MATH 7241 Fall 2020: Problem Set #6

Due date: Sunday November 8

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

$\boxed{\textbf{Exercise 1}}$ In each case below, determine whether or not the chain is reversible (note: the condition for reversibility is $w_i p_{ij} = w_j\, p_{ji}$ for all states i, j).

$$(a) \qquad P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

$$(b) \qquad P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$\boxed{\textbf{Exercise 2}}$ A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

$\boxed{\textbf{Exercise 3}}$ Grinstead and Snell p.423, #7.

$\boxed{\textbf{Exercise 4}}$ Grinstead and Snell p.423, #9.

$\boxed{\textbf{Exercise 5}}$ Grinstead and Snell p.427, #24.

$\boxed{\textbf{Exercise 6}}$ Consider a Markov chain on the set $S = \{0, 1, 2, \dots\}$ with transition probabilities

$$p_{i,i+1} = a_i, \quad p_{i,0} = 1 - a_i$$

for $i \geq 0$, where $\{a_i \,|\, i \geq 0\}$ is a sequence of constants which satisfy $0 < a_i < 1$ for all i. Let $b_0 = 1$, $b_i = a_0 a_1 \cdots a_{i-1}$ for $i \geq 1$. Show that the chain is

(a) persistent if and only if $b_i \to 0$ as $i \to \infty$ [Hint: compute $f_{00} = \sum_n f_{00}(n)$]
(b) positive persistent if $\sum_i b_i < \infty$ [Hint: compute mean return time to state 0, namely $\sum_n n f_{00}(n)$]. Compute the stationary distribution if this condition holds.