MATH 7241: Probability 1 FALL 2020 MAKEUP TEST 1

Important:

- This Makeup Test will be available at **6pm on Friday November 6. You must start the Test at 6pm**.
- This Test must be **completed within 2 hours** – you will not be able to upload your answer after that time.
- You must **upload your answer as a pdf file**. Photos, jpg files etc will not be accepted. You may wish to install and use a **scanner app on your phone**.
- You must put your **full name and student ID** at the top of your answer.
- Send me an email if you have any questions or encounter any problems.
- You may use any material from the class, including notes, problem sets and recordings. **You may not access material from any other source, and you may not discuss these problems with anyone until they have been submitted**.

Questions:

1) Five people play a game, where each person rolls a die once. Find the probability that the number 6 is rolled by at least one person. [Hint: look at the complementary event].

$A = \{\text{at least one person rolls } 6\}$

$A^c = \{\text{nobody rolls } 6\}$

$$\mathbb{P}(A^c) = \mathbb{P}(\text{nobody rolls } 6)$$

$$= \mathbb{P}(\text{person does not roll } 6)^5$$

$$= \left(\frac{5}{6}\right)^5$$

$$\Rightarrow P(A) = 1 - \left(\frac{5}{6}\right)^5$$

2) Four balls are shared between box #1 and box #2. At each step a biased coin is tossed which comes up Heads with probability p. If the coin comes up Heads and box #1 is not empty, a ball is removed from box #1 and placed in box #2. If the coin comes up Heads and box #1 is empty, no balls are moved. If the coin comes up Tails and box #2 is not empty, a ball is removed from box #2 and placed in box #1. If the coin comes up Tails and box #2 is empty, no balls are moved. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$.

State space $\{0, 1, 2, 3, 4\}$

$$P = \begin{array}{cc} \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & 1-p \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

3) Consider the following transition probability matrix for a Markov chain on 5 states:

$$\begin{array}{cc} P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first return to state 1.

Find stationary distribution:

$$\left.\begin{array}{l} w_1 = \frac{1}{2} w_2 \\ w_2 = w_1 + \frac{1}{2} w_3 \end{array}\right\} \Rightarrow w_2 = w_3 \Rightarrow w_1 = \frac{1}{2} w_3$$

$$\left.\begin{array}{l} w_4 = \frac{1}{2} w_3 + w_5 \\ w_5 = \frac{1}{2} w_4 \end{array}\right\} \Rightarrow w_4 = w_3 \Rightarrow w_5 = \frac{1}{2} w_3$$

$$\Rightarrow w = w_3 \left(\frac{1}{2}, 1, 1, 1, \frac{1}{2}\right) \Rightarrow w_3 = \frac{1}{4}$$

$$\Rightarrow w = \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$$

$\Rightarrow$ mean return time to state $1 = w_1^{-1}$

$= 8$

4) The continuous random variables X and Y are independent. X is uniform on $[2, 4]$, and Y is uniform on $[0, 1]$. Compute $\mathrm{P}(X \le Y + 2)$. [Hint: first condition on Y to compute $\mathrm{P}(X \le Y + 2 \,|\, Y = y)$, then undo the conditioning on Y].

$$X \sim U[2,4] \Rightarrow f_X(x) = \frac{1}{2} \quad (2 \le x \le 4)$$

$$Y \sim U[0,1] \Rightarrow f_Y(y) = 1 \quad (0 \le y \le 1)$$

$$\mathbb{P}(X \le Y+2 \,|\, Y=y) = \mathbb{P}(X \le y+2)$$

$$= \int_2^{y+2} \frac{1}{2}\, dx$$

$$= \frac{y}{2}$$

$$\mathbb{P}(X \le Y+2) = \int_0^1 \mathbb{P}(X \le Y+2 \,|\, Y=y)\, f_Y(y)\, dy$$

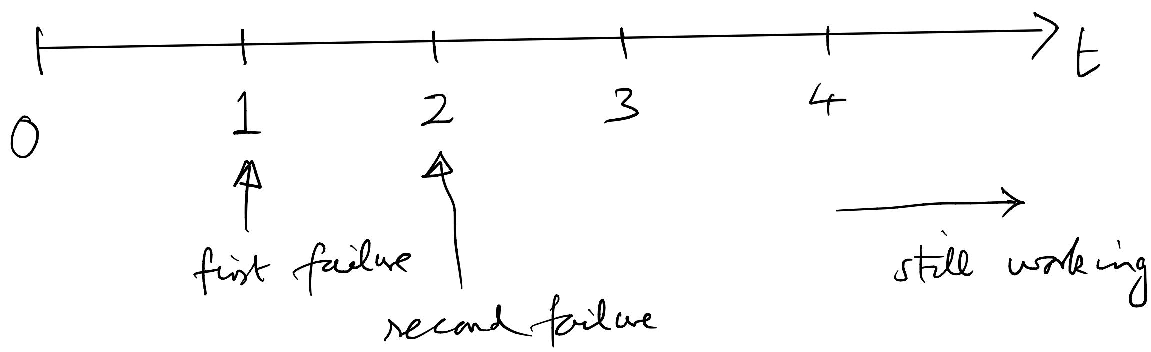
$$= \int_0^1 \frac{y}{2} \cdot 1 \, dy$$

