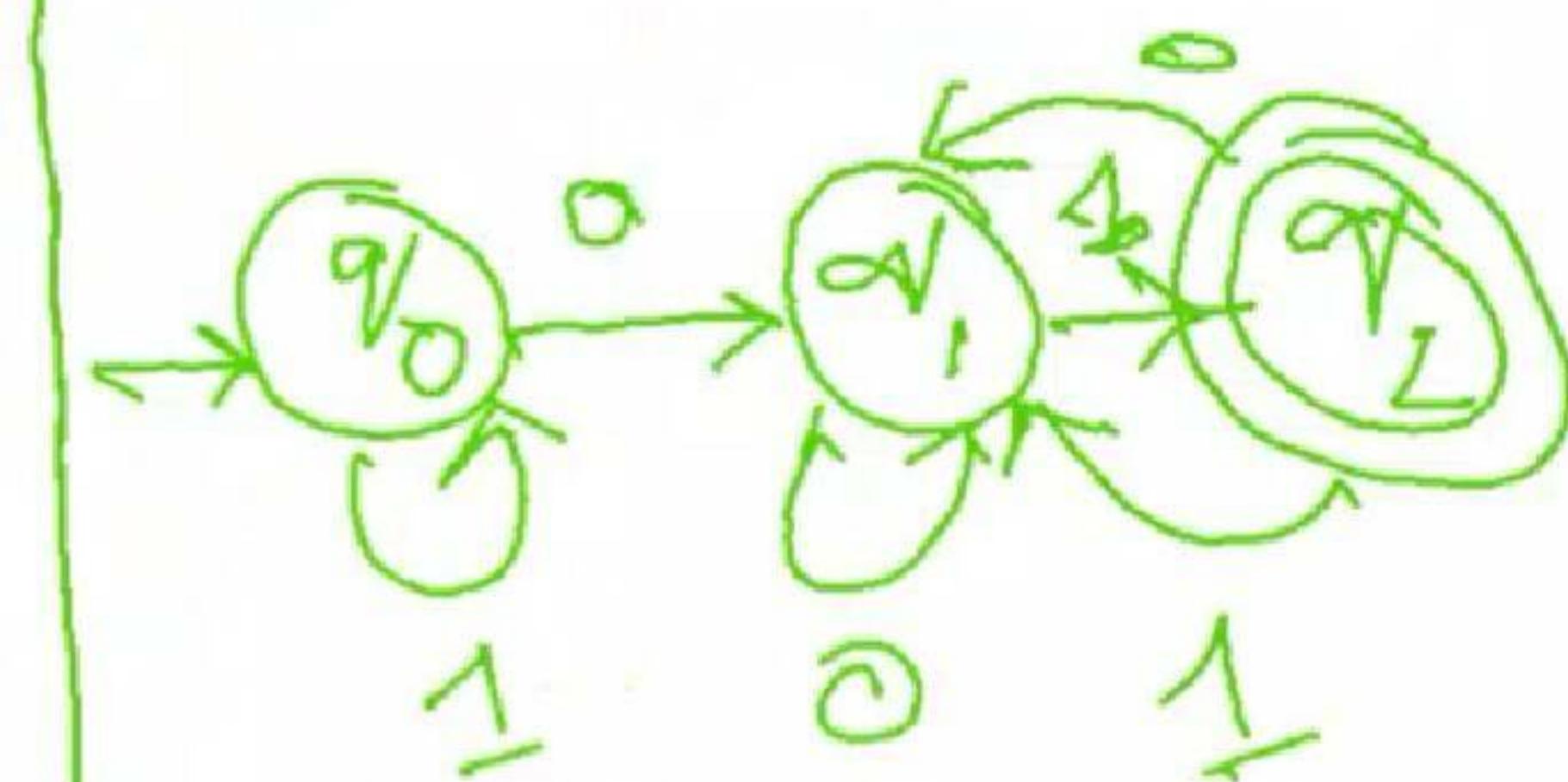
Example:



Determine acceptability 0/0
of 101011, 111010



Define the language



test for the following

011011	1
0111110	
0100010	
1110001	
1111000	

Properties of transition function

1. $\delta(\alpha, \lambda) = \alpha$, in a finite automaton

2. for all strings w and an input symbol a

$$\delta(\alpha, aw) = \delta(\delta(\alpha, a), w)$$

$$\delta(\alpha, wa) = \delta(\delta(\alpha, w), a)$$

first one gives a state after consuming the first symbol of aw .

Second property gives a state after consuming the prefix of the string wa .

problem: Show that for a transition function δ and any two strings x & y

$$\delta(\alpha, xy) = \delta(\delta(\alpha, x), y)$$

Proof: Let $|y|=1$ and $y=\alpha \in \mathbb{Z}$

from second property

$$\text{L.H.S } \delta(\alpha, x\alpha) = \delta(\delta(\alpha, x), \alpha) = \text{R.H.S}$$

Let the rule is true for $|y_1|=n$, where $y=y_1$ i.e.

$$\begin{aligned} |y| &= n+1 \\ \text{L.H.S } \delta(\alpha, x_{y_1}, \alpha) &= \delta(\alpha, x_1, \alpha) \\ &= \delta(\alpha, x_1, \alpha) \text{ where } x_1 = x_{y_1} \end{aligned}$$

from property 2,

$$\begin{aligned} \text{L.H.S. } \delta(\delta(\alpha, x_1), \alpha) \\ &= \delta(\delta(\alpha, x_{y_1}), \alpha) \text{ from induction hypothesis} \end{aligned}$$

$$\text{L.H.S } \delta(\delta(\delta(\alpha, x), y_1), \alpha)$$

$$\text{R.H.S } \delta(\delta(\alpha, x), y) = \delta(\delta(\alpha, x), y_1, \alpha)$$

$$\text{From property 2, } = \delta(\delta(\delta(\alpha, x), y_1), \alpha)$$

Problem: PROVE that if $\delta(\alpha, \gamma) = \delta(\gamma, \gamma)$, then $\delta(\alpha, \gamma z) = \delta(\alpha, \gamma z)$

Sol:
$$\begin{aligned}\delta(\alpha, \gamma z) &= \delta(\delta(\alpha, \gamma), z) \\ &= \delta(\delta(\alpha, \gamma), z) \\ &= \delta(\alpha, \gamma z)\end{aligned}$$

Acceptability of string by a finite automaton

Definition: A string is said to be accepted by a finite automaton $M = (Q, \Sigma, \delta, \alpha_0, F)$

if $\delta(\alpha_0, x) = \gamma$, for some $\gamma \in F$.

This is basically the acceptability of a string by the final state.

