

SOLUTIONS MANUAL FOR

Mathematical Modelling with
Case Studies - A Differential
Equations Approach using
Maple and MATLAB,
Second Edition

_____ by _____

Belinda Barnes
Glenn Fulford



CRC Press

Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number: 978-1-4398-3441-1 (Paperback)

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2 Chapter 2 Solutions

2.1. Atmospheric Pressure.

- (a) $p(h) = 1013e^{-0.1504h}$
- (b) Approximately 0.55 millibars
- (c) Approximately 0.79 kilometres

2.2. The Rule of 72.

$$t = \frac{100 \ln(2)}{r} \approx \frac{69}{r} < 72r.$$

2.3. Dating a sea shell.

Approximately -4103 years

2.4. Oldvai Gorge.

(a)

$$\begin{aligned} K(t) &= k_0 e^{-k_3 t}, \\ A(t) &= \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}), \\ C(t) &= \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}). \end{aligned}$$

(b) The sum of K , A and C is k_0 , which represents the total number of atoms in the system. This number is constant because the number of particles in the system must be conserved. (This result also implies that both potassium-40 and argon-40 are stable isotopes.)

(f) $A/K \approx 1.008 \times 10^{-4}$.

2.5. Tracers in the body.

Approximately 284 days.

2.6. Lake Burley Griffin.

(a) The concentration is reduced to 5% of the original concentration after approximately 21 months. The safety threshold of 4×10^6 parts/m³ is reached after approximately 6 months.

(b) See Figure 6.

2.7. North American lake system.

(a)

$$\frac{dc_1}{dt} = \frac{F_1}{V_1} c_{1,in} - \frac{F_1}{V_1} c_1, \quad \frac{dc_2}{dt} = \frac{F_2}{V_2} c_1 - \frac{F_2}{V_2} c_2.$$

