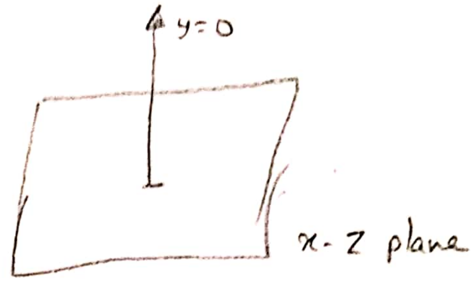
(14)

(a) Eq. of $xz$-plane

$$y = 0$$




$\{(1,0,0), (0,0,1)\}$ is an orthormal basis to $y=0$

$$p(x,y,z) = \langle (x,y,z), (1,0,0) \rangle (1,0,0) + \langle (x,y,z), (0,0,1) \rangle (0,0,1)$$

$$= (x,0,0) + (0,0,z)$$

$$= (x,0,z)$$

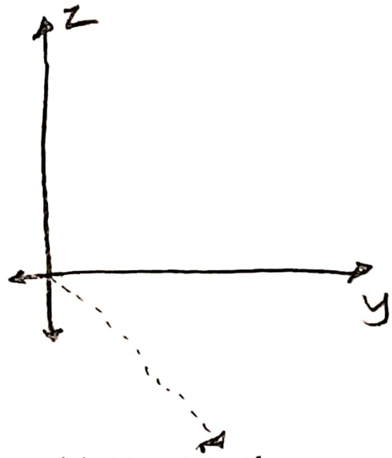
So,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

Counterclockwise rotation about

x-axis $\Rightarrow$




$(x', y', z') =$

$$(x,\ y\cos 45^\circ - z\sin 45^\circ,\ y\sin 45^\circ + z\cos 45^\circ)$$

$$= \left(x,\ \frac{y-z}{\sqrt{2}},\ \frac{y+z}{\sqrt{2}}\right)$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{bmatrix}$$

(c) $P(x', y', z') = \left(x,\ 0,\ \frac{y+z}{\sqrt{2}}\right)$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(d) $R = R_z R_y R_x$

where,

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is inclined at $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ w.r.t x, y, z axes-

rotating this by $120^\circ$ w.r.t origin.

gives, $\theta = 120^\circ + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow R = R_z R_y R_x$ where

$$\boxed{\theta = 120^\circ + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

(15) (1) $(\vec{x} \cdot \vec{y})_n = \sum_{n=1}^{N} x_m y_{n-m}$

let $z \in R$,

$$((x+z) * y)_n = \sum_{m=1}^{N} (x+z)_m \, y_{n-m}$$

$$= \sum_{m=1}^{N} x_m y_{n-m} + \sum_{m=1}^{N} z_m y_{n-m}$$

$$= (x * y)_m + (z * y)_m$$

let $\alpha \in \mathbb{R}$,

$$((\alpha x) * y)_n = \sum_{m=1}^{N} (\alpha x)_m \, y_{n-m}$$

$$= \alpha \sum_{m=1}^{N} (x_m) \, y_{n-m}$$

$$= \alpha (x * y)_m$$

Also, $\vec{0} \in L$

$\therefore$, if $y \in \mathbb{R}^N$ is fixed, then

$L: \vec{x} \to \vec{x} * \vec{y}$ is linear.

(2) $L: \vec{x} \to \vec{x} * \vec{y}$

$F(e_k)_n = y_{n-k}$ , $n = 0, 1, \dots, N$

then $F(e_0) = (y_0, y_1, \dots, y_N)$

$F(e_1) = (y_{N-1}, y_0, \dots y_{N-2})$

$$A = \begin{bmatrix} y_0 & y_{N-1} & y_{N-2} & \cdots & y_1 \\ y_1 & y_0 & & \cdots & y_2 \\ \vdots & & & & \vdots \\ & & & \cdots & y_0 \\ y_{N-1} & y_{N-2} & & & \end{bmatrix}$$

