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$$A^k = 0$$

$$Ax = \lambda x$$

$$A^2 x = A(Ax) = \lambda Ax = \lambda^2 x$$

$$\vdots$$

$$A^k x = \lambda^k x$$

$$\Rightarrow \lambda^k = 0$$

$$\Rightarrow \boxed{\lambda = 0}$$

2 (1)

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

For eigen values,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (\lambda - a)(\lambda - c) - b^2 = 0$$

$$\lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\Delta = (a+c)^2 - 4(ac-b^2)$$

$$= (a-c)^2 + 4b^2 \geq 0$$

$\therefore$ "λ" is real.

(2) Multiple eigen values

$\Rightarrow \quad \Delta > 0$

$\Rightarrow \quad a \neq c$ (or) $b \neq 0$ (or) both

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3 $A x = \lambda x$

$x \to$ eigenvector of "A"

$\lambda \to$ eigen value of 'A'

Multiply with "A^{-1}" from left on both sides

$$I x = \lambda A^{-1} x$$

$$\Rightarrow \quad A^{-1} x = \left(\frac{1}{\lambda}\right) x$$

$x \to$ eigen vector of A^{-1}

$\frac{1}{\lambda} \to$ eigen value of A^{-1}

$\therefore A^{-1}$ has same eigen vectors as A.

Also, Eigen values of A^{-1} are of the form "$\frac{1}{\lambda}$".

④

(1) $\lambda_1 = 1$ (multiplicity $= 2$)

$\lambda_2 = -1$ (multiplicity $= 1$)

For $\lambda_1 = 1$,

basis of eigenspace $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$

For $\lambda_2 = -1$, $\left\{ \begin{pmatrix} \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right\}$

(2) $\lambda_1 = 1$ (multiplicity $= 1$)

$\lambda_2 = 2$ (multiplicity $= 2$)

For $\lambda_1 = 1$, basis of eigenspace $= \left\{ \begin{pmatrix} -\frac{5}{9} \\ -\frac{5}{9} \\ 1 \end{pmatrix} \right\}$

For $\lambda_2 = 2$, basis $= \left\{ \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$

(3) $\lambda_1 = 2$ (mulplicity $= 3$)

For $\lambda_1 = 2$, basis $= \left\{ \begin{pmatrix} \frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix} \right\}$

(4) $\lambda_1 = 1-2i$ (multiplicity = 1)

$$\text{basis} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

$\lambda_2 = 1+2i$ (multiplicity = 1)

$$\text{basis} = \left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

(5) $\lambda_1 = -1$ (multiplicity = 2)

$$\text{basis} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\lambda_2 = 2$ (multiplicity = 1)

$$\text{basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

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5 (1)

$$A = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{4}{7} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -14 & 0 & 8 \\ 14 & 0 & -7 \end{bmatrix}}_{P^{-1}}$$

(2) B is not diagonalizable.

No "3" linearly independent eigen vectors.

(3) C is not diagonalizable.

No "3" linearly independent eigen vectors.

(4) $$D = \underbrace{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}}_{P} \cdot \underbrace{\begin{pmatrix} 1-2i & 0 \\ 0 & 1+2i \end{pmatrix}}_{D} \cdot \underbrace{\begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}}_{P^{-1}}$$

(5) $$E = \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{P.} \cdot \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \cdot \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{P^{-1}}$$

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$$E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$T_{EE} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial, $\det(T_{EE} - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$$

$$= -\lambda^3 + \lambda + \lambda + 1 + \lambda + 1$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= -(\lambda + 1)^2 (\lambda - 2)$$

$\therefore$ Eigen values are $-1, 2$

$$T_{EE} = \underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{B} \times \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{T_{BB}} \times \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{B^{-1}}$$

$$\therefore T_{BB} = B^{-1} T_{EE} B$$

$\therefore$ Basis (B) $= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, $T_{BB} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

⑦

(1) $AB x = \lambda x$

let, $Bx = y \quad (\lambda \neq 0 \Rightarrow y \neq 0)$ — (i)

Consider, $BAy = BABx$

$= B\lambda x$

$= \lambda (Bx)$

$= \lambda y$

$\therefore BAy = \lambda y$ — (ii)

$\therefore$ From (i) & (ii), "λ" is a non-zero eigen value of "BA".

Since all λ are same, algebraic multiplicities are also same.

(2) let, $\alpha_1 \neq 0, \alpha_2 \neq 0, \dots \alpha_p \neq 0$ are p-different non-zero eigenvalues of AB and BA.

$\alpha_0 = 0$ has algebraic multiplicity "k" for A.

