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MATHEMATICAL MODELS OF INSURANCE AND REINSURANCE

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Abstract: *The topic of the paper is the insurance and reinsurance of enterprises' capital. The notion of capital insurance i.e. capital reinsurance, their classification, as well as the mathematical models used in insurance and reinsurance are discussed here. Reinsurance represents the most efficient form of risk sharing among insurance companies.*

Keywords: *insurance, reinsurance, risk, retention, mathematical models, retention function, compensation function.*

1. Introduction

The term insurance refers to security and trust. Therefore, the purpose of insurance is providing security. In broader sense, insurance is joining of all those open to the same danger i.e. risk, together [1]. Risk is an uncertain future event that may cause damaging consequences. Risk is narrowly related to insurance since the main prerequisite for the existence of insurance is the presence of risk. Accordingly, insurance represents the protection of the ownership interests of physical and legal persons during risk realization i.e. insured case, by the insurance funds founded through premium collection from these persons.

In economic terms, insurance represents economic activity that has as its objectives: to estimate the existence of risk that carries material damage to the economic subject (insured) for the realization of an insured event; to cover possible damage caused in the realization of an insured event; to perform the redistribution of damage in time; to realize the recovery of insurance premium in terms of the compensation for damage from the insurance fund.

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Risk represents the uncertainty in terms of the outcome of an accident, the danger against the damaging consequences of which, the economic protection through insurance is organized. In order that the insurance could cover the risk, it must have the following features: the possibility of risk realization, the uncertainty of events to be insured, the danger of the realization of economically damaging event, repetition, that is the risk mustn't be isolated, independence from the will of an insurer and insured, legality of risk, limited scope of risk in space and time and homogeneity.

In practice, it is rare for only one risk event to be insured. It is usually spoken of a risk portfolio. In economic terms, the risk portfolio represents the total ownership of insurance company over insurance policies related to different insured events. In mathematical terms, the risk portfolio represents an ordered n-tuples of n elements of different insurance policies.

The insurance company accepts risk from its clients i.e. the insured for compensation called premium. A premium is the amount of money paid by the insured to the insurer in accordance with an insurance contract.

If the risk or risk portfolio is extremely high for an insurance company, it can divide it into smaller parts and pass it over to other companies, its reinsurers. They can further share the risk between their reinsurers. Part of the risk that remains with the insurance company that has taken on the risk from its insured is called retention, [2]. So, the starting risk is covered by an entire net of the insurance and reinsurance of more insurance companies, whereby each has its retention. Determining the size of retention depends on many parameters, that being the issue of actuarial mathematics.

In practice, insurance companies come across different problems, the most common being: evaluation i.e. calculation of a premium; evaluation of risk capital; evaluation i.e. calculation of retention; behavior of the company's management regarding the kinds of risk; maximally acceptable amount of risk for a company; and other. Each problem is very significant for the business of an insurance company. It is in the interest of the company to collect a higher insurance premium in order not to have losses, however, in that case, most of insured would go to another insurance company that has a favorable insurance premium.

The evaluation of risk capital, retention and maximally acceptable amount of risk that a company can cover with its capital are the issues associated with reinsurance. Every insurance company disposes of a certain amount of capital used for claim payment to its clients. If a company insures an event of high risk whose realization can have catastrophic consequences, the size of the claim for the realization of that event can exceed all the available capital of the company. In that case, the company will share the surplus of the risky capital among its reinsurers and will retain the part that can be covered with available capital. In order to be without the loss, it is in the interest of the insurance company to decide precisely on the part of the risky capital it wants to keep for itself.

2. Basic Mathematical Models of Insurance

From the theoretical point of view, a mathematical model can be described as a reservoir. The characteristic feature of this model is that the capital inflow is regular, on one side, while on the other, the capital outflow is unknown. The capital outflow can be very irregular depending on the unpredictable events such as accidents and natural catastrophes. It can be noticed that the stochastic nature of the capital outflow is dual i.e. it is neither known when it will come to a claim, nor what is the size of a claim. This simple model describes the most important aspects of insurance problems.

Mathematical model of insurance depends on four elements, two being stochastic and other two deterministic.The basic elements are:

1) The initial reserves u defining the beginning point of the process;
2) The premium return c determining capital (profit) accretion;
3) The time sequence when it comes to the claim T_1, T_2, T_3, ..., where T_1 refers to the time interval between the beginning moment $t = 0$ and the moment when it comes to the first claim, T_2 the time interval between the first and second claim.
4) Sequence of the claim variables X_1, X_2, X_3, ... determining the capital decrease.

It should be pointed out that the sequence (T_i) and (X_i) are the sequences of independent random values.

Apart from the above mentioned values, in insurance theory, the following values are also important :

1) numerical process *N(t)* represents the number of claims up to time *t* and we get

$$N(t) = \sup\{n \geq 1 \mid T_n \leq t\}, \text{ за } t \geq 0,$$

2) the total amount of claims up to time t is defined by the following expression:

$$S(t) = \sum_{i=1}^{N(t)} X_i .$$

When $U(t)$ represents the profit of an insurance company we get the following formula:

$$U(t) = u + ct - S(t).$$

Here, the claims X_1, X_2, X_3, ..., and the premium income c are stochastic i.e. random variables.