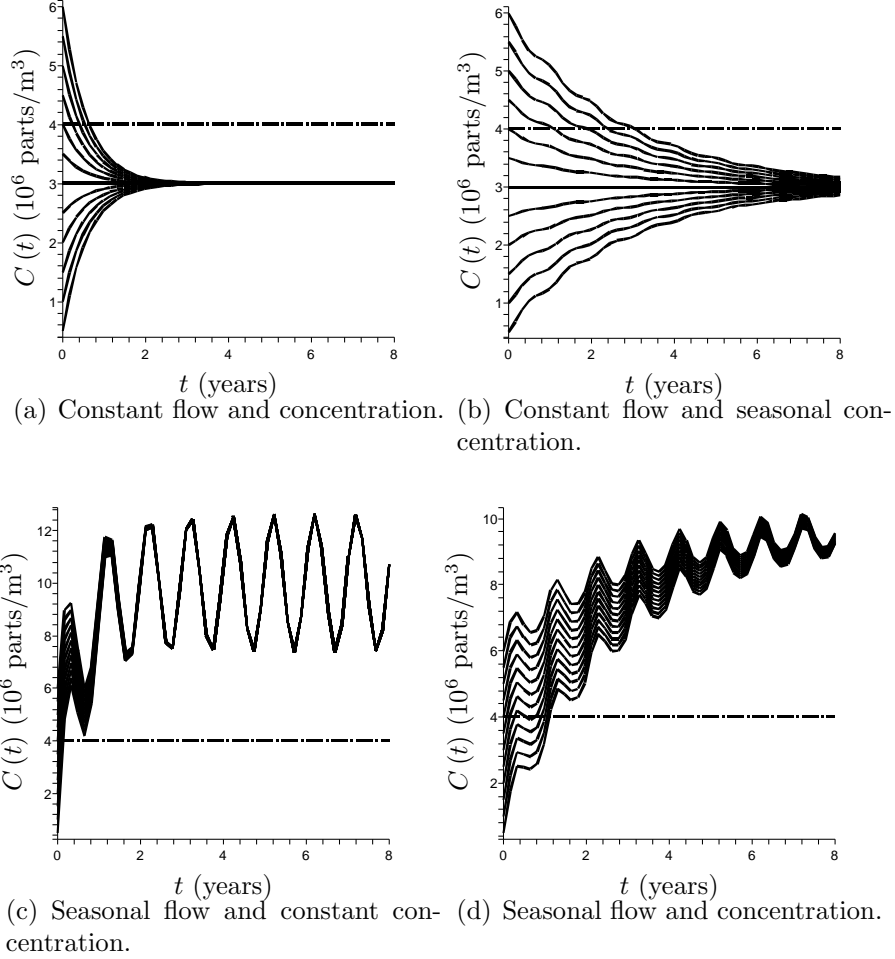


Figure 1: The effect of incorporating a seasonal inflowing pollutant concentration and flow rate on pollution levels in Lake Burley Griffin (Question 6). The constant concentration is $c_{in} = 4 \times 10^6$ parts/ m^3 and the variable concentration is $c_{in} = 10^6 (10 + 10 \cos(2\pi t))$ parts/ m^3 . The constant flow rate is $F = 4 \times 10^6$ m^3 /month and the variable flow rate is $F = 10^6 (1 + 6 \sin(2\pi t))$ m^3 /year. In all plots $V = 28$ m^3 . The grey line is the pollution threshold level.

(b)

$$\frac{dc_1}{dt} = -\frac{F_1}{V_1}c_1, \quad \frac{dc_2}{dt} = \frac{F_2}{V_2}c_1 - \frac{F_2}{V_2}c_2.$$

(c)

$$c_1(t) = \alpha e^{-\frac{F_1 t}{V_1}}, \quad c_2(t) = \frac{F_2 \alpha}{V_2 \left(\frac{F_2}{V_2} - \frac{F_1}{V_1} \right)} e^{-\frac{F_1 t}{V_1}} + \beta e^{-\frac{F_2 t}{V_2}},$$

where α and β are arbitrary constants.

(e) As the initial concentration of pollution in Lake Erie increases, the peak concentration of pollution in Lake Ontario increases as well. It takes longer to reach this peak level and for the pollution concentration to decrease in Lake Ontario. Increasing the initial concentration of pollution in Lake Ontario increases the rate at which the pollution is initially removed.

2.8. Smoke in the bar.

(a) The compartment model is shown below.



The corresponding word equation is below.

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of amount of} \\ \text{carbon monoxide} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate carbon} \\ \text{monoxide produced} \\ \text{by smoke} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate carbon} \\ \text{monoxide removed} \\ \text{by ventilators} \end{array} \right\}$$

The appropriate differential equation is:

$$c'(t) = \frac{F_{in}}{V}c_{in} - \frac{F_{out}}{V}c(t), \quad c(0) = 0$$

and its solution is

$$c(t) = \frac{F_{in}c_{in}}{F_{out}} \left(1 - e^{-(F_{out}/V)t} \right).$$

(b) Approximately 10 hours.

2.9. Detecting art forgeries.

(a) The original disintegration rate of Lead-210 is

$$\lambda n_0 = \lambda N e^{\lambda(t-t_0)} - R \left(e^{\lambda(t-t_0)} - 1 \right),$$

Assuming 'Washing of Feet' is approximately 300 years old,

$$\lambda n_0 = 8.2 \times 2^{150/11} - 0.26 \left(2^{150/11} - 1 \right) > 101\,000.$$

This original rate of disintegration is too high. The painting is modern.

(b) With $\lambda N = 5.2$ and $R = 4$ and assuming the painting is 300 years old the equation gives

$$\lambda n_0 = 5.2 \times 2^{150/11} - 4 \left(2^{150/11} - 1 \right) \approx 15284.$$

The painting could be authentic.

2.10. Cold pills.

(b) Cannot divide by $k_1 - k_2$

$$y(t) = k_1 x_0 t e^{-k_1 t}.$$

(c) Lowering the clearance coefficient does not change the GI-tract drug concentration since the differential equation describing the amount in the tract is independent of k_2 . It does, however, increase the time taken for the drug to be removed from the blood stream. Accordingly, the peak concentration in the blood stream is higher.

