MATH 7241 Fall 2020: Problem Set #8

Due date: Sunday November 22

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

$\boxed{\textbf{Exercise 1}}$ Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \dots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

$$P_k = \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad\qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Starting at $X_0 = k$, let R_k be the probability to reach state N without returning to state k. Use the Gambler's Ruin result to compute R_k for all $k = 0, \dots, N$, and for all $0 < p < 1$. [Hint: condition on the first step and use the formula given above].

$\boxed{\textbf{Exercise 2}}$ Consider a biased random walk X_n on the semi-infinite line $\{0, 1, 2, \dots\}$, where at each step the walker either goes left with probability q or goes right with probability p, where $p + q = 1$. The point 0 is absorbing. Recall the solution of the Gambler's Ruin problem:

$$Q_k = \frac{(q/p)^N - (q/p)^k}{(q/p)^N - 1}, \quad q \neq p, \qquad\qquad Q_k = 1 - \frac{k}{N} \quad \text{for } p = q$$

was the probability of being absorbed at 0 before being absorbed at N, starting at k. By taking appropriate limits, use this result to find the probability that the walk X_n on the semi-infinite line is absorbed at 0, after starting at k (your answer will depend on q and p – be sure to include the case $q = p$).

$\boxed{\textbf{Exercise 3}}$ The Markov chain $X = \{X_n\}$ is defined on the state space $S = \{0, 1, 2, \dots\}$. The chain is irreducible, aperiodic and positive persistent, with stationary distribution $\{w_k\}$ $(k = 0, 1, 2, \dots)$. Let $Y = \{Y_n\}$ be an independent copy of X, and define $Z = (X, Y)$.

a). Write down the transition matrix for Z, and compute its stationary distribution (your answer will depend on w).

b). Given that the chain Z starts at the state (k, k) (so that $X_0 = Y_0 = k$), find an expression for the expected number of steps until the first return to (k, k).

$\boxed{\textbf{Exercise 4}}$ Let $X_1, X_2, \dots$ be IID random variables, where the moment generating function is

$$\mathbb{E}[e^{tX}] = \frac{4}{(2-t)^2}, \qquad t < 2$$

a). Find the mean $\mathbb{E}[X]$.

b). Let $Y_n = (1/n) \sum_{i=1}^n X_i$. For $x > \mathbb{E}[X]$ use Cramer's Theorem to compute

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n > x)$$

$\boxed{\textbf{Exercise 5}}$ A fair coin is tossed n times, coming up Heads N_H times and Tails $N_T = n - N_H$ times. Let $S_n = N_H - N_T$. Use Cramer's Theorem to show that for $0 < a < 1$,

$$\lim_{n \to \infty} P(S_n > an)^{1/n} = \left[(1+a)^{1+a} \, (1-a)^{1-a} \right]^{-1/2}$$

$\boxed{\textbf{Exercise 6}}$ Let $X_1, X_2, \dots$ be IID r.v.'s, and $Y_n = n^{-1} \sum_{i=1}^n X_i$. For $x > \mathrm{E}[X]$ use Cramer's Theorem to compute

$$\lim_{n \to \infty} 1/n \, \log P(Y_n > x)$$

when X is a) exponential with rate λ, and b) uniform on $[0, 1]$. [Note: for part (b) you will not achieve an explicit solution; instead produce a plot of the result as a function of x]

$\boxed{\textbf{Exercise 7}}$ For a branching process, calculate the probability of extinction when $p_0 = 1/6$, $p_1 = 1/2$, $p_2 = 1/3$.

$\boxed{\textbf{Exercise 8}}$ The number of offspring Z in a branching process has the following distribution:

$$p_0 = \mathbb{P}(Z = 0) = p, \quad p_1 = \mathbb{P}(Z = 1) = q, \quad p_2 = \mathbb{P}(Z = 2) = 2p - \frac{1}{6}$$

where $0 \leq q \leq 1$ and $1/9 \leq p \leq 4/9$. Also $\mathbb{P}(Z > 2) = 0$.

a). Compute q as a function of p.

b). Find the mean $\mathbb{E}[Z]$ (your answer should depend on p, but not on q).

c). Find the largest value of p for which extinction is certain (your answer should be a number).

d). Let p_m be the value computed in (c). Calculate the probability of extinction for $p > p_m$ (your answer should depend on p, but not on q).