Lab 2a - Areas and Determinants

MATH 5110: Applied Linear Algebra and Matrix Analysis

Northeastern University

Author: Chris King

1 Background

Suppose that L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then the map L sends any line segment to another line segment. To see this, consider a line segment whose endpoints are the vectors $\vec{u}, \vec{v}$. The line segment connecting $\vec{u}$ and $\vec{v}$ is described by the set of vectors

$$\{t\,\vec{u} + (1-t)\,\vec{v}\}, \quad 0 \le t \le 1$$

The image of this set under the map L is the set

$$\{L(t\,\vec{u} + (1-t)\,\vec{v})\} = \{t\,L(\vec{u}) + (1-t)\,L(\vec{v})\}\,, \quad 0 \le t \le 1,$$

which is the line segment with endpoints at the two vectors $L(\vec{u})$ and $L(\vec{v})$.

As a consequence the linear map L sends a polygon into a polygon. In particular L maps a triangle into a triangle. Note that it is possible that the image triangle could 'collapse' down to a line segment or even to a point (consider the case where L maps every vector to $\vec{0}$!). But this can only happen if L is singular, so we assume henceforth that L is non-singular. This means that the image of a triangle is a genuine triangle with three non-colinear corners.

For any set C in $\mathbb{R}^2$, let $L(C)$ denote its image under the map L. Suppose that C is the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$. Then $L(C)$ is the parallelogram spanned by the vectors $L(\vec{u})$ and $L(\vec{v})$. The areas of these sets are

$$\text{Area}(C) = |\text{Det}(\vec{u}\,\vec{v})|, \quad \text{Area}(L(C)) = |\text{Det}(L(\vec{u})\,L(\vec{v}))|$$

Now suppose that L is represented by the 2×2 matrix A. Then

$$(L(\vec{u})\,L(\vec{v})) = (A\vec{u}\,A\vec{v}) = A\,(\vec{u}\,\vec{v})$$

and therefore

$$\text{Det}(L(\vec{u})\,L(\vec{v})) = \text{Det}(A)\,\text{Det}(\vec{u}\,\vec{v})$$

It follows that

$$\text{Area}(L(C)) = |\text{Det}(A)|\,\text{Area}(C)$$

This relation between the area of C and of its image $L(C)$ holds for all regions in the plane, not just parallelograms. This can be demonstrated by breaking up the region C into many small rectangles and applying the formula to each small piece. Or it can be shown by writing the area as a double integral and computing the Jacobian of the map L, which turns out to be just $|\mathrm{Det}(A)|$.

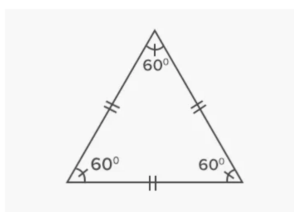
So to summarize: for any region C in the plane $\mathbb{R}^2$, the areas of C and $L(C)$ are related by

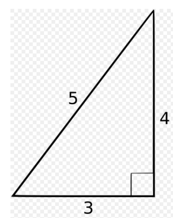
$$\mathrm{Area}(L(C)) = |\mathrm{Det}(A)|\, \mathrm{Area}(C)$$

2 Problems

2.1 Task 1

Let C denote the equilateral triangle with corners at $(0,0), (2,0), (1, \sqrt{3})$, and let D denote the $3-4-5$ triangle with corners at $(0,0), (3,0), (3,4)$ (see pictures below). Consider the linear map L from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps C to D, that is $L(C) = D$. Find the matrix A which represents L in the standard basis. [Hint: note that two sides of the triangle C meet at the endpoint $(0,0) = \vec{0}$. The map L should send these two sides to the two sides of the triangle D which meet at $(0,0)$; use the same method as in Lab 1b].





$C \xrightarrow{L} D$

2.2 Task 2

Calculate the area of the largest circle that can be inscribed inside the triangle C (note that this is also the largest ellipse that can be inscribed in C).

2.3 Task 3

A linear map L from $\mathbb{R}^2$ to $\mathbb{R}^2$ sends an ellipse into another ellipse. If L is non-singular then every ellipse is the image under L of an ellipse. Use this fact together with the result from Task 2 to compute the area of the largest ellipse which can be inscribed in the triangle D.

2.4 Task 4

Find the equation of the largest ellipse described in Task 3.