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Integrating factors and first integrals for ordinary differential equations

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We show how to find all the integrating factors and corresponding first integrals for any system of Ordinary Differential Equations (ODEs). Integrating factors are shown to be all solutions of both the adjoint system of the linearised system of ordinary differential equations and a system that represents an extra adjoint-invariance condition. We present an explicit construction formula to find the resulting first integrals in terms of integrating factors, and discuss techniques for finding integrating factors. In particular, we show how to utilize known first integrals and symmetries to find new integrating factors. Illustrative examples are given.

1 Introduction

For first-order scalar Ordinary Differential Equations (ODEs), Sophus Lie (cf. Lie, 1874) showed how to construct an integrating factor from each admitted point symmetry. Conversely, Lie showed that each integrating factor yields an admitted point symmetry.

In general, for systems of one or more ODEs, an integrating factor is a set of functions, multiplying each of the ODEs, which yields a first integral. If the system is self-adjoint, then its integrating factors are necessarily solutions of its linearized system. Such solutions are the symmetries of the given system of ODEs. If a given system of ODEs is not self-adjoint, then its integrating factors are necessarily solutions of the adjoint system of its linearized system. Such solutions are known as *adjoint symmetries* (Gordon, 1986; see also Sarlet *et al.*, 1987, 1990) of the given system of ODEs.

In this paper, we introduce an *adjoint-invariance condition* which is a necessary and sufficient condition for an admitted adjoint symmetry to be an integrating factor. We present an explicit formula for the first integral corresponding to each integrating factor. These results are the counterparts of our work on Partial Differential Equations (PDEs) (Anco & Bluman, 1997, 1998).

For a first-order scalar ODE, a first integral is a quadrature. For an *n*th-order scalar ODE, a first integral is an expression relating the independent variable, the dependent variable and derivatives to order $n-1$, which is constant for *all* solutions of the ODE. First integrals are defined analogously for systems of ODEs.

If *r* independent first integrals are known, then an *n*th-order scalar ODE can be reduced to one or more $(n-r)$th-order ODEs *in terms of r essential constants*[1] *and the given*

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[1] Constants are essential if none of them can be reduced in terms of function combinations of the others.

dependent and independent variables. In particular, n independent first integrals yield the general solution involving n essential constants.

Sophus Lie (cf. Lie, 1888; Bluman, 1990) showed that if an nth-order scalar ODE admits an r-parameter solvable group of point symmetries, then it can be reduced to an $(n-r)$th-order ODE plus r quadratures.[2] Lie's reduction uses derived independent and dependent variables, given by invariants and differential invariants to order $n-r$, arising from the admitted point symmetries. Consequently, the 'reduced' ODE *is not* an $(n-r)$th-order ODE in terms of the given dependent and independent variables. Thus, Lie's reduction is not as useful as a reduction in terms of first integrals.

In §2 we establish our framework. We define integrating factors and first integrals for systems of ODEs. We show that each integrating factor must be an adjoint symmetry, and derive the adjoint-invariance condition for an adjoint symmetry to be an integrating factor. We give the explicit formula for the first integral arising from an integrating factor. Finally, we show how our framework treats the well-known situation for first-order scalar ODEs.

In §3 we treat the case of second-order scalar ODEs, and make some remarks about the situation for higher-order scalar ODEs. In §4 we discuss techniques for finding and utilizing adjoint symmetries in conjunction with the adjoint-invariance condition. We show how to use an adjoint symmetry and functions of known first integrals to obtain *new* first integrals. Finally, in §5 we consider various examples.

2 The basic framework

Consider any nth-order system of one or more ODEs

$$G_\sigma(x, y, y', \dots, y^{(n)}) = 0, \quad \sigma = 1, \dots, N \tag{2.1}$$

with any number of dependent variables $y = \{y^1, \dots, y^M\}$ and one independent variable x; y' represents the first-order derivative of y; $y^{(j)}$ represents the jth-order derivative of y. For arbitrary functions $Y = \{Y^1, \dots, Y^M\}$, let $G_\sigma[Y] = G_\sigma(x, Y, Y', \dots, Y^{(n)})$. The aim is to find all factors $\Lambda^\sigma[Y] = \Lambda^\sigma(x, Y, Y', \dots, Y^{(n-1)})$ and functions $\Phi[Y] = \Phi(x, Y, Y', \dots, Y^{(n-1)})$ so that

$$\Lambda^\sigma[Y] G_\sigma[Y] = \frac{d}{dx} \Phi[Y] \tag{2.2}$$

holds for all $Y(x)$ for which $\Lambda^\sigma[Y] G_\sigma[Y]$ is finite. (Throughout this paper, we use the index notation $\sigma = 1, \dots, N$; $\rho = 1, \dots, M$; and the convention that summation is assumed over any repeated index in all expressions.)

From equation (2.2), it follows that

$$\Phi[y] = \text{const} \tag{2.3}$$

on the solutions $y(x)$ of system (2.1) for which each $\Lambda^\sigma[y]$ is finite. In particular, if $\Lambda^\sigma[Y]$ is finite for arbitrary $Y(x)$, then $\Phi[y] = \text{const}$ holds for all solutions of system (2.1).

We allow $\Lambda^\sigma[Y]$ and $\Phi[Y]$ to depend at most upon $Y^{(n-1)}$, since we assume that the system (2.1) determines $y^{(n)}$ in terms of lower-order derivatives of y.

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[2] Lie's method can be extended to invariance under r-parameter solvable groups of higher order symmetries.

Definition 2.1 *A set of factors* $\{\Lambda^\sigma[Y]\}$ *satisfying* (2.2) *is an* integrating factor *of system* (2.1) *and, correspondingly,* $\Phi[y] = $ const *is a* first integral *of system* (2.1).

Before defining adjoint symmetries and introducing our adjoint-invariance condition, we first consider the linearized system, and its adjoint, obtained from equation (2.1).

The linearized system is given by

$$L_{\sigma\rho}[y] v^\rho = 0 \tag{2.4}$$

where

$$L_{\sigma\rho}[Y] V^\rho = G_{\sigma\rho}[Y] + G^1_{\sigma\rho}[Y]\frac{dV^\rho}{dx} + \cdots + G^n_{\sigma\rho}[Y]\frac{d^n V^\rho}{dx^n} \tag{2.5}$$

with

$$G_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y^\rho}, \quad G^1_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y'^\rho}, \ldots, G^n_{\sigma\rho}[Y] = \frac{\partial G_\sigma[Y]}{\partial Y^{(n)\rho}}.$$

In equation (2.4), $v = \{v^1, \ldots, v^M\}$ is a solution of the linearized system holding for *all* solutions $y(x)$ of system (2.1); in equation (2.5), $V = \{V^1, \ldots, V^M\}$ and $Y = \{Y^1, \ldots, Y^M\}$ are *arbitrary* functions of x.

