Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 4.

1. Reading: [Gockenbach], Chapters 4 and 5.

2. Questions: (You can use Matlab if needed, e.g. eigenvalues by eig(A))

The following questions are about eigenvalues and eigenspaces.

Question 1. An $n \times n$ matrix A is called nilpotent if there exists an integer k such that $A^k = 0$. Find all possible eigenvalues of A.

Question 2. Let $A \in \mathbb{R}^{2\times 2}$ defined by

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. (Notice that A is symmetric, that is, $A^T = A$.)

(1) Prove that A has only real eigenvalues.

(2) Under what conditions on a, b, c does A have a multiple eigenvalue?

Question 3. Let $A \in \mathbb{F}^{n\times n}$ be an invertible matrix. Show that every eigenvector of A is also an eigenvector of A^{-1}. What is the relationship between the eigenvalues of A and A^{-1}?

Question 4. For each of the following real matrices, find the eigenvalues and a basis for each eigenspace. (Use Matlab)

(1) $A = \begin{bmatrix} -15 & 0 & 8 \\ 0 & 1 & 0 \\ -28 & 0 & 15 \end{bmatrix}$

(2) $B = \begin{bmatrix} -4 & -4 & -5 \\ -6 & -2 & -5 \\ 11 & 7 & 11 \end{bmatrix}$

(3) $C = \begin{bmatrix} 6 & -1 & 1 \\ 4 & 1 & 1 \\ -12 & 3 & -1 \end{bmatrix}$

(4) $D = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(5) $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Question 5. Which matrices in the above question are diagonalizable? If it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Question 6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}.$$

Find a basis $\mathscr{B}$ for $\mathbb{R}^3$ such that $[T]_{\mathscr{B},\mathscr{B}}$ is diagonal. What is the matrix $[T]_{\mathscr{B},\mathscr{B}}$?

Question 7. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times m}$ and $n \geq m$.

(1) Show that AB and BA has the same non-zero eigenvalues with the same algebraic multiplicities.

(2) If 0 is an eigenvalue of AB with algebraic multiplicity k, what is the algebraic multiplicity of 0 as eigenvalue of BA.

Question 8. (1) Find the characteristic polynomial of $B = \begin{bmatrix} 0 & a \\ -1 & b \end{bmatrix}$.

(2) Shows that every monic polynomial

$$f(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$$

is the characteristic polynomial of some matrix B. (Hint: look at (1))

The following two questions are about Cayley-Hamilton Theorem and Jordan normal forms.

Question 9. Let A and B be 2×2 matrices such that $(AB)^2 = \mathbf{0}$. Prove that $(BA)^2 = \mathbf{0}$.

Question 10. (1) Let A be a 3×3 matrix such that the traces $tr(A^k)$=0 for $k = 1, 2, 3$. Show that all eigenvalues of A are zeros.

(2) Is there a 3×3 nilpotent matrix such that $A^3 \neq \mathbf{0}$?