The greatest importance for the risk have the variables S and N. When their values are known, the size of the risk is also known. All the other variables can be deduced from these two series of basic variables. This means that the claims distribution X_i should be defined as well as that for time moments T_i, when the claims are paid.

The most famous model of modeling risk in insurance theory is the Cramer-Lundberg model. It is a simple model that quite realistically examines the process of the total amount of claims S. For the application of this model in practice, it is necessary to meet the following conditions:

1) it comes to the claims at intervals $0 \le T_1 \le T_2 \le \ldots$ that make a homogeneous Puasson process $N(t) = \sup\{n \ge 1 \mid T_n \le t\},\ t \ge 0$;
2) the claim at time T_i amounts to X_i. Sequence (X_i) represents the sequence of independent, identically distributed, non-negative random variables;
3) the sequences (T_i) and (X_i) are independent among themselves, as well as the process N and the sequence of random variables (X_i).

The Cramer-Lundberg model, as one of the simplest model of the total amount of claims will be considered in modeling capital reinsurance.For more information on the model se [3], [4], [6].

3. Mathematical Model of Reinsurance

Reinsurance represents repeated insurance i.e. insurance of insurance. The insurer can transfer part of the insurance obligations to other companies, the reinsurers, if there is no sufficient capital to cover the total risk taken on from the insured. Part of the obligations that an insurance company can keep for itself, without jeopardizing its liquidity, is called retention. Liquidity represents the ability of an enterprise to pay its money obligations till the due date [5]. The reinsurer is not in direct obligation to the insurer. If it comes to the realization of the insured case, the total claim is paid to the insured by the insurer an then the insurer from the reinsurer compensates the part of the claim that the reinsurer has taken on through reinsurance. Risk distribution of the insurer and the insured is not final. The reinsurer can share the obligation with his/her reinsurers.

One of the most important tasks of actuarial mathematics is the calculation of retention. In that sense, the factors influencing the size of retention should be understood. They are: financial force, i.e. the capital of an insurance company; the readiness to take on the risk on part of the management of an insurance company; profitability, i.e. the possibility of earning from insurance business; (non)alignment of risk. Financially strong, sound, company can afford a larger amount of risk in comparison to a poor company, so retention directly depends on the economic power of a company. The size of retention depends on the company's management, a conservative management will show little interest for taking on the risk in contrast to a brave and competent management. The size of the premium is such that it brings profit to the company. Also, the very nature of risk affects the size of retention.

If we compare the share of these factors in the calculation of retention, in notation (*Ret*), it is clear that its size depends on the following parameters: ”capital”, shortly (*C*), “readiness for taking on the risk”, shortly (*Risk*), “the possibility of earning”, shortly (*Earn*). If any of the values is higher the retention is higher. On the other side, the size of retention depends on the (non) alignment of risk, shortly (*nonAR*). The notion of non-alignment of risk supposes the random character of risk. So, it is clear that if the alignment of risk is higher the retention is lower and vice versa. If we have in mind all the mentioned parameters, their relation to the retention can be presented with the following formula:

$$\mathrm{Re}\,t = \frac{C * Risk * Earn}{nonAR} \qquad (1.1.)$$

The formula (1.1.) offers the basis for mathematical modeling of retention that includes all the previously mentioned terms: claims, premiums, etc.

In mathematical terms, the reinsurance agreement for the claim X defines part of the claim $X - h(X)$ that will be covered by the reinsurers of the viewed insurance company.

The function $h : R^{+} \to R^{+}$ is called the retention function. The retention function has the following features:

1) $h(x)$ and $x - h(x)$ are monotonously increasing functions
2) $h(0) = 0$
3) $0 \leq h(x) \leq x$, $(\forall x \in R^{+})$

The function $k(x) = x - h(x)$ is called the compensation function. The assumption that the functions h and k are monotonously increasing is correct since the increase of claims supposes the share increase of an insurance company, but also the share of its reinsurers in covering the amount of the claim.

When it comes to proportional insurance, a good choice of the retention function would be the formula

$$h(x) = ax, \quad 0 < a \leq 1,$$

where, a defines part of the claim that the insurance company would cover with its own means.

On the other side, when it comes to the reinsurance of the excess of loss, a good choice is the function

$$h(x) = \min(a, x), \quad a > 0 .$$

Let the formula

$$S = \sum_{i=1}^{N} X_i$$

stand for the risk that represents the amount of individual risks X_i. The individual reinsurance formula $h_i(x)$ can be modeled in the following way: let the value of the ith claim be X_i and let its part $X_i - h_i(X_i)$ be covered by insurance contract. It is supposed that the local retention function has the same features as the retention function h itself. Let $u(x)$ be a profit (usefulness) function and p the value of the premium paid to the reinsurer. Then the reinsurer efficiency function is as follows

$$u\left(p-\sum_{i=1}^{N}k_i(X_i)\right).$$

At the entire portfolio level, the reinsurer efficiency is shown in the function

$$u(p-k(S)).$$

The reinsurer is often in the position to choose whether he will cover only a part of the offered risk and give over the remaining part to other reinsurers or take on the risk himself.

