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Sai Nikhil

NUID: 001564864

Favorable events

Case#	C_1	C_2
1	H T	T T
2	T H	T T
3	H H	T T
4	H H	H T
5	H H	T H

$$P(\text{case \#1}) = P(\text{case\#2}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3}$$

$$= \frac{1}{36}$$

$$P(\text{case \#3}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{36}$$

$$P(\text{Case \#4}) = P(\text{case \# 5}) = \frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{36}$$

$$P(X>Y) = \sum_{i=1}^{5} P(\text{case\# } i) = \frac{7}{36}$$

② $P(A) = 0.7 \quad P(B) = 0.9$

$$P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$$

$$\text{Max}\left(P(A \cup B) - P(A \cap B)\right) = P(A) + P(B) - 2\min\left(P(A \cap B)\right)$$

Minimum occurs for independent events

$$= P(A) + P(B) - 2P(A) \times P(B)$$

$$= 0.7 + 0.9 - 2 \times 0.7 \times 0.9$$

$$= \boxed{0.34}$$

$$\text{Min}\left(P(A \cup B) - P(A \cap B)\right) = P(A) + P(B) - 2\max\left(P(A \cap B)\right)$$

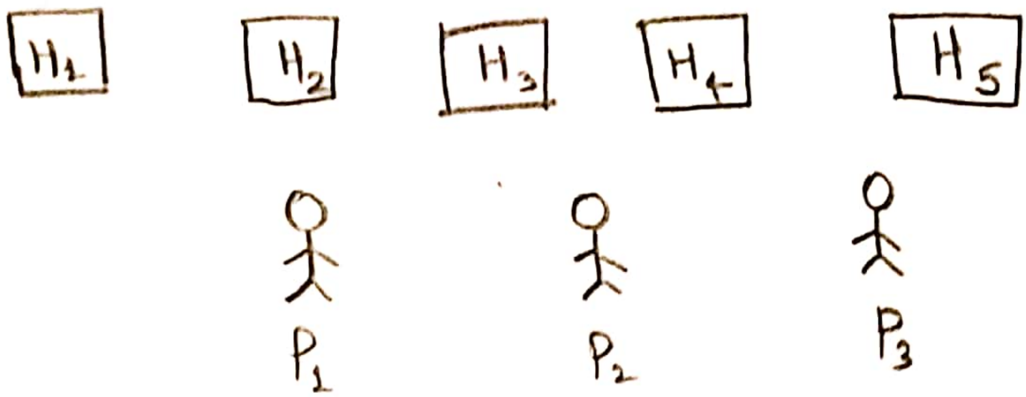
Maximum occurs when least probable event is inside most probable event

$$= P(A) + P(B) - 2P(A)$$

$$= P(B) - P(A)$$

$$= 0.9 - 0.7 = \boxed{0.2}$$

(3)



$$P\left(\begin{array}{c}\text{exactly two}\\ \text{in Same Hotel}\end{array}\right) = P\left(\begin{array}{r}P_1, P_2 \text{ in Same hotel and}\\ P_3 \text{ in different}\end{array}\right)$$

$$+ P\left(\begin{array}{r}P_2, P_3 \text{ in Same hotel and}\\ P_1 \text{ in different}\end{array}\right)$$

$$+ P\left(\begin{array}{r}P_3, P_1 \text{ in Same hotel and}\\ P_2 \text{ in different}\end{array}\right)$$

$$= 3 \times \left[\frac{\cancel{5} \times 4}{\underbrace{\cancel{5} \times 5 \times 5}_{\text{Total number of possible ways}}} \right]$$

$$= \boxed{\frac{12}{25}}$$

4

Geometric Distribution

$$P(X=n) = (1-p)^{n-1} p \, , \quad n = 1, 2, \ldots$$

$$E(X) = \sum_{i=1}^{\infty} i \cdot P(X=i) = \sum_{i=1}^{\infty} i (1-p)^{i-1} p$$

$$= p \times \sum_{i=1}^{\infty} -\frac{d}{dp}\left[(1-p)^{i}\right]$$

$$= -p \times \frac{d}{dp} \sum_{i=1}^{\infty} (1-p)^{i}$$

$$= -p \times \frac{d}{dp}\left[(1-p) + (1-p)^2 + \cdots \infty\right]$$

$$= -p \times \frac{d}{dp}\left(\frac{(1-p)}{1-(1-p)}\right)$$

$$= -p \times \frac{d}{dp}\left(\frac{(1-p)}{p}\right)$$

$$= -p \times \frac{\left[p(-1) - (1-p)\right]}{p^2}$$

$$= -\left[\frac{-p-1+p}{p}\right] = \boxed{+\frac{1}{p}}$$

$$Var[x] = E[x^2] - (E[X])^2$$

$$E[X^2] = \sum_{i=1}^{\infty} i^2 (1-p)^{i-1} p = p \sum_{i=1}^{\infty} i^2 (1-p)^{i-1}$$

