Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 1. (Due: Monday, September 21)

1. Reading: [Gockenbach], Chapter 1 and Chapter 2.

Notations of **column** vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (v_1, v_2, v_3)$. The right-side used in our book is a point notation. It is different from 1×3 matrix $[v_1\ v_2\ v_3]$.

2. Questions:

Rules of answering the questions: (1.) Write reason or proof for each conclusion of your answer.

(2.) For calculation "by hand" questions, write down all steps of calculations. For calculation by Matlab questions write down (copy) the input and useful output.

(3.) You can scan and submit your handwriting answers. However, it is highly recommended that you use **LaTex** to write your answers. (At least for some homework.) You can either use the online version https://www.overleaf.com/ or download the local disc version https://www.latex-project.org/get/ on Mac or PC. *Warning:* Texmaker or Texworks are just editors. You need to download the full tex first. I recommend to use Texmaker.

A basic template can be (copy the following text and run tex.) There are many packages for tex. For example, using "tikz" you can draw many beautiful pictures. A template I used for lecture notes is also on Canvas.

```
\documentclass[11pt]{paper}
\usepackage{amssymb,amscd,amsmath}
\usepackage[all]{xy}

\textwidth=17cm \textheight=23cm
\voffset=-0.4in
\hoffset=-0.9in

\begin{document}
\begin{center}
\textbf{Math 5110- Applied Linear Algebra-Homework 1 }

\textbf{Name:  Your name}
\end{center}

Write your answers Here. For example

\textbf{Answer of Question 1:}
If you don't know how to write formulas in Latex, just Google: ''Latex  ...."

\end{document}
```

For all questions, if there is no particular instruction, the field is real number field $\mathbb{R}$.

Question 1. Write down the two operations on field $\mathbb{Z}_3$.

+	[0]	[1]	[2]
[0]			
[1]			
[2]			

×	[0]	[1]	[2]
[0]			
[1]			
[2]			

Solution:

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

□

Question 2. Determine which of the matrices below are in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

Solution: B, D □

Question 3. We says that two $m \times n$ matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form.

(2) How many types of 2×3 matrices in reduced row-echelon form.

(3) Find all 4×1 matrices in reduced row-echelon form.

List all of them. (Use $*$ to denote any real number. Group them by rank)

Solution: (1) 3×2 **rref**:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(2) 2×3 **rref**:

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(3) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

□

Question 4. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form? $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$

Solution: $e = 0$; $c = 1$; $d = 0$; $b = 0$; a any real number □

Question 5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$.

(1) Calculation **rref**(A) over $\mathbb{R}$ by hand. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

(2) Calculation **rref**(A) over field $\mathbb{Z}_7$ by hand.

(3) Using Matlab verify your result and calculation **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Matlab function is uploaded on Canvas, put the rrefgf.m file in the same folder with your calculation file.)

(4) Is it possible that a matrix M has different rank over different fields $\mathbb{Z}_p$? (By calculation in (3))

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-R_1]{R_3-2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \xrightarrow{R_3/7} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \xrightarrow[R_1-3R_3]{R_2-3R_3}$$

$$\begin{bmatrix} 1 & 2 & 0 & 22/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} = \textbf{rref}(A)$$

□

Solution:

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-R_1]{R_3-2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3/2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-4R_3]{R_2-2R_3}$
$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{rref}(A)$

(3) See Matlab out put:

(4) Compare Ar2 and Ar7, we can see that , the first three columns different rank. So $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ has different rank over $\mathbb{Z}_2$ and $\mathbb{Z}_7$.

