MTH 7241: Fall 2020

Practice Problems for Test 2

1). Consider an irreducible chain on 3 states. Prove or give a counterexample: $p_{jj}(3) > 0$ for every state j.

2). Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \dots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, let M_k be the expected number of steps until the walk reaches either 0 or N.

a). By conditioning on X_1, derive a recursion formula for M_k.

b). Compute the boundary conditions M_0 and M_N.

c). Show that the recursion formula from (a) is satisfied by the special solution $M_k = c\,k$ and compute the value of the constant c (for this part you should ignore the boundary conditions in (b)).

d). Show that the recursion formula from (a) is satisfied by the solution $M_k = c\,k + A + B(q/p)^k$ where c was computed in (c), and A, B are constants (again ignore the boundary conditions in (b)).

e). Use the boundary conditions from (b) to find the values of A and B in (d).

3). A reduced 4×4 chessboard has 16 squares, labeled $i = 1, \dots, 16$. Let $d(i)$ denote the number of available nearest neighbors to the square i, where we allow vertical, horizontal and diagonal moves (so for example the corner has 3 neighbors; a square in the middle has 8 neighbors). A random walk is constructed on the board as follows:

$$p_{i,j} = \begin{cases} \dfrac{1}{d(i)} & \text{if } j \text{ is an available nearest neighbor to } i \\ 0 & \text{otherwise} \end{cases}$$

a). Find the stationary distribution of the chain. [Hint: write down and solve the time reversible equations for the chain].

b). Starting in a corner, find the expected number of steps until the first return to the same corner.

4). An irreducible persistent Markov chain X_n is defined on the infinite state space $S = \{1, 2, 3, \ldots\}$. The chain has a stationary positive vector $v = (v_1, v_2, \ldots)$ where

$$v_k = \frac{1}{1+k}$$

a). Suppose $u = (u_1, u_2, \ldots)$ is another stationary positive vector for the chain. Calculate u_1/u_2.

b). Determine whether the chain X_n is null persistent or positive persistent.

5). Let $X_1, X_2, \ldots$ be IID random variables, where the moment generating function is

$$\mathrm{E}[e^{tX}] = e^{t^2 - 3t}$$

a). Find the mean $\mu = \mathrm{E}[X]$.

b). Let $Y_n = (1/n) \sum_{i=1}^{n} X_i$. Use Cramer's Theorem to compute

$$\lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}(Y_n > -2)$$

6). For a branching process, calculate the probability of extinction when

$$p_0 = \mathrm{P}(Z = 0) = 1/12, \quad p_1 = \mathrm{P}(Z = 1) = 1/2, \quad p_2 = \mathrm{P}(Z = 2) = 5/12.$$

7). Suppose that the distribution of Z for a branching process is geometric, so that

$$p_k = \mathrm{P}(Z = k) = x^k (1 - x) \quad \text{for} \quad k = 0, 1, 2, \ldots$$

where $0 < x < 1$.

a). Find the largest value of x for which extinction is guaranteed. Call this value x_m.

b). For $x > x_m$ compute the probability of extinction.