MATH 7241 Fall 2020: Problem Set #3

Due date: Sunday October 11

Reading: relevant background material for these problems can be found in the class notes, and in Rosenthal Chapter 3.

$\boxed{\textbf{Exercise 1}}$ Let $X_1, X_2, \ldots$ be a sequence of random variables (not necessarily independent), and suppose that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[(X_n)^2] = 1$ for all $n \geq 1$. Prove that

$$\mathbb{P}(X_n \geq n \;\text{ i.o.}) = 0$$

where i.o. means 'infinitely often'. [Hint: use the Borel-Cantelli Lemma and Markov's inequality]

$\boxed{\textbf{Exercise 2}}$ Let $X_1, X_2, \ldots$ be independent random variables and suppose that X_n is uniform on the set $\{1, 2, \ldots, n\}$ for each $n \geq 1$. Compute $\mathbb{P}(X_n = 5 \;\text{ i.o.})$. [Hint: use the Borel-Cantelli Lemma]

$\boxed{\textbf{Exercise 3}}$ Define the sequence X_n inductively by setting $X_0 = 1$, and selecting X_{n+1} randomly and uniformly from the interval $[0, X_n]$. Prove that there is a number c such that for every $\epsilon > 0$, $\mathbb{P}\left(|n^{-1} \log X_n - c| \geq \epsilon\right)$ converges to zero, and evaluate the number c.
[Hint: Note that $U_n = X_n X_{n-1}^{-1}$ is a uniform random variable for each $n \geq 2$. Use the Law of Large Numbers.]

$\boxed{\textbf{Exercise 4}}$ **Fact**: For any sequence of *pairwise disjoint* events $\{C_n\}$ we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{P}(C_n)$$

Definition: A sequence of events $\{A_n\}$ is *decreasing* if $A_{n+1} \subset A_n$ for all n. A sequence of events $\{B_n\}$ is *increasing* if $B_n \subset B_{n+1}$ for all n.

a) Let $\{B_n\}$ be an increasing sequence, let $C_1 = B_1$, and let $C_n = B_n \cap B_{n-1}^c$ for all $n \geq 2$. Show that the events $\{C_n\}$ are disjoint, and that for every $N \geq 1$

$$B_N = \bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} C_n.$$

b) Show that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n.$$

[Hint: let x be any element of the left side, show that this implies that x belongs to the right side. Then show that the converse is also true].

c) Using the results of (b), the Fact, and (a), show that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{N\to\infty} \mathbb{P}(B_N)$$

d) Let $\{A_n\}$ be a decreasing sequence. By taking the complement in the result (c) show that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} \mathbb{P}(A_N)$$

e) Use the result of (d) to show that the cdf of any random variable X is right continuous, that is

$$F(a) = \lim_{h\to 0,\, h>0} F(a+h)$$

for all a. [Hint: Recall that the cdf of X is the function $F(a) = P(X \leq a)$. Define appropriate sets $\{A_n\}$ and use the result of part (d). Note that it is sufficient to prove convergence along any decreasing sequence of points $\{h_1, h_2, \dots\}$ which converge to 0.]