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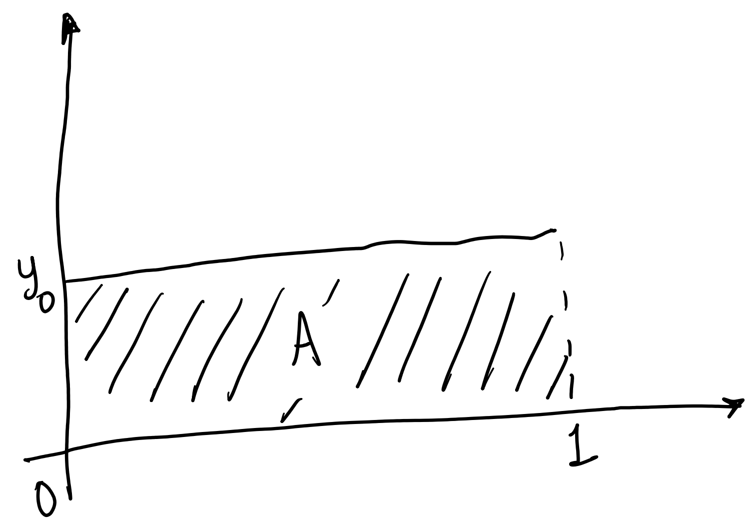
a) $f_0 \rightarrow$ uniform over $[0, 1]$



$$A = 1 = y_0 \times 1$$

$$\Rightarrow y_0 = 1$$

$$\therefore f_0(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{else} \end{cases}$$

b) Data : { Continuous r.v. X is uniform over $[0, \theta]$ }

$$0 \leq \theta \leq 1$$

$X \sim$ uniform on $[0, \theta]$

$\therefore$ PDF of X is

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{else} \end{cases}$$

$\therefore$ Likelihood,

$$f(D|\theta) = f(x|\theta)$$

$$= \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{else} \end{cases}$$

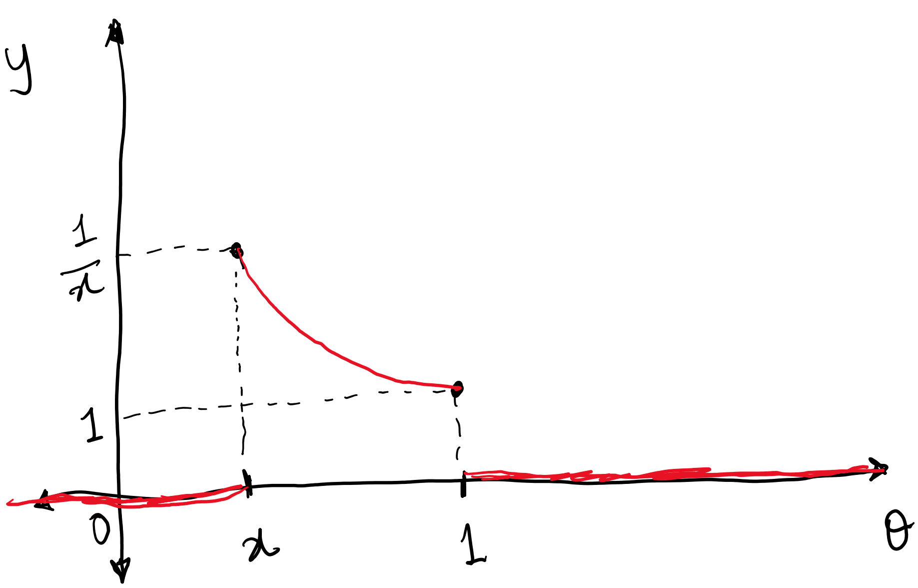
c) Posterior PDF

$$f_1(\theta|D) = \frac{f(x_1|\theta)\, f_0(\theta)}{Z}$$

$$= \begin{cases} \dfrac{\frac{1}{\theta} \times 1}{Z}, & 0 \le x_1 \le \theta \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{1}{Z\theta}, & 0 \le x_1 \le \theta \\ 0, & \text{else} \end{cases}$$

$$y = f(x_1|\theta)\, f_0(\theta) = \frac{1}{\theta}$$



d)

$$Z = \int_0^1 f(x_1|\theta)\, f_0(\theta)\, d\theta$$

$$= \int_0^x f(x_1|\theta) \times \underset{\substack{\uparrow \\ \theta \geq x}}{0} \times d\theta + \int_x^1 f(x_1|\theta) \times 1 \times d\theta$$

$$= \int_x^1 \frac{1}{\theta} \times 1 \times d\theta$$

$$= \frac{-1}{\ln(x)}$$

$$\therefore f_1 = \begin{cases} \dfrac{1}{Z\theta} = -\dfrac{1}{\theta \ln(x)} & x \leq \theta \leq 1 \\ 0 \quad , & \text{else} \end{cases}$$

