

①

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 3 \\ 3 & -2 & 13 \end{bmatrix}$$

Sai Nikhil

NUID: 001564864

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2 < 3$$

\Rightarrow linearly dependent

②

$$A = \begin{bmatrix} 1 & 1 & 17 & 1 & 3 \\ 5 & 5 & 85 & 5 & 16 \\ 4 & 3 & 56 & 2 & 13 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & 12 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) $\text{rank}(A) = 3 \Rightarrow A$ spans \mathbb{R}^3 .

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

(3)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 9 \\ 5 & 9 & 7 \\ 1 & 0 & -5 \end{bmatrix}$$

(a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

\Rightarrow linearly independent

(b) Basis of $\mathbb{R}^4 \leftarrow \left[\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{4^{\text{th}} \text{ vector}} \right\} \right]$

(4)

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 \\ -x_1 + 2x_2 \end{bmatrix}$$

$$2(x_1 + x_2) - 3(x_1) - 1(-x_1 + 2x_2) = 0$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

(5)

(1)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 4 \\ 0 & 0 & 1 \\ -3 & -4 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

\rightarrow linearly independent

(2)

Basis of \mathbb{R}^5

$$\left(\begin{bmatrix} 1 \\ 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{extension}} \right)$$

⑥

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 2 & 3 & 7 & -2 & 10 \\ 1 & 1 & 3 & 1 & 6 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow u_1, u_2, u_4$ is a basis

⑦

$$[A|b] = \left[\begin{array}{cccc|c} 1 & -5 & 10 & -1 & 2 \\ 2 & 11 & -23 & 2 & -4 \\ -4 & -23 & 49 & -4 & 8 \\ 1 & 2 & -1 & 1 & -2 \end{array} \right] \xrightarrow{\substack{R_2 + 2R_1 \\ R_3 - 4R_1 \\ R_4 + R_1}} \left[\begin{array}{cccc|c} -1 & -5 & 10 & -1 & 2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & -3 & 9 & 0 & 0 \\ 0 & -3 & 9 & 0 & 0 \end{array} \right]$$

$$\text{rref} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & -2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-R_1 \\ R_1 - 5R_2 \\ R_3 + 3R_2 \\ R_4 + 3R_2}}$$

$$\therefore \text{rank}(A) = \text{rank}([A|b]) = 2 < 4 \text{ (number of variables)} \\ \Rightarrow \text{infinitely many solutions}$$

$$\text{Nullity} = 4 - \text{rank}(A) = 4 - 2 = 2.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \ker(A) = \text{Nul}(A) = \text{Span} \left(\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$A^T = \begin{pmatrix} -1 & 2 & -4 & 1 \\ -5 & 11 & -23 & 2 \\ 10 & -23 & 49 & -1 \\ -1 & 2 & -4 & 1 \end{pmatrix}$$

$$\begin{aligned} R_2 - 5R_1 \\ R_3 + 10R_1 \\ R_4 - R_1 \end{aligned}$$

$$\begin{pmatrix} -1 & 2 & -4 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & -3 & 9 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} -R_1 \\ R_1 + 2R_2 \\ R_3 + 3R_2 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & -2 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

← rref

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{im}(A) = \text{col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -3 \end{bmatrix} \right\}$$

⑧

$$L(x) = x_1 L(e_1) + x_2 L(e_2) + x_3 L(e_3),$$

because "L" is Linear

$$\Rightarrow L(x) = \begin{bmatrix} x_1 + 5x_2 + 7x_3 \\ 2x_1 + 2x_2 - 3x_3 \\ 3x_1 + x_2 + 9x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix}$$

⑨

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 0 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

From Home work 1,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$$

rank = 3 = dimension of Codomain \mathbb{R}^3

\Rightarrow "L" is Surjective

$$\text{nullity} = 4 - 3 = 1 \neq 0$$

\Rightarrow "L" is not injective

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \\ 1 \end{bmatrix} = t \begin{bmatrix} 6 \\ 8 \\ 2 \\ -7 \end{bmatrix}$$

where $-\infty < t < \infty$

$$\text{Basis}(\ker(T)) = \left\{ \begin{bmatrix} 6 \\ 8 \\ 2 \\ -7 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \\ 4 & -2 & 2 \end{bmatrix} \Rightarrow \text{ref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis}(\text{Im}(A)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(10)

$$M: \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \text{rank}(A) = 3$$

$$\therefore \text{nullity} = 4 - \text{rank}(A) \\ = 4 - 3 = \boxed{1}$$

(11)

$$A = \begin{bmatrix} -1 & -2 & -1 & 1 & -1 \\ 2 & 4 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⌊ (1)

$$\text{rank}(A) = 3$$

image

$$A^T = \begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 4 & 2 & 0 \\ -1 & 5 & 4 & 0 \\ 1 & 1 & 4 & 2 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis of } \text{Im}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

From (1),

$$x_1 + 2x_2 + x_3 + 2x_5 = 0$$

$$+ x_3 - \frac{x_5}{2} = 0$$

$$+ x_4 + \frac{x_5}{2} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ +1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\text{Basis}(\text{Ker}(A)) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ +1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$$\text{ref}((A)_{Z_3}) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

└ (2)

$$(A^T)_{z_3} = \begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 4 & 2 & 0 \\ -1 & 5 & 4 & 0 \\ 1 & 1 & 4 & 2 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

$$\text{ref}((A^T)_{z_3}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Basis}(\text{Im}((A^T)_{z_3})) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

From (2),

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis of kernel} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(12)

Consider A with columns $\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n$

then

$$\vec{L} = A\vec{x} = x_1\vec{A}_1 + x_2\vec{A}_2 + \dots + x_n\vec{A}_n$$

$n > m$, $\{A_1, A_2, \dots, A_n\}$ is linearly dependent

$$\Rightarrow x_1\vec{A}_1 + x_2\vec{A}_2 + \dots + x_n\vec{A}_m = \vec{0}$$

has a non-trivial solution.

which is a non-zero vector

\Rightarrow Linear transformation is not injective.

(13)

a)

Consider 2 standard basis vectors

$$e_1 = (1, 0) \text{ \& } e_2 = (0, 1)$$

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

gives the vector after counter

clockwise rotation

2)

$$\vec{b} = \begin{bmatrix} x \\ y \end{bmatrix}$$

we observe, $e_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e_2 \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3)

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \rightarrow \begin{bmatrix} x + \gamma y \\ y \end{bmatrix}$$

$$e_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 \rightarrow \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

4)

$$L: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$L = r\vec{x}, \quad r > 0$$

$$L(e_i) = re_i$$

$$A = \begin{bmatrix} r & 0 & 0 & 0 & \dots & 0 \\ 0 & r & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \dots & r \end{bmatrix}$$

$$L = A x = r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$