The linearized system (2.4) is the set of determining equations for the symmetries of system (2.1). In particular, a solution v of system (2.4) is a symmetry of the system (2.1) with infinitesimal generator $v^\rho \partial/\partial y^\rho$.

The adjoint of the linearized system (2.4) is given by

$$L^*_{\rho\sigma}[y] w^\sigma = 0, \tag{2.6}$$

where

$$L^*_{\rho\sigma}[Y] W^\sigma = G_{\sigma\rho}[Y] W^\sigma - \frac{d}{dx}(G^1_{\sigma\rho}[Y] W^\sigma) + \cdots + (-1)^n \frac{d^n}{dx^n}(G^n_{\sigma\rho}[Y] W^\sigma). \tag{2.7}$$

In system (2.6), $w = \{w^1, \ldots, w^N\}$ is a solution of the adjoint system holding for all solutions $y(x)$ of the given system of ODEs (2.1); in system (2.7), $W = \{W^1, \ldots, W^N\}$ and $Y = \{Y^1, \ldots, Y^M\}$ are arbitrary functions of x.

Definition 2.2 *The adjoint system* (2.6) *is the set of determining equations for the* adjoint symmetries *of system* (2.1). *In particular, a solution w of the adjoint system* (2.6) *is an adjoint symmetry of the system* (2.1).

Definition 2.3 *System* (2.1) *is* self-adjoint *if and only if* $L^*_{\sigma\rho}[Y] \equiv L_{\sigma\rho}[Y]$.

Theorem 2.4 *Every integrating factor of system* (2.1) *satisfies the* adjoint-invariance condition

$$L^*_{\rho\sigma}[Y]\Lambda^\sigma[Y] = -\Lambda^\sigma_\rho[Y] G_\sigma[Y] + \frac{d}{dx}(\Lambda^{1\sigma}_\rho[Y] G_\sigma[Y]) + \cdots + (-1)^{n-2}\frac{d^{n-1}}{dx^{n-1}}(\Lambda^{(n-1)\sigma}_\rho[Y] G_\sigma[Y]) \tag{2.8}$$

for arbitrary $Y(x)$ *where*

$$\Lambda^\sigma_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y^\rho}, \quad \Lambda^{1\sigma}_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y'^\rho}, \ldots, \Lambda^{(n-1)\sigma}_\rho[Y] = \frac{\partial \Lambda^\sigma[Y]}{\partial Y^{(n-1)\rho}}.$$

Proof Since system (2.2) holds for arbitrary $Y(x)$, it also holds with $Y^\rho(x)$ replaced by the one-parameter (λ) family of functions $Y^\rho(x;\lambda) = Y^\rho(x) + \lambda V^\rho(x)$, where $Y^\rho(x), V^\rho(x)$ are arbitrary functions of x. Thus, we have

$$A^\sigma[Y(x;\lambda)]\, G_\sigma[Y(x;\lambda)] = \frac{d}{dx}\Phi[Y(x;\lambda)]. \tag{2.9}$$

Now differentiate system (2.9) with respect to λ and set $\lambda = 0$. Then use

$$\left.\frac{\partial G_\sigma[Y(x;\lambda)]}{\partial \lambda}\right|_{\lambda=0} = L_{\sigma\rho}[Y]\, V^\rho,$$

given by the linearizing expression (2.5). This leads to

$$\frac{d}{dx}\left(\frac{\partial}{\partial\lambda}\Phi[Y(x;\lambda)]\bigg|_{\lambda=0}\right)$$

$$= A^\sigma[Y](L_{\sigma\rho}[Y]\, V^\rho) + G_\sigma[Y]\left(A_\rho^\sigma[Y]\, V^\rho + A_\rho^{1\sigma}[Y]\frac{dV^\rho}{dx} + \cdots + A_\rho^{(n-1)\,\sigma}[Y]\frac{d^{n-1}V^\rho}{dx^{n-1}}\right). \tag{2.10}$$

Now apply the Euler operators

$$E_{V^\rho} = \frac{\partial}{\partial V^\rho} - \frac{d}{dx}\frac{\partial}{\partial V'^\rho} + \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\frac{\partial}{\partial V^{(n-1)\rho}} \tag{2.11}$$

to each side of equation (2.10), which is an expression in terms of the arbitrary functions $\{Y^\rho(x)\}, \{V^\rho(x)\}$. Since Euler operators annihilate total derivatives, the left-hand side of equation (2.10) vanishes upon action by the Euler operators (2.11). On the right-hand side of equation (2.10), the Euler operators (2.11) applied to $A^\sigma[Y](L_{\sigma\rho}[Y]\, V^\rho)$ yield $L_{\rho\sigma}^*[Y]\, A^\sigma[Y]$, given by system (2.7) with $W^\sigma = A^\sigma[Y]$. The Euler operators (2.11) applied to the rest of the right-hand side of equation (2.10) yield

$$A_\rho^\sigma[Y]\, G_\sigma[Y] - \frac{d}{dx}(A_\rho^{1\sigma}[Y]\, G_\sigma[Y]) + \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(A_\rho^{(n-1)\,\sigma}[Y]\, G_\sigma[Y]).$$

Thus the adjoint-invariance condition (2.8) is obtained. $\quad\square$

Corollary 2.5 *If $\Phi[y] = \mathrm{const}$ is a first integral of the system of ODEs (2.1), then its integrating factor $\{A^\sigma[Y]\}$ satisfies the adjoint system*

$$L_{\rho\sigma}^*[y]\, A^\sigma[y] = 0, \tag{2.12}$$

holding for all solutions $y(x)$ of system (2.1).

The proof of Corollary 2.5 follows immediately from the adjoint-invariance condition (2.8) with $Y(x) = y(x)$ given by any solution of system (2.1).

An important consequence of Corollary 2.5 is that *all* first integrals arise from solutions of the adjoint system (2.12). If system (2.1) is self-adjoint, then solutions of the adjoint system (2.12) are symmetries of system (2.1). If system (2.1) is not self-adjoint, the solutions

of the adjoint system (2.12) are not symmetries of (2.1) but *adjoint symmetries* (Gordon, 1986; Sarlet *et al.*, 1987, 1990) of system (2.1). However, as will be shown in the examples in §5, an adjoint symmetry does not always satisfy the adjoint-invariance condition (2.8), i.e. an adjoint symmetry does not always give rise to a first integral.

For any adjoint symmetry that satisfies the adjoint-invariance condition (2.8), we now derive a formula which yields the corresponding first integral. To proceed, we first need to establish the following identity.