(2) $X \sim$ uniform on $[0, \theta]$

$$f_0(\theta) = \begin{cases} 1, & \text{if } \begin{array}{l} 0 \leq x_1 \leq \theta \\ 0 \leq x_2 \leq \theta \end{array} \\ 0, & \text{else} \end{cases} \qquad 0 \leq \theta \leq 1$$

Likelihood:

$$f(D|\theta) = f(x_1|\theta) \cdot f(x_2|\theta)$$

because independent measurements

$$= \begin{cases} \dfrac{1}{\theta^2}, & \begin{array}{l} 0 \leq x_1 \leq \theta \\ 0 \leq x_2 \leq \theta \end{array} \\ 0, & \text{else} \end{cases}$$

Posterior:

$$f_1(\theta|D) = \begin{cases} \dfrac{1}{Z\theta^2}, & \begin{array}{l} 0 \leq x_1 \leq \theta \\ 0 \leq x_2 \leq \theta \end{array} \\ 0, & \text{else} \end{cases}$$

let, $x = \max(x_1, x_2)$

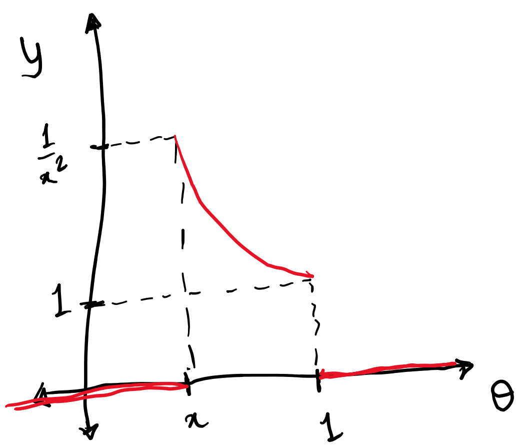
$$Z = \int_0^1 f(x_1|\theta) \times f(x_2|\theta) \times f_0(\theta) \times d\theta$$

$$Z = \int_x^1 \frac{1}{\theta^2} \times 1 \times d\theta$$

$x = \max(x_1, x_2)$

$$= \frac{1}{x} - 1$$

$$y = f(x_1|\theta) \times f(x_2|\theta) \times f_0(\theta) = \begin{cases} \frac{1}{\theta^2} & 0 \le x_1 \le \theta \\ & 0 \le x_2 \le \theta \\ 0 \end{cases}$$



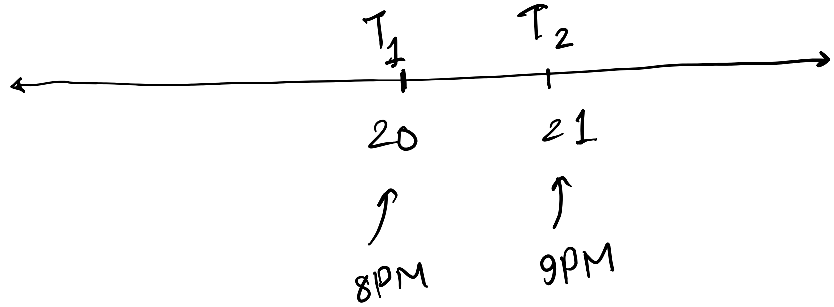
$$f_1(\theta|D) = \begin{cases} \frac{y}{Z}, & x \le \theta \le 1 \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{\theta^2} \times \frac{x}{1-x}, & x \le \theta \le 1 \\ 0, & \text{else} \end{cases}$$

3

$$\lambda = 2 \text{ hr}^{-1}$$

(a)

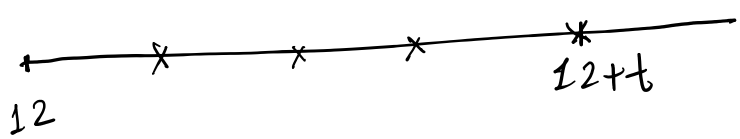


$$T_2 - T_1 = 1 \text{ hr}$$

$$\therefore\ P\left(N(1) = 0\right) = \frac{(\lambda \times 1)^0 e^{-\lambda \times 1}}{0!}$$

$$= e^{-\lambda}$$

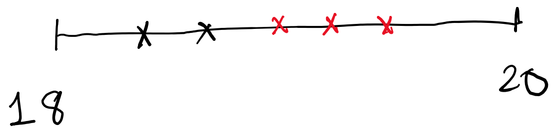
(b)



$$E\left(T_4\right) = 12 + \frac{4}{\lambda} = 12 + \frac{4}{2} = 14 \text{ hr}$$

