Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 2.

1. Reading: [Gockenbach], Chapters 2 and 3.

Reminder: Two notations of **column** vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (v_1, v_2, v_3)$.

2. Questions: (You can use Matlab to calculate **rref**)

Question 1. Show that $\{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 13 \end{bmatrix}\} \in \mathbb{R}^3$ is linearly dependent by writing one of the vectors as a linear combination of the others.

Question 2. Consider the following vectors in $\mathbb{R}^3$:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 17 \\ 85 \\ 56 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 3 \\ 16 \\ 13 \end{bmatrix}$

(a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ spans $\mathbb{R}^3$.

(b) Find a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ that is a basis for $\mathbb{R}^3$.

Question 3. Consider the following vectors in $\mathbb{R}^4$:

$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 7 \\ -5 \end{bmatrix}$

(a) Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent.

(b) Find a vector $\vec{u}_4 \in \mathbb{R}^4$ such that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a basis for $\mathbb{R}^4$.

Question 4. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Show that $S = \text{Span}\{\vec{u}, \vec{v}\}$ is a plane in $\mathbb{R}^3$ by showing there exist constants $a, b, c \in \mathbb{R}$ such that

$$\text{Span}\{\vec{u}, \vec{v}\} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid ax_1 + bx_2 + cx_3 = 0 \right\}$$

Question 5. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}; \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}; \vec{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ be vectors in $\mathbb{R}^5$.

(1) Show that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is linearly independent.

(2) Extend $\vec{u}_1, \vec{u}_2, \vec{u}_3$ to a basis for $\mathbb{R}^5$.

Question 6. Consider the following vectors in $\mathbb{R}^5$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_5 = \begin{bmatrix} 2 \\ 10 \\ 3 \\ 6 \\ 2 \end{bmatrix},$$

Let $S = \mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$. Find a subset of $\mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$ that is a basis for S.

Question 7. Apply the row reduction algorithm to solve each of the following systems of equations. In each case, state whether the system has no solution, exactly one solution, or infinitely many solutions. Also, state the rank and nullity of A, where A is the coefficient matrix of the system, and find a basis for $\ker(A) = Nul(A)$ and a basis for $\mathrm{im}(A) = \mathrm{Col}(A)$, where possible.

$$\begin{aligned} -x_1 - 5x_2 + 10x_3 - x_4 &= \ 2 \\ 2x_1 + 11x_2 - 23x_3 + 2x_4 &= \ -4 \\ -4x_1 - 23x_2 + 49x_3 - 4x_4 &= \ 8 \\ x_1 + 2x_2 - x_3 + x_4 &= \ -2 \end{aligned}$$

Question 8. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the following conditions:

(a) L is linear; (b) $L(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; (c) $L(\vec{e}_2) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$; (d) $L(\vec{e}_3) = \begin{bmatrix} 7 \\ -3 \\ 9 \end{bmatrix}$;

Here $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Prove that there is a 3×3 matrix A such that $L(x) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. What is the matrix A?

Question 9. Find bases of the **kernel** and the **image** of the linear map $L : \mathbb{R}^4 \to \mathbb{R}^3$ described by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 0 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

(with respect to the standard bases). Is L injective or surjective? (We already have **rref**(A) in homework1.)

Question 10. Define $M : \mathbb{R}^4 \to \mathbb{R}^3$ by $M(x) = \begin{bmatrix} x_1 + 3x_2 - x_3 - x_4 \\ 2x_1 + 7x_2 - 2x_3 - 3x_4 \\ 3x_1 + 8x_2 - 3x_3 - 16x_4 \end{bmatrix}$.

Find the rank and nullity of M.

Question 11. Consider the 4×5 matrix

$$A := \begin{bmatrix} -1 & -2 & -1 & 1 & -1 \\ 2 & 4 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

over a field $\mathbb{F} = \mathbb{R}$.

(a) Find the row reduced echelon form for A.

(b) Find the rank of A.

(c) Find a basis of $\mathrm{im}(f_A)$, where the linear mapping $f_A : \mathbb{F}^5 \to \mathbb{F}^4$ is defined by $f_A(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{F}^5$.

(d) Find a basis of the solution set of $f_A(\vec{x}) = 0$, with f_A as in part (c).

(e*) Solve problems (a)(b)(c)(d) for the case: $\mathbb{F} = \mathbb{Z}_3$.

Question 12. Let A be an $m \times n$ matrix with real entries, and suppose $n > m$. Prove the linear transformation defined by A is not injective. (That is, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$.)

Question 13. Find matrix of each linear operator: (Hint: using theorem on matrix of linear transformation.)

(1.) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** of angle θ about the origin (a positive θ indicates a counterclockwise rotation). Find the matrix A such that $R(x) = Ax$ for all $x \in \mathbb{R}^2$.

(2.) Consider the linear operator mapping $\mathbb{R}^2$ into itself that sends each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to its **projection** onto the x-axis, namely, $\begin{bmatrix} x \\ 0 \end{bmatrix}$. Find the matrix representing this linear operator.

