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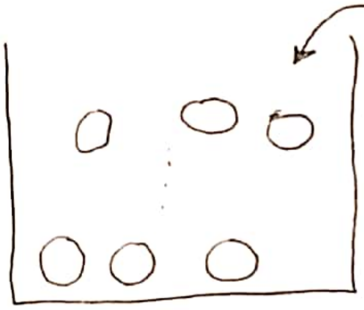
(1)



$X_n \rightarrow$ number of red balls

Let the number of red balls at a certain instance $= k$

$$p_{ij} = P(x_1 = j \mid x_0 = i) = \begin{cases} p\left(\frac{N-k}{N}\right) & , \text{if } j = i+1 \\ (1-p)\frac{k}{N} & , \text{if } j = i-1 \\ \frac{p \times k + (1-p)(N-k)}{N} & , \text{if } j = i \end{cases}$$

Let us assume the chain is reversible.
If we get a valid stationary distribution, then our assumption is true

$$\pi_{k-1} \cdot P_{k-1,k} = \pi_k \cdot P_{k,k-1}$$

$$\therefore \pi_k = \frac{P_{k-1,k}}{P_{k,k-1}} \times \pi_{k-1}$$

$$= \frac{p}{(1-p)} \times \left(\frac{N-k+1}{k}\right) \pi_{k-1}$$

$$= \left(\frac{p}{1-p}\right)^2 \frac{(N-k+1)\ (N-k+2)}{k \times (k-1)} \pi_{k-2}$$

$$\vdots$$

$$= \left(\frac{p}{1-p}\right)^k \left(\frac{(N-k+1)(N-k+2)\cdots N}{k \times (k-1) \times \cdots 1}\right) \pi_0$$

$$\therefore \boxed{\pi_k = \left(\frac{p}{1-p}\right)^k {}^N C_k \ \pi_0} \quad \text{—(1)}$$

$k = 0, 1, \cdots N$

$$\sum_{k=0}^{N} \pi_k = 1 \quad (\text{Normalization}) \quad \text{—(2)}$$

From (1) & (2),

$$\sum_{k=0}^{N} \pi_k = \sum_{k=0}^{N} \left(\frac{p}{1-p}\right)^k {}^N C_k \ \pi_0 = 1$$

$$\Rightarrow \pi_0 \times \left(\frac{1}{1-p}\right)^N = 1$$

$$\Rightarrow \pi_0 = (1-p)^N$$

$$\therefore \pi_k = \left(\frac{p}{1-p}\right)^k {}^N C_k \ (1-p)^N$$

$$\Rightarrow \boxed{\pi_k = p^k (1-p)^{N-k} \ {}^N C_k}$$

This is a biromial distribution

$$\therefore \lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \sum_{k=0}^{N} k\, P(X_n = k)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{N} k\, \pi_k$$

$$= \lim_{n\to\infty} \sum_{k=0}^{N} k\, p^k (1-p)^{N-k}\, {}^{N}C_k$$

$$= Np \lim_{n\to\infty} \sum_{t=0}^{N} p^t (1-p)^{N-t}\, {}^{N-1}C_t$$

$$= Np$$

This Could have been guessed with intuition because tossing up a coin follows binomial distibution and in the long run number of red balls depends solely on the number of heads that show up after "N" tosses, whose expected value is "Np".