□

Matlab Input

```
A=[1   2   3   4;
1   1   0   2;
2   0   1   2]
S=sym(A)
rref(S)
rref(A)
Ar2 = rrefgf(A,2)
Ar3 = rrefgf(A,3)
Ar7 = rrefgf(A,7)
```

Matlab Output

```
Ar2 =
      1       0       0       0
      0       1       0       0
      0       0       1       0

Ar3 =
      1       0       0       0
      0       1       0       2
      0       0       1       2

Ar7 =
      1       0       4       0
      0       1       3       0
      0       0       0       1
```

Question 6. (Use Matlab) Solve the linear system

$$\begin{cases} 3x_1 + 11x_2 + 19x_3 & = -2 \\ 7x_1 + 23x_2 + 39x_3 & = 10 \\ -4x_1 - 3x_2 - 2x_3 & = 6 \end{cases}$$

and write solutions in parametric vector forms.

Solution: $\textbf{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. No solution. $\square$

Question 7. (Use Matlab) Solve the linear system

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

and write solutions in parametric vector forms.

Solution: Let A be the augmented matrix.

$$\textbf{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $\begin{cases} x_1 = 6 - 2x_2 - 3x_3 - 5x_5 \\ x_4 = 7 - 2x_5 \end{cases}$ and

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, where x_2, x_3, x_5 are any real numbers.

$\square$

Question 8. (Use Matlab) Solve the linear system

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 & = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 & = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 & = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 & = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 & = 24 \end{cases}$$

and write solutions in parametric vector forms. (Hint: In Matlab, if you want precise value, use symbolic calculation A=sym(A))

Solution: Let A be the augmented matrix and calculate rref(A) in Matlab.

$$\begin{cases} x_1 = -8221/4340 \approx -1.89 \\ x_2 = 8591/8680 \approx 0.99 \\ x_3 = 4695/434 \approx 10.82 \\ x_4 = -459/434 \approx -1.06 \\ x_5 = 699/434 \approx 1.61 \end{cases}$$

$\square$

Question 9. (1) If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B are invertible?

Solution: (1) By invertible theorem, A and C are invertible and $A^{-1} = BC$ and $C^{-1} = AB$. Then $AB = C^{-1}$, then $CAB = I$. So, B is invertible and $B^{-1} = CA$.
(2) If AB is invertible, then there exist a matrix C such that $ABC = I$. Then by (1) each matrix is invertible.

□

Question 10. Provide a counter-example to the statement: For any 2×2 matrices A and B, $(AB)^2 = A^2B^2$.

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
$(AB)^2 = \begin{bmatrix} 0 & 0 \\ 12 & 16 \end{bmatrix}$
$A^2B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

□

Question 11. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

Solution: Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then, $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ We want $A^{-1} = A^T$. We may set $ad - bc = 1$, then we need $a = d$ and $b = -c$.
So, examples will be $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ such that $a^2 + b^2 = 1$.
For example $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Later, we will see more examples like this for $n \times n$ matrices, called orthogonal matrix.

□

Question 12. Here are a couple of new definitions: An $n \times n$ matrix A is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.
(2) What can you say about the main diagonal of a skew-symmetric matrix?
(3) Give an example of a matrix that is both symmetric and skew-symmetric.
(4) Prove that for any $n \times n$ matrix A, the matrices $A + A^T$, AA^T, and A^TA are symmetric and $A - A^T$ is skew-symmetric.
(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices. Hint: Did you do part (4) yet?

Solution: (1)
(2) All zeros. Since $a_{ii} = -a_{ii}$.
(3) zero matrix.
(4) $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.
$(AA^T)^T = (A^T)^T A^T = AA^T$
$(A^T A)^T = A^T (A^T)^T = A^T A$.
So, all above three are symmetric.
$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.
(5) $A = \dfrac{(A + A^T)}{2} + \dfrac{(A - A^T)}{2}$ □

Question 13. Let V be a vector space over $\mathbb{R}$ and let $\vec{v} \in V$ be a nonzero vector. Is the subset $\{0, \vec{v}\}$ is a subspace of V? Prove your result.

Solution: No. $2\vec{v}$ is not in the subset. So the set $\{0, \vec{v}\}$ is not closed under scalar product. □

Question 14. Determine whether or not the following set a subspace of $\mathbb{R}^2$. Prove your result.