Problem: Consider a machine where δ is defined

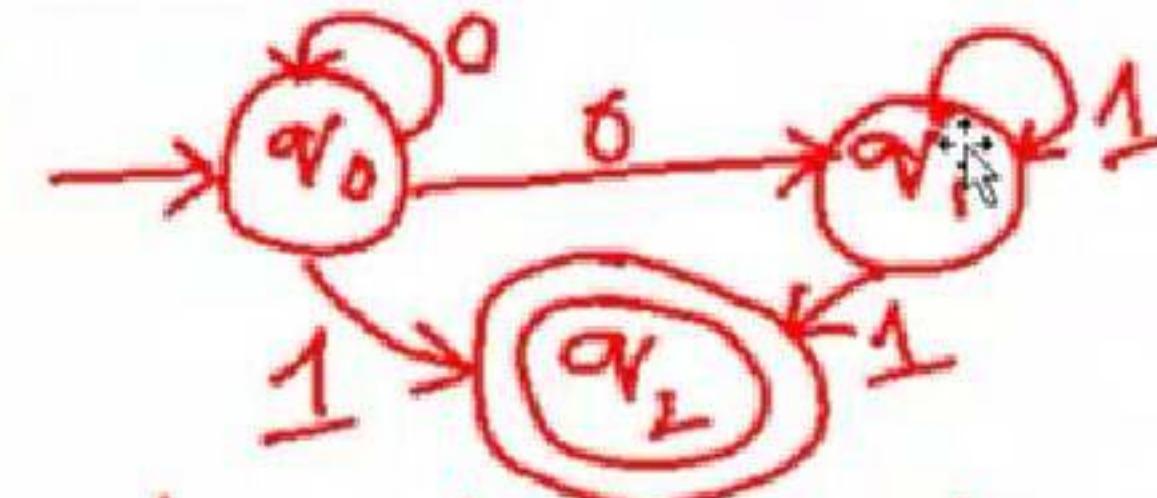
as

state	input	
	0	1
α_0	α_1	α_1
α_1	α_2	α_0
α_2	α_3	α_0
α_3	α_1	α_3
α_4	α_2	α_2

determine whether string 11001 is accepted or not
 $F = \{\alpha_0\}$, $\Sigma = \{0, 1\}$

Non-deterministic finite state machine

Let us consider the following transition diagram (transition system)



- If the automaton is in a state $\{q_0\}$ and the input symbol is 0, what will be the next state?
- Thus some moves of the machine cannot be determined uniquely by the input symbol and present state.

Definition

A non-deterministic finite automaton (N DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$. The only difference is in δ .

(i) Q is a finite non empty set of states.

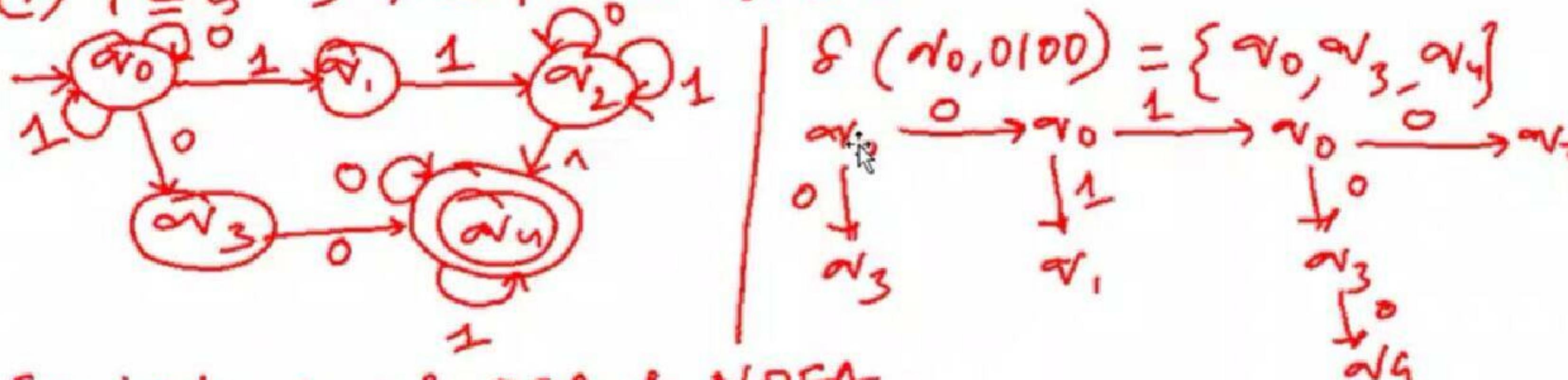
(ii) Σ is a finite non empty set of input symbols.

(iii) δ is the transition function defined as

$$\delta: Q \times \Sigma \longrightarrow 2^Q$$

(iv) $\alpha_0 \in Q$ is the initial state.

(v) $F \subseteq Q$ is the final state.



Equividence of DFA & NDFA

-A DFA can simulate the behaviours of NDFA by increasing the number of states.

- In other words, a DFA $(Q, \Sigma, \delta, \alpha_0, F)$ can be viewed as an NDFA $(Q, \Sigma, \delta, \alpha_0, F)$ by defining $\delta'(\alpha, a) = \{\delta(\alpha, a)\}$

NDFA is more general machine without being more powerful.

The transition for NDFA

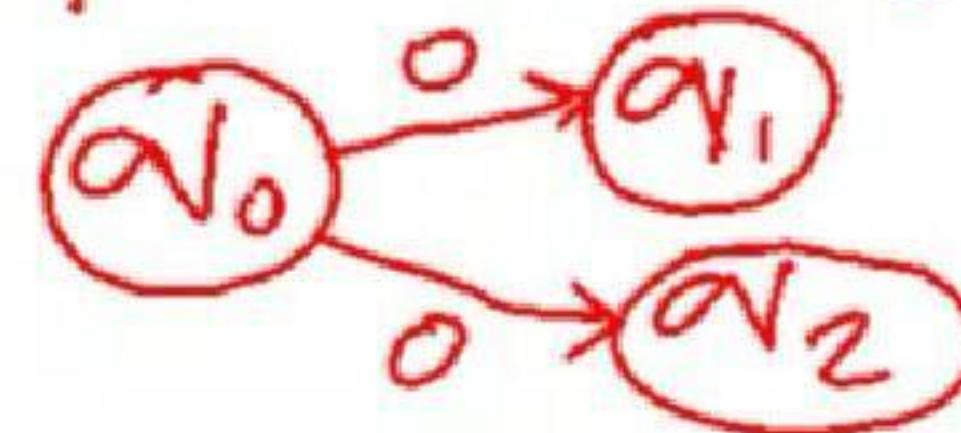
$$\delta : \Sigma \times Q \rightarrow 2^Q$$

The corresponding DFA

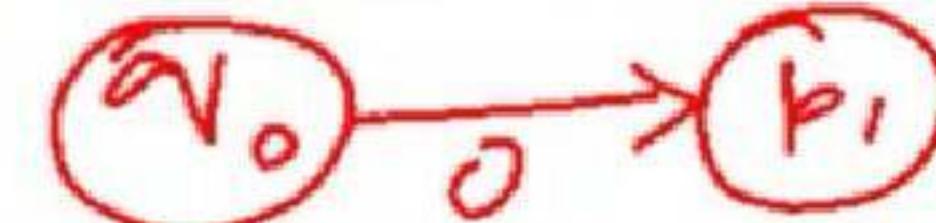
$$\delta' : \Sigma \times Q' \rightarrow Q'$$

$$Q' \subseteq 2^Q$$

gf for an NDFA



The the corresponding DFA will have



$$\text{where } p_1 = [q_1, q_2]$$

Theorem: For every NDFA, there exists a DFA which simulates the behaviour of NDFA.

