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(a) Absorbing $\Rightarrow P_{ii} = 1$

$\therefore 1 - a = 1 \Rightarrow a = 0$ | $1 - b = 1 \Rightarrow b = 0$

$\therefore a = 0$ or $b = 0$

(b) Ergodic but nor regular

$\Rightarrow$ Ehrenfest urn model

i.e., all elements on anti-diagonal $= 1$
Other elements $= 0$

$\therefore a = 1$ and $b = 1$

(c) If no entries are zero, then matrix is regular

$\Rightarrow \boxed{0 < a < 1 \text{ and } 0 < b < 1}$

If $\underline{a = 1} \Rightarrow P = \begin{pmatrix} 0 & 1 \\ b & 1-b \end{pmatrix} \Rightarrow P^2 = \begin{pmatrix} b & 1-b \\ b(1-b) & b^2-b+1 \end{pmatrix}$

$\therefore \boxed{0 < b < 1}$

By symmetry,
If $\underline{b = 1} \Rightarrow \boxed{0 < a < 1}$

(2) (a) $WP = W$

let, $W = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$; $\omega_1 + \omega_2 = 1$

$$\left[\begin{array}{cc|c} -0.25 & 0.5 & 0 \\ 0.25 & -0.5 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad W = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cc|c} -0.1 & 0.1 & 0 \\ 0.1 & -0.1 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad W = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

(c)

$$\left[\begin{array}{ccc|c} -1/4 & 0 & 1/4 & 0 \\ 1/4 & -1/3 & 1/4 & 0 \\ 0 & 1/3 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad W = \begin{bmatrix} 2/7 \\ 3/7 \\ 2/7 \end{bmatrix}$$

(3) Doubly stochastic

$$\Rightarrow \sum_{i=1}^{M} P_{ij} = 1 \quad \text{—— (1)}$$

$$\pi P = \pi$$

$$\Rightarrow \pi_j = \sum_{i=1}^{M} \pi_i P_{ij} \quad \text{—— (2)}$$

$$(1)-(2) \Rightarrow (1-\pi_j) = \sum_{i=1}^{M} (1-\pi_i) P_{ij}$$

$$\left(K(1-\pi_1),\ K(1-\pi_2),\ \ldots\ K(1-\pi_M)\right)$$

is another form of stationary distribution.

$$\therefore \pi_j = k(1-\pi_j)$$

$$\pi_j (K+1) = K$$

$$\pi_j = \frac{K}{K+1}$$

Also, $\sum_{j=1}^{M} \pi_j = 1 \Rightarrow \frac{K}{K+1} \times M = 1$

$$K(M-1) = 1 \Rightarrow K = \frac{1}{M-1}$$

$$\therefore \pi_j = \frac{\frac{1}{M-1}}{\frac{1}{M-1}+1} = \frac{1}{M}$$

$$\therefore \pi = \left[\frac{1}{M}, \frac{1}{M}, \ldots \frac{1}{M}\right]$$

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$$P(Y_n = (i,j) \mid Y_{n-1} = (a,b)_{n-1}, Y_{n-2} = (a,b)_{n-2} \cdots Y_1 = (a,b)_1)$$

$$= P((X_{n-1}, X_n) = (i,j) \mid (X_{n-2}, X_{n-1}) = (a,b)_{n-1}, \cdots (X_0, X_1) = (a,b)_1) \quad (1)$$

X_n is regular finite Markov chain

$$\Rightarrow (1) = P((X_{n-1}, X_n) = (i,j) \mid (X_{n-2}, X_{n-1}) = (a,b)_{n-1})$$

which is equivalent to

$$P(Y_n = (i,j) \mid Y_{n-1} = (a,b)_{n-1})$$

$\Rightarrow$ Y_n is a markov chain

$$\lim_{n\to\infty} X_n = \vec{\omega}$$

$$\Rightarrow \lim_{n\to\infty} P(Y_n = (i,j)) = \lim_{n\to\infty} P(X_{n-1} = i, X_n = j)$$

$$= \lim_{n\to\infty} P(X_n = j \mid X_{n-1} = i)\, P(X_{n-1} = i) = P_{ij}\, \vec{\omega}$$