(3.) A (horizontal) **shear** acting on the plane maps a **point** $\begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\begin{bmatrix} x + ry \\ y \end{bmatrix}$, where r is a real number. Find the matrix representing this operator.

(4.) A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = r\vec{x}$ is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$. What is the matrix of L?

Question 14. Consider the following geometrically defined linear maps of $\mathbb{R}^3$ to itself. Describe each of them by a matrix with respect to the canonical basis of $\mathbb{R}^3$. (Hint: using theorem on matrix of linear transformation.)

(a) Orthogonal projection onto the xz-plane.

(b) Counterclockwise rotation by 45°about the x-axis.

(c) The map (rotation) of part (b) then followed by the map(projection) of part (a).

(d) Rotation by 120° about the main diagonal in space (spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, taken counterclockwise as you look towards the origin.

Question 15. Let $x \in R^N$ be denoted as $x = (x_1, x_2, ..., x_N)$. Given $\vec{x}, \vec{y} \in \mathbb{R}^N$, the **convolution** of $\vec{x}$ and $\vec{y}$ is the vector $\vec{x} * \vec{y} \in \mathbb{R}^N$ defined by

$$(\vec{x} * \vec{y})_n = \sum_{m=1}^{N} x_m y_{n-m}, \quad \text{for } n = 1, 2, ..., N.$$

In this formula, $\vec{y}$ is regarded as defining a periodic vector of period N; therefore, if $n - m \leq 0$, we take $y_{n-m} = y_{N+n-m}$. For instance, $y_0 = y_N, y_{-1} = y_{N-1}, y_{-2} = y_{N-2}$, and so forth.

(1) Prove that if $y \in \mathbb{R}^N$ is fixed, then the mapping

$$L : \vec{x} \to \vec{x} * \vec{y}$$

is linear. (2) Find the matrix representing this operator L.

Question 16. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, $\vec{b} \in \mathbb{R}^3$ is given, and $\vec{u} = (1, 0, 1)$, $\vec{v} = (1, 1, -1)$ are two solutions to $L(x) = b$. Find two more solutions to $L(\vec{x}) = \vec{b}$.

Question 17. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ has kernel $\ker(T) = \mathrm{Span}\{\vec{u} = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \vec{v} = \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\}$. Suppose further that $T(\vec{y}) = \vec{b}$, where $\vec{y} = \begin{bmatrix}1\\2\\-1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\1\\-2\\-1\end{bmatrix}$. (1) Find all possible solutions to $T(\vec{x}) = b$. Explain the reason.

(2) Is $\vec{z} = \begin{bmatrix}0\\4\\0\\1\end{bmatrix}$ a solution of $T(\vec{x}) = \vec{b}$?

Question 18. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy $\ker(L) = \mathrm{Span}\{(1, 1, 1)\}$ and $L(\vec{u}) = \vec{v}$, where $\vec{u} = (1, 1, 0)$ and $\vec{v} = (2, -1, 2)$. Which of the following vectors is a solution of $L(\vec{x}) = \vec{v}$?

(a) $\vec{x} = (1, 2, 1)$

(b) $\vec{x} = (3, 3, 2)$

(c) $\vec{x} = (-3, -3, -2)$

Question 19. Consider the linear subspaces U and W of $\mathbb{R}^4$ spanned by $\vec{u}_1 := \begin{bmatrix}-1\\3\\1\\0\end{bmatrix}, \vec{u}_2 := \begin{bmatrix}1\\0\\1\\-1\end{bmatrix}, \vec{u}_3 := \begin{bmatrix}2\\2\\1\\-3\end{bmatrix}$ and $\vec{w}_1 := \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}, \vec{w}_2 := \begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \vec{w}_3 := \begin{bmatrix}2\\-2\\-1\\-1\end{bmatrix}, \vec{w}_4 := \begin{bmatrix}2\\2\\1\\-1\end{bmatrix}$ respectively.

Find the **dimensions** of the sum $U + W$, the intersection $U \cap W$, and the quotient spaces $\mathbb{R}^4/U$ and $\mathbb{R}^4/W$.

Question 20. Let V be a vector space over a field $\mathbb{K}$, and let $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors of V such that any $n - 1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent. Show:

(a) There exist scalars $\alpha_1, \ldots, \alpha_n$ in $\mathbb{K}$, **all** nonzero, such that $\sum_{j=1}^{n} \alpha_j \vec{v}_j = \vec{0}$.

(b) If $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are two sets of nonzero scalars in $\mathbb{K}$ such both $\sum_{j=1}^{n} \alpha_j \vec{v}_j = 0$ and $\sum_{j=1}^{n} \beta_j \vec{v}_j = 0$ then there exists a nonzero scalar γ in K such that $\beta_j = \gamma \alpha_j$ for each $j = 1, ..., n$.