Name: Sai Nikhil

NUID: 001564864

① Z_3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

② B, D are in reduced row-echelon form

A, C, E are not because for rref we need to have all zeroes before leading 1 and if there is a leading 1, then all other elements in that column must be zeroes.

③ (1) 3×2

Rank = 2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Rank = 1 $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank = 0 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(2) 2×3

Rank = 2

$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rank = 1

$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank = 0 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(3) 4×1

Rank = 1 $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Rank = 0 $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(4) $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$

$c = 1$, as it is the leading co-efficient and it can't be zero because $A[0][3] = 3$

$\Rightarrow b = 0, e = 0, d = 0$

$\Rightarrow a \in \mathbb{C}$

---

(5) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

$\downarrow R_3 - 2R_1$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$

$\xleftarrow[R_3 + 4R_2]{R_1 + 2R_2,\ -R_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$\xleftarrow{R_3/7} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$\downarrow R_1 + 3R_3,\ R_2 - 3R_3$

$\begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6/7 \\ 8/7 \\ 2/7 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{(R_2+6R_1)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

$$\xrightarrow{(R_3+5R_1)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{(R_3+3R_1)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{(4R_3)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(6R_2)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(R_1+5R_2)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(R_2+5R_3)_{\mathbb{Z}_7}} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) verified

```
>> A=[1 2 3 4; 1 1 0 2; 2 0 1 2]

>> field=2

>> rrefgf(A, field)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
>> field=3

>> rrefgf(A,field)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(4) I guess No. It is not possible that matrix M can have different rank over two different fields Z_p.

(6) >>A = [3 11 19 -2; 7 23 39 10; -4 -3 -2 6]

>> rref(A)

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution exists for the set of equations.

(7) >> A = [3 6 9 5 25 53; 7 14 21 9 53 105; -4 -8 -12 5 -10 11]

>> rref(A)

$$\begin{array}{c} \begin{array}{cccccc} & s & t & & u & \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_2 = s$, $x_3 = t$, $x_5 = u$ are free variables.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(8)

>> A= [2 4 3 5 6 37;
4 8 7 5 2 74;
-2 -4 3 4 -5 20;
1 2 2 -1 2 26;
5 -10 4 6 4 24]

>> sym(rref(A))

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -197/104 \\ 0 & 1 & 0 & 0 & 0 & 193/195 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -257/243 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -197/104 \\ 193/195 \\ 4695/434 \\ -257/243 \end{bmatrix}$$

(9) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & -1 \\ -2 & +2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix}$$

$$= -5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$(AB)^2 = 25 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 25 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= 50 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{---} (1)$$

$$A^2 B^2 = 9 \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 9 \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$$= 45 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{---} (2)$$

$\therefore$ From (1) & (2),

$$(AB)^2 \neq A^2 B^2$$

10

(1) $A(BC) = I_n \Rightarrow$ "A" is invertible

$(AB)C = I_n \Rightarrow$ "C" is invertible

$\therefore$ Multiplying with A^{-1} from left, C^{-1} from right, we get,

$$B = A^{-1}C^{-1}$$

$$\Rightarrow B = (CA)^{-1}$$

Multiplying with "CA" from right,

$$B(CA) = I_n$$

$\Rightarrow$ "B" is invertible

(2) $(AB)(AB)^{-1} = I_n$

$\Rightarrow A \times B \times (AB)^{-1} = I_n$

$\Rightarrow A\left(B \times (AB)^{-1}\right) = I_n$ [Using law of associativity]

$\Rightarrow$ "A" is invertible.

$(AB)^{-1} \times A \times B = I_n$

$\Rightarrow \left((AB)^{-1} \times A\right) \times B = I_n$

$\Rightarrow$ "B" is invertible.

(11) Example of 2×2 whose inverse is same as the transpose $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \forall\, \theta \in C$

For $\theta = \frac{\pi}{2} \Rightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^T = A^{-1}$$

---

(12)

$A^T = A \Rightarrow$ Symmetric

(1) $A^T = -A \Rightarrow$ Skew-symmetric

Symmetric Examples

2×2 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ 3×3 $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$

4×4 $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 5 & 2 & 8 & 9 \\ 6 & 8 & 3 & 10 \\ 7 & 9 & 10 & 4 \end{bmatrix}$

Skew-Symmetric Examples

$2\times 2 \quad \begin{bmatrix} 0 & -2 \\ +2 & 0 \end{bmatrix} \qquad 3\times 3 \quad \begin{bmatrix} 0 & -1 & -2 \\ +1 & 0 & -3 \\ +2 & +3 & 0 \end{bmatrix}$

$4\times 4 \quad \begin{bmatrix} 0 & -1 & -2 & -3 \\ +1 & 0 & -4 & -5 \\ +2 & +4 & 0 & -6 \\ +3 & +5 & +6 & 0 \end{bmatrix}$