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(16) $L: R^3 \to R^3$

$\vec{u} = (1, 0, 1)$ $\vec{v} = (1, 1, -1)$

$\vec{u} - \vec{v} = (0, -1, 2)$

we have, $L(\vec{u} - \vec{u}) = 0$

$\therefore L(\vec{v} + \alpha(\vec{u} - \vec{v})) = L(\vec{v}) + \alpha \times 0$

$= b \quad \forall \alpha \in \mathbb{R}$

$\therefore \vec{v} + 2(\vec{u} - \vec{v}) = (1, -1, 3)$
$\vec{v} + 5(\vec{u} - \vec{v}) = (1, -4, 9)$

(17) $T: R^4 \to R^4$

$$\ker(T) = \text{Span}\left( \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$T(\vec{y}) = \vec{b} \quad \vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix}$$

(1)

$\dim(T) = 2$

$\therefore$ All possible solutions are linear combinations of $\vec{u}$ + $\vec{v}$

i.e., $\vec{y} + \alpha \vec{u} + \beta \vec{v}$

because $\vec{u}$ & $\vec{v}$ $\in \ker(T)$

i.e., null space

(2)

$$\begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

NO $\alpha, \beta$ Satisfy

$\Rightarrow$ No Solution

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$\vec{u} = (1,1,0)$ | $\vec{v} = (2,-1,2)$

$\ker(L) = \text{Span}\{(1,1,1)\}$

$L(\vec{u}) = \vec{v}$

a) $\vec{x} = (1,2,1)$

$\vec{x} - \vec{u} = (0,1,1) \neq \alpha \ker(L)$

$\Rightarrow$ not a Solution

b) $\vec{x} - \vec{u} = (2,2,2) = 2(1,1,1) \in \ker(L)$

This is a solution

c) $\vec{x} - \vec{u} = (-4,-4,-2) \neq \alpha \ker(L)$

$\Rightarrow$ Not a Solution.

(19)

$$\vec{U} = \begin{bmatrix} -1 & 3 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 2 & 1 & -3 \end{bmatrix}$$

$$rref(U) = \begin{bmatrix} 1 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & -3/3 \\ 0 & 0 & 1 & 1/7 \end{bmatrix}$$

$dim(U) = 3$

$$W = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 3 & 1 & -1 \\ 2 & -2 & -1 & -1 \\ 2 & 2 & 1 & -1 \end{bmatrix}$$

$$rref(W) = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$dim(W) = 3$

$$\Rightarrow rref(U+W) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$dim(U+W) = 4$ ; $dim(U \cap W) = 3+3-4 = 2$

$\dim(U) = 3$ ; $\dim(R^4) = 4$

$\therefore \dim(R^4) - \dim(U) = 4 - 3 = 1$

$\therefore$ 1 more vector is needed

$\therefore (-1, 3, 1, 0), (1, 0, 1, -1),$
$(2, 2, 1, -3)$ and $(0, 0, 0, 1)$
form a basis of
$R^4/U$

$\dim(V) = 3$ ; $\dim(R^4) = 4$

$\dim(R^4) - \dim(U) = 4 - 3 = 1$

$\therefore$ 1 more vector is needed.

$\therefore (-1, 1, 1, 1), (1, 3, 1, -1), (2, -2, -1, -1)$
and $(0, 0, 0, 1)$
form a basis of
$R^4/W$

(20) $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$

are lineerly dependent

(a) By definition,

$$v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n$$

i.e.,

$$\alpha_1 v_1 + \cdots (-1) v_i + \cdots \alpha_n v_n = 0$$

$\Rightarrow$ At least 1 scalar non-zero multiple.

Hence, Proved.

(b)

Given,

$$\sum_{j=1}^{n} \alpha_j \vec{v}_j = 0$$

Let $\gamma \neq 0 \in K$,

Multiply with $\gamma$ on both sides

$$\Rightarrow \sum_{j=1}^{n} (\gamma \alpha_j) \vec{v}_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} (\beta_j) \vec{v}_j = 0,$$

such that $\beta_j = \gamma \alpha_j$

Hence, proved.