Lemma 2.6 *The operators $L_{\sigma\rho}[Y]$ and $L_{\rho\sigma}^*[Y]$ satisfy the identity*

$$W^\sigma L_{\sigma\rho}[Y] V^\rho - V^\rho L_{\rho\sigma}^*[Y] W^\sigma \equiv \frac{d}{dx} S[W, V; G[Y]] \tag{2.13}$$

for arbitrary functions $Y^\rho(x), V^\rho(x), W^\sigma(x)$, where

$$S[W, V; G[Y]] = V^\rho W^\sigma G_{\sigma\rho}^1 + \left(\frac{dV^\rho}{dx} - V^\rho \frac{d}{dx}\right)(W^\sigma G_{\sigma\rho}^2) + \cdots$$

$$+ \left(\frac{d^{n-1} V^\rho}{dx^{n-1}} + \sum_{l=1}^{n-2} (-1)^l \frac{d^{n-l-1} V^\rho}{dx^{n-l-1}} \frac{d^l}{dx^l} + (-1)^{n-1} V^\rho \frac{d^{n-1}}{dx^{n-1}}\right)(W^\sigma G_{\sigma\rho}^n). \tag{2.14}$$

Proof The identity (2.13) follows from a direct expansion of both sides of (2.13), using the definitions of $L_{\sigma\rho}[Y]$ and $L_{\rho\sigma}^*[Y]$ given by equations (2.5) and (2.7), respectively. $\square$

We are now ready to establish the converse of Theorem 2.4.

Theorem 2.7 *Suppose $\{\Lambda^\sigma[Y]\}$ satisfies the adjoint-invariance condition (2.8). Then $\{\Lambda^\sigma[Y]\}$ is an integrating factor for the system of ODEs (2.1). In particular,*

$$\Lambda^\sigma[Y] G_\sigma[Y] = \frac{d}{dx} \Phi[Y] \tag{2.15}$$

with $\Phi[Y] = \Phi_1(x, Y, Y', \ldots, Y^{(2n)}) + \Phi_2(x)$ given by the formulae:

$$\Phi_1 = \int_0^1 d\lambda \left(S\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial \lambda}; G[Y(x;\lambda)]\right] + N\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial \lambda}; G[Y(x;\lambda)]\right]\right), \tag{2.16}$$

$$\Phi_2 = \int k(x)\, dx, \tag{2.17}$$

where

$$S\left[\Lambda[Y(x;\lambda)], \frac{\partial Y(x;\lambda)}{\partial \lambda}; G[Y(x;\lambda)]\right] = \frac{\partial Y^\rho(x;\lambda)}{\partial \lambda} \Lambda^\sigma[Y(x;\lambda)] G_{\sigma\rho}^1[Y(x;\lambda)]$$

$$+ \left(\frac{d}{dx}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial \lambda}\right) - \frac{\partial Y^\rho(x;\lambda)}{\partial \lambda} \frac{d}{dx}\right)(\Lambda^\sigma[Y(x;\lambda)] G_{\sigma\rho}^2[Y(x;\lambda)]) + \cdots$$

$$+ \left(\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial \lambda}\right) + \sum_{l=1}^{n-2} (-1)^l \frac{d^{n-l-1}}{dx^{n-l-1}}\left(\frac{\partial Y^\rho(x;\lambda)}{\partial \lambda}\right) \frac{d^l}{dx^l} + (-1)^{n-1} \frac{\partial Y^\rho(x;\lambda)}{\partial \lambda} \frac{d^{n-1}}{dx^{n-1}}\right)$$

$$\times (\Lambda^\sigma[Y(x;\lambda)] G_{\sigma\rho}^n[Y(x;\lambda)]); \tag{2.18}$$

$$N\left[A[Y(x;\lambda)],\frac{\partial Y(x;\lambda)}{\partial\lambda};G[Y(x;\lambda)]\right]=\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}G_{\sigma}[Y(x;\lambda)]A_{\rho}^{1\sigma}[Y(x;\lambda)]$$

$$+\left(\frac{d}{dx}\left(\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}\right)-\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}\frac{d}{dx}\right)(G_{\sigma}[Y(x;\lambda)]A_{\rho}^{2\sigma}[Y(x;\lambda)])+\cdots$$

$$+\left(\frac{d^{n-2}}{dx^{n-2}}\left(\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}\right)+\sum_{l=1}^{n-3}(-1)^{l}\frac{d^{n-l-2}}{dx^{n-l-2}}\left(\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}\right)\frac{d^{l}}{dx^{l}}+(-1)^{n-2}\frac{\partial Y^{\rho}(x;\lambda)}{\partial\lambda}\frac{d^{n-2}}{dx^{n-2}}\right)$$

$$\times(G_{\sigma}[Y(x;\lambda)]A_{\rho}^{(n-1)\sigma}[Y(x;\lambda)]);\tag{2.19}$$

$$k(x)=A^{\sigma}[\tilde{Y}(x)]G_{\sigma}[\tilde{Y}(x)].\tag{2.20}$$

Here $\tilde{Y}(x)=\{\tilde{Y}^{1}(x),\ldots,\tilde{Y}^{M}(x)\}$ *are any fixed functions such that the function* $k(x)$ *is finite, and* $Y(x;\lambda)$ *is the one-parameter* (λ) *family of functions* $Y^{\sigma}(x;\lambda)=\lambda Y^{\sigma}(x)+(1-\lambda)\tilde{Y}^{\sigma}(x)$, *for arbitrary* $Y^{\sigma}(x)$, $\sigma=1,\ldots,M$.

Proof Let $V(x)=(\partial Y(x;\lambda))/\partial\lambda=Y(x)-\tilde{Y}(x)$. From the adjoint-invariance condition (2.8), we obtain

$$V^{\rho}(x)\left(L_{\rho\sigma}^{*}[Y(x;\lambda)]A^{\sigma}[Y(x;\lambda)]+A_{\rho}^{\sigma}[Y(x;\lambda)]G_{\sigma}[Y(x;\lambda)]-\frac{d}{dx}(A_{\rho}^{1\sigma}[Y(x;\lambda)]G_{\sigma}[Y(x;\lambda)])+\cdots\right.$$

$$\left.+(-1)^{n}\frac{d^{n-1}}{dx^{n-1}}(A_{\rho}^{(n-1)\sigma}[Y(x;\lambda)]G_{\sigma}[Y(x;\lambda)])\right)=0.\tag{2.21}$$

