Math 5110- Applied Linear Algebra-Fall 2020

Instructor: He Wang

Test 2.

Student Name: ____________________ ________/**50**

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to look at notes or textbook. However, you are **not** allowed to asked help using any online platform.

4. You have 60 minutes for the exam and 15 minutes to scan your solutions, merge into **one .pdf**, and upload. This is plenty of time to use a scanner or scanning app and clearly scan every page.

5. If you have any technical difficulty with the upload or scan, contact me immediately. Do not wait until the end of exam to contact me about a technical difficulty.

1. (8 points) Let P_n be the $n \times n$ matrix whose entries are all ones, except for zeros directly below the main diagonal. For example,

$$P_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(1) Find the determinant of P_n for all n.

(2) Is P_n invertible? (Explain the reason)

(3) Is zero an eigenvalue for the matrix P_n? (Explain the reason)

2. (4 points) Suppose $\vec{v}$ is an eigenvector of A corresponding to eigenvalue λ. (1) Is $\vec{v}$ an eigenvector of $A^4 + 2A + I_n$? (2) What is the corresponding eigenvalue? Prove your result.

3. (6 points) The matrix A has eigenvalues $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ with corresponding eigenvectors $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 0$.

(1) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$P = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \text{ and } D = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}.$$

(2) The trace of A is $tr(A) =$ __________

(3) What are the eigenvalues of $A^2 - 3A$?

(4) The determinant of $A^2 - 3A$ is $\det(A^2 - 3A) =$ __________

(5) The rank of $A^2 - 3A$ is $\text{rank}(A^2 - 3A) =$ __________

4. (3 points each) Answer the following questions. (**Explain the reason for your answer**).

(1) If the determinant of an $n \times n$ matrix A is 3, what is the determinant of $A^2 A^T A^{-1}$?

(2) Does there exist real invertible 3×3 matrices A and S such that $S^{-1}AS = 2A$.

5. (3 points each) Answer the following questions. (**Explain the reason for your answer**).
(1) Is there a real $n \times n$ matrix such that $A^2 + A + I_n = \mathbf{0}$?

(2) Recall that an $n \times n$ matrix A is called skew-symmetric if $A^T = -A$. Does there exist a skew-symmetric matrix such that $\det(A) \neq 0$? If yes, find an example. If no, explain the reason.

6. (6 points) Suppose $N \in \mathbb{R}^{n \times n}$ is a nilpotent matrix. Find the determinant $\det(N + 2I)$ and prove your result. (Hilt: Use Jordan canonical form)

7. (4 points) Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. The standard basis $\mathscr{B}$ of $\mathbb{R}^{2\times 2}$ is given by

$$\mathscr{B} = \{E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

(1) Find the coordinate of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ relative to the standard basis $\mathscr{B}$.

(2) Suppose the coordinate of a matrix C relative $\mathscr{B}$ is $[C]_{\mathscr{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. What is the matrix C?

(If time is short, you can complete the test with 30 extra minutes. But you will receive 80% of the next question. (Only 80% for question 8. You will still receive full points for questions 1-7.))

8. (10 points) Consider the transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by

$$T(A) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} A - A \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

for any 2×2 matrices $A \in \mathbb{R}^{2\times 2}$.

(1) Show that T is a linear transformation.

(2) Find the matrix M of the linear transformation T with respect to the standard basis $\mathscr{B}$ of $\mathbb{R}^{2\times 2}$.

(3) Find the rank of T by $\text{rank}(M)$. Is T an isomorphism?

(4) Find the dimension of $\ker(T)$ and the dimension of $\text{im}(T)$.

(5) (2 bonus) Find a basis for $\ker(T)$ and a basis for $\text{im}(T)$. (Add paper if needed.)