Sai Nikhil

NUID: 001564864

1

$$P_n = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \end{bmatrix}}_{n} \Bigg\} n$$

(1)

$$\det(P_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & 0 & 1 \end{vmatrix}$$

$$\xrightarrow[\substack{R_4 - R_1 \\ \vdots \\ R_n - R_1}]{R_3 - R_1} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & 0 \cdots & 0 \; 0 \\ 0 & 0 & -1 & 0 \cdots & 0 \; 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & -1 \; 0 \end{vmatrix}$$

$$\xrightarrow[\substack{\cancel{R_4 + R_2} \\ \vdots \\ R_n + R_2}]{R_3 + R_2} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

This is an upper triangular matrix

$$\therefore \det(P_n) = \prod_{i=1}^{n} a_{ii} = 1 \times 1 \times \cdots 1 = 1$$

(2)

$$\det(P_n) = 1 > 0$$

$\Rightarrow$ P_n is invertible

(3) Zero cannot be an eigenvalue because,

$$\det(P_n) = \prod_{i=1}^{k} \lambda_i^{p_i}$$ (where $\sum_{i=1}^{k} p_i = n$ and $\lambda_1, \lambda_2 \dots \lambda_k$ are "k" different eigen values with algebraic multiplices, $p_1, p_2 \dots p_k$)

$$\therefore \prod_{i=1}^{k} \lambda_i^{p_i} = 1$$

$\therefore$ All λ_i must be non-zero.

---

(2)

Given, $A\vec{v} = \lambda\vec{v}$

(1) Let, $\vec{v}$ is an eigen vector of $A^4 + 2A + I_n$

$$\therefore (A^4 + 2A + I_n)\vec{v} = (\lambda^4 + 2\lambda + 1)\vec{v}$$

$$\therefore B\vec{v} = (\lambda')\vec{v}$$

$\Rightarrow$ λ' is an eigen value of B and $\vec{v}$ is eigen vector of B"

Our assumption is Correct.

(2) Corresponding eigen value.

$$\lambda' = \lambda^4 + 2\lambda + 1$$

③ (1) $P = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$

$\lambda_1 = 6,\ \lambda_2 = 3,\ \lambda_3 = 0$

All 3 eigen values are different & $\vec{u}, \vec{v}, \vec{w}$ are linearly independent

$\Rightarrow \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(2) $A = PDP^{-1}$

$tr(A) = \sum_{i=1}^{3} \lambda_i = 6+3+0 = 9$

(3) Eigen values of A^2-3A are,

$\lambda_i^2 - 3\lambda_i$; $i = 1, 2, 3$

$\therefore\ 6^2 - 3\times6,\ 3^2-3\times3,\ 0^2-3\times0$

$18,\ 0,\ 0$

(4) $\det(A^2-3A) = \det(A)\underbrace{\det(A-3I)}_{0}$

(because, 3 is a eigen value of A)

$= \boxed{0}$

(5) rank (A^2-3A) = # non-zero eigen values

$= \boxed{1}$

---

④

(1) $\det(A^2 A^T A^{-1})$

$= [\det(A)]^2 \times [\det(A)] \times \frac{1}{[\det(A)]}$

$= 3^2 \times 3 \times \frac{1}{3}$

$= \boxed{9}$

(2)

$S^{-1} A S = 2A$

Since S^{-1} exists, let's multiply by "S" on both sides.

$S \times S^{-1} A S = 2SA$

$AS = 2SA$

$\therefore \det(A) \times \det(S) = 0$

$\Rightarrow \det(A) = 0$ or $\det(S) = 0$

But it is given that,

"A" & "S" are invertible.

This is a contradiction. $\therefore$ There are no such matrices.

(5) (1) $A^2 + A + I_n = 0$

$f(A) = 0$

$\Rightarrow f(\lambda)$ is characteristic polynomial.