Alternatively,

if L be the set of strings accepted by NDFA, then there exists a DFA which also accepts L .

Proof: Let $M = (Q, \Sigma, S, \alpha_0, F)$ be an NDFA accepting L .
Let us construct a DFA as follows

$$M' = (Q', \Sigma, S', \alpha'_0, F')$$

where $Q' = 2^Q$ (Any state in Q' is denoted by $(\alpha_1, \alpha_2, \dots, \alpha_j)$)

(i) $\alpha'_1 = 2^Q$ (Where $\alpha_1, \alpha_2, \dots, \alpha_j \in Q$)

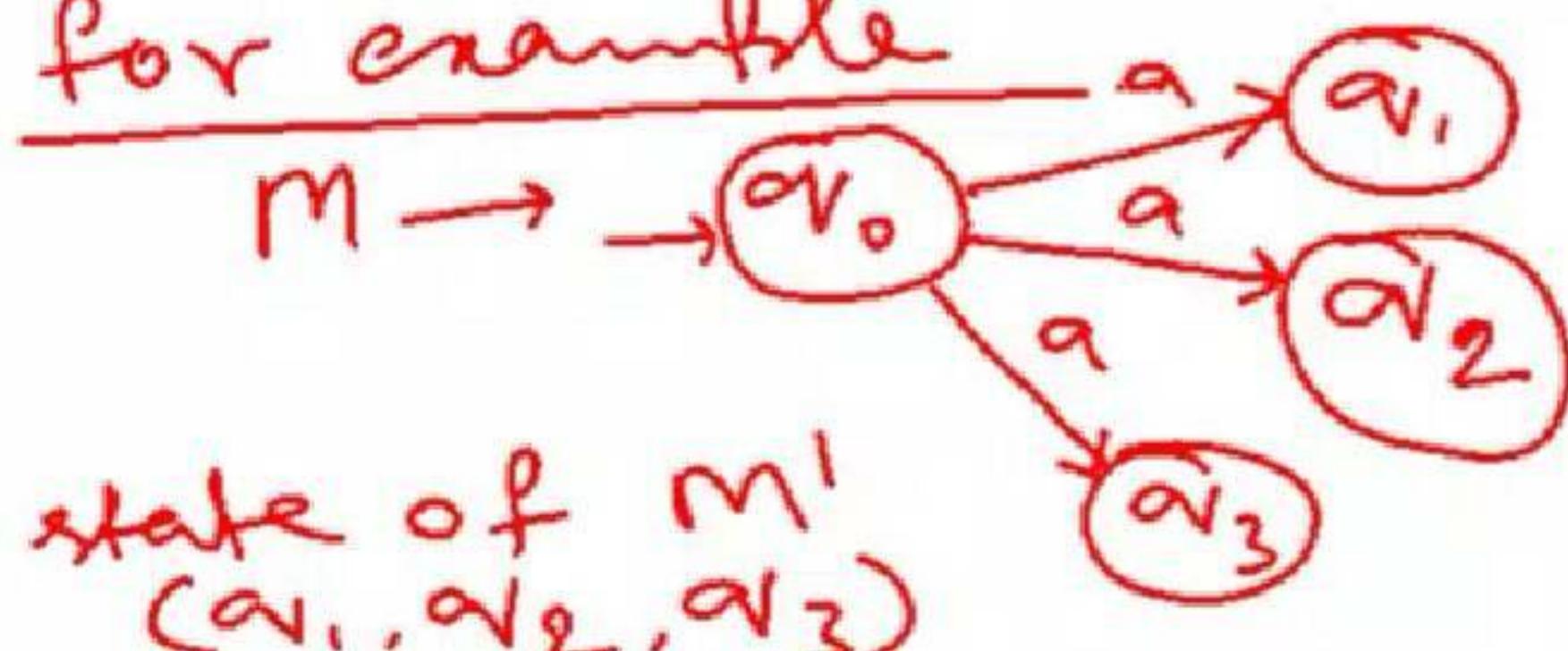
$$(ii) \alpha'_0 = [\alpha_0]$$

(iii) F' is the set of all subsets of Q containing an element of F .

- On applying any input a to M at state α_0 , M can reach to state $S(\alpha_0, a)$.
- The state of corresponding DFA, M' on input a at

state α'_0 is a set of all states that can be reached by $\delta(\alpha_0, a)$ on M .

for example



- Hence the states of M' are defined as subsets of Q . The M' should also start with the initial state of α_0 , Hence $\alpha'_0 = [\alpha_0]$
- On processing any string w by M , if M reaches a final state then $w \in T(M)$
- Hence final state in M' should be any of the final states of M .

(iv) therefore, δ' can be defined as

$$\delta'([\alpha_1, \alpha_2, \dots, \alpha_j], a) = \delta(\alpha_1, a) \cup \delta(\alpha_2, a) \cup \dots \cup \delta(\alpha_j, a)$$

Now let us prove that

$$\delta'(\alpha'_0, x) = [\alpha_1, \alpha_2, \dots, \alpha_i] \text{ iff } \delta(\alpha_0, x) = [\alpha_1, \alpha_2, \dots, \alpha_i]$$

- we prove it by induction on $|x|$

when $|x|=0$, $\delta(\alpha_0, \lambda) = \alpha_0$

and by definition of δ'

$$\delta'(\alpha'_0, \lambda) = \alpha'_0 = \alpha_0$$

hence $\delta'(\alpha'_0, x) = \alpha_i$ if $\delta(\alpha_0, x) = \alpha_i$ for $|x|=0$

Let the hypothesis is true for all strings y with $|y| \leq m$

Let $\delta(\alpha_0, y) = [p_1, p_2, \dots, p_i]$ & $\delta(\delta(\alpha_0, y), a) = [r_1, r_2, \dots, r_k]$

where $|y| = m+1$ and $|y| = m$

$$\delta([p_1, p_2, \dots, p_i], a) = [r_1, r_2, \dots, r_k]$$

Now let us see

$$\delta'(\alpha'_0, \gamma a) = ?$$

We have $\delta'(\alpha'_0, \gamma) = [p_1, p_2, \dots, p_i] = \delta(\alpha_0, \gamma)$

from hypothesis

$$\begin{aligned}\delta'(\alpha'_0, \gamma a) &= \delta'(\delta'(\alpha'_0, \gamma), a) \\&= \delta'([p_1, p_2, \dots, p_i], a) \\&= \delta'(p_1, a) \cup \delta'(p_2, a) \dots \cup \delta'(p_i, a) \\&= \delta(p_1, a) \cup \delta(p_2, a) \dots \cup \delta(p_i, a) \\&= [\tau_1, \tau_2, \dots, \tau_K]\end{aligned}$$

Hence it is proved for $\kappa = \gamma a$.