$$= \frac{1}{4}$$

5) A supply depot has three working computers. The lifetimes of the three computers are independent, and all have exponential distributions with the same mean equal to 1 year. One of the computers fails after year 1, and another computer fails after year 2. Find the probability that the third computer is still working after year 4.

T_1, T_2, T_3 = lifetimes of computers.

$\sim$ exp., mean = 1.



Third computer is still working at $t=2$.

$\Rightarrow$ lifetime "starts over" again at $t=2$;

$$\mathbb{P}(T>4 \mid T>2) = \mathbb{P}(T>2)$$ memoryless

$$= e^{-2\lambda}$$

$$= e^{-2} \quad (\lambda=1)$$

6) Let $A_1, A_2, \ldots$ be a sequence of independent events, and suppose that

$$\sum_{k=1}^{n} \mathrm{P}(A_k) \geq \sqrt{n+4} \qquad \text{for all } n \geq 1.$$

Compute

$$\mathrm{P}(A_n \text{ i.o.})$$

where i.o. means 'infinitely often'. [Hint: use the Borel-Cantelli Lemma]

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(A_k)$$

$$\geq \lim_{n\to\infty} \sqrt{n+4} = \infty$$

Events $\{A_n\}$ are independent

$\Rightarrow$ by Borel - Can~~tt~~elli 2,

$$\mathbb{P}(A_n \text{ i.o.}) = 1$$

7) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) = 3 - \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

$$\sum_{k=1}^{\infty} p_{ii}(k) = \lim_{n\to\infty} \sum_{k=1}^{n} p_{ii}(k)$$

$$= \lim_{n\to\infty} 3 - \frac{9}{\sqrt{n+8}}$$

$$= 3$$

$\Rightarrow$ state i is transient.

8) A biased coin has probability p of coming up Heads. The coin is tossed repeatedly. Let N_3 be the number of tosses until the first occurrence of the sequence (Heads, Tails, Heads), and let N_4 be the number of tosses until the first occurrence of the sequence (Heads, Tails, Heads, Tails).

a) Compute the conditional probability $\mathrm{E}[N_4 \,|\, N_3 = k]$ for any $k \geq 3$ (your answer should involve k and also $\mathrm{E}[N_4]$).

b) We have

$$\mathrm{E}[N_4] = \sum_{k=3}^{\infty} \mathrm{E}[N_4 \,|\, N_3 = k] \, \mathrm{P}(N_3 = k)$$

Substitute your answer from part (a) into this formula and compute $\mathrm{E}[N_4]$ (your answer should depend on p, but nothing else). **NOTE:** you should use the result that

$$\mathrm{E}[N_3] = \frac{1}{p} + \frac{1}{p^2(1-p)}.$$

a) Condition on $(k+1)^{st}$ toss

$$\mathbb{E}[N_4 \,|\, N_3 = k] = \mathbb{E}[N_4 \,|\, N_3 = k, H_{k+1}] \cdot p + \mathbb{E}[N_4 \,|\, N_3 = k, T_{k+1}] \cdot (1-p)$$

$$= (k+1 + \mathbb{E}[N_4])\, p + (k+1)(1-p)$$

$$= (k+1) + p\, \mathbb{E}[N_4]$$

b)

$$\mathbb{E}[N_4] = \sum_{k=3}^{\infty} \mathbb{E}[N_4 \,|\, N_3 = k] \, \mathbb{P}(N_3 = k)$$

$$= \sum_{k=3}^{\infty} \left(k+1+p\,\mathbb{E}[N_4]\right)\mathbb{P}(N_3=k)$$

$$= \sum_{k=3}^{\infty} k\;\mathbb{P}(N_3=k)$$

$$+ \left(1+p\;\mathbb{E}[N_4]\right)\sum_{k=3}^{\infty}\mathbb{P}(N_3=k)$$

$$= \mathbb{E}[N_3] + 1 + p\;\mathbb{E}[N_4]$$

$$\Rightarrow \quad (1-p)\;\mathbb{E}[N_4] = 1 + \mathbb{E}[N_3]$$

$$\Rightarrow \quad \mathbb{E}[N_4] = \frac{1}{1-p} + \frac{1}{p(1-p)}$$

$$+ \frac{1}{p^2(1-p)^2}$$