Math 5110- Applied Linear Algebra-Fall 2020

Instructor: He Wang

Final Exam

Student Name: ____________________ __________/100

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to look at notes or textbook. However, you are **not** allowed to asked help using any online platform.

4. Scan your solutions, merge into **one .pdf**, and upload. This is plenty of time to use a scanner or scanning app and clearly scan every page.

1. (8 points) Let V_n be subspace of $\mathbb{R}^{n\times n}$ defined by $V_n = \{A \in \mathbb{R}^{n\times n} \mid \text{trace}(A) = 0\}$. (1) Find a basis for V_n. (2). What is the dimension for V_n?

2. (8 points) Suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real vector space V. Let $S = \{\vec{u}, \vec{v}, \vec{w}\}$ and $T = \{\vec{u} + \vec{v} + \vec{w}, \vec{v} + 2\vec{w}, 3\vec{v} + 4\vec{w}\}$. Show that S is independent if and only if T is independent.

3. (14 points)

(1) Suppose A is an $n \times n$ diagonalizable real matrix. Show that $\operatorname{rank}(A - \lambda I) = \operatorname{rank}(A - \lambda I)^2$ for every $\lambda \in \mathbb{R}$.

(2) Let A be an $n \times n$ real matrix with n real eigenvalues (counting algebraic multiplicities). Suppose that $\operatorname{rank}(A - \lambda I) = \operatorname{rank}(A - \lambda I)^2$ for every $\lambda \in \mathbb{R}$. Prove that A is diagonalizable. (Hint: Jordan normal form.)

4. (15 points) Let $P_3 = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 3. Let $T : P_3 \to P_3$ be a transformation defined by $T(p) = p + p'$ for any $p \in P_3$.

(1) Show that T is a linear transformation.

(2) Find the matrix representation of T with respect to the basis $\{1, t, t^2, t^3\}$ of P_3.

(3) What is the determinant of T?

(4) What are the eigenvalues and eigenvectors of T?

(5) What can you say about existence and uniqueness of solutions of $T(p) = q$ for $q \in P_3$?

5. (5 points) Let A be a matrix in $\mathbb{R}^{n\times n}$. Suppose $W = \text{Span}\{I_n, A, A^2, A^3, ...\}$ is the subspace of $\mathbb{R}^{n\times n}$. Show that $\dim W \leq n$.

6. (12 points) Compute by hand, **without** matlab.

(1) Find the eigenvalues of the matrix of $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. (Hint: Think $\det(A)$, $\ker A$, $tr(A)$)

(6. continue)

(2) Find the eigenvalues of the matrix of $B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$. (Hint: Relation with A in (1))

(3) Find the determinant of the matrix of $M = \begin{bmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{bmatrix}$

(4) For which real values of the constants a, b is the matrix M positive definite.

Calculate the following questions 7, 8, 9, 10 with the help of MATLAB.

Write the results clearly for the following questions. Only sketch the steps and the method of your calculation. Write **2 decimals** for the digital calculation.

7. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 6 \\ 1 & 2 & 7 \\ 1 & 3 & 8 \\ 1 & 4 & 9 \\ 1 & 5 & 0 \end{bmatrix}$

(1). Find a basis for the vector space $(\operatorname{im} A)^{\perp}$.

(2) Decompose $A = U\Sigma V^T$, where U and V are orthogonal matrices and Σ is a 5×3 diagonal matrix.

8. (8 points) Find an **orthogonally diagonalization** of matrix $B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$

9. (10 points) Let $M = \frac{1}{7}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$. Calculate $\lim_{n\to\infty} M^n$.

(Hint: Eigenvalues of M are 7 and 2.) Clearly write the theorem you used. (Write down the precise result. Do not use decimal numbers. Only use Matlab for rref.)

10. (10 points) Suppose we have the following data points

x	1	1	2	3	3
y	1	2	1	4	2
z	3	6	9	3	0

(1) Determine a plane $z = a + bx + cy$ that best fits the data in the least-squares sense.

(2) Calculate the squared error of the fitting.