(2) All elements in diagonal are zeroes

(3) A zero-matrix is both symmetric & skew-symmetric

$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(4) $(A+A^T)^T = A^T + A = A + A^T$

$(AA^T)^T = (A^T)^T \cdot A^T = AA^T$

$(A^TA)^T = (A^T) \cdot (A^T)^T = A^TA$

$\Rightarrow (A+A^T),\ AA^T,\ A^TA$ are symmetric

$(A-A^T)^T = A^T - (A^T)^T = -(A-A^T)$

$\Rightarrow (A-A^T)$ is skew-symmetric

(5) $\frac{(A+A^T)}{2}$ is Symmetric

$\frac{(A-A^T)}{2}$ is skew-symmetric

$$A = \frac{(A+A^T)}{2} + \frac{(A-A^T)}{2}$$

Hence, Proved.

---

(13) subset : $S = \{\vec{0}, \vec{v}\}$ $\vec{v} \in V$ (vector space)

1) we have $\vec{0}$ (zero vector)
2) Sum of two elements $(\vec{0}+\vec{v}) = \vec{v}$, $(\vec{v}+\vec{0}) \in S$
3) scalar multiplication: $3\vec{v} \notin S$

Hence, "S" is not a subspace

---

(14) (1) $x_1 x_2 = 0 \Rightarrow$ x-axis and Y-axis

$\vec{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \in S$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \in S$

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \notin S$$

Hence, not a subspace.

(2) $T = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in T$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin T \left(\text{as } 1^2 + 1^2 > 1\right)$$

---

15 (1)

i) $\vec{0} \in U_{3\times3}$

ii) $\vec{v}_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{pmatrix} \in U_{3\times3}$

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} (x_{11}+y_{11}) & (x_{12}+y_{12}) & (x_{13}+y_{13}) \\ 0 & (x_{22}+y_{22}) & (x_{23}+y_{23}) \\ 0 & 0 & (x_{33}+y_{33}) \end{pmatrix}$$

$\in U_{3\times3}$

iii) $\vec{v} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix}$

$$c\vec{v} = \begin{pmatrix} cx_{11} & cx_{12} & cx_{13} \\ 0 & cx_{22} & cx_{23} \\ 0 & 0 & cx_{33} \end{pmatrix} \in U_{3\times3}$$

Hence, $U_{3\times3}$ is a sub-space

(2) $T_{3\times3} \rightarrow$ set of 3×3 triangular matrices

$$\vec{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in T_{3\times3}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \notin T_{3\times3}$$

Hence, not a subspace

(3) $W = \{t + at^2\}$

$$p(t) = t + at^2$$

i) zero vector is not in the subspace except for $t=0$.

ii) $\vec{v}_1 = t \quad \vec{v}_2 = t + t^2 \in W$

$$\vec{v}_1 + \vec{v}_2 = 2t + t^2 \notin W$$

iii) $\vec{v}_1 = t \in W$

$$3\vec{v}_1 = 3t \notin W$$

$\Rightarrow$ W is not a subspace.

16

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$T = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{rref}(S) = \text{rref}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{rref}(T) = \text{rref}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rref}(S) \sim \text{rref}(T) \Rightarrow$ They are Same Subspace of $\mathbb{R}^3$

(11)

$$S = \text{Span}\left\{ \vec{b_1} = \begin{bmatrix} -1 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix} \right\}$$

$\text{rank}(S) = 2$

(1)

Consider, $M = \left[\begin{array}{cc|c} -1 & 0 & -1 \\ -2 & 1 & 0 \\ 4 & -5 & -6 \\ -2 & 4 & 6 \end{array}\right]$ $\vec{v}$

$$\text{rank}(M) = \text{rank}(\text{rref}(M))$$

$$= \text{rank}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= 2$$

$\therefore \text{rank}(M) = \text{rank}(S) = 2$

Hence, $\vec{v} \in S$

(2) $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $M = \left[\begin{array}{cc|c} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 4 & -5 & 1 \\ -2 & 4 & 1 \end{array}\right]$

$$\text{rank}(M) = \text{rank}(\text{rref}(M)) = \text{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 3$$

$\text{rank}(M) > \text{rank}(S) \Rightarrow \vec{w} \notin S$

$\vec{b_1}, \vec{b_2}, \vec{w}$ Span another subspace.

(18)

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -3 & -4 & -1 \\ 3 & 3 & 4 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-3R_1} \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & +2 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\downarrow R_2 - 2R_3$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow[-R_2]{R_1-R_2} \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow R_1 + R_3$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the vectors are linearly independent

Hence, they span $\mathbb{R}^3$.

$$\left.\begin{matrix}\text{Dimension of} \\ \text{Subspace}\end{matrix}\right\} = 3 = \left\{\begin{matrix}\text{Dimension of} \\ \mathbb{R}^3\end{matrix}\right.$$

$\therefore$ This is NOT a proper subspace.

(19) (1) Union of two Subspaces is NOT a subspace

Consider X-axis and Y-axis.

Their union is just two lines

Consider a point of X-axis $\rightarrow$ $(2,0)$

Consider another point on Y-axis $\rightarrow$ $(0, 5)$

Their Sum $= (2,0) + (0,5) = (2,5) \notin$ Union

Therefore, sum is NOT closed and

hence union is NOT a subspace.

---

(2) Intersection of two subspaces is a subspace.

Consider U, V are subspaces of a vectorspace W.

i) As U, V are subspaces they contain zero vector (0)

$\Rightarrow U \cap V$ contains 0

ii) Suppose $x, y \in U \cap V$

$\Rightarrow x, y \in U$ as well as V

Since U is a subspace

$x+y \in U$ and Similarly $x+y \in V$

Therefore, $x+y \in U \cap V$

iii) $x \in U \cap V$, $w \in W$

$x \in U$ and $x \in V$

As U and V are subspaces

$w \cdot x \in U$ and $w \cdot x \in V$

$\Rightarrow w \cdot x \in U \cap V$

Hence, $U \cap V$ is a subspace.