Now we manipulate the terms in equation (2.21) as follows. From identity (2.13), the first term in (2.21) becomes $V^{\rho}L_{\rho\sigma}^{*}A^{\sigma}=A^{\sigma}L_{\sigma\rho}V^{\rho}-dS/dx$, where S is given by expression (2.18). Using the Leibniz rule for d/dx, the third term of equation (2.21) becomes

$$-V^{\rho}\frac{d}{dx}(A_{\rho}^{1\sigma}G_{\sigma})=-\frac{d}{dx}(V^{\rho}A_{\rho}^{1\sigma}G_{\sigma})+\left(\frac{dV^{\rho}}{dx}\right)A_{\rho}^{1\sigma}G_{\sigma},$$

and the other terms of (2.21) become

$$(-1)^{q}V^{\rho}\frac{d^{q}}{dx^{q}}(A_{\rho}^{q\sigma}G_{\sigma})$$

$$=(-1)^{q}\frac{d}{dx}\left\{\left[V^{\rho}\frac{d^{q-1}}{dx^{q-1}}+\sum_{l=1}^{q-2}(-1)^{l}\left(\frac{d^{l}V^{\rho}}{dx^{l}}\right)\frac{d^{q-l-1}}{dx^{q-l-1}}+(-1)^{q-1}\left(\frac{d^{q-1}V^{\rho}}{dx^{q-1}}\right)\right]A_{\rho}^{q\sigma}G_{\sigma}\right\}$$

$$+\left(\frac{d^{q}V^{\rho}}{dx^{q}}\right)A_{\rho}^{q\sigma}G_{\sigma}$$

for $q=1,\ldots,n-1$.

Hence equation (2.21) becomes

$$A^{\sigma}L_{\sigma\rho}V^{\rho}+\left(A_{\rho}^{\sigma}V^{\rho}+A_{\rho}^{1\sigma}\frac{dV^{\rho}}{dx}+\cdots+A_{\rho}^{(n-1)\sigma}\frac{d^{(n-1)}V^{\rho}}{dx^{(n-1)}}\right)G_{\sigma}-\frac{d}{dx}(S+N)=0,\tag{2.22}$$

where N is given by expression (2.19).

Now observe that $L_{\sigma\rho}[Y(x;\lambda)]V^{\rho}(x)=(\partial G_{\sigma}[Y(x;\lambda)])/\partial\lambda$, and that

$$A_{\rho}^{\sigma}[Y(x;\lambda)]V^{\rho}(x)+A_{\rho}^{1\sigma}[Y(x;\lambda)]\frac{dV^{\rho}(x)}{dx}+\cdots+A_{\rho}^{(n-1)\sigma}[Y(x;\lambda)]\frac{d^{n-1}V^{\rho}(x)}{dx^{n-1}}=\frac{\partial}{\partial\lambda}A^{\sigma}[Y(x;\lambda)].$$

Then equation (2.22) becomes

$$\Lambda^\tau[Y(x;\lambda)]\left(\frac{\partial}{\partial\lambda}G_\sigma[Y(x;\lambda)]\right)+\left(\frac{\partial}{\partial\lambda}\Lambda^\tau[Y(x;\lambda)]\right)G_\sigma[Y(x;\lambda)] = \frac{\partial}{\partial\lambda}(\Lambda^\tau[Y(x;\lambda)]\,G_\sigma[Y(x;\lambda)])$$

$$= \frac{d}{dx}(S[\Lambda[Y(x;\lambda)], V(x); G[Y(x;\lambda)]] + N[\Lambda[Y(x;\lambda)], V(x); G[Y(x;\lambda)]]), \quad (2.23)$$

where S and N are given by expressions (2.18) and (2.19), respectively.

Now integrate equation (2.23) with respect to λ from $\lambda = 0$ to $\lambda = 1$. Then we obtain

$$\Lambda^\tau[Y]\,G_\sigma[Y] - \Lambda^\tau[\tilde{Y}]\,G_\sigma[\tilde{Y}] = d\Phi_1/dx,$$

where Φ_1 is given by expression (2.16). To complete the proof, we observe that $\Lambda^\tau[\tilde{Y}]\,G_\sigma[\tilde{Y}] = k(x) = d\Phi_2/dx$. $\square$

Note that, if $\Lambda^\tau[Y]$, $G_\sigma[Y]$, $\sigma = 1,\ldots,n$ are finite for $Y^\rho = 0$, $\rho = 1,\ldots,M$, then we can choose $\tilde{Y}^\rho = 0$, $\rho = 1,\ldots,M$, and thus simplify the integral for Φ_1. Moreover, if $y^\rho = 0$, $\rho = 1,\ldots,M$, is a solution of system (2.1), then Φ_2 vanishes.

As a consequence of Theorems 2.4 and 2.7, we see that for any system of ODEs, all first integrals arise from adjoint symmetries that satisfy the adjoint-invariance condition.

2.1 First-order ODEs

We now consider the classical problem of finding the integrating factor for any first-order scalar ODE written in solved form

$$G(x,y,y') = y' - g(x,y) = 0. \quad (2.24)$$

Here the linearized ODE is

$$L[y]v = \frac{dv}{dx} - g_y v = 0, \quad (2.25)$$

and the corresponding adjoint ODE is given by

$$L^*[y]w = -\frac{dw}{dx} - g_y w = 0. \quad (2.26)$$

The symmetries of ODE (2.24) are the solutions of (2.25), while the adjoint symmetries of ODE (2.24) are the solutions of (2.26), which hold for all solutions $y(x)$ of ODE (2.24).

For arbitrary $Y = Y(x)$, each integrating factor $\Lambda(x,Y)$ of ODE (2.24) satisfies the adjoint-invariance condition

$$-\frac{d\Lambda(x,Y)}{dx} - g_Y(x,Y)\,\Lambda(x,Y) = -(Y' - g(x,Y))\,\Lambda_Y(x,Y), \quad (2.27)$$

which reduces to

$$g_Y\,\Lambda + \Lambda_x + g\,\Lambda_Y = 0. \quad (2.28)$$

Theorem 2.8 *If $\Lambda(x,y)$ is an adjoint symmetry of ODE* (2.24), *then $\Lambda(x,Y)$ is an integrating factor of ODE* (2.24).

