Sai Nikhil
NUID: 001564864

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(1)

$$A = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{array}\right]$$

$$\text{rref}(A) = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(2)

$$\vec{y} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

2

$$T(p) = 2p' + p''$$

(1)

$$\vec{0} \in T(P)$$

$$\begin{aligned} T(x+y) &= 2(x+y)' + (x+y)'' \\ &= (2x' + x'') + (2y' + y'') \\ &= T(x) + T(y) \end{aligned}$$

$$\begin{aligned} T(\alpha x) = 2(\alpha x)' + (\alpha x)'' &= \alpha(2x' + x'') \\ &= \alpha T(x) \end{aligned}$$

$\Rightarrow T(P)$ is a Linear transformation

(2) $\mathcal{B} = \{t, t^2, t^3\}$ $\quad \mathcal{C} = \{1, t, t^2\}$

$$T(t) = 2(1) + 0 = 2 = 2(1) + 0(t) + 0(t^2)$$

$$\therefore \vec{T_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

||^{ly} $T(t^2) = 4t + 2 = 2(1) + 4(t) + 0(t^2)$

$$\vec{T_2} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

||^{ly} $T(t^3) = 2(3t^2) + 6t = 0(1) + 6(t) + 6(t)$

$$\vec{T_3} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}$$

$$\therefore [T]_{\mathcal{B}\mathcal{C}} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

(3)

$$\text{rref}(T) = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $\ker T = \{0\}$

$\Rightarrow$ 'T' is an isomorphism.

(3) (1) Let,

$$S = [\vec{s}_1 \ \vec{s}_2 \ \cdots \ \vec{s}_s]$$

Let, $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_s]$, $C = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_s]$

$$C \times \vec{s}_1 = \vec{b}_1$$

$$\Rightarrow \vec{s}_1 = C^{-1} \times \vec{b}_1$$

$$\therefore s_i = C^{-1} \times \vec{b}_i$$

$$\Rightarrow S = [C^{-1} \times \vec{b}_1 \quad C^{-1} \times \vec{b}_2 \quad \cdots \quad C^{-1} \times \vec{b}_s]$$

$$S = C^{-1} \times B$$

$$\therefore S = [id]_{\mathcal{B}}^{\mathcal{C}} = C^{-1} \times B$$

(2)

$$S = C^{-1} \times B$$

$$\Rightarrow B = C \times S$$

$$\Rightarrow [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_s] = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_s] S$$

Hence, proved

4

(1) $$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{rref}\left(\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & -1 & -3 \end{array} \right] \right)$$

$$\Rightarrow \vec{S}_1 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

||ly $$\text{rref}\left(\left[\begin{array}{cc|c} 0 & 1 & 4 \\ 1 & 0 & -1 \\ -1 & -1 & -3 \end{array} \right] \right) \Rightarrow \vec{S}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\therefore S = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

(2)

$$[\vec{v}_1 \vec{v}_2] S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ -3 & -3 \end{bmatrix} = [\vec{b}_1 \ \vec{b}_2]$$

(5)

$$A_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 2 \end{bmatrix}$$

$$\downarrow \; R_2 \to R_2 + \frac{R_1}{2}$$

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ 0 & \frac{3}{2} & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 2 \end{bmatrix}$$

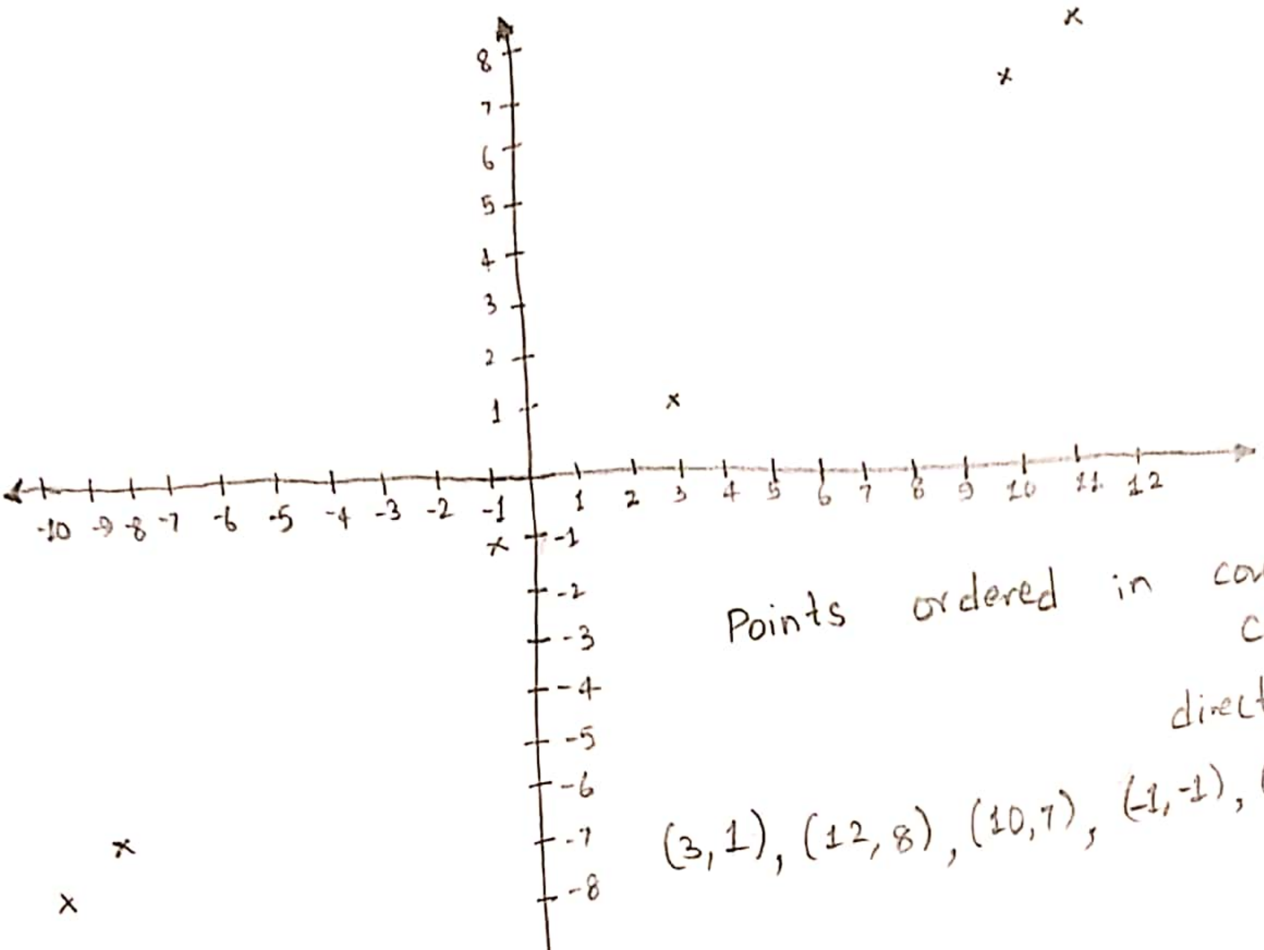
$$\downarrow \; R_3 \to R_3 + \frac{2}{3} R_2$$

