Math 5110 Applied Linear Algebra –Homework 5.

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1. Reading: [Gockenbach], Chapters 6

2. Questions:

Question 1. Let $\mathbb{R}^5$ be the Euclidean space. Let V be a subspace spanned by

$$\vec{u}_1 = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}$$

(1) Apply the Gram-Schmidt process to find the orthonormal basis of V. (2) Find the orthogonal complement of V. (3) Compute $\text{proj}_V \vec{y}$. (4) Write $\vec{y} = \begin{bmatrix}2\\2\\-1\\1\\1\end{bmatrix}$ as $\vec{y} = \vec{y}_1 + \vec{y}_2$ such that $\vec{y}_1 \in V$ and $\vec{y}_2 \in V^{\perp}$

(5) Write a Matlab function projection(y, A) to compute $\text{proj}_V \vec{y}$ (6) Write a Matlab function OBasis(A) to compute orthogonal basis and then a function NBasis(A) to compute orthonormal basis.

Question 2. Consider the following vectors in $\mathbb{R}^4$:

$$\vec{u}_1 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \vec{u}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(a) Prove that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

(b) Express $\vec{v} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$.

Question 3. Notice that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-1\\2\\1\\0\end{bmatrix},$$

is an orthogonal subset of $\mathbb{R}^4$.

(1) Find a fourth vector $\vec{v}_4 \in \mathbb{R}^4$ that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

(2) Find the orthogonal projection of $\vec{y} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ onto $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Question 4. Find scalars a, b, c, d, e, f, g such that the vectors

$$\vec{u} = \begin{bmatrix} a \\ d \\ f \end{bmatrix}, \vec{v} = \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \vec{w} = \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$$

are orthonormal.

Question 5. Let P be the plane in $\mathbb{R}^3$ defined by the equation $-3x + y + z = 0$.

(a) Find an orthogonal basis for P.

(b) Find the shortest distance from $(1, 1, 1)$ to the plane P.

Question 6. Show that an orthogonal transformation L from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves angles: The angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ equals the angle between $L(\vec{v})$ and $L(\vec{w})$. Conversely, is any linear transformation that preserves angles orthogonal?

Question 7. True or False. (Prove or explain the reason or provide a counter example for the false statement.)

(1) If a matrix A is orthogonal, then A^2 is orthogonal.
(2) If a matrix A is orthogonal, then A^T is orthogonal.
(3) Rows of an orthogonal matrix A are orthogonal.
(4) There exists an orthogonal transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

(5) If A is any $m \times n$ matrix, then $A^T A$ and AA^T are symmetric.
(6) If $n \times n$ matrices A and B are symmetric, then AB is symmetric.
(7) If an $n \times n$ matrix A is symmetric, then A^k is symmetric.
(8) If an $n \times n$ invertible matrix A is symmetric, then A^{-1} is symmetric.
(9) If matrices A and S are orthogonal, then $S^{-1}AS$ is orthogonal.
(10) the entries of orthogonal matrices are less than or equal to 1.
(11) If A is invertible and $A = A^{-1}$, then A is orthogonal.

Question 8. Let $c_1, ..., c_n$ be positive numbers. The weighted dot product on $\mathbb{R}^n$ is

$$\langle \vec{v}, \vec{w} \rangle := \sum_{i=1}^{n} c_i v_i w_i.$$

Verify that weighted dot product satisfies all axioms of definition of inner product..

Question 9. Does the formula $\|\vec{x}\| := \sum_{i=1}^{n} x_i^2$ define a norm on $\mathbb{R}^n$?

Question 10. Among all the unit vectors $\vec{x} \in \mathbb{R}^4$, find the one for which the sum $x_1 + 2x_2 + 3x_3 + 4x_4$ is minimal. What is the minimal value?

Question 11. Show that the following formula defines an inner product on the vector space of all $m \times n$ matrices.

$$\langle A, B \rangle := \text{trace}(A^T B).$$