Sai Nikhil
NUID: 001564864

1

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\mathrm{rref}(A^T) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\therefore \mathrm{im}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$e_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ 0 \\ \sqrt{6}/3 \end{bmatrix}$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{\|u_1\|^2} u_1$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \\ -1/3 \end{bmatrix}$$

$$e_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/2 \\ -\sqrt{3}/6 \end{bmatrix}$$

orthogonal basis $= \{e_1, e_2\}$

$$\operatorname{Proj}_{im(A)} \vec{y} = (\vec{e_1} \cdot \vec{y})\vec{e_1} + (\vec{e_2} \cdot \vec{y})\vec{e_2}$$

$$= \boxed{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}}$$

(2)

$$\vec{y}_2 = \vec{y} - \operatorname{proj}_{im(A)} \vec{y}$$

$$= \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

Shortest distance $= \| \vec{y_2} \|$

$$= \sqrt{2^2 + (-1)^2 + (-1)^2}$$

$$= \boxed{\sqrt{6}}$$

3

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} ; B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} ; B^T = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\langle A, B \rangle = tr\left(A^T B\right)$$

$$A^T B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$B^T B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\langle A, B \rangle = \mathrm{tr}(A^T B) = 5$$

$$\langle A, A \rangle = \mathrm{tr}(A^T A) = 5$$

$$\langle B, B \rangle = \mathrm{tr}(B^T B) = 9$$

$$\theta_{(A,B)} = \cos^{-1}\left(\frac{\langle A, B \rangle}{\sqrt{\langle A, A \rangle} \cdot \sqrt{\langle B, B \rangle}}\right)$$

$$= \cos^{-1}\left(\frac{5}{\sqrt{5} \cdot \sqrt{9}}\right)$$

$$= \boxed{\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)}$$

4

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$$AA^T \neq A^T A$$

$\boxed{\text{False}}$

(2) $A \to$ orthogonal $\Rightarrow A^T A = AA^T = I_n$

$B = A^T$

$B^T B = (A^T)^T A^T = AA^T = I_n$

$\therefore B \to$ orthogonal $\Rightarrow$ rows of "A" form orthonormal basis

$\boxed{\text{True}}$

(5)

$$A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}, \quad 0 \leq a, b \leq 1$$

From MATLAB,

Eigen values $= 1, b-a$

Eigen vectors $= \left\{ \begin{bmatrix} -\frac{b}{a-1} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$\therefore A = P D P^{-1}$$

$$= \begin{bmatrix} -b/a-1 & -1 \\ 1 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix} \times$$

$$\frac{1}{b-a+1} \begin{bmatrix} -(a-1) & -(a-1) \\ & b \\ (a-1) & \end{bmatrix}$$

Case-1:

$$0 \le a, b, < 1$$

$$\therefore \boxed{-1 < (a-b) < 1}$$

$$\therefore \lim_{n\to\infty} A^n = \lim_{n\to\infty} P D^n P^{-1}$$

$$\lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \lim_{n\to\infty} A^n = \boxed{\frac{1}{b-a+1} \begin{bmatrix} b & b \\ -a+1 & -a+1 \end{bmatrix}}$$

Case-2: $a=0, \quad b=1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A^2 = I, \quad A^3 = A \; \ldots$$

$$\therefore A^{2k} = I, \quad A^{2k+1} = A$$

This is oscillating. So, limit doesn't exist in this case

Case -3:

$a = 1, \quad b = 0$

$$\lim_{n \to \infty} A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \lim_{n \to \infty} A^n = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

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(2)

$$\|\vec{u} + a\vec{v}\|^2 = \langle \vec{u} + a\vec{v}, \vec{u} + a\vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle + 2a \underbrace{\langle \vec{u}, \vec{v} \rangle}_{= 0} + a^2 \langle \vec{v}, \vec{v} \rangle$$

$$= \|\vec{u}\|^2 + a^2 \|\vec{v}\|^2 \geq \|\vec{u}\|^2$$

$$\Rightarrow \boxed{\|\vec{u} + a\vec{v}\| \geq \|\vec{u}\|}$$

(equality comes when $a = 0$)

Let

$$\|\vec{u}\| \leq \|\vec{u} + a\vec{v}\|$$

$$\Rightarrow \|\vec{u}\|^2 \leq \|\vec{u} + a\vec{v}\|^2$$

$$\Rightarrow a^2\|\vec{v}\|^2 + 2a\langle\vec{u}, \vec{v}\rangle \geq 0$$

$$\Rightarrow 2a\langle\vec{u}, \vec{v}\rangle \geq -a^2\|\vec{v}\|^2$$

Case-1:

$a > 0$

$$\langle\vec{u}, \vec{v}\rangle \geq -\frac{a}{2}\|\vec{v}\|^2$$

Let $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a = 1$

$\therefore \quad 8 \geq -1$ [above condition is satisfied]

But, $\langle\vec{u} \cdot \vec{v}\rangle \neq 0$

Case-2:

$a < 0$

$$\langle \vec{u}, \vec{v} \rangle \leq -\frac{a}{2} \|\vec{v}\|^2$$

Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $a = -2$

$\therefore \quad -5 \leq 25$ [above Condition is satisfied]

But $\langle \vec{u}, \vec{v} \rangle \neq 0$

Case-3:

$a = 0$ Given equation is clearly satisfied

$$\|\vec{u}\| \leq \|\vec{u} + a\vec{v}\|$$

$\vec{v}$ can be any random vector

$$\therefore \|\vec{u}\| \leq \|\vec{u} + a\vec{v}\| \not\Rightarrow \langle \vec{u} \cdot \vec{v} \rangle = 0$$