MATH 7241: Probability 1 FALL 2020 TEST 1

Important:

- This Test will be available at **6pm on Thursday October 29. You must start the Test at 6pm**.
- This Test must be **completed within 2 hours** – you will not be able to upload your answer after that time.
- You must **upload your answer as a pdf file**. Photos, jpg files etc will not be accepted. You may wish to install and use a **scanner app on your phone**.
- You must put your **full name and student ID** at the top of your answer.
- Send me an email if you have any questions or encounter any problems.
- You may use any material from the class, including notes, problem sets and recordings. **You may not access material from any other source, and you may not discuss these problems with anyone until they have been submitted**.

$\boxed{\textbf{Questions:}}$

1) A town has five hotels, numbered $1, 2, 3, 4, 5$. Seven people arrive and each person randomly and independently selects a hotel. Find the probability that nobody selects hotel 1 or hotel 2.

2) Four balls are shared between box #1 and box #2. At each step one of the two boxes is randomly selected. Let B_n be the box selected at the nth step. If the box B_n is not empty, a ball is removed from B_n and is then placed in the other box. If the box B_n is empty, a ball is removed from the other box and is placed in B_n. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$.

3) Consider the following transition probability matrix for a Markov chain on 4 states:

$$P = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0 & 0.25 \\ 0.75 & 0.25 & 0 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first visit to state 2.

4) The continuous random variables X and Y are independent. X is uniform on the interval $[0, 2]$, and Y has the pdf $f_Y(y) = 2y$ for $0 \le y \le 1$. Compute $\mathrm{P}(X \le Y)$. [Hint: first write down the pdf of X, then condition on Y to compute $\mathrm{P}(X \le Y \,|\, Y = y)$, then undo the conditioning on Y].

5) A bank has two ATM machines. Three customers A,B, and C enter the bank at the same time. A and B go directly to the ATM's; C waits until either A or B completes their service, and then immediately goes to the ATM. Assume that all service times are exponential with mean 3 minutes. Find the probability that C completes service before A. [Hint: use the memoryless property of the exponential distribution]

6) Let $X_1, X_2, \ldots$ be a sequence of random variables, and suppose that

$$\mathrm{E}[\,|X_n|\,] \leq \frac{1}{n} \qquad \text{for all } n \geq 1.$$

Compute

$$\mathrm{P}(X_n \geq n \ \text{ i.o.}\,)$$

where i.o. means ‘infinitely often’. [Hint: use the Borel-Cantelli Lemma]

7) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$p_{ii}(n) = \mathrm{P}(X_n = i \,|\, X_0 = i) \geq \frac{1}{n+7} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

8) A biased coin has probability p of coming up Heads. The coin is tossed repeatedly. Let N_2 be the number of tosses until the first occurrence of the sequence (Heads, Tails), and let N_3 be the number of tosses until the first occurrence of the sequence (Heads, Tails, Heads).

a) Compute the conditional probability $\mathrm{E}[N_3 \,|\, N_2 = k]$ for any $k \geq 2$ (your answer should involve k and also $\mathrm{E}[N_3]$).

b) We have

$$\mathrm{E}[N_3] = \sum_{k=2}^{\infty} \mathrm{E}[N_3 \,|\, N_2 = k] \, \mathrm{P}(N_2 = k)$$

Substitute your answer from part (a) into this formula and compute $\mathrm{E}[N_3]$ (your answer should depend on p, but nothing else). **NOTE:** you should use the result from Practice Problem #3, where it was shown that $\mathrm{E}[N_2] = 1/p(1-p)$.

$\boxed{\text{EXTRA CREDIT CHALLENGE: only attempt this if you are bored!!}}$

9) Consider an irreducible chain on 3 states. **Either** prove that $p_{jj}(6) > 0$ for every state j, **or** give an example where $p_{jj}(6) = 0$ for some state j.