$$= 2\text{PM}$$

(c)



$$P(N(20-18) \geq 2) = 1 - P(N(2) < 2)$$

$$= 1 - P(N(2) = 0) - P(N(2) = 1)$$

$$= 1 - \frac{e^{-2\times 2} \times (2\times 2)^0}{0!} - \frac{e^{-2\times 2} \times (2\times 2)^1}{1!}$$

$$= 1 - 5e^{-4}$$

4

$P(Y=1) = 0.4$ $\quad$ $P(Y=2) = 0.3$

$P(Y=3) = 0.2$ $\quad$ $P(Y=4) = 0.1$

a) $P(Y=4) = 0.1$

b) $\lambda' = \text{rate of fullcars} = \lambda P(Y=4)$

$= 10 \times 0.1$ per minute

$= 1$ per minute

$$P(N(1)=2) = \frac{e^{-1\times 1} \times (1)^2}{2!}$$

$$= \frac{1}{2e}$$

c) $$E\left(\sum_{i=1}^{(1)} Y_i\right) = E(N(1)) \times E(Y)$$

$$= (10 \times 1) \times \left(1\times 0.4 + 2\times 0.3 + 3\times 0.2 + 4\times 0.1\right)$$

$$= 20$$

d) $P\left(N\left(\frac{1}{6}\right) \geq 2\right) = 1 - P\left(N\left(\frac{1}{6}\right) < 2\right)$

$$= 1 - P\left(N\left(\tfrac{1}{6}\right) = 0\right) - P\left(N\left(\tfrac{1}{6}\right) = 1\right)$$

$$= 1 - \frac{e^{-10\times\frac{1}{6}} \times \left(10\times\frac{1}{6}\right)^0}{0!} - \frac{e^{-10\times\frac{1}{6}} \times \left(10\times\frac{1}{6}\right)^1}{1!}$$

$$= 1 - \frac{8}{3}e^{-\frac{5}{3}}$$

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(5)

$$p_1 = \frac{\mu}{\mu+\lambda} = \frac{3}{5}$$

$$p_2 = \frac{\lambda}{\mu+\lambda} = \frac{2}{5}$$

This can be mapped to coin-tossing problem

$$P(\text{at least 2 men before 2 women})$$

$$= 1 - P(\text{1 man before 2 women})$$

$$- P(\text{0 men before 2 women})$$

$$= 1 - \underbrace{\left(P(MWW) + P(WMW)\right)}_{\text{1 man}} - \underbrace{P(WW)}_{\text{0 men}}$$

$$= 1 - \left(2 \times \left(\frac{2}{5}\right)^2 \times \left(\frac{3}{5}\right)\right) - \left(\frac{2}{5}\right)^2$$

$$= \frac{81}{125}$$

6

(a) $$P\left(N\left(\tfrac{1}{3}\right)=2 \mid N(1)=2\right)$$

$$= \frac{P\left(N\left(\frac{1}{3}\right)=2 \cap N(1)=2\right)}{P(N(1)=2)}$$

$$= \frac{P\left(N\left(\frac{1}{3}\right)=2 \cap N\left(\frac{2}{3}\right)=0\right)}{P(N(1)=2)}$$

memory less property

$$= \frac{P\left(N\left(\frac{1}{3}\right)=2\right) \; P\left(N\left(\frac{2}{3}\right)=0\right)}{P(N(1)=2)}$$

$$= \frac{\frac{e^{-\lambda/3} \times \left(\frac{\lambda}{3}\right)^2}{2!} \times \frac{e^{-\frac{2\lambda}{3}} \times \left(\frac{2\lambda}{3}\right)^0}{0!}}{e^{-\lambda} \frac{\lambda^2}{2!}}$$

$$= \boxed{\frac{1}{9}}$$

(b)

$$P\left(N\left(\tfrac{1}{3}\right) \geq 1 \,\middle|\, N(1)=2\right) = 1 - P\left(N\left(\tfrac{1}{3}\right)=0 \,\middle|\, N(1)=2\right)$$

$$= 1 - \frac{P\left(N\left(\frac{1}{3}\right)=0 \cap N\left(\frac{2}{3}\right)=2\right)}{P\left(N(1)=2\right)}$$

$$= 1 - \frac{\dfrac{e^{-\frac{\lambda}{3}} \times \left(\frac{\lambda}{3}\right)^0}{0!} \times \dfrac{e^{-\frac{2\lambda}{3}} \times \left(\frac{2\lambda}{3}\right)^2}{2!}}{\dfrac{e^{-\lambda} \times \lambda^2}{2!}}$$

$$= 1 - \frac{4}{9} = \boxed{\frac{5}{9}}$$