Proof From (2.26)–(2.27) it follows that $\Lambda(x,Y)$ is a solution of (2.28) if and only if $w = \Lambda(x,y)$ is a solution of (2.26). $\square$

Theorem 2.9 *Each symmetry $v(x,y)$ of ODE (2.24) yields an adjoint symmetry $\Lambda(x,y) = 1/(v(x,y))$ of ODE (2.24). Conversely, each adjoint symmetry $\Lambda(x,y)$ of ODE (2.24) yields a symmetry $v(x,y) = 1/(\Lambda(x,y))$ of ODE (2.24).*

Proof From (2.24)–(2.25), it follows that any symmetry $v(x,y)$ of ODE (2.24) satisfies

$$v_x(x, Y) + g(x, Y) v_Y(x, Y) - g_Y(x, Y) v(x, Y) = 0, \qquad (2.29)$$

for arbitrary $Y(x)$. In turn, by direct substitution, one can show that $v(x, Y)$ satisfies (2.29) if and only if $\Lambda(x, Y) = 1/(v(x, Y))$ satisfies

$$-g_Y(x, Y)\Lambda(x, Y) - \Lambda_x(x, Y) + g(x, Y)\Lambda_Y(x, Y) = 0. \qquad (2.30)$$

Hence $\Lambda(x, Y)$ satisfies the adjoint-invariance condition (2.28).   □

For any integrating factor $\Lambda(x, Y)$, the first integral formula (2.16)–(2.20) yields

$$\Phi_1(x,y) + \Phi_2(x) = \text{const},$$

which gives the general solution of ODE (2.24). In terms of any fixed function $\tilde{y}(x)$, one has

$$S = (y - \tilde{y})\Lambda(x, \lambda(y - \tilde{y}) + \tilde{y}),$$
$$N = 0,$$
$$k(x) = \Lambda(x, \tilde{y})(\tilde{y}' - g(x, \tilde{y})),$$

which leads to

$$\Phi_1(x,y) = \int_0^1 S\, d\lambda = \int_{\tilde{y}}^y \Lambda(x,z)\, dz, \quad \Phi_2(x) = \int k(x)\, dx. \qquad (2.31)$$

From the above, we see that for any first-order ODE each adjoint symmetry is an integrating factor and, conversely, each integrating factor is an adjoint symmetry. In the next section, we will show that this is not the case for higher-order ODEs.

3 Second-order and higher-order scalar ODEs

We now show how the framework presented in §2 applies to any second-order scalar ODE

$$y'' - g(x, y, y') = 0 \qquad (3.1)$$

and higher-order scalar ODEs

$$y^{(n)} - g(x, y, y', y'', \dots, y^{(n-1)}) = 0. \qquad (3.2)$$

3.1 Second-order ODEs

The linearized ODE for equation (3.1) is given by

$$L[y]\,v = \frac{d^2 v}{dx^2} - g_{y'}\frac{dv}{dx} - g_y v = 0, \qquad (3.3)$$

and the corresponding adjoint ODE is

$$L^*[y]\,w = \frac{d^2 w}{dx^2} + \frac{d}{dx}(g_{y'}\, w) - g_y w = \frac{d^2 w}{dx^2} + g_{y'}\frac{dw}{dx} + (g_{xy'} + y'\, g_{yy'} + g g_{y'y'} - g_y)\, w = 0. \ (3.4)$$

The solutions $w = \Lambda(x, y, y')$ of ODE (3.4), holding for any $y(x)$ satisfying the second-order

ODE (3.1), are the adjoint symmetries of (3.1). Explicitly, the determining equation for an adjoint symmetry $\Lambda(x, y, y')$ is given by

$$L^*[y]\,\Lambda(x, y, y') = \Lambda_{xx} + 2y'\,\Lambda_{xy} + 2g\,\Lambda_{xy'} + (y')^2\,\Lambda_{yy} + 2y'g\Lambda_{yy'} + g^2\,\Lambda_{y'y'}$$

$$+ (g_x + y'g_y + 2gg_{y'})\,\Lambda_{y'} + (g + y'g_{y'})\,\Lambda_y + g_{y'}\,\Lambda_x + (g_{xy'} + y'g_{yy'} + gg_{y'y'} - g_y)\Lambda = 0, \quad (3.5)$$

which must hold for arbitrary x, y, y'. In turn, an adjoint symmetry $\Lambda(x, y, y')$ of ODE (3.1) yields an integrating factor $\Lambda(x, Y, Y')$ of (3.1) if and only if $\Lambda(x, Y, Y')$ satisfies the adjoint-invariance condition

$$L^*[Y]\,\Lambda(x, Y, Y') = -(Y'' - g)(\Lambda_{xY'} + Y'\,\Lambda_{YY'} + g\,\Lambda_{Y'Y'} + 2g_{Y'}\,\Lambda_{Y'} + 2\Lambda_Y + g_{Y'Y'}\,\Lambda),$$
$$(3.6)$$

which must hold for arbitrary x, Y, Y', Y'' with $g = g(x, Y, Y')$. Thus, the adjoint-invariance condition for $\Lambda(x, Y, Y')$ to be an integrating factor of (3.1) reduces to $\Lambda(x, Y, Y')$, solving the linear system of PDEs

$$\Lambda_{xx} + 2Y'\Lambda_{xy} + 2g\Lambda_{xy'} + (Y')^2\,\Lambda_{yy} + 2Y'g\Lambda_{yy'} + g^2\Lambda_{y'y'} + (g_x + Y'g_y + 2gg_{y'})\,\Lambda_{y'}$$

$$+ (g + Y'g_{y'})\,\Lambda_y + g_{y'}\,\Lambda_x + (g_{xy'} + Y'g_{yy'} + gg_{y'y'} - g_y)\Lambda = 0 \quad (3.7)$$

given by equation (3.5) with y replaced by Y, and

$$\Lambda_{xY'} + Y'\,\Lambda_{YY'} + g\,\Lambda_{Y'Y'} + 2g_{Y'}\,\Lambda_{Y'} + 2\Lambda_Y + g_{Y'Y'}\,\Lambda = 0. \quad (3.8)$$

given by (3.6). Equations (3.7)–(3.8) must hold for arbitrary values of x, Y, Y'.

Since every second-order ODE (3.1) has an infinite number of integrating factors, it follows that there must exist an infinite number of solutions of the system (3.7)–(3.8). Unlike the situation for a first-order ODE, where each adjoint symmetry yields an integrating factor, solutions of (3.7) are not always integrating factors, since they must also satisfy condition (3.8).