Similarly,

$$\text{if } \delta'(\alpha'_0, \kappa) = [\alpha_1, \alpha_2, \dots, \alpha_i]$$

$$\text{then } \delta'(\alpha_0, \kappa) = [\alpha_1, \alpha_2, \dots, \alpha_i]$$

$$\delta'(\alpha'_0, \gamma) = [p_1, p_2, \dots, p_i]$$

from hypothesis

$$\delta(\alpha_0, \gamma) = [b_1, b_2, \dots, b_i]$$

$$\delta'(\delta'(\alpha_0, \gamma), \alpha) = \delta'([b_1, b_2, \dots, b_i], \alpha) = [r_1, r_2, \dots, r_k] - 0$$

① is possible only if

$$\delta(b_1, \alpha) = r_1, \delta(b_2, \alpha) = r_2, \dots, \delta(b_i, \alpha) = r_k$$

Since from definition

$$\delta'([b_1, b_2, \dots, b_i], \alpha) = \delta(b_1, \alpha) \cup \delta(b_2, \alpha) \cup \dots \cup \delta(b_i, \alpha)$$

Problem: construct a deterministic automaton equivalent to
 $M = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, [q_0]), S$ is given as follows

state / Σ	0	1
$\rightarrow q_0$	q_0	q_1

- Sol : For deterministic automaton M_1 ,
- (i) The states are subsets of $\{q_0, q_1\}$ i.e. $\phi, [q_0], [q_1], [q_0, q_1]$.
 - (ii) Final states are $[q_0]$ and $[q_0, q_1]$.
 - (iii) $[q_0]$ is the initial state.
 - (iv) δ_1 for M_1 is defined as,

$s_{\text{state}} / \Sigma$	0	1
\emptyset	\emptyset	\emptyset
a_0	a_0	a_1
a_1	a_1	$[a_0, a_1]$
$[a_0, a_1]$	$[a_0, a_1]$	$[a_0, a_1]$

$$\delta_s([a_0, a_1], 0) = \delta(a_0, 0) \cup \delta(a_1, 0) = [a_0, a_1]$$

Similarly

$$\delta_s([a_0, a_1], 1) = \delta(a_0, 1) \cup \delta(a_1, 1) = [a_1, a_0, a_1] = [a_0, a_1]$$

No new states are there, hence we stop here.

When M has n states, the corresponding deterministic automaton has 2^n states. However, we need not construct transition (δ_s) for all 2^n states, but only for those states reachable from $[a_0]$.

efate/ Σ

\emptyset

a_0

a_1

$[a_0, a_1]$

0

\emptyset

a_0

a_1

$[a_0, a_1]$

1

\emptyset

a_1

$[a_0, a_1]$

$[a_0, a_1]$

$$\delta([a_0, a_1], 0) = \delta(a_0, 0) \cup \delta(a_1, 0) = [a_0, a_1]$$

Similarly

$$\delta_1([a_0, a_1], 1) = \delta(a_0, 1) \cup \delta(a_1, 1) = [a_1, a_0, a_1] = [a_0, a_1]$$

No new states are there, hence we stop here.

When M has n states, the corresponding deterministic automaton has 2^n states. However, we need not construct transition (δ_1) for all 2^n states, but only for those states reachable from $[a_0]$.

Problem: Find a deterministic automaton (acceptor) equivalent to $M = (\{v_0, v_1, v_2\}, \{a, b\}, \delta, v_0, \{v_2\})$ and δ is defined as

state / Σ	a	b
v_0	v_0, v_1	v_2
v_1	v_0	v_1
v_2	\emptyset	v_0, v_1

Sol: set the deterministic automaton M' equivalent to M is defined as

$$M' = (2^S, \{a, b\}, \delta', [v_0], F')$$

where 'possible F' is $\{[v_2], [v_0, v_2], [v_1, v_2], [v_0, v_1, v_2]\}$

$$\delta'(\alpha_0, a) = \delta(\alpha_0, a) = [\alpha_0, \alpha_1]$$

$$\delta'(\alpha_0, b) = \delta(\alpha_0, b) = [\alpha_2]$$

$$\delta'(\alpha_1, a) = \delta(\alpha_1, a) = \alpha_0$$

$$\delta'(\alpha_1, b) = \delta(\alpha_1, b) = \alpha_1$$

$$\delta'(\alpha_2, a) = \delta(\alpha_2, a) = \emptyset$$

$$\delta'(\alpha_2, b) = \delta(\alpha_2, b) = [\alpha_0, \alpha_1]$$

$$\delta'([\alpha_0, \alpha_1], a) = \delta(\alpha_0, a) \cup \delta(\alpha_1, a) = [\alpha_0, \alpha_1] \cup [\alpha_0]$$

$$= [\alpha_0, \alpha_1]$$

$$\delta'([\alpha_0, \alpha_1], b) = \delta(\alpha_0, b) \cup \delta(\alpha_1, b) = [\alpha_2] \cup [\alpha_1]$$