$$= p \cdot \frac{1+(1-p)}{[1-(1-p)]^2} = \frac{(2-p)}{p^2}$$

$$Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \boxed{\frac{1-p}{p^2}}$$

(5)

$$R.H.S. = \sum_{n=1}^{\infty} P(N \geq n)$$

$$= P(N \geq 1) + P(N \geq 2) + P(N \geq 3) + \cdots \infty$$

$$= \left[\underline{P(N=1)} + \underline{\underline{P(N=2)}} + \underline{\underline{\underline{P(N=3)}}} + \cdots \infty\right] +$$

$$\left[\underline{\underline{P(N=2)}} + \underline{\underline{\underline{P(N=3)}}} + P(N=4) + \cdots \infty\right] +$$

$$\left[\underline{\underline{\underline{P(N=3)}}} + P(N=4) + P(N=5) + \cdots \infty\right]$$

$$= 1 \times P(N=1) + 2 \times P(N=2) + 3 \times P(N=3) + \cdots \infty$$

$$= \sum_{i=1}^{\infty} i \, P(N=i) = E(N) = L.H.S.$$

Hence, proved.

(6)

$$x_1 + x_2 + x_3 + \cdots x_n = r$$

Total number of ways to distribute r balls in n boxes $\Big\} = {}^{n+r-1}C_{n-1}$

Case #1: First box is empty

Total number of ways to distribute r balls in boxes labeled 2 to n $\Big\} = {}^{n+r-2}C_{n-2}$

$$\text{Probability} = \frac{{}^{n+r-2}C_{n-2}}{{}^{n+r-1}C_{n-1}} = \frac{(n+r-2)! \times \cancel{r!} \times (n-1)!}{\cancel{r!} \times (n-2)! \times (n+r-1)!}$$

$$= \boxed{\frac{(n-1)}{(n+r-1)}}$$

Case #2: First two boxes are empty

Total number of ways to distribute r balls in boxes labelled 3 to n $\Big\} = {}^{n+r-3}C_{n-3}$

$$\text{Probability} = \frac{{}^{n+r-3}C_{n-3}}{{}^{n+r-1}C_{n-1}} = \frac{(n+r-3)! \times \cancel{r!} \times (n-1)!}{\cancel{r!} \times (n-3)! \times (n+r-1)!}$$

$$= \boxed{\frac{(n-1)(n-2)}{(n+r-1)(n+r-2)}}$$

(7)

$$f(x) = \begin{cases} 0 & , x < 1 \\ e^{-(x-1)} & , x \geq 1 \end{cases}$$

$$Y = x^2 \Rightarrow Y \in [1, \infty)$$

C.D.F. $F_Y(y)$ for $y \in [1, \infty)$

$$F_Y(y) = P(Y \leq y)$$

$$= P(x^2 \leq y)$$

$$= P(-\sqrt{y} \leq x \leq \sqrt{y})$$

$$= \underbrace{P(-\sqrt{y} \leq x \leq 1)}_{=0} + P(1 \leq x \leq \sqrt{y})$$

$$= \int_1^{\sqrt{y}} e^{-(x-1)} dx = 1 - e^{(1-\sqrt{y})}$$

$\therefore$ C.D.F. of Y, $F_Y(y) = \begin{cases} 0, & y < 1 \\ 1 - e^{(1-\sqrt{y})} & , y \geq 1 \end{cases}$

P.D.F. of $Y = f_Y(y) = F_Y'(y) = \begin{cases} 0 & , y < 1 \\ \dfrac{e^{(1-\sqrt{y})}}{2\sqrt{y}} & , y \geq 1 \end{cases}$

(8) $$\mathrm{Var}[X_1+X_2+\cdots+X_n] = E\left[\left((X_1+X_2+\cdots X_n) - E(X_1+X_2+\cdots+X_n)\right)^2\right]$$

$$= E\left[\left((X_1+X_2+\cdots+X_n) - (E(X_1)+E(X_2)+\cdots+E(X_n))\right)^2\right]$$

$$= E\left[\left((X_1-E(X_1))+(X_2-E(X_2))+\cdots(X_n-E(X_n))\right)^2\right] \quad (1)$$

From Algebra, we know that,

$$\boxed{\sum_{i=1}^{N} a_i^2 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} a_i a_j = \left(\sum_{i=1}^{n} a_i\right)^2}$$