Correspondingly, for each integrating factor the construction formula (2.16)–(2.20) yields the first integral

$$\Phi[y] = \Phi_1(x, y, y') + \Phi_2(x) = \text{const}$$

of equation (3.1). In terms of any fixed function $\tilde{y}(x)$, with $r = \lambda y + (1 - \lambda)\,\tilde{y}$, $\Lambda = \Lambda(x, r, r')$, one has

$$S = ((y' - \tilde{y}') - (y - \tilde{y})g_r(x, r, r'))\,\Lambda$$

$$- (y - \tilde{y})(\Lambda_x + (\lambda y' + (1 - \lambda)\,\tilde{y}')\,\Lambda_r + (\lambda g(x, y, y') + (1 - \lambda)\,\tilde{y}'')\,\Lambda_r),$$

$$N = (y - \tilde{y})(\lambda g(x, y, y') + (1 - \lambda)\,\tilde{y}'' - g(x, r, r'))\,\Lambda_{r'},$$

so that

$$S + N = (y' - \tilde{y}')\,\Lambda - (y - \tilde{y})(g(x, r, r')\,\Lambda_{r'} + (\lambda y' + (1 - \lambda)\,\tilde{y}')\,\Lambda_r + \Lambda_x + g_r(x, r, r')\,\Lambda), \quad (3.9)$$

$$k(x) = [\tilde{y}'' - g(x, \tilde{y}, \tilde{y}')]\,\Lambda(x, \tilde{y}, \tilde{y}'). \quad (3.10)$$

Consequently,

$$\Phi_1(x, y, y') = \int_0^1 (S + N)\,d\lambda, \quad (3.11)$$

$$\Phi_2(x) = \int k(x)\,dx. \quad (3.12)$$

One chooses $\tilde{y}$ so that $k(x)$ is finite. If both $g(x,0,0)$ and $\Lambda(x,0,0)$ are finite, one can set $\tilde{y} = 0$ provided the corresponding integral $\int_0^1 (S+N) \, d\lambda$ converges. In this case,

$$S+N = y' \Lambda(x, \lambda y, \lambda y') - y[g(x, \lambda y, \lambda y')(\Lambda_r(x, \lambda y, r')|_{r'=\lambda y'}) + \lambda y'(\Lambda_r(x, r, \lambda y')|_{r=\lambda y})$$

$$+ \Lambda_x(x, \lambda y, \lambda y') + (g_r(x, \lambda y, r')|_{r'=\lambda y'}) \Lambda(x, \lambda y, \lambda y')].$$

3.2 Higher-order ODEs

For higher-order scalar ODEs (3.2), the adjoint-invariance condition for an integrating factor $\Lambda(x, Y, Y', \ldots, Y^{(n-1)})$ yields a linear determining equation which is a relation involving $x, Y, Y', \ldots, Y^{(2n-2)}$, where each of the $2n$ quantities $x, Y, Y', \ldots, Y^{(2n-2)}$ are to be treated as independent variables. This relation is a polynomial expression in terms of $Y^{(n)}$, $Y^{(n+1)}, \ldots, Y^{(2n-2)}$, whose coefficients depend on $x, Y, Y', \ldots, Y^{(n-1)}$. The coefficient of the term independent of $Y^{(n)}, Y^{(n+1)}, \ldots, Y^{(2n-2)}$, yields the determining equation for the adjoint symmetries. The coefficients of the other terms in the polynomial expression yield further linear PDEs satisfied by $\Lambda(x, Y, Y', \ldots, Y^{(n-1)})$. For $n = 2$, as shown in (3.6), this splitting yields one such linear PDE (from the coefficient of the Y'' term). For $n = 3$, one can show that this splitting yields three such linear PDEs from the coefficients of the terms involving $Y^{(4)}, (Y''')^2$ and Y'''. For $n = 4$, the splitting yields five such linear PDEs from the coefficients of the terms involving $Y^{(6)}, Y^{(4)} Y^{(5)}, Y^{(5)}, (Y^{(4)})^2$ and $Y^{(4)}$.

4 Techniques for obtaining adjoint symmetries yielding first integrals

For *any* system of ODEs (2.1), there is an infinite number of linearly independent solutions of its corresponding adjoint system (2.6). Hence, a system of ODEs (2.1) always has an infinite number of adjoint symmetries. Consequently, in practice one must resort to specific ansatze in order to find adjoint symmetries.

We now focus on nth-order scalar ODEs. Here, one such ansatz is to seek solutions of the form $w = \Lambda(x, y, y', \ldots, y^{(n-2)})$, which depend upon derivatives of order at most $n-2$ rather than $n-1$, for the corresponding adjoint symmetry determining equation (2.6).

More importantly, if one knows an adjoint symmetry and one or more first integrals arising from other adjoint symmetries, then one can use a second ansatz to seek further first integrals as follows. For a given nth-order scalar ODE, suppose that

$$\Phi_1[y] = C_1, \ldots, \Phi_m[y] = C_m$$

are m functionally independent first integrals corresponding to the m integrating factors $\Lambda_1[Y], \ldots, \Lambda_m[Y]$, respectively. Note that

$$\Lambda[y] = \frac{\partial \Gamma}{\partial C_1} \Lambda_1[y] + \cdots + \frac{\partial \Gamma}{\partial C_m} \Lambda_m[y]$$

for any function $\Gamma(C_1, \ldots, C_m)$, generates an inessential first integral $\Gamma(C_1, \ldots, C_m) = \text{const.}$

Now suppose $w = \Lambda[y]$ is an adjoint symmetry such that

$$\Lambda[y] \neq \frac{\partial \Gamma}{\partial C_1} \Lambda_1[y] + \cdots + \frac{\partial \Gamma}{\partial C_m} \Lambda_m[y], \tag{4.1}$$

for all functions $\Gamma(C_1, \ldots, C_m)$. We observe that for an arbitrary function $F(C_1, \ldots, C_m)$,

$$w = \Lambda_F[y] = F(C_1, \ldots, C_m)\, \Lambda[y] \tag{4.2}$$

is also an adjoint symmetry.

If we substitute $w = \Lambda_F[Y]$, given by equation (4.2) with y replaced by Y, into the adjoint-invariance condition (2.8), then we obtain a linear determining equation for F. Each solution, if any, of this determining equation yields a *new* integrating factor for the nth-order ODE. This will be illustrated through examples in §5.

A third ansatz which can lead to finding further adjoint symmetries is suggested by the following observation. If a given nth-order ODE admits a point symmetry, then each integrating factor of the ODE can always be expressed as a product of a multiplier expression, and some function of the invariants/differential invariants of the point symmetry. Consequently, the ODE admits adjoint symmetries of such a product form. One can then try using a known adjoint symmetry or integrating factor as the multiplier expression in a trial form in order to seek new adjoint symmetries. In particular, suppose a given nth-order ODE admits an integrating factor $\Lambda[Y]$ and a point symmetry with corresponding invariants/differential invariants $u(x, y), v_1(x, y, y'), \ldots, v_{n-1}(x, y, y', \ldots, y^{(n-1)})$. Let

$$\Lambda_f[y] = f(u, v_1, \ldots, v_{n-1})\, \Lambda[y], \tag{4.3}$$

for an arbitrary function $f(u, v_1, \ldots, v_{n-1})$. If we substitute $w = \Lambda_f[y]$ into the adjoint symmetry determining equation (2.6), then we obtain a linear determining equation for f. Each solution $f \neq \text{const}$ of this determining equation yields a new adjoint symmetry of the nth order ODE. In turn, we feed such a new adjoint symmetry into the second ansatz to seek further first integrals.