Intuitively, it would be logical for the reinsurer to take on the risk himself. The following theorem shows that this conclusion is not entirely correct.

Theorem 1. *Let S be the risk defined by the formula* $S=\sum_{i=1}^{N}X_i$ *and let* $u:R^+\to R^+$ *be an increasing and concave function. For the local reinsurance with a compensation function* k_i*, there is a function* $k(x)$*,and so it follows that*

$$Ek(S)=E\left(\sum_{i=1}^{N}k_i(X_i)\right)$$

and

$$Eu\left(p-\sum_{i=1}^{N}k_i(X_i)\right)\leq Eu(p-k(S)).\ \square$$

In the theorem's proof, we use the inequality that is known in the theory of probability as the Jensen's inequality for conditional mathematic expectation and it states: Let X be an integrable random variable in a probability space (Ω,Φ,P) and let $G\in\Phi$ be arbitrary σ-algebra. If

$\varphi : R^{+} \to R^{+}$ is a convex function and $\varphi(X)$ an integrable random variable, then

$$\varphi(E(X \mid G)) \leq E(\varphi(X) \mid G).$$

If φ is a concave function, the above inequality is valid and vice versa.

Proof: Let for each $s \geq 0$ be defined the function

$$k(s) = E\left(\sum_{i=1}^{N} k_i(X_i) \mid S = s\right). \tag{1.2}$$

If we apply the feature of the conditional mathematic expectation $EX = E(E(X \mid Y))$ to the function (1.2.) we indirectly get $Ek(S) = E\left(\sum_{i=1}^{N} k_i(X_i)\right)$. So, the first part of the theorem is valid.

Further, as u is a convex function, we can apply the Jensen's inequality to it and we indirectly get

$$Eu\left(p - \sum_{i=1}^{N} k_i(X_i)\right) = E\left(E\left(u\left(p - \sum_{i=1}^{N} k_i(X_i)\right) \mid S\right)\right) \leq$$

$$\leq Eu\left(p - E\left(\sum_{i=1}^{N} k_i(X_i) \mid S\right)\right) = Eu(p - k(S))$$

and so, the second part of the theorem has been proved. □

The above proof has been deduced under the assumption that the compensation function is in the form of the formula (1.2.). However, that is not valid for the compensation function in general. Nevertheless, with some additional assumptions, the formula (1.2.) gives a compensation function. The assumptions are stated in the following theorem. In order that the theorem could be as clear as possible, we introduce the following definition:

Definition 1. *If the function* $f : R \to R^{+}$ *is given. The function f belongs to the class PF*$_2$[1] *if*

$$\det\begin{pmatrix} f(x_1 - y_1) & f(x_1 - y_2) \\ f(x_2 - y_1) & f(x_2 - y_2) \end{pmatrix} \geq 0,$$

for each $x_1 \leq x_2$ *and* $y_1 \leq y_2$.□

We can state the conditions under which the function $k(x)$ defined by the formula (1.3) can be considered as a compensation function.

Theorem 2. *Let* $n \geq 1$ *be a fixed real number in* k_1, k_2, ..., k_n, *arbitrary compensation functions. If* $S = \sum_{i=1}^{N} X_i$ *,where* $N = n$ *is a deterministic value and* X_1, X_2, ..., X_n, *are independent random variables of an absolutely constant type with the density* f_{X_1}, f_{X_2}, ..., f_{X_n}, *of the class PF*$_2$*, then the function k(x) defined by the formula (1.2,) is a compensation function.*□

On the basis of Theorem 2, we can model the compensation function, according to the given conditions, as well as the retention function. This way the problem of modeling reinsurance, with the above assumptions and limitations, is entirely theoretically solved.In more complex cases, in stead of deterministic value $N = n$, a numerical process, for instance, the homogeneous and inhomogeneous Puasson process $\{N(t) \mid t \geq 0\}$ can be viewed. Surely, in concrete situations the validity of mathematical modeling depends on the right choice of the compensation function k_i.

4. Conclusion

The theory of insurance represents a very important and interesting field for exploration and research, because insurance companies in the world, and recently, in our country have become important participants in the financial market. In this paper, the models of the functioning of insurance and reinsurance companies are described. The viewed models are theoretically realized under ideal conditions, however, in practice, the

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[1] PF$_2$ - P:lya frequency function of order 2

situation is more complex, since the cash flows in insurance are affected by a great number of unpredictable factors. This is the reason why the researches in the field of finance are current and attractive.

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MATEMATIČKI MODELI OSIGURANJA I REOSIGURANJA

Rezime. Tema ovog rada je osiguranje i reosiguranje kapitala preduzeća. Razmatra se sam pojam osiguranja, odnosno reosiguranja kapitala, njihove podele, kao i matematički modeli koji se koriste u osiguranju i reosiguranju. Reosiguranje predstavlja najefikasniji oblik deljenja rizika medju osiguravajućim kompanijama.

Ključne reči: osiguranje, reosiguranje, rizik, retencija, matematički modeli, funkcija retencije, funkcija kompenzacije.