$$= [\alpha_1, \alpha_2]$$

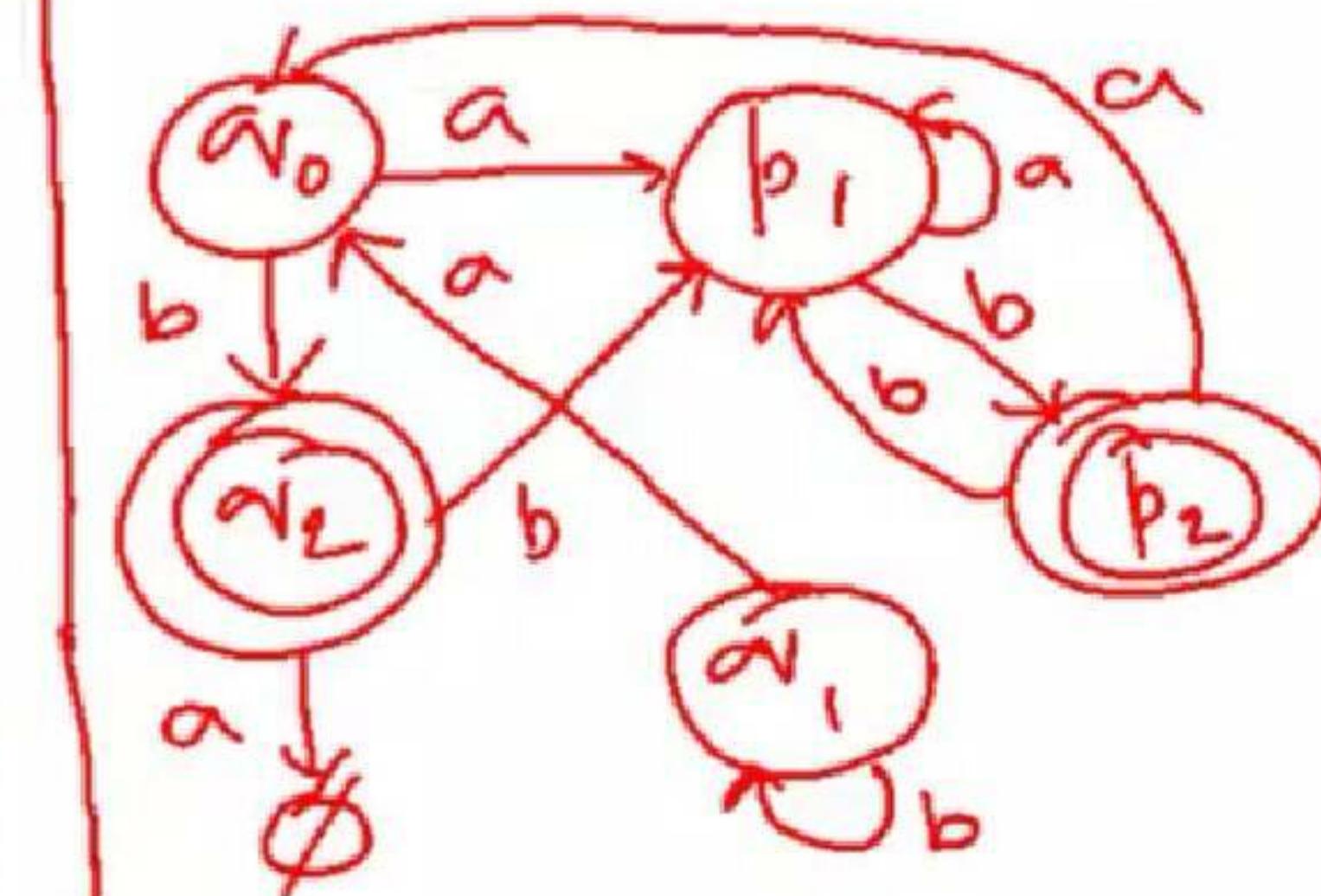
$$\delta'([\alpha_1, \alpha_2], a) = \delta(\alpha_1, a) \cup \delta(\alpha_2, a) = [\alpha_0] \cup \emptyset = [\alpha_0]$$

$$\delta'([\alpha_1, \alpha_2], b) = \delta(\alpha_1, b) \cup \delta(\alpha_2, b) = [\alpha_1] \cup [\alpha_0, \alpha_1] \\ = [\alpha_0, \alpha_1]$$

So the δ' can be defined as states/ Σ

	a	b
ϕ	ϕ	ϕ
α_0	$[\alpha_0, \alpha_1]$	α_2
α_1	α_0	α_1
α_2	ϕ	$[\alpha_0, \alpha_1]$
$[\alpha_0, \alpha_1]$	$[\alpha_0, \alpha_1]$	$[\alpha_1, \alpha_2]$
$[\alpha_1, \alpha_2]$	$[\alpha_0]$	$[\alpha_0, \alpha_1]$

Let $[\alpha_0, \alpha_1] = p_1$ &
 $[\alpha_1, \alpha_2] = p_2$



Finite automata with output

- The acceptability/rejection these two outputs were possible for the machines we considered so far.
- Now, let us consider the machines which do not have restriction over reachability to final state and output can be chosen from some alphabet
- This output function is dependent either on current state and current input both or current state only.

i.e. if λ is the output function then

either $\lambda(s(t), x(t)) \rightarrow$ (Mealy machine generalized)
or $\lambda(s(t)) \rightarrow$ (moore machine restricted)

where $s(t)$ is the current state
and $x(t)$ is current input.

Moore Machine

moore machine is defined using 6-tuples

$$(\mathcal{Q}, \Sigma, \Delta, \delta, \gamma, q_0)$$

\mathcal{Q} → finite state

Σ → input alphabet

Δ → output alphabet

δ → transition function

$$\Sigma \times \mathcal{Q} \rightarrow \mathcal{Q}$$

γ → output function

$$\gamma: \mathcal{Q} \rightarrow \Delta$$

q_0 → initial state

Example

Present state

Next state

output

	$a=0$	$a=1$	
$\rightarrow \alpha_0$	α_3	α_1	0
α_1	α_1	α_2	1
α_2	α_2	α_3	0
α_3	α_3	α_0	0

for input string 0111 the transition states given

by
output is $\left| \begin{array}{ccccc} \alpha_0 \rightarrow \alpha_3 \rightarrow \alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right.$

Mealy Machine

The Mealy machine also have 6-tuples, the only difference is in γ defined as

$$\gamma: Q \times \Sigma \rightarrow \Delta$$

Example

Present state

Next state

Present state			Next state	
	$a=0$	$a=1$	state	o/p
	state	o/p	state	o/p
$\rightarrow q_1$	q_3	0	q_2	0
q_2	q_1	1	q_4	0
q_3	q_2	1	q_1	1
q_4	q_4	1	q_3	0

for the i/p 0011 we have

$$q_1 \rightarrow q_3 \rightarrow q_2 \rightarrow q_4 \rightarrow q_3$$

o/p 0 1 0 0

Note: In Mealy machine we get the output only
on the input to the machine state

- A finite automaton can be converted into a Moore machine by introducing $\Delta = \{a, b\}$ and defining $\gamma(a) = a$, if $a \in F$ and $\gamma(a) = b$ if $a \notin F$
- Moore machine has output string of length m+1 for i/p of length n.
- Mealy has the same o/p length as the i/p length.

for the i/p 0011 we have

$$q_1 \rightarrow q_3 \rightarrow q_2 \rightarrow q_4 \rightarrow q_3$$

o/p 0 1 0 0

Note: In Mealy machine we get the output only
on the input to the machine state

- A finite automaton can be converted into a Moore machine by introducing $\Delta = \{a, b\}$ and defining $\gamma(a) = \begin{cases} a, & \text{if } a \in F \\ \gamma_{**}(a), & \text{if } a \notin F \end{cases}$
- Moore machine has output string of length m+1 for i/p of length n.
- Mealy has the same o/p length as the i/p length.

associated with each state a_i in the next state column.
The state a_i then split into the number of states
equal to the outputs associated with a_i in Mealy
machine.

In the given Mealy machine a_1, a_3 are associated
outputs 1 and 0 respectively.

a_2, a_4 are associated with 0,1.

