Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 3.

1. Reading: [Gockenbach], Chapters 4.

2. Questions: (You can use Matlab if needed.)

The following questions 1-4 are about Section 4.

Question 1. Let $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $V = \text{Span}\{\vec{b}_1, \vec{b}_2\}$.

(1). Find the coordinate of $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ relative to $\mathscr{B}$.

(2). Suppose the coordinate of $\vec{y} \in V$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the vector $\vec{y}$.

Question 2. Let $V = \{a_1 t + a_2 t^2 + a_3 t^3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$ with basis $\mathscr{B} = \{t, t^2, t^3\}$. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ with basis $\mathscr{C} = \{1, t, t^2\}$. Let $T : V \to P_2$ be a transformation defined by derivatives $T(p) = 2f' - f''$.

(1) Prove that T is a linear transformation. (using properties of derivative.)

(2) Find the matrix $[T]_{\mathscr{B}\mathscr{C}}$ of the transformation T respective to the bases $\mathscr{B}$ and $\mathscr{C}$.

(3) Is T an isomorphism?

Question 3. Let V be a subspace of $\mathbb{R}^n$. Suppose $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_s\}$ and $\mathscr{C} = \{\vec{v}_1, \ldots, \vec{v}_s\}$ are two bases of V.

(1) Find the $\mathscr{B} - \mathscr{C}$-matrix $S = [\text{id}]_{\mathscr{B}\mathscr{C}}$ of the identity map from V to V. This matrix is also called **change of coordinate matrix** from $\mathscr{B}$ to $\mathscr{C}$.

(2) Show that $[\vec{b}_1 \ \ldots \ \vec{b}_s] = [\vec{v}_1 \ \ldots \ \vec{v}_s]S$.

Question 4. Let V be a subspace of $\mathbb{R}^3$. Suppose $\mathscr{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right\}$ and $\mathscr{C} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ are two bases of V. (Example for the above question.)

(1) Find change of coordinate matrix S from $\mathscr{B}$ to $\mathscr{C}$.

(2) Verify that $[\vec{b}_1 \ \vec{b}_2] = [\vec{v}_1 \ \vec{v}_2]S$.

The following questions 5-9 are about determinant in Section 5.

Question 5. Consider the real $n \times n$ matrix $A_n = (a_{ij})_{i,j=1,\dots,n}$ which has 2's on the main diagonal, -1's on the two diagonals next to the main diagonal, and 0's elsewhere. For example $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

Compute $\det(A_n)$ in terms of n.

Question 6. Compute the area of the hexagon with vertices (3, 1), (12, 8), (10, 7), (-1,-1), (-10,-8) and (-8,-7). Compute by hand (using determinant) and verify by Matlab use polyshape.

Question 7. True or False: (Briefly explain the reason.)

(1) $\det(A + B) = \det(A) + \det(B)$ for all 5×5 matrices A and B.
(2) The equation $\det(-A) = \det(A)$ holds for all 6×6 matrices.
(3) If all the entries of a 7×7 matrix A are 7, then $\det(A)$ must be 7^7
(4) An 8×8 matrix fails to be invertible if (and only if) its determinant is nonzero.
(5) If B is obtained be multiplying a column of A by 9, then the equation $\det(B) = 9 \det(A)$ must hold.
(6) If A is any $n \times n$ matrix, then $\det(AA^T) = \det(A^T A)$
(7) There is an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}$
(8) If A is an invertible $n \times n$ matrix, then $\det(A^T)\det(A^{-1}) = 1$.
(9) $\det(4A) = 4\det(A)$ for all 4×4 matrices A.
(10) There is a nonzero 4×4 matrix A such that $\det(4A) = 4\det(A)$.
(11) $\det(AB) = \det(BA)$ for all $n \times n$ matrices A and B.

Question 8. Is there a 3×3 matrix such that $A^2 + I = \mathbf{0}$? Show your reason.

Question 9. Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 2 & 4 & 6 & 10 \\ 1 & 5 & 10 & 9 \end{bmatrix}$. Compute by hand the **determinant** of A. Write down all steps you are using. (Hint: using row operations together with cofactor expansion)