(1) $S = \{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$.

(2) $T = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 0\}$ the unit disc in $\mathbb{R}^2$.

Solution: (1) No, the set S is not closed under sum. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in S, but their sum is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in S.

(2) No. the set T is not closed under scalar product. For example. $3 \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$ which is not in T. □

Question 15. (1) Let $U_{3\times3}$ be the set of all 3×3 upper triangular matrices with real entries. Is $U_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$? Prove your result.

(2) Let $T_{3\times3}$ be the set of all 3×3 triangular matrices with real entries. Is $T_{3\times3}$ a subspace of $\mathbb{R}^{3\times3}$?

(3) Let W be the set of all polynomials in the form $\{t + at^2\}$ where a is any real number. Is W a subspace of P the vector space of all polynomials.

Solution: (1) Yes. Verify three conditions.
(2) No. Sum is not closed.
(3) No. Not include zero.
□

Question 16. (Allow to use Matlab for **rref**) Let S be the following subspace of $\mathbb{R}^4$:

$$S = \operatorname{Span}\left\{\vec{b}_1 = \begin{bmatrix}-1\\-2\\4\\-2\end{bmatrix},\ \vec{b}_2 = \begin{bmatrix}0\\1\\-5\\4\end{bmatrix}\right\}.$$

Determine if each vector belongs to S:

(1.) $\vec{v} = \begin{bmatrix}-1\\0\\-6\\6\end{bmatrix}$; (2.) $\vec{w} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

Solution: It is the same question as whether or not $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{v}$ or $x_1\vec{b}_1 + x_2\vec{b}_2 = \vec{w}$ has a solution. Set up augmented matrix $[\vec{b}_1\ \vec{b}_2|\vec{v}]$ and $[\vec{b}_1\ \vec{b}_2|\vec{w}]$ and find their **rref**.
(1) Yes. (2) No. □

Question 17. Show that $S = \operatorname{Span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\}$ and $T = \operatorname{Span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\}$ are the same subspace of $\mathbb{R}^3$.

Solution: It is clear that $S \subset T$.

We only need to show that $\begin{bmatrix}1\\0\\1\end{bmatrix} \in S$. That is show that $\vec{v}_3 = x_1\vec{u}_1 + \vec{u}_2$ has a solution. □

Question 18. Let S be the subspace of $\mathbb{R}^3$ defined by $S = \operatorname{Span}\{\begin{bmatrix}-1\\-3\\3\end{bmatrix}, \begin{bmatrix}-1\\-4\\3\end{bmatrix}, \begin{bmatrix}-1\\-1\\4\end{bmatrix}\}$. Is S a **proper** subspace of R^3 or not? In other words, do there exist vectors in $\mathbb{R}^3$ that do not belong to S, or is S all of $\mathbb{R}^3$?

Solution: For any $\vec{b} \in \mathbb{R}^3$, solve $\vec{b} = x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$ Denote $A = \begin{bmatrix}-1 & -1 & -1\\-3 & -4 & -1\\3 & 3 & 4\end{bmatrix}$ then $\textbf{rref}(A) = I_3$.

$A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$. So, $\vec{b} \in S$. □

Question 19. Suppose U and V are two subspaces of a vector space W.

(1) Is the union of two subspace $U \cup V$ a subspace?

(2) Is the intersection $U \cap V$ is a subspace?

Solution: (1) No. Sum is not closed.
(2) Yes. Verify three conditions:
1. $\vec{0} \in U$ and $\vec{0} \in V$, so $\vec{0} \in U \cap V$
2. If $\vec{u}, \vec{v} \in U \cap V$, then $\vec{u} + \vec{v} \in U$ and $\vec{u} + \vec{v} \in V$. So, $\vec{u} + \vec{v} \in U \cap V$.
2. If $\vec{u} \in U \cap V$, then $k\vec{u} \in U$ and $k\vec{v} \in V$ for any $k \in F$. So, $k\vec{u} \in U \cap V$.

$\square$