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①

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 3 \\ 3 & -2 & 13 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$rank(A) = 2 < 3$

$\Rightarrow$ linearly dependent

②

$$A = \begin{bmatrix} 1 & 1 & 17 & 1 & 3 \\ 5 & 5 & 85 & 5 & 16 \\ 4 & 3 & 56 & 2 & 13 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & 12 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) $rank(A) = 3 \Rightarrow A$ Spans R^3.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

(3)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 9 \\ 5 & 9 & 7 \\ 1 & 0 & -5 \end{bmatrix}$$

(a)

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ linearly independent

(b)

$$\left[\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{4^{th} \text{ vector}} \right] \right]$$

Basis of R^4

(4)

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 \\ -x_1 + 2x_2 \end{bmatrix}$$

$$2(x_1+x_2) - 3(x_1) - 1(-x_1+2x_2) = 0$$

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

5

(1)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 4 \\ 0 & 0 & 1 \\ -3 & -4 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ linearly independent

(2)

Basis of $\mathbb{R}^5$

$$\left(\begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{extension}} \right)$$

⑥

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 2 \\ 2 & 3 & 7 & -2 & 10 \\ 1 & 1 & 3 & 1 & 6 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ u_1, u_2, u_4 is a basis

(7)

$$[A|b] = \left[\begin{array}{cccc|c} 1 & -5 & 10 & -1 & 2 \\ 2 & 11 & -23 & 2 & -4 \\ -4 & -23 & 49 & -4 & 8 \\ 1 & 2 & -1 & 1 & -2 \end{array}\right] \xrightarrow[R_4+R_1]{R_2+2R_1,\ R_3-4R_1} \left[\begin{array}{cccc|c} -1 & -5 & 10 & -1 & 2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & -3 & 9 & 0 & 0 \\ 0 & -3 & 9 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3+3R_2,\ R_4+3R_2]{-R_1,\ R_1-5R_2}$$

$$\text{rref} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & -2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $\text{rank}(A) = \text{rank}([A \,\vdots\, b]) = 2 < 4$ (number of variables)

$\Rightarrow$ infinitely many solutions

$\text{Nullity} = 4 - \text{rank}(A) = 4 - 2 = 2.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \ker(A) = \text{Nul}(A) = \text{Span}\left(\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$A^T = \begin{pmatrix} -1 & 2 & -4 & 1 \\ -5 & 11 & -23 & 2 \\ 10 & -23 & 49 & -1 \\ -1 & 2 & -4 & 1 \end{pmatrix}$$

$$\xrightarrow{\substack{R_2 - 5R_1 \\ R_3 + 10R_1 \\ R_4 - R_1}} \begin{pmatrix} -1 & 2 & -4 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & -3 & 9 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{-R_1 \\ R_1 + 2R_2 \\ R_3 + 3R_2}} \begin{pmatrix} 1 & 0 & -2 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{rref}$$

$$\operatorname{im}(A) = \operatorname{col}(A) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -3 \end{bmatrix} \right\}$$

(8)

$$L(x) = x_1 L(e_1) + x_2 L(e_2) + x_3 L(e_3),$$

because "L" is Linear

$$\Rightarrow L(x) = \begin{bmatrix} x_1 + 5x_2 + 7x_3 \\ 2x_1 + 2x_2 - 3x_3 \\ 3x_1 + x_2 + 9x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix}$$

(9)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 0 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

From Homework 1,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$$

rank = 3 = dimension of Codomain R^3

$\Rightarrow$ "L" is surjective

$$\text{nullity} = 4 - 3 = 1 \neq 0$$

$\Rightarrow$ "L" is not injective

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \\ 1 \end{bmatrix} = t \begin{bmatrix} 6 \\ 8 \\ 2 \\ -7 \end{bmatrix}$$

where $-\infty < t < \infty$

$$\text{Basis}(\ker(T)) = \left\{ \begin{bmatrix} 6 \\ 8 \\ 2 \\ -7 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \\ 4 & -2 & 2 \end{bmatrix} \Rightarrow \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis}(\text{Im}(A)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(10)

$M : R^4 \to R^3$

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{rank}(A) = 3$$

$$\therefore \text{nullity} = 4 - \text{rank}(A)$$
$$= 4 - 3 = \boxed{1}$$

(11)

$$A = \begin{bmatrix} -1 & -2 & -1 & 1 & -1 \\ 2 & 4 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1

$$\text{rank}(A) = 3$$

image

$$A^T = \begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 4 & 2 & 0 \\ -1 & 5 & 4 & 0 \\ 1 & 1 & 4 & 2 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

$$\operatorname{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis of } \operatorname{Im}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

From 1,

$$x_1 + 2x_2 \qquad\qquad + 2x_5 = 0$$

$$+ x_3 \qquad - \frac{x_5}{2} = 0$$

$$+ x_4 + \frac{x_5}{2} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ +1/2 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\text{Basis}\left(\text{Ker}(A)\right) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ +1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$$\text{rref}\left((A)_{\mathbb{Z}_3}\right) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2

$$\left(A^T\right)_{\mathbb{Z}_3} = \begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 4 & 2 & 0 \\ -1 & 5 & 4 & 0 \\ 1 & 1 & 4 & 2 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

$$\operatorname{rref}\left(\left(A^T\right)_{\mathbb{Z}_3}\right) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Basis}\left(\operatorname{Im}\left(\left(A^T\right)_{\mathbb{Z}_3}\right)\right) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

From 2,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis of kernel} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(12) Consider A with columns $\vec{A}_1, \vec{A}_2, \cdots \vec{A}_n$

then

$$\vec{L} = A\vec{x} = x_1 \vec{A}_1 + x_2 \vec{A}_2 + \cdots x_n \vec{A}_n$$

$n > m$, $\{A_1, A_2, \cdots A_n\}$ is linearly dependent

$$\Rightarrow x_1 \vec{A}_1 + x_2 \vec{A}_2 + \cdots x_n \vec{A}_m = 0$$

has a non-trival Solution.

which is a non-zero vector

$\Rightarrow$ Linear transformation is not injective.

(13) 1)

Consider 2 Standard basis vectors

$$e_1 = (1,0) \;\&\; e_2 = (0,1)$$

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

gives the vector after Counter Clockwise rotation

2)

$$\vec{b} = \begin{bmatrix} x \\ y \end{bmatrix}$$

we observe,

$$e_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3)

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \rightarrow \begin{bmatrix} x + \gamma y \\ y \end{bmatrix}$$

$$e_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 \rightarrow \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

4)

$L: R^n \to R^n$

$L = r\vec{x}, \quad r > 0$

$L(e_i) = r e_i$

$$A = \begin{bmatrix} r & 0 & 0 & 0 & \cdots & 0 \\ & & & & & 0 \\ 0 & r & \cdots & & & \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & & & r \end{bmatrix}$$

$$L = Ax = r\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$