Math 5110- Applied Linear Algebra-Fall 2020

Instructor: He Wang

Test 1.

Student Name: ____________________ __________/50

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to look at notes or textbook. However, you are **not** allowed to asked help using any online platform.

4. You have 75 minutes for the exam and 15 minutes to scan your solutions, merge into **one .pdf**, and upload. This is plenty of time to use a scanner or scanning app and clearly scan every page.

5. If you have any technical difficulty with the upload or scan, contact me immediately. Do not wait until the end of exam to contact me about a technical difficulty.

1. (10 points) Let $A = \begin{bmatrix} 1 & 3 & 6 & 2 \\ 2 & 4 & 10 & 2 \\ -1 & -1 & -8 & -4 \\ 1 & 3 & 2 & -2 \end{bmatrix}$. The reduced echelon form of A is $\textbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^4$.

(1) Find a **basis** for the kernel of T and a **basis** for the image of T.

(2) Suppose the columns of A are $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$. Is $\vec{a}_4$ a redundant vector? If it is, write $\vec{a}_4$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

(3) Is $\vec{a}_2$ a linear combination of $\vec{a}_1, \vec{a}_3, \vec{a}_4$? Explain the reason.

(4) Suppose $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a **solution** for $A\vec{x} = \vec{b}$. Write down all solutions for $A\vec{x} = \vec{b}$. (Hint: we need to use the result in (1).)

2. (6 points) (1) Determine whether each of the following sets of vectors is linearly dependent or independent. (Explain the reason.)

(a) $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$, (b) $\left\{\begin{bmatrix}-1\\-1\\4\end{bmatrix}, \begin{bmatrix}-4\\-4\\17\end{bmatrix}, \begin{bmatrix}1\\1\\-3\end{bmatrix}\right\}$, (c) $\left\{\begin{bmatrix}-1\\3\\-2\end{bmatrix}, \begin{bmatrix}0\\-10\\7\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}$, (d) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\3\\-2\end{bmatrix}, \begin{bmatrix}4\\-2\\7\end{bmatrix}, \begin{bmatrix}5\\2\\-1\end{bmatrix}\right\}$

(2) Determine which set in the above question a basis for $\mathbb{R}^3$.

3. (3 points) Let A be a 4×3 matrix. We are told that $A\vec{x} = \vec{0}$ only has trivial solution.

(1) What is the reduced row-echelon form of A?

(2) Can $A\vec{x} = \vec{c}$ has no solution for some $\vec{c} \in \mathbb{R}^4$? (Explain your reason)

(3) Can $A\vec{x} = \vec{c}$ has infinitely many solutions for some $\vec{c} \in \mathbb{R}^4$? (Explain your reason)

4. (5 points) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{bmatrix}$ with reduced row echelon form $\textbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 7 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix}$

Let $M = \begin{bmatrix} 7 & -2 & -1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$.

(1) What is the inverse of M? (Using the above information.)

(ii) Find all solutions for $M\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ using the result in (i).

(iii) Does $M^5\vec{x} = \vec{b}$ have a unique solution for any $\vec{b} \in \mathbb{R}^3$? (Explain the reason.)

5. (6 points) Let S_n set of all $n \times n$ symmetric matrices with entries in real numbers. (Recall: A is symmetric if and only if $A = A^T$.)

(1) Is S_n is a vector space with matrix sum and scalar product? Prove your result.

(2) Find a **basis** for S_2={all 2×2 symmetric matrices} and find the **dimension** of S_2. Prove your result.

6. (3 points) Recall that elementary matrices are obtained from identity matrix I_n by only one elementary row operation, i.e., $I_n \xrightarrow{R_i \leftrightarrow R_j} E_{ij}$, $I_n \xrightarrow{rR_i} E_i(r)$, and $I_n \xrightarrow{R_i + kR_j} E_{ij}(k)$.

Find the **transpose** of elementary matrices E_{ij}, $E_i(r)$ and $E_{ij}(k)$. (It is ok to only write down the solutions.)

7. (3 point) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be an linear transformation. We know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}3\\5\\3\end{bmatrix}$.

Find the matrix A such that $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^2$. (Show all your work)

8. (3 points) Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be linearly independent vectors. Find all value of k such that the vectors $\vec{u} + k\vec{v}$, $\vec{u} + \vec{w}$, $\vec{v} + \vec{w}$ are linearly independent? (Show all your work)

9. (8 points) Let A, B, C be **any** $n \times n$ matrices. Determine whether or not each of the following statements is true or false in general. (Reason is not required.)

(1.) If $AB = \mathbf{0}$, then $BA = \mathbf{0}$, where $\mathbf{0}$ is the $n \times n$ zero matrix.

(T) True (F) False

(2.) If A and B are invertible, then A^2B^T is invertible and $(A^2B^T)^{-1} = (B^{-1})^T(A^{-1})^2$.

(T) True (F) False

(3.) If rank $A = n$, then the equation $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b} \in \mathbb{R}^n$.

(T) True (F) False

(4.) $(A - I_n)(A + I_n) = A^2 - I_n$, where I_n is the $n \times n$ identity matrix.

(T) True (F) False

(5.) If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are solutions of $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$. Then $\vec{v} + \vec{w}$ is also a solution for $A\vec{x} = \vec{b}$.

(T) True (F) False

(6.) $\text{rank}(A) \leq \text{rank}(A^2) \leq n$.

(T) True (F) False

(7.) $\ker(A) \subseteq \ker(A^2)$

(T) True (F) False

(8.) $T(\vec{x}) = A^2B\vec{x}$ defines a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(T) True (F) False

10. (3 points) (Challenge question. Do this in the end.) Let $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ be linearly independent vectors in $\mathbb{R}^n$. Find the rank and nullity of the matrix $M = \vec{a}_1\vec{a}_1^T + \vec{a}_2\vec{a}_2^T + \vec{a}_3\vec{a}_3^T$. Prove your result.

(Suggestions of first step: Consider write M as matrix product, or write M explicitly, or other methods.)