Therefore, (1) becomes,

$$= E\left[\sum_{i=1}^{n}(X_i-E(X_i))^2 + 2\sum_{j<i}(X_i-E(X_i))(X_j-E(X_j))\right]$$

$$= \sum_{i=1}^{n} E(X_i-E(X_i))^2 + 2\sum_{j<i} E\left((X_i-E(X_i))(X_j-E(X_j))\right)$$

$$= \sum_{i=1}^{n}\mathrm{Var}[X_i] + 2\sum_{j<i}\mathrm{cov}(X_i,X_j)$$

Hence, Proved

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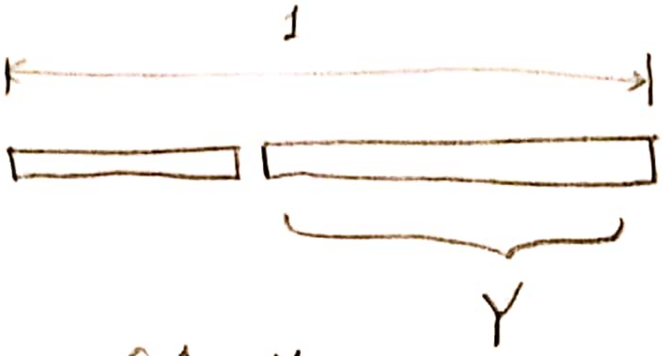

$$Y = \begin{cases} 1 - X & \text{if } X < \frac{1}{2} \\ X & \text{if } X \geq \frac{1}{2} \end{cases}$$

$$E(Y) = \int_0^{0.5} (1-x)\, dx + \int_{0.5}^{1} x\, dx$$

$$= \left. x - \frac{x^2}{2} \right]_0^{0.5} + \left. \frac{x^2}{2} \right]_{0.5}^{1} = \boxed{\frac{3}{4}}$$

$$Var(Y) = \int_0^{0.5} (1-x)^2\, dx + \int_{0.5}^{1} x^2\, dx$$

$$= \left. -\frac{(1-x)^3}{3} \right]_0^{0.5} + \left. \frac{x^3}{3} \right]_{0.5}^{1}$$

$$= \frac{(1 - 0.125) + (1 - 0.125)}{3}$$

$$= \boxed{\frac{7}{12}}$$

10)

Number of runs = 10^6

Language : Python

Method : Monte Carlo Simulation

$$\left.\begin{array}{l}\text{Volume under} \\ \text{curve } z = \frac{1}{3}\cosh\sqrt{x^2+y^2}\end{array}\right\} = P(\text{Success}) \times \text{Volume of Cube}$$

$$= \frac{\text{inside\_points}}{\text{total\_points}} \times 1$$

$$\approx \boxed{0.45}$$

(1.1) 'n' Red ; 'm' Black

(i) replacement

$X = R_1 + R_2 + \cdots R_K$

$E[X] = E[R_1 + R_2 + \cdots R_K]$

$i = 1, \ldots, k$

$$R_i = \begin{cases} 1 & \text{if the } i^{th} \text{ ball is Red} \\ 0 & \text{if the } i^{th} \text{ ball is Black} \end{cases}$$

$$P(R_i) = \frac{n}{m+n}$$

$$E[X] = \frac{Kn}{m+n}.$$

$$Var[X] = Var[R_1 + R_2 + \cdots + R_K] = \sum_{i=1}^{k} Var(R_i) + 2\underbrace{\sum_{p<q} \underset{0}{\underset{\shortparallel}{Cov(R_p, R_q)}}}_{\text{independent events}}$$

$$= [Var(R_1) + Var(R_2) + \cdots + Var(R_K)]$$

$$= K \times Var(R_1)$$

$$= K \times \left(E(R_1^2) - [E(R_1)]^2 \right)$$

$$= K \times \left(1^2 \times \frac{n}{m+n} - \frac{n^2}{(m+n)^2} \right)$$

$$= K \left(\frac{nm + \cancel{n^2} - \cancel{n^2}}{(m+n)^2} \right) = \boxed{\frac{Kmn}{(m+n)^2}}$$

(ii) without replacement

$$E[X] = \sum_{i=1}^{k} E[R_i]$$

$$E[R_i] = \frac{n}{m+n}$$

$$\therefore E[X] = \frac{kn}{m+n}$$

$$Var[X] = Var[R_1 + R_2 + \cdots + R_k]$$

$$= \sum_{i=1}^{k} \underset{\substack{\uparrow \\ \text{all same} \\ \text{for all } i}}{Var[R_i]} + 2 \sum_{1 \le i < j \le k} Cov(R_i, R_j)$$

$$Var[R_i] = E[R_i^2] - (E[R_i])^2$$

$$= \frac{1^2 \times n}{m+n} - \left(\frac{n}{m+n}\right)^2 = \frac{mn}{(m+n)^2}$$

$$Cov(R_i, R_j) = P(R_i = 1, R_j = 1) - P(R_i = 1)\, P(R_j = 1)$$

$$=$$