$\therefore \lambda^2 + \lambda + 1 = 0$

$\left(\lambda + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 0$

$\Rightarrow \boxed{\left(-\frac{1}{2} - \lambda\right)^2 + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = 0}$ —— (i)

Let, $A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$\det(A - \lambda I) \Rightarrow$ (i)

$\therefore$ "A" annhailates charackristic polynomial $f(\lambda) = 0$.

(2)

$A^T = -A$

$\det(A^T) = \det(-A)$

$\det(A) = (-1)^n \det(A)$

$$\det(A)\left[1-(-1)^n\right] = 0$$

$\therefore$ $\det(A) = 0$ (or) $1-(-1)^n = 0$

"n" is odd "n" is even

$\therefore$ "n" is even $\Rightarrow$ $\det(A)$ may not be zero.

For example,

$$A = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$$

$\det(A) = c^2 \neq 0$ (assuming $c \neq 0$)

---

(6) The characteristic polymial of Nilpotent matrix is,

$f(\lambda) = \lambda^n$ (As all eigenvalues of nilpotent are zero).

$$\det(\lambda I - N) = \lambda^n$$

Put $\lambda = -2$

$$\det(-2I - N) = (-2)^n$$

$$(-1)^n \det(N+2I) = (-2)^n$$

$\therefore$ $\boxed{\det(N+2I) = 2^n}$

⑦ (1)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore [C]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(2)

$$C = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

⑧ (1) (i) $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore T(\mathbf{0}) = \mathbf{0}$

$\mathbf{0} \in T$

(ii) Let, $X, Y \in R^{2\times 2}$

$T(X+Y) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}(X+Y) - (X+Y)\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$= \left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}X - X\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right) +$

$\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}Y - Y\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right)$

$= T(X) + T(Y)$

(iii) Let, $\alpha \in R$ be a scalar.

$T(\alpha X) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}(\alpha X) - (\alpha X)\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$= \alpha\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}X - X\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right)$

$= \alpha T(X)$

From (i), (ii) & (iii) above,

"T" is a linear transformation

(2) $T(A) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} A - A \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

Let, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$

$$T(A) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} - \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2a_1 & 2a_2 \end{bmatrix} - \begin{bmatrix} 2a_2 & 0 \\ 2a_4 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} -2a_2 & 0 \\ 2a_1 - 2a_4 & 2a_2 \end{bmatrix}$$

$T: R^{2\times 2} \to R^{2\times 2}$ can be re-written as,

$$L: R^{4\times 1} \to R^{4\times 1}$$

$$\therefore L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -2a_2 \\ 0 \\ 2a_1 - 2a_4 \\ 2a_2 \end{pmatrix}$$

standard Basis of "L" would be,

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$L\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\0\end{pmatrix}, \quad L\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\0\\2\end{pmatrix}$$

$$L\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad L\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-2\\0\end{pmatrix}$$

$$\therefore \quad M = \begin{pmatrix}0 & -2 & 0 & 0\\0 & 0 & 0 & 0\\2 & 0 & 0 & -2\\0 & 2 & 0 & 0\end{pmatrix}$$

(3)

$$\begin{aligned}\text{rank}(M) &= \text{rank}(\text{rref}(M))\\ &= 2 \quad (\because 2 \text{ independent rows})\\ &< 4\end{aligned}$$

$\therefore$ L or T is not an isomorphism

(4)
$$\dim(\ker(T)) = \dim(\ker(L))$$
$$= 4 - \text{rank}(M)$$
$$= 4 - 2$$
$$= 2$$

$$\dim(\text{im}(T)) = \dim(\text{im}(L))$$
$$= \text{rank}(M)$$
$$= 2$$

(5) $$\text{rref}(M) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

$\therefore$ basis of $\ker(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Re-writing in matrix form,

basis of $\ker(T) = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

we have leading co-efficient $= 1$ in columns 1 & 2.

$$\therefore \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

are independent vectors for "L".

$$\therefore \text{ basis of } im(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Re-writing in matrix form,

$$\text{basis of } im(T) = \left\{ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$