The above discussion extends naturally to systems of ODEs.

5 Examples

We now use three examples to illustrate our procedure for obtaining first integrals.

5.1 Harmonic oscillator

Consider the harmonic oscillator equation

$$y'' + y = 0. \tag{5.1}$$

The ODE (5.1) is self-adjoint, so that its adjoint symmetries are symmetries. The corresponding determining equation (3.5) for an adjoint symmetry $w = \Lambda(x, y, y')$ becomes

$$\Lambda_{xx} + 2y'\Lambda_{xy} - 2y\Lambda_{xy'} + (y')^2\Lambda_{yy} - 2yy'\Lambda_{yy'} + y^2\Lambda_{y'y'} - y'\Lambda_y - y\Lambda_y + \Lambda = 0. \tag{5.2}$$

Here the extra adjoint-invariance determining equation (3.8) for $w = \Lambda(x, y, y')$ to yield an integrating factor becomes

$$\Lambda_{xy'} + y'\Lambda_{yy'} - y\Lambda_{y'y'} + 2\Lambda_y = 0. \tag{5.3}$$

Obviously ODE (5.1) admits translations in x and scalings in y which respectively yield adjoint symmetries $\Lambda_1 = y'$ and $\Lambda_2 = y$ satisfying (5.2).

Clearly, $\Lambda = y'$ satisfies the adjoint-invariance condition (5.3). Since $y'' + y$ and y' are non-singular for $y = 0$, we can set $\tilde{y} = 0$ in our construction formula (3.9)–(3.12). Then we have

$$S + N = \lambda[(y')^2 + y^2],$$

and hence the corresponding first integral is the energy

$$\Phi = \int_0^1 \lambda((y')^2 + y^2)\, d\lambda = \tfrac{1}{2}((y')^2 + y^2) = C_1. \tag{5.4}$$

It is easy to check that the adjoint symmetry $\Lambda = y$ does not satisfy the adjoint-invariance condition (5.3). Now we try the second ansatz presented in §4, using the previously-obtained first integral (5.4). Let

$$\Lambda = \Lambda_F = F(C_1)\, Y, \tag{5.5}$$

where $C_1 = \tfrac{1}{2}((Y')^2 + Y^2)$. Substituting (5.5) into the adjoint-invariance condition (5.3), we find that $F(C_1)$ satisfies the ODE $C_1 F' + F = 0$. This yields the integrating factor $\Lambda = Y/((Y')^2 + Y^2)$. Since Λ is singular for $Y = 0$, we choose $\tilde{y} = 1$ in our construction formula (3.9)–(3.12). Correspondingly, $r = \lambda(y - 1) + 1, r' = \lambda y'$, so that

$$S + N = \frac{y'r - (y - 1)r'}{r^2 + (r')^2}, \quad k(x) = 1.$$

This leads to the first integral

$$\Phi = x + \int_0^1 \frac{y'}{\lambda^2(y')^2 + [\lambda(y - 1) + 1]^2}\, d\lambda = x + \frac{\pi}{2} - \tan^{-1}\left(\frac{y}{y'}\right) = C_2, \tag{5.6}$$

which is the phase.

The first integrals (5.4) and (5.6) lead to the complete reduction $y = \sqrt{2C_1} \sin(x - C_2 + \pi/2)$.

5.2 Frequency-damped oscillator

As a second example, we use the frequency-damped oscillator equation

$$y'' + y(y')^2 = 0, \tag{5.7}$$

considered by Gordon (1986), Sarlet *et al.* (1987) and Mimura & Nôno (1994). The ODE (5.7) is not self-adjoint, so that its adjoint symmetries are not symmetries. Here the adjoint symmetry determining equation (3.5) for $w = \Lambda(x, y, y')$ is

$$\Lambda_{xx} + 2y'\Lambda_{xy} - 2y(y')^2\Lambda_{xy'} + (y')^2\Lambda_{yy} - 2y(y')^3\Lambda_{yy'} + y^2(y')^4\Lambda_{y'y'}$$
$$+ (4y^2 - 1)(y')^3\Lambda_{y'} - 3y(y')^2\Lambda_y - 2yy'\Lambda_x + (2y^2 - 1)(y')^2\Lambda = 0. \tag{5.8}$$

The extra adjoint-invariance determining equation (3.8) becomes

$$\Lambda_{xy'} + y'\Lambda_{yy'} - y(y')^2\Lambda_{y'y'} - 4yy'\Lambda_{y'} + 2\Lambda_y - 2y\Lambda = 0. \tag{5.9}$$

We try the first ansatz $\Lambda = \Lambda(x, y)$. Then equation (5.8) leads to the adjoint symmetry

$$\Lambda = axe^{y^2/2} + l(y), \tag{5.10}$$

with $a =$ const, and $l(y)$ satisfying the ODE $l'' - 3yl' + (2y^2 - 1)l = 0$. Substituting (5.10) into the adjoint-invariance condition (5.9), we obtain $l' - yl = 0$, and thus $l(y) = be^{y^2/2}$, $b =$ const. Hence we get two integrating factors, $\Lambda_1 = e^{y^2/2}$, $\Lambda_2 = xe^{y^2/2}$.

Next, we construct the first integral arising from $\Lambda = \Lambda_1 = e^{y^2/2}$. Clearly, we can set $\tilde{y} = 0$ in the construction formulae (3.9)–(3.12). This leads to the first integral

$$\Phi = y' \int_0^1 [1 + \lambda^2 y^2] e^{\lambda^2 y^2/2} \, d\lambda = y' e^{y^2/2} = C_1, \tag{5.11}$$

after integration by parts on the second term.

Now the first integral arising from $\Lambda = \Lambda_2 = xe^{y^2/2}$ is easy to construct since, again with $\tilde{y} = 0$, the construction formulae (3.9)–(3.12) reduces to

$$\Phi = C_1 x - \int_0^1 y e^{\lambda^2 y^2/2} \, dy = C_1 x - \int_0^y e^{u^2/2} \, du = C_2. \tag{5.12}$$

This yields the general solution $\int_0^y e^{u^2/2} \, du = C_1 x - C_2$ of the ODE (5.7).

