MATH 7241 Fall 2020: Problem Set #6

Due date: Tuesday November 3

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

Grinstead and Snell: see pages 442-443 and pages 444, 467-468 on Canvas. The text is available online (free!) at

http://www.dartmouth.edu/∼chance/

Click on the link "A GNU book".

$\boxed{\textbf{Exercise 1}}$ A box contains N balls, some red and some blue. At each step, a coin is flipped with probability p of coming up Heads, and probability $1-p$ of coming up Tails. If the coin comes up Heads, a ball is chosen at random from the box and is replaced by a red ball; if the coin comes up Tails then a ball is chosen randomly from the box and replaced by a blue ball. Let X_n denote the number of red balls in the box after n steps. Find the transition matrix for the chain $\{X_n\}$, and find the stationary distribution. [Hint: is the chain reversible?] Compute $\lim_{n\to\infty} \mathrm{E}[X_n]$, and explain why you could have guessed your answer without doing the calculation.

$\boxed{\textbf{Exercise 2}}$ A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

Ex. 1

$X_n = \#$ red balls in Box 1.

State space $\{0, 1, 2, \ldots, N-1\}$.

Transition matrix:

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p\,\frac{N-i}{N} & \text{if } j = i+1 \\ (1-p)\,\frac{i}{N} & \text{if } j = i-1 \\ 1 - p\,\frac{N-i}{N} - (1-p)\frac{i}{N} & \text{if } j = i \end{cases}$$

Note: if $j = i+1$, then must toss Heads (prob $= p$) and must select a blue ball (prob $= \frac{N-i}{N}$). etc.

Try to solve reversible equations:

$$w_i P_{ij} = w_j P_{ji}.$$

In this case we only have the equations

$$W_i P_{i,i+1} = W_{i+1} P_{i+1,i}$$

$$\Rightarrow \quad W_{i+1} = W_i \frac{P_{i,i+1}}{P_{i+1,i}}$$

$$= W_i \frac{p}{1-p} \frac{N-i}{i+1}$$

$$= W_{i-1} \left(\frac{p}{1-p}\right)^2 \frac{N-i}{i+1} \frac{N-i+1}{i}$$

$$= \cdots\cdots$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \frac{(N-i)(N-i+1)\cdots(N)}{(i+1)(i)\cdots(1)}$$

$$= W_0 \left(\frac{p}{1-p}\right)^{i+1} \binom{N}{i+1}$$

$$\Rightarrow \quad W_i = W_0 \left(\frac{p}{1-p}\right)^{i} \binom{N}{i} \qquad (i = 0, 1, 2, \dots, N)$$

Normalize to find w_0:

$$\sum_{i=0}^{N} w_i = w_0 \sum_{i=0}^{N} \left(\frac{p}{1-p}\right)^i \binom{N}{i}$$

$$= w_0 \left(1 + \frac{p}{1-p}\right)^N$$

$$= w_0 (1-p)^{-N}.$$

$$\Rightarrow \quad w_0 = (1-p)^N.$$

$$\Rightarrow \quad w_i = p^i (1-p)^{N-i} \binom{N}{i}$$

This is the binomial dist. w/ prob. p.

$$\Rightarrow \quad \lim_{N \to \infty} \mathbb{E}[X_n] = \sum_{i=0}^{N} i\, w_i$$

$$= Np. \quad \text{b/c binomial.}$$

We could have expected this; as $N \to \infty$ the balls are randomly mixed so that the fraction of Red balls is p, and the fraction of Blue balls is $1-p$.