# MATH 7241 Fall 2020: Problem Set #4

Due date: Sunday October 18

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

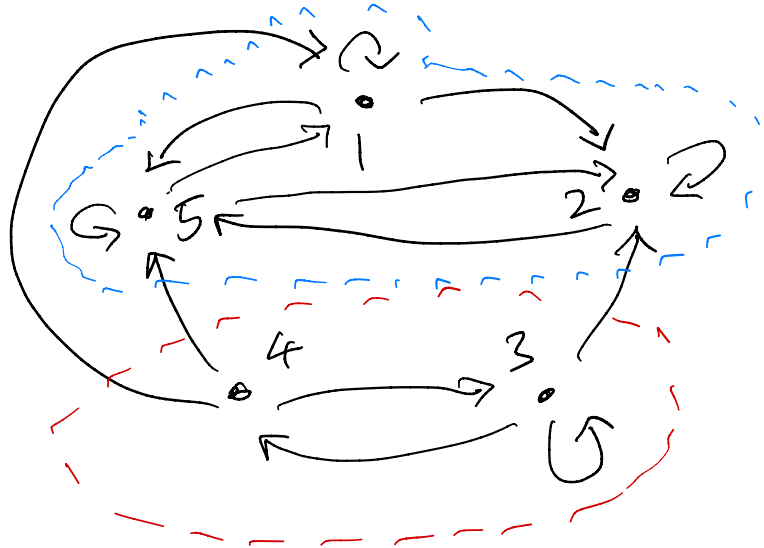**Exercise 1** 'Finite Markov Chains - Problems', Exercise 4.

Hint: draw a graph with 6 nodes to represent the states of the chain, and draw a directed edge between each pair of nodes (i, j) for which the transition matrix entry p_{ij} is positive. You can identify the set of transient states as the group of nodes from which edges exit, but into which there are no entering edges. Once you find the transient states, the remaining states are all irreducible, and break up into subsets which intercommunicate.
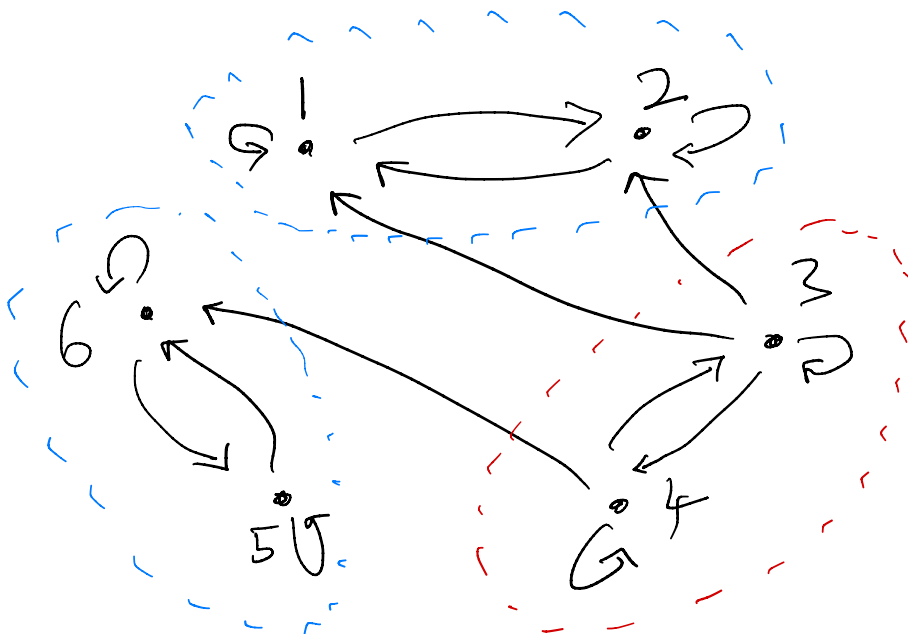
$\{3,4\}$ transient,

$\{1,2\}, \{5,6\}$ closed irreducible classes

Exercise 2 'Finite Markov Chains - Problems', Exercises 5 a), 5 b).

a)

$\{3,4\}$ transient

$\{1,2,5\}$ closed   irreducible

b)   Stationary   dist. $(W_1, W_2, W_3, W_4, W_5) = W$:

$W_1 = 0.5\, w_1 + 0.3\, w_4 + 0.5\, w_5$

$W_2 = 0.3\, W_1 + 0.5\, W_2 + 0.4\, W_3 + 0.2\, w_5$

$W_3 = 0.4\, W_3 + 0.2\, w_4$

$W_4 = 0.2\, W_3$

$W_5 = 0.2\, W_1 + 0.5\, W_2 + 0.5\, W_4 + 0.3\, w_5$

$\Rightarrow\ W = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \tfrac{1}{3}\right).$ — unique solution.

Exercise 3 'Finite Markov Chains - Problems', Exercise 2.

Hint: to represent this by a Markov chain you must use the result of two successive trials as your state. So there are four states: SS, SF, FS, FF where S is success and F is failure, and SF means success on trial n and failure on trial $n + 1$. Then if your current state is SF, your next state must be either FS or FF.

$$X_n = \text{result of } n^{th} \text{ trial} \quad, \quad X_n \in \{S, F\}$$

$$Y_n = (X_n, X_{n-1}) \quad \text{results of } n^{th}, (n-1)^{st} \text{ trial}$$

Y_n is a Markov chain w/ transition matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} \begin{matrix} SS \\ SF \\ FS \\ FF \end{matrix} \quad Y_0 = \text{input}$$

$$\underbrace{\quad SS \qquad SF \qquad FS \qquad FF \quad}_{Y_1 = \text{output}}$$

Stationary distribution: $W = \left(\dfrac{5}{11}, \dfrac{2}{11}, \dfrac{2}{11}, \dfrac{2}{11} \right)$.

Long run proportion of successful trials is

$$\lim_{n \to \infty} \mathbb{P}(X_n = S) = \lim_{n \to \infty} \left(\mathbb{P}(Y_n = SS) + \mathbb{P}(Y_n = SF) \right)$$

$$= W_{SS} + W_{SF} = \dfrac{7}{11}$$

Note: see pages 442-443 from Grinstead and Snell on Canvas. The text is available online (free!)
at

http://www.dartmouth.edu/~chance/

Click on the link "A GNU book".

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

a) $$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} \qquad P^3 = \begin{pmatrix} & & \\ & \times & \\ & & \end{pmatrix} \begin{matrix} \text{all} \\ \text{non-zero} \end{matrix}$$

$$\Rightarrow P \text{ is regular.}$$

b) $$P(X_2 = 3 \mid X_0 = 1) = (P^2)_{13} = \frac{1}{6}$$

c) $$W_j = \sum_{i=1}^{3} W_i P_{ij} \qquad \text{for} \qquad j = 1, 2, 3$$

$$\Rightarrow W = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right)$$