MATH 7241: Problems for Notes #1

Due date: Tuesday September 22

Reading: relevant background material for these problems can be found in the class notes, and in Ross (Sections 2.1 2.2, 2.3, 2.4) and in Grinstead and Snell (Chapters 1,2 3, 6).

Exercise 1 You have two coins, one is unbiased, the other is biased with probability of Heads equal to $2/3$. You toss both coins twice, X is the number of Heads for the fair coin, Y is the number of Heads for the biased coin. Find $P(X > Y)$.

Exercise 2 Let A and B be events such that $P(A) = 0.7$ and $P(B) = 0.9$. Find the largest and smallest possible values of $P(A \cup B) - P(A \cap B)$ (note: the event $A \cup B$ means either A or B or both are true, the event $A \cap B$ means both A and B are true).

Exercise 3 A town has five hotels; three people arrive and each randomly and independently selects a hotel. Find the probability that exactly two of them stay in the same hotel.

Exercise 4 Find the mean and variance of the geometric distribution:

$$P(X = n) = (1 - p)^{n-1} p, \quad n = 1, 2, \dots$$

Exercise 5 If $\mathrm{Ran}(N) = \{1, 2 \dots\}$ show that

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} P(N \geq n)$$

Exercise 6 Randomly distribute r balls in n boxes. Find the probability that the first box is empty. Find the probability that the first two boxes are both empty.

$\boxed{\textbf{Exercise 7}}$ The current in a resistor is a random variable X. The pdf of X is $f(x) = e^{-(x-1)}$ for $x \geq 1$. The power dissipated in the resistor is $Y = X^2$. Find the pdf of Y.

$\boxed{\textbf{Exercise 8}}$ Derive the formula

$$\mathrm{VAR}[X_1 + X_2 + \cdots + X_n] = \sum_{k=1}^{n} \mathrm{VAR}[X_k] + 2 \sum_{i<j} \mathrm{COV}(X_i, X_j) \tag{1}$$

$\boxed{\textbf{Exercise 9}}$ We start with a stick of length 1, and break it in two pieces at a randomly chosen position (chosen uniformly over its length). Find the mean and variance of the longer piece.

$\boxed{\textbf{Exercise 10}}$ Find a random number generator that generates uniformly on $[0, 1]$ (for example the command *rand* in Matlab). Using this generator, estimate the volume of the region under the surface

$$z = \frac{1}{3} \cosh \sqrt{x^2 + y^2}$$

and above the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. [Note: generate three independent uniform random variables for each run, corresponding to the three coordinates of a random point in the unit cube. Do enough runs to be confident that you have an accurate estimate of the first two decimal places].

$\boxed{\textbf{Exercise 11}}$ In class we considered this problem: "An urn contains n Red balls and m Black balls. Suppose that k balls are withdrawn from the urn, and let X be the number of Red balls among these. Find $\mathbb{E}[X]$ assuming (i) replacement, and (ii) no replacement." Using the same reasoning as in class, compute $\mathrm{VAR}[X]$ assuming (i) replacement, and (ii) no replacement. [Hint: use the formula from Exercise 8 above. The answers will be different for the two cases].