Sai Nikhil
NUID: 001564864

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a)

$$M_k = \gamma(1+M_k) + \left(\frac{1-\gamma}{2}\right)(1+M_{k-1}) + \left(\frac{1-\gamma}{2}\right)(1+M_{k+1})$$

$$\boxed{M_k = 1 + \gamma M_k + \left(\frac{1-\gamma}{2}\right) M_{k-1} + \left(\frac{1-\gamma}{2}\right) M_{k+1}}$$

b)

$$M_k = A + Bk + Ck^2$$

$$M_0 = M_N = 0$$

$M_0 = 0$ | $M_N = 0$

$\Rightarrow \boxed{A = 0}$ — (1) | $\Rightarrow \boxed{A + BN + CN^2 = 0}$ — (2)

(1) & (2) $\Rightarrow$

$\therefore\ B + CN = 0$

$$\boxed{B = -CN} \text{ — (3)}$$

$$\therefore \boxed{M_k = -CNk + Ck^2} \text{ — (4)}$$

Substituting (4) in recursion derived in part a),

we get,

$$(-CNk + ck^2) = 1 + \gamma(-CNk + Ck^2) + \left(\frac{1-\gamma}{2}\right)\left(-CN(k-1) + c(k-1)^2\right)$$
$$+ \left(\frac{1-\gamma}{2}\right)\left(-CN(k+1) + c(k+1)^2\right)$$

$$(-CNk + ck^2) = 1 + \gamma(-CNk + Ck^2)$$
$$+ \left(\frac{1-\gamma}{\cancel{2}}\right)\left[-\cancel{2}kCN + \cancel{2}c(k^2+1)\right]$$

$$(1-\gamma)\left[-C\cancel{N}k + C\cancel{k^2} + k\cancel{CN} - c\cancel{k^2} - c\right] = 1$$

$$c = \frac{-1}{(1-\gamma)}$$

$$\therefore \quad B = -CN = +\frac{N}{(1-\gamma)}$$

$$\therefore \quad M_k = \frac{N}{(1-\gamma)}k - \frac{1}{(1-\gamma)}k^2 = \boxed{\frac{(N-k)k}{(1-\gamma)}}$$

2

$d(1) = 2$ $\quad d(2) = 2$ $\quad d(3) = 1$

$d(4) = 4$ $\quad d(5) = 6$ $\quad d(6) = 2$

$d(7) = 1$ $\quad d(8) = 2$ $\quad d(9) = 2$

a) $$\omega_i = \frac{d(i)}{\sum_{j=1}^{9} d(j)}$$

$$\sum_{j=1}^{9} d(j) = 22$$

$$\therefore \boxed{\omega = \frac{1}{22}(2, 2, 1, 4, 6, 2, 1, 2, 2)}$$

b) $$N_1 = \frac{1}{\omega_1} = \frac{1}{\frac{2}{22}} = \boxed{11}$$

c)

$$N_1' = \frac{1}{\omega_1'}$$

$$\min(N_1') \Rightarrow \max(\omega_1')$$

$$\omega_1 = \frac{2}{22}$$

In order to maximize ω_1,

we need,

new edge between 1 & 5.

$$\therefore \omega_1' = \frac{3}{23} > \frac{2}{22}$$

$$\therefore N_1' = \frac{1}{\frac{3}{23}} = \boxed{\frac{23}{3}}$$

(3)

$$E\left[e^{tX}\right] = e^{2e^{3t}-2}$$

a) $$\mu = \frac{d}{dt}\left(E\left[e^{tX}\right]\right)_{t=0}$$

$$= \frac{d}{dt}\left(e^{2e^{3t}-2}\right)_{t=0}$$

$$= \left[\left(e^{2e^{3t}-2}\right) \times 6e^{3t}\right]_{t=0}$$

$$= \boxed{6}$$

b) $$\Lambda(t) = \ln\left(E\left[e^{tX}\right]\right) = 2e^{3t}-2$$

$$\Lambda^{*}(x) = \sup_{t\in\mathbb{R}} \left\{xt - 2e^{3t} + 2\right\}$$

$$\frac{d}{dt}\left(xt - 2e^{3t} + 2\right) = 0$$

$$x - 6e^{3t} = 0$$

$$\boxed{t = \frac{1}{3}\ln\left(\frac{x}{6}\right)}$$

$$\therefore \Lambda^*(x) = x t^* - 2e^{3t^*} + 2$$

$$= x \times \frac{1}{3}\ln\left(\frac{x}{6}\right) - 2 \times \frac{x}{6} + 2$$

$$= \frac{x}{3}\left[\ln\left(\frac{x}{6}\right) - 1\right] + 2$$

Cramer's Theorem:

$$\lim_{n \to \infty} \frac{1}{n}\ln(\mathbb{P}(Y_n > 12)) = -\Lambda^*(12)$$

$$= -\left[\frac{12}{3}\left(\ln\left(\frac{12}{6}\right) - 1\right) + 2\right]$$

$$= \boxed{-4\ln(2) + 2}$$

(4)

$$p_0 = \frac{1}{6}, \quad p_1 = \frac{5}{12}, \quad p_2 = \frac{5}{12}$$

a)

$$\mu = E[Z] = \sum_{k=0}^{2} k\, P(z=k)$$

$$= 0 \times \frac{1}{6} + 1 \times \frac{5}{12} + 2 \times \frac{5}{12}$$

$$= \frac{15}{12}$$

$$= \boxed{\frac{5}{4}}$$

b) For probability of extinction,

$$s = \phi(s)$$

$$s = E(s^Z) = \sum_{k=0}^{2} s^k \cdot p_k$$

$$\Rightarrow s = 1 \times \frac{1}{6} + s \times \frac{5}{12} + s^2 \times \frac{5}{12}$$

$$\Rightarrow \frac{5}{12}s^2 - \frac{7}{12}s + \frac{1}{6} = 0$$

Let,

$\alpha = 1, \beta$ are the roots

$$\alpha + \beta = \frac{+7}{5}$$

$$\beta = +\frac{7}{5} - 1 = \frac{2}{5}$$

$$\rho = \text{Smallest positive root} = \boxed{\frac{2}{5}}$$

c)

$$P\left(\begin{array}{c}\text{at least one population}\\ \text{does not become}\\ \text{extinct}\end{array}\right) = 1 - P\left(\begin{array}{c}\text{population ever}\\ \text{becomes extinct}\end{array}\right)$$

$$= 1 - \rho^3$$

$$= 1 - \left(\frac{2}{5}\right)^3$$

$$= 1 - \frac{8}{125}$$

$$= \boxed{\frac{117}{125}}$$

5 a) $P_0 + P_1 + P_3 = 1$

for probability of entinction,

$s = \phi(s)$

$s = E(s^z) = 1 \times P_0 + S \times P_1 + s^3 P_3$

This equation is satisfied by $\rho = \frac{1}{4}$

$$\therefore \boxed{P_0 + \frac{1}{4} P_1 + \frac{1}{64} P_3 = \frac{1}{4}}$$

b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{64} & \frac{1}{4} \end{array}\right]$$

$$P_0 = \frac{5}{16} P_3$$

$$P_1 = 1 - \frac{21}{16} P_3$$

T

To maximize p_0, we need to maximize p_3

$\Rightarrow \quad p_1 = 0$

$\Rightarrow \quad \boxed{p_{3\,(max)} = \frac{16}{21}}$

$\therefore \quad p_{0\,(max)} = \frac{5}{\cancel{16}} \times \frac{\cancel{16}}{21} = \frac{5}{21}$

$\therefore \quad \boxed{p_{0\,(max)} = \frac{5}{21}}$