5.3 Wave-speed equation

For a third example, we consider the fourth-order wave-speed equation

$$G(y, y', y'', y''' y^{(4)}) = (yy'(y/y')')' = 0, \tag{5.13}$$

which arises when one seeks potential symmetries for a wave equation with a variable wave speed $y(x)$ (see Bluman & Kumei, 1987). The ODE (5.13) is not self-adjoint. Its adjoint symmetry determining equation for $w = \Lambda(x, y, y', y'', y''')$ is given by

$$((\Lambda' yy')'' y/(y')^2)' + (\Lambda' yy')''/y' - (\Lambda' y(y/y'')')' + \Lambda' y'(y/y')'' = 0. \tag{5.14}$$

The adjoint-invariance condition is

$$((\Lambda' YY')'' Y/(Y')^2)' + (\Lambda' YY')''/Y' - (\Lambda' Y(Y/Y'')')' + \Lambda' Y'(Y/Y')''$$
$$= -((G\Lambda_{Y'''})')''' - (G\Lambda_{Y''})'' + (G\Lambda_{Y'})' - G\Lambda_Y), \tag{5.15}$$

with $\Lambda = \Lambda(x, Y, Y', Y'', Y''')$ and $G = (YY'(Y/Y')')'$.

By inspection, $\Lambda = 1$ satisfies (5.14)–(5.15), which leads to the first integral

$$\Phi = yy'(y/y')'' = C_1. \tag{5.16}$$

Since the ODE (5.13) admits translations in x with corresponding invariants y and y', we employ the third ansatz of §4, in conjunction with the integrating factor $\Lambda = 1$ and these invariants, and seek adjoint symmetries of the form $\Lambda = \Lambda_f = f(y, y')$. Then (5.14) becomes a polynomial in y''', y''. The coefficient of $(y''')^2$ gives $y' f_{y'y'} + 3f_{y'} = 0$. This yields $f_{y'} = h(y)/(y')^3$ for some function $h(y)$. Then the coefficient of y''' gives the equation $12y(y')^2 f_{y'y'y} + 3y^2 f_{y'y} + 41yy' f_{y'y} + 9y' f_y + 12yf_y = 0$. This leads to $h =$ const, and hence $f = 1/(y')^2$. Once can check that $\Lambda = 1/(y')^2$ satisfies both the adjoint symmetry determining equation (5.14) and the adjoint-invariance condition equation (5.15). The singularity of Λ at $y' = 0$ leads us to choose $\tilde{y} = x$ in our construction formula (3.9)–(3.12). The resulting first integral is

$$\Phi = -y''' y^2/(y')^3 - yy''/(y')^2 + (yy'')^2/(y')^4 = C_2. \tag{5.17}$$

The first integrals (5.16)–(5.17) reduce the ODE (5.13) to the second-order ODE

$$(y'')^2 = ((C_1 - C_2(y')^2)(y')^2)/y^2. \tag{5.18}$$

Now we again use the third ansatz in conjunction with the integrating factors $\varLambda = 1$ and $\varLambda = 1/(y')^2$ together with the differential invariant $\alpha = yy''/(y')^2$ arising from the invariance of ODE (5.13) under scalings in both x and y.

Using the integrating factor 1, we try $\varLambda = \varLambda_f = f(\alpha)$. The adjoint symmetry determining equation (5.14) yields $f = \alpha^2$. Feeding this into the second ansatz, we first substitute $\varLambda = \varLambda_F = F(C_1)\alpha^2$ into the adjoint-invariance restriction (5.15). Unfortunately, this yields $F = 0$. Next we substitute $\varLambda = \varLambda_F = F(C_2)\alpha^2$ into (5.15), which then becomes a polynomial with terms $y^{(6)}, y^{(4)}y^{(5)}, y^{(5)}, (y^{(4)})^3, (y^{(4)})^2, y^{(4)}$. The coefficient of $y^{(6)}$ yields $F = 1/(C_2)^2$. One can then check that

$$\varLambda = \left(\frac{yy''}{(y')^2 C_2}\right)^2$$

satisfies (5.15). From our construction formula we obtain the corresponding first integral

$$\varPhi = (yy''/y')^2/C_2 + (y')^2 = C_3.$$

However, one can show that the first integral (5.19) is inessential, since $C_3 = C_1/C_2$.

Finally, using the integrating factor $1/(y')^2$, we try $\varLambda = \varLambda_f = f(\alpha)/(y')^2$. In this case, the adjoint symmetry determining equation (5.14) leads to

$$f = \tan^{-1}(c/\alpha) + \frac{c/\alpha}{1 + (c/\alpha)^2} \tag{5.20}$$

with $c = $ const. Here c arises from the scaling symmetry $\alpha \to \alpha/c$ admitted by the determining ODE satisfied by $f(\alpha)$.

One can check that $\varLambda = f(\alpha)/(y')^2$ does not satisfy the adjoint-invariance determining equation (5.15). Now we try a variant of the second ansatz as follows. We substitute $\varLambda = \varLambda_F = F(C_2)f/(y')^2$, where f is given by equation (5.20) with $c = H(C_2)$, into the adjoint-invariance determining equation (5.15). This leads to $F = (C_2)^{-3/2}$ and $H = (C_2)^{1/2}$. Consequently, we obtain the integrating factor

$$\varLambda = (C_2)^{-3/2}\left(\tan^{-1}(\sqrt{C_2}/\alpha) + \frac{\sqrt{C_2}/\alpha}{1 + (\sqrt{C_2}/\alpha)^2}\right),$$

and our construction formula yields the first integral

$$\varPhi = (C_2)^{-1/2}\tan^{-1}(\sqrt{C_2}/\alpha) - \ln y = C_4. \tag{5.21}$$

The first integrals (5.16), (5.17) and (5.21) reduce the ODE (5.13) to a first order ODE. In particular, we have

$$y'\sqrt{C_1/C_2 - (y')^2} = y\cot(\sqrt{C_2}(C_4 + \ln y)).$$

Isolating y', we obtain

$$y' = \sqrt{C_3}\sin(\sqrt{C_2}(C_4 + \ln y)).$$

6 Conclusion

For any system of ODEs, we have derived determining equations which are necessary and sufficient conditions satisfied by its integrating factors. In particular, the solutions of these determining equations yield all integrating factors. We have also derived a simple explicit formula which yields a first integral for each solution. For an nth-order scalar ODE the determining equations are a linear system of $2n - 2$ PDEs consisting of the adjoint of the determining equation for symmetries of the nth-order ODE and an additional $2n - 3$ equations when $n \geq 2$. No additional equations arise in the case of a first-order scalar ODE.

We have introduced special techniques to seek solutions of the determining equations. These techniques involve the use of known first integrals, eliminations of variables and symmetry considerations. We have exhibited several examples illustrating combinations of these techniques.

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