(5) $X_1, X_2, \dots\dots X_n \rightarrow$ outcomes

$$S_n = X_1 + X_2 + \dots + X_n$$

let, $Y_n = S_n \% 7$

$$Y_n = (S_{n-1} + X_n) \% 7 = (Y_{n-1} + X_n) \% 7$$

$\Rightarrow$ Y_n is a markov chain

$$\therefore P_{i,j} = P(Y_n = j \mid Y_{n-1} = i)$$

$$= P((i + X_n) \% 7 = j)$$

$$\forall i, j \in \{0, 1, 2, 3, 4, 5, 6\}$$

$$X_n \in \{1, 2, 3, 4, 5, 6\}$$

$$P(X_n) = m = \frac{1}{6} ; \quad 1 \leq m \leq 6$$

$$\therefore P_{i,j} = \frac{1}{6} \quad \forall\ 0 \leq i, j \leq 6$$

$\therefore$ proportion of first "n" values of S_n $\Big\} = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}(Y_k = 0)$

From large of large numbers, $P\left(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}(Y_k = 0) = \pi(0)\right)$

$$= \boxed{1}$$

⑥ State i is transient

$$\Rightarrow \quad \sum_{l=1}^{\infty} P_{ii}(l) < \infty$$

Lets consider going $i \to j$ in k steps,

$j \to i$ in m steps

$i \to i$ in l steps

we have,

$$P_{ij}(n) \times P_{ji}(m) \leq P_{ii}^{(n+m)} = P_{ii}(l)$$

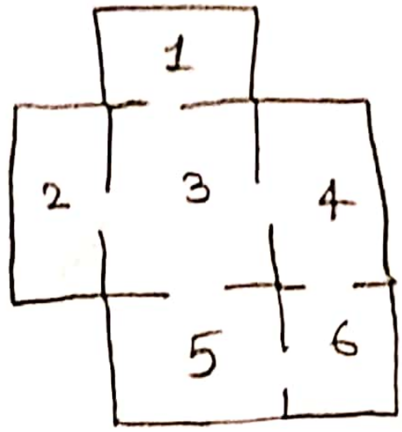
$$\Rightarrow P_{ij}(n) = \frac{P_{ii}(l)}{P_{ji}(m)}$$

$$\Rightarrow \sum_{n=1}^{\infty} P_{ij}(n) \leq \frac{\sum_{l=1}^{\infty} P_{ii}(l)}{P_{ji}(m)} < \infty$$

$\left(\text{as } P_{ji}(m) > 0 \text{ and } \sum_{l=1}^{\infty} P_{ii}(l) < \infty\right)$

$$\therefore \boxed{P_{ij}(n) \to 0}$$

7 (a)



$$P = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 4 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 5 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 6 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{array}$$

(b) Chain is ergodic because $p_{ij} > 0$ for at least one j, $\forall\ 1 \le i \le 6$

A MATLAB program shows that,

$$n \rightarrow \text{even} \Rightarrow p_{6,1}^{(n)} = 0$$

$$n \rightarrow \text{odd} \Rightarrow p_{1,1}^{(n)} = 0$$

$\therefore$ chain is not regular

(c)

$\pi P = \pi$

$$\left[\begin{array}{cccccc|c} -1 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad \pi = \begin{bmatrix} 1/12 \\ 1/12 \\ 1/3 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$$

(d)

Let $N(i) = E\left(\text{number of steps to reach state } 5 \mid X_0 = i\right)$

we have,

$$N(1) = 1 + N(3)$$

$$N(2) = 1 + N(3)$$

$$N(3) = 1 + \frac{1}{4}\left(N(1) + N(2) + N(4) + N(5)\right)$$

$$N(4) = 1 + \frac{1}{2} N(6) + \frac{1}{2} N(3)$$

$$N(5) = 0$$

$$N(6) = 1 + \frac{1}{2} \times N(5) + \frac{1}{2} N(4)$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\left[\begin{array}{cccccc|c}
1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & +1 & -1 & 0 & 0 & 0 & 1 \\
-\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \\
0 & 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 1
\end{array}\right]
\end{array}
$$

$$
N = \begin{bmatrix} 7 \\ 7 \\ 6 \\ 6 \\ 0 \\ 4 \end{bmatrix}
$$

$$
\therefore \boxed{N(1) = 7}
$$