$\therefore$ characteristic equation of AB,

$\lambda^k \cdot (\lambda - \alpha_1)^{n_1} \cdot (\lambda - \alpha_2)^{n_2} \cdot \dots (\lambda - \alpha_p)^{n_p}$ — (iii)

characteric eqation of BA,

$\lambda^{z} \cdot (\lambda - \alpha_1)^{n_1} \cdot (\lambda - \alpha_2)^{n_2} \cdot \dots (\lambda - \alpha_p)^{n_p}$ — (iv)

From (iii) & (iv),

$$k + n_1 + n_2 + \cdots n_p = m \quad -(v)$$

$$x + n_1 + n_2 + \cdots n_p = n \quad -(vi)$$

$$(vi) - (v) \Rightarrow (x - k) = n - m$$

$$\therefore \boxed{x = n - m + k}$$

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8 (1) $|B - \lambda I|$ $\qquad B = \begin{bmatrix} 0 & a \\ -1 & b \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} -\lambda & a \\ -1 & b-\lambda \end{vmatrix}$$

$$= \lambda(\lambda - b) + a$$

$$= \lambda^2 - b\lambda + a$$

(2) $f(t) = t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0$

$n = 2 \Rightarrow f(t) = t^2 + c_1 t + c_0$

$$\therefore B_2 = \begin{bmatrix} 0 & c_0 \\ -1 & -c_1 \end{bmatrix}$$

$$n = 3 \Rightarrow f(t) = t^3 + c_2 t^2 + c_1 t + c_0$$

let,

$$B_3 = \begin{bmatrix} 0 & 1 & c_1 - c_0 \\ 0 & 0 & c_0 \\ -1 & -1 & -c_2 \end{bmatrix}$$

Characteristic polynomial, $\det(B_3 - tI) = t^3 + c_2 t^2 + c_1 t + c_0$

Extending the idea,

it is clear that,

every monic of the form,

$$f(t) = t^n + c_{n-1} t^{n-1} + \dots c_1 t + c_0$$

is a characteristic polynomial of some matrix B.

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Let, $C = AB$

$$C^2 - tr(C)\, C + det(C)\, I_2 = 0 \quad \text{— (i)}$$

Given, $(AB)^2 = 0$ — (ii)

$\therefore\ det(C^2) = 0$

$\Rightarrow det(C) = 0$ — (iii)

$\therefore$ From (i), (ii) & (iii),

$tr(C) \times C = 0$

$\therefore\ tr(C) = 0$ (or) $C = 0$

$\Rightarrow tr(C) = 0$

$det(BA) = det(AB) = det(C) = 0$ — (iv)

$tr(BA) = tr(AB) = tr(C) = 0$ — (v)

$\therefore$ According to Cayley-Hamilton Theorem,

$$(BA)^2 - tr(BA) \times BA + det(BA)\, I_2 = 0 \quad \text{— (vi)}$$

From (iv), (v) & (vi),

$$(BA)^2 = 0$$

(10)

Let,

(1) $\lambda_1, \lambda_2 \cdots \lambda_s \rightarrow$ non-zero eigenvalues of "A".

$\downarrow \quad \downarrow \quad \downarrow$

$n_1, n_2 \cdots n_s \rightarrow$ albegraic multiplicities.

Given, $tr(A^k) = 0, \quad k = 1, 2, 3$

$$\begin{cases} n_1\lambda_1 + n_2\lambda_2 + \cdots + n_s\lambda_s = 0 \\ n_1\lambda_1^2 + n_2\lambda_2^2 + \cdots + n_s\lambda_s^2 = 0 \\ \vdots \\ n_1\lambda_1^s + n_2\lambda_2^s + \cdots + n_s\lambda_s^s = 0 \end{cases} \quad \text{— (1)}$$

$$\Rightarrow \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^s & \lambda_2^s & \cdots & \lambda_s^s \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

we know,

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^s & \lambda_2^s & \cdots & \lambda_s^s \end{pmatrix} = \lambda_1 \cdots \lambda_s \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \vdots & \vdots & & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \cdots & \lambda_s^{s-1} \end{pmatrix} \neq 0$$

$$\therefore \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix}\text{Only possible eigen} \\ \text{value}\end{matrix}\right\} = 0$$

(2) $n \times n$ nilpotent matrix A

Let

$$f(t) = t^n$$

$$f(\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 0}$$

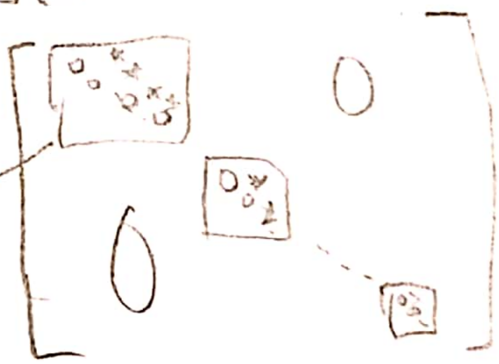
$\therefore$ All eigen values, $\lambda_i = 0$ (algebraic multiplicity $= n$)

$\Rightarrow$ A annhaibtes $f(t)$

$$\Rightarrow A^n = 0$$

Method 2:-

Let $A = P\, J(0)\, P^{-1}$

$J(0) =$ 

Let, the largest block (B) is of size $k \times k$ $(k \leq n)$

$\therefore B^k = 0$

$$\Rightarrow A^k = P\, J(0)^k\, P^{-1} = POP^{-1} = 0$$

$\therefore A^k = 0$

As $k \leq n \Rightarrow A^n = 0$.

$\therefore$ There is no 3×3 nilpotent matrix s.t. $A^3 \neq 0$