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ 0 & \frac{3}{2} & -1 & \cdots & 0 \\ 0 & 0 & \frac{4}{3} & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & \cdots & & & \\ 0 & \frac{3}{2} & \cdots & & \\ 0 & 0 & 4/3 & \cdots & \\ 0 & 0 & 0 & 5/4 & \\ & & \cdots & & \ddots \\ 0 & 0 & 0 & 0 \cdots & \frac{n+1}{n} \end{bmatrix}$$

$$\therefore \quad \det(A_n) = 2 \times \frac{3}{2} \times \frac{4}{3} \cdots \frac{n+1}{n} = (n+1)$$

(6) $$A = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} \right)$$



Points ordered in counter-clockwise direction are,

(3,1), (12,8), (10,7), (-1,-1), (-10,-8), (-8,-7)

$$\therefore A = \frac{1}{2}\left(\begin{vmatrix} 3 & 12 \\ 1 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} 10 & -1 \\ 7 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -10 \\ -1 & -8 \end{vmatrix} + \begin{vmatrix} -10 & -8 \\ -8 & -7 \end{vmatrix} + \begin{vmatrix} -8 & 3 \\ -7 & 1 \end{vmatrix} \right)$$

$$= \frac{1}{2}\left(12 + 4 - 3 - 2 + 6 + 13 \right)$$

$$= \boxed{15}$$

⑦ (1) False

Let, $A = B = I$

$\det(A+B) = 2^5 = 32$

$\det(A) + \det(B) = 1 + 1 = 2$

(2) $\det(-A) = (-1)^n \det(A)$

Here, $n = 6$

$\therefore \det(-A) = \det(A)$

True

(3) False

If any row is linear combination of other rows,

then $\det(A) = 0$

(4) False

If $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)} (\text{adj}(A))$

$\Rightarrow$ "A" is invertible.

(5) True

calculate determinant with the column that is multiplied by 9.

Then take "9" Common from every term.

$\therefore \det(B) = 9 \times \det(A)$

(6) True

$$\det(A \times A^T) = \det(A) \cdot \det(A^T)$$
$$= \det(A^T) \times \det(A)$$
$$= \det(A^T \times A)$$

(7) False

Because, 2 and 4 are linearly dependent.

(8) True

$$\det(A^T) = \det(A)$$

$$\therefore \det(A^T) \cdot \det(A^{-1}) = \det(A) \times \frac{1}{\det(A)} = 1$$

(9) False

$\det(4A) = 4^4 \det(A)$, for a 4x4 matrix

(10) True

$$\det(4A) = 4^4 \det(A) = 256 \det(A)$$

$\therefore$ If $\det(A) = 0$, we have

$$256 \det(A) = 4 \det(A)$$

There are infinitely many matrices apart from 0 matrix.

(11) True

$$\det(AB) = \det(A) \cdot \det(B)$$
$$= \det(B) \cdot \det(A)$$
$$= \det(BA)$$

(8)

$$A^2 = -I$$

$$\Rightarrow [\det(A)]^2 = (-1)^3$$

$$\Rightarrow \det(A) = \pm i$$

$\therefore$ det(A) is complex

$\Rightarrow$ "A" has complex entries

A has two complex eigen values

$i, -i$

$\therefore$ A is similar to,

$$B = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_n \end{pmatrix},$$

Such that,

$m+n = 3$

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(9)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{vmatrix} \xrightarrow{2R_1} \frac{1}{2} \begin{vmatrix} 2 & 4 & 6 & 8 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{vmatrix}$$

$$\xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ \frac{R_1}{2}}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 7 & 5 \end{vmatrix} \xrightarrow{R_3 \leftrightarrow R_4} - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$\downarrow R_3 - 3R_2$$

$$- \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$\therefore \Delta = -(1 \times 1 \times 1 \times 2)$$

$$= \boxed{-2}$$