Hence we split a_2 into a_{20}, a_{21} and a_4 into a_{40}
and a_{41} .

Hence the table is modified as shown in the next
slide.



Present state

Next state

	input estate	$a = 0$ output		input estate	$a = 1$ output
$\rightarrow \alpha_1$	α_3	0		α_{20}	0
α_{20}	α_1	1		α_{40}	0
α_{21}	α_1	1		α_{40}	0
α_3	α_{12}	1		α_1	1
α_{40}	α_{41}	1		α_3	0
α_{41}	α_3	1		α_3	0

The previous table can be rearranged as

Present state

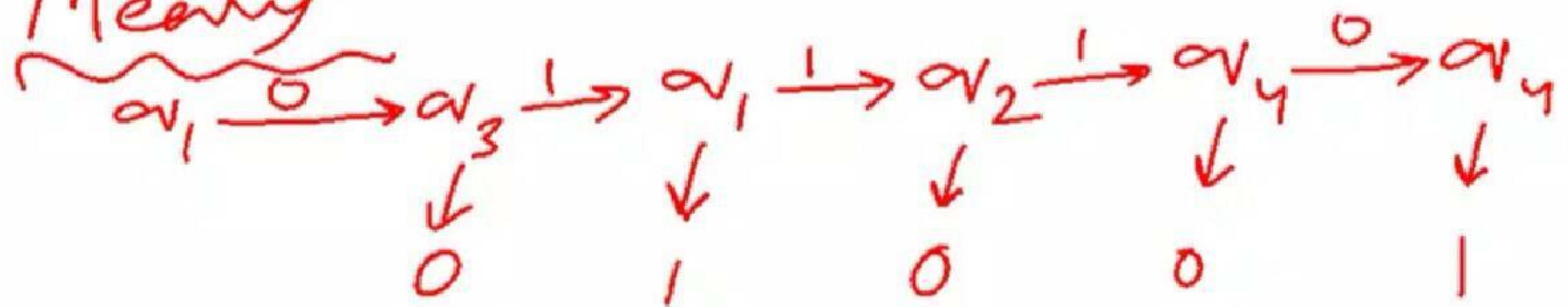
Next state

Present state	Input $a = 0$		Input $a = 1$		Output
	input	$a = 0$	$a = 1$	output	
$\rightarrow a_1$		a_3	a_{20}	1	
a_{20}		a_1	a_{40}	0	
a_{21}		a_1	a_{40}	1	
a_3		a_2	a_1	0	
a_{40}		a_{41}	a_3	0	
a_{41}		a_{41}	a_3^*	1	

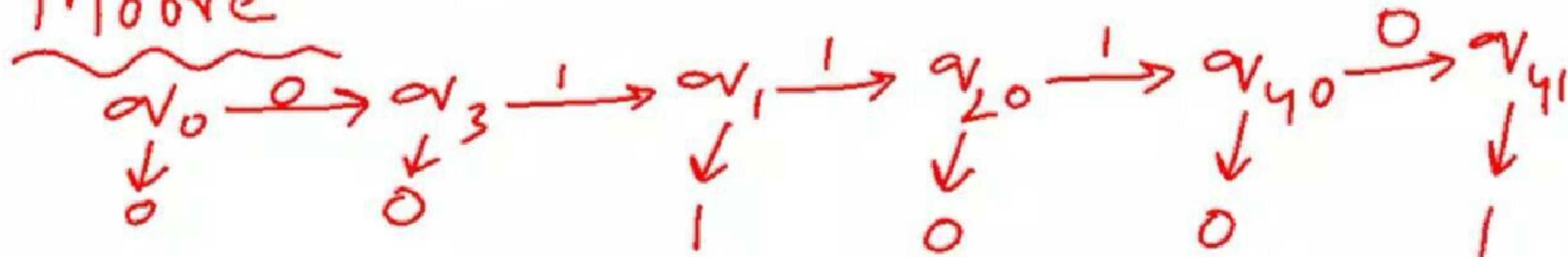
Now this Moore machine accepts a null string as it outputs 1 for λ with initial state a_1 . In Mealy machine λ string is not allowed. Hence, either we can neglect the response for

Example: Show how output is generated for the input 01110 by Mealy and the corresponding Moore machines of previous example.

Mealy



Moore



Procedure to transform Moore into Mealy

A moore machine M' is equivalent to a Mealy machine M ,

$$\text{if } bZ_M(w) = Z_{M'}(w)$$

where $Z(\cdot)$ is the output string function for M and M' for input string w , and b is the output for a string accepted by Moore machine.

Let the Moore machine

$$M' = (\Sigma, \Delta, \delta, s_0, \lambda, t_0)$$

construction

We need to define the output function γ of Mealy machine using output function λ of Moore machine as
$$\gamma(a, a) = \lambda(s(a), a)$$

Where $\delta \rightarrow$ is the transition function of M' .

Problem: Construct an equivalent Mealy machine
for Moore machine given below.

P. State	M. State		O/P
	$a=0$	$a=1$	
$\xrightarrow{a_0}$	a_3	a_1	0
a_1	a_1	a_2	1
a_2	a_2	a_3	0
a_3	a_3	a_0	0

Solution: For a_0 in Moore machine and
next state a_3 output is

$$\gamma'(a_0, 0) = \gamma(\delta(a_0, 0)) = \gamma(a_3) = 0$$

As α_3 is the state for input 0 at α_0 .

Similarly,

$$\gamma'(\alpha_0, 1) = \gamma(\delta(\alpha_0, 1)) = \gamma(\alpha_1) = 1$$

For present state α_1 in Moore machine

$$\gamma'(\alpha_1, 0) = \gamma(\delta(\alpha_1, 0)) = \gamma(\alpha_1) = 1$$

$$\gamma'(\alpha_1, 1) = \gamma(\delta(\alpha_1, 1)) = \gamma(\alpha_2) = 0$$

For present state α_2 in Moore machine

$$\gamma'(\alpha_2, 0) = \gamma(\delta(\alpha_2, 0)) = \gamma(\alpha_2) = 0$$

$$\gamma'(\alpha_2, 1) = \gamma(\delta(\alpha_2, 1)) = \gamma(\alpha_3) = 0$$

For α_3 in Moore

$$\gamma'(\alpha_3, 0) = \gamma(\delta(\alpha_3, 0)) = \gamma(\alpha_3) = 0$$

$$\gamma'(\alpha_3, 1) = \gamma(\delta(\alpha_3, 1)) = \gamma(\alpha_0) = 0$$

So the transition of the corresponding Mealy machine

P. State

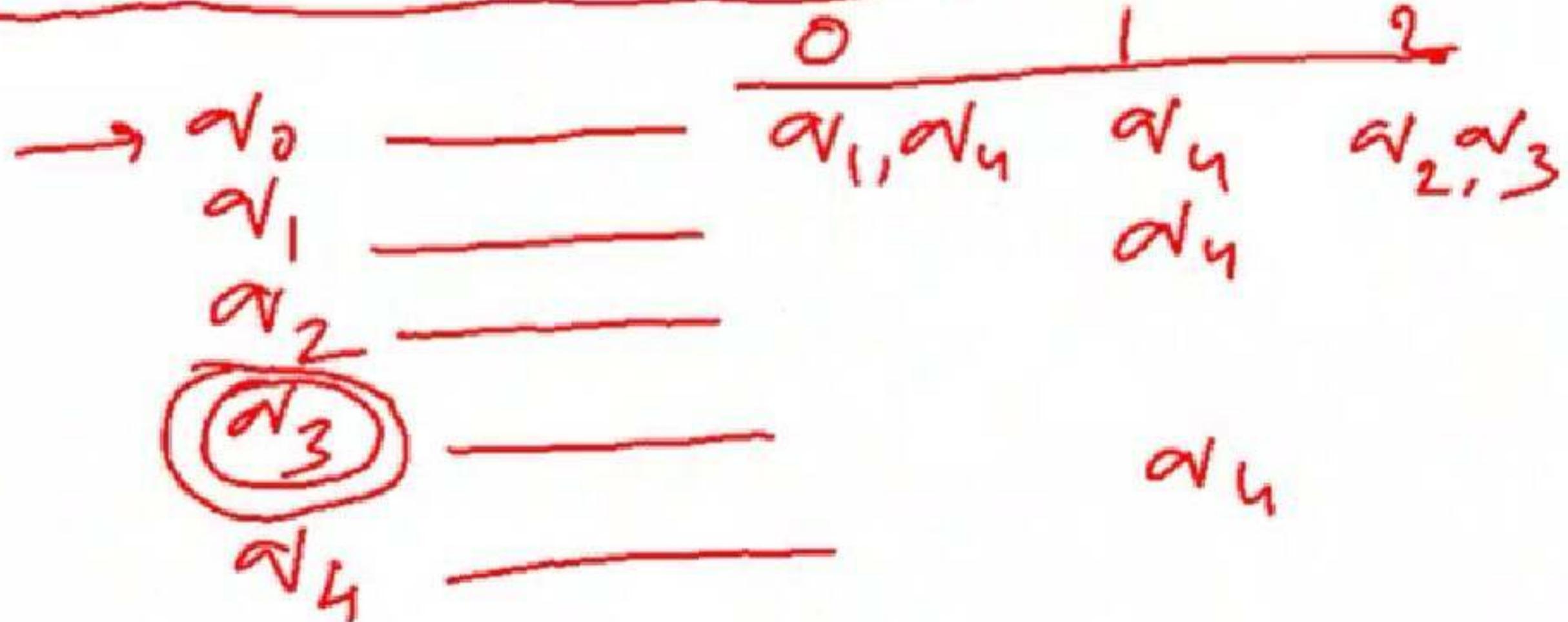
M. State

input a=0		input a=1	
state	o/p	state	o/p
α_0	α_3	α_1	1
α_1	1	α_2	0
α_2	0	α_3	0
α_3	0	α_0	0

- Q) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton.
Let R be a relation in Q , defined by
 $q_1 R q_2$, if $\delta(q_1, a) = \delta(q_2, a)$, for some $a \in \Sigma$.
Is R an equivalent relation.
- Q) Construct an N DFA accepting $\{ab, ba\}$ and
use it to find a deterministic automaton
accepting the same string.
- Q) The transition table for an N DFA is
in the next slide. construct the equiv -
ivalent DFA

state

input



State minimization in automaton

The objective is to construct an automaton M' with minimum number of states from the given automaton M .

Definition: Two states are called equivalent i.e. α_1 and α_2 are $\alpha_1 \equiv \alpha_2$, if either both $\delta(\alpha_1, x) \neq \delta(\alpha_2, x)$ are final states, or both are non final states,
 $\forall x \in \Sigma^*$.

Note: It is very difficult to check for all $x \in \Sigma^*$ whether $\delta(\alpha_1, x) \neq \delta(\alpha_2, x)$ are final or non final states.

modified definition

Two states a_1 and a_2 are K equivalent ($\epsilon \geq 0$) if both $\delta(a_1, x)$ and $\delta(a_2, x)$ are either non final or final states, $x, x \in \Sigma^*$ such that $|x| \leq K$.

Base case: Every final or non final states are 0-equivalent, i.e. any two final or non final states are 0-equivalent.

Properties of K -equivalence relations

1. These K -equivalence relations are equivalence relations, i.e. reflexive, symmetric, & transitive.

2. These k -equivalence relations create partition over Ω . These partitions can be denoted as Π or Π_k where Π_k denotes partitions of k -equivalence classes.
3. If $a_1, f a_2$ are k equivalent for all $K > 0$, then they are equivalent.
4. If $a_1, f a_2$ are $(k+1)$ equivalent then they are k -equivalent.
5. $\Pi_n = \Pi_{n+1}$, $\exists n$ denotes that there can be no further partitions.
(Π_n denotes the set of equivalence classes under n -equivalence)

Theorem: If α_1 & α_2 are (a) K-equivalent &
(b) $s(\alpha_1, \alpha)$ & $s(\alpha_2, \alpha)$ are also equivalent, $\forall \alpha \in \Sigma$
Then α_1 & α_2 are $K+1$ equivalent.

Proof: We prove this by contradiction.
Let us consider that α_1 , and α_2 are not
 $(K+1)$ equivalent.

Let $|aw_1| = K+1$, where $|w_1| = K$.
As α_1 & α_2 are not $K+1$ equivalent, hence
we can assume

$s(\alpha_1, aw_1) \rightarrow$ gives a non final state.

$s(\alpha_2, aw_1) \rightarrow$ gives a final state.

As we know
 $\delta(a, xy) = \delta(\delta(a, x), y)$

hence
 $\delta(\alpha_1, aw_1) = \delta(\delta(\alpha_1, a), w_1)$

&
 $\delta(\alpha_2, aw_2) = \delta(\delta(\alpha_2, a), w_2)$

So, $\delta(\underline{\delta(\alpha_1, a)}, w_1)$ & $\delta(\underline{\delta(\alpha_2, a)}, w_2)$ are

\downarrow \uparrow
 p_1 p_2
non final & final states respectively.

As $|w_1| = k$, $\delta(\alpha_1, a)$ & $\delta(\alpha_2, a)$ are not
K-equivalent, which is a contradiction.
Hence α_1 & α_2 has to be $k+1$ equivalent for
 $\delta(\alpha_1, a)$ & $\delta(\alpha_2, a)$ to be K equivalent.

Algorithm to construct minimum state automation

Step 1: construct Π_0 , where $\Pi_0 = [Q_1^0, Q_2^0]$ and Q_1^0 is the set of all final states & $Q_2^0 = Q - Q_1^0$.

Step 2: Let Q_i^K be K -equivalent class i.e.

$Q_i^K \subseteq \Pi_K$. If o_1 and o_2 are in Q_i^K they are $(K+1)$ equivalent provided $s(o_1, a)$ & $s(o_2, a)$ are k -equivalent i.e.

$$\{s(o_1, a), s(o_2, a)\} \subseteq \Pi_K$$

Step 3: construct T_m for $m = 1, 2, 3, \dots$ until $T_m = T_{m+1}$
 The minimum automaton M is the m -equivalent partition i.e. T_m

(i) construct a minimum automaton from the given automaton.

States / Σ	0	1
a_0	a_1	a_5
a_1	a_b	a_2
a_2	a_0	a_2
a_3	a_2	a_6
a_4	a_7	a_5
a_5	a_2	a_6
a_6	a_6	a_4
a_7	a_6	a_2

Solution: By applying step 1

$$\mathcal{Q}_1^0 = F = \{\alpha_2\}$$

$$\mathcal{Q}_2^0 = Q - \mathcal{Q}_1^0 = \{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$$

$$\Pi_0 = \{\mathcal{Q}_1^0, \mathcal{Q}_2^0\} = \{\{\alpha_2\}, \{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}\}$$

$\{\alpha_2\}$ in 0th partition Π_0 cannot be partitioned further. So $\{\alpha_2\}$ will remain same in 1st partition Π_1 , i.e. 1st equivalent class

$$\text{in } \Pi_1 \text{ is } \mathcal{Q}_1^1 = \{\alpha_2\}$$

Consider α_0, α_1 in \mathcal{Q}_2^0 for 0-column in transition table, we have α_1, α_6 and for 1-Column we have α_5, α_2 .

Where $\alpha_1 \in Q_2^0$ & $\alpha_6 \in Q_2^0$

Similarly $\alpha_5 \in Q_2^0$ & $\alpha_2 \in Q_1^0$.

Hence, $s(\alpha_0, 1)$ & $s(\alpha_1, 1)$ are not 0-equivalent,
so they are not 1-equivalent.

Similarly, α_0 is not 1-equivalent with α_3 ,
 α_5 & α_7 .

Consider α_0, α_4 for input-0, we have $\alpha_1 \in Q_2^0$ and
 $\alpha_7 \in Q_2^0$.

for input-1, we have $\alpha_5 \in Q_2^0$ for both α_0, α_4 .
hence α_0, α_4 are 1-equivalent as $s(\alpha_0, 0)$,
 $s(\alpha_0, 1)$, $s(\alpha_4, 0)$ & $s(\alpha_4, 1)$ are 0-equivalent.

With similar arguments we can say

α_0, α_6 are 1-equivalent

as $0 - \alpha_1 \in \mathcal{Q}_2^\circ$ & $\alpha_6 \in \mathcal{Q}_2^\circ$

$1 - \alpha_4 \in \mathcal{Q}_2^\circ$ & $\alpha_5 \in \mathcal{Q}_2^\circ$

So $\{\alpha_0, \alpha_4, \alpha_6\} \in \Pi_1$, say $\mathcal{Q}'_2 = \{\alpha_0, \alpha_4, \alpha_6\}$

Now, let us consider $\alpha_1, \alpha_3, \alpha_5$, & α_7 .

Amongst these α_1, α_3 , & α_5 are not 1-equivalent
 α_1, α_7 are 1-equivalent.

So, $\mathcal{Q}'_3 = \{\alpha_1, \alpha_7\} \in \Pi_1$

Now α_3, α_5 are 1-equivalent, as

$0 - \{\alpha_2\} \in \mathcal{Q}_1^\circ$

$1 - \{\alpha_5\} \in \mathcal{Q}_2^\circ$

$$\begin{aligned} \text{So } \Pi_1 &= \{\mathcal{Q}_1^1, \mathcal{Q}_2^1, \mathcal{Q}_3^1, \mathcal{Q}_4^1\} \\ &= \{\{\alpha_2\}, \{\alpha_0, \alpha_4, \alpha_6\}, \{\alpha_1, \alpha_7\}, \{\alpha_3, \alpha_5\}\} \end{aligned}$$

Now try to partition \mathcal{Q}_2^1 ,

α_0, α_4 gives us

$0 - \alpha_1, \alpha_7$ and

$1 \rightarrow \alpha_5$

$\alpha_1, \alpha_7 \in \mathcal{Q}_3^1$ hence they are 1-equivalent

Similarly, $\alpha_5 \in \mathcal{Q}_4^1$

As $s(\alpha_0, 0), s(\alpha_0, 1), s(\alpha_4, 0) \notin s(\alpha_4, 1)$ are
1-equivalent, $\alpha_0, \alpha_4 \in \Pi_2$

Let us take α_0, α_6
 $\alpha_0 - \alpha_1, \alpha_6$, which are not 1-equivalent, hence

hence $\{\alpha_0, \alpha_6\} \notin \Pi_2$, $\{\alpha_0, \alpha_4, \alpha_6\}$ in Π_1 breaks to $\{\alpha_0, \alpha_4\} \notin \{\alpha_6\}$ in Π_2 .

α_1, α_7 have

$0 - \alpha_6 \in Q_2^1$, $1 - \alpha_2 \in Q_1^1$, hence $\{\alpha_1, \alpha_7\} \in \Pi_2$

Similarly,

$\{\alpha_3, \alpha_5\} \in \Pi_2$

$\therefore \Pi_2 = \{\{\alpha_2\}, \{\alpha_0, \alpha_4\}, \{\alpha_6\}, \{\alpha_1, \alpha_7\}, \{\alpha_3, \alpha_5\}\}$

we can compute Π_3 further as

$\Pi_3 = \{\{\alpha_2\}, \{\alpha_0, \alpha_4\}, \{\alpha_6\}, \{\alpha_1, \alpha_7\}, \{\alpha_3, \alpha_5\}\}$.

So the minimum state automaton is Π_2
 where $\delta' \rightarrow \Pi_2$
 The transition table will be modified
 by replacing α_i with $[\overset{*}{\alpha_i}, \alpha_i]$

α_1 with $[\alpha_1, \alpha_7]$

α_3 with $[\alpha_3, \alpha_5]$

States/ Σ

0

1

$[\alpha_0, \alpha_4]$	$[\alpha_1, \alpha_7]$	$[\alpha_3, \alpha_5]$
$[\alpha_1, \alpha_7]$	$[\alpha_6]$	$[\alpha_2]$
$[\alpha_2]$	$[\alpha_0, \alpha_4]$	$[\alpha_2]$
$[\alpha_3, \alpha_5]$	$[\alpha_2]$	$[\alpha_6]$
$[\alpha_6]$	$[\alpha_6]$	$[\alpha_0, \alpha_4]$