2.11. Cold pills. To solve the 1st-order linear equation $dx/dt + k_1 x = I$ integrating factor is $\mu(t) = e^{k_1 t}$ so

$$e^{k_1 t} x(t) = \int e^{k_1 t} I dt = \frac{I}{k_1} e^{k_1 t} + a_1$$

where a_1 is an arbitrary constant. Dividing by $e^{k_1 t}$ and applying $x(0) = 1$ gives the required solution

$$x(t) = \frac{I}{k_1} (1 - e^{-k_1 t}).$$

Now solve

$$\frac{dy}{dt} + k_2 y = I (1 - e^{-k_1 t}).$$

Now integrating factor is $\mu(t) = e^{k_2 t}$ and

$$e^{k_2 t} y = \int I (e^{k_2 t} - e^{(k_2 - k_1)t}) dt = \frac{I}{k_2} e^{k_2 t} - \frac{I}{k_2 - k_1} e^{(k_2 - k_1)t} + a_2$$

where a_2 is an arbitrary constant. Now $y(0) = 0$ implies that $a_2 = (I/(k_2 - k_1) - I/k_2)$. Hence

$$y(t) = \frac{I}{k_2} (1 - e^{-k_2 t}) + \frac{I}{k_2 - k_1} (e^{-k_2 t} - e^{-k_1 t}).$$

Note: solution for y that was given in question and text is INCORRECT.

2.12. Antibiotics.

(a)

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug in} \\ \text{GI tract} \end{array} \right\} = - \left\{ \begin{array}{l} \text{rate drug} \\ \text{leaves GI tract} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug} \\ \text{in blood} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate drug} \\ \text{enters blood} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate drug} \\ \text{leaves blood} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug in} \\ \text{urinary tract} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate drug enters} \\ \text{urinary tract} \end{array} \right\}$$

(b) Let $x(t)$ be the amount of the drug in the GI-tract, $y(t)$ be the amount of drug in the blood stream and $z(t)$ be the amount of drug in the urinary tract at time t . Then

$$\begin{aligned}x(t) &= -k_1x, \\y(t) &= k_1x - k_2y, \\z(t) &= k_2y,\end{aligned}$$

with

$$x(0) = x_0, \quad y(0) = 0, \quad z(0) = 0,$$

and where k_1 , and $k_2 > 0$ represent the rates at which the drug is removed from or enters different parts of the system.

(c)

$$\begin{aligned}x(t) &= 0.0001e^{-0.72t}, \\y(t) &= 0.000126(e^{-.15t} - e^{-.72t}), \\z(t) &= 0.000026e^{-.72t} - 0.000126e^{-.15t} + 0.0001.\end{aligned}$$

The maximum level in the bloodstream is approximately 6.6×10^{-5} milligrams and that this maximum is reached after about 2.8 hours.

(d)

$$\begin{aligned}x(t) &= I - k_1x, \quad x(0) = 0, \\y(t) &= k_1x - k_2y, \quad y(0) = 0, \\z(t) &= k_2y, \quad z(0) = 0,\end{aligned}$$

where I represents the rate of ingestion of the drug.

The amount of tetracycline in the GI-tract levels out at just over 1 milligram after approximately 5 hours. The amount in the bloodstream rises in the first 24 hours, although this too appears to tend towards a constant value after 24 hours. The urinary tract is considered to be an absorbing compartment and so it is not surprising that the amount of tetracycline continues to rise.

2.13. Alcohol consumption.

(b) The BAL of a man is slightly lower than that of a woman, regardless of the number of standard drinks consumed, but tends to rise and fall in a qualitatively similar manner.

(c) To remain strictly under the legal limit men and woman can only consume one drink at the start of the first hour. If the man consumes two drinks his BAL just exceeds the legal limit of 0.05 in the second half of the first hour but then falls below this threshold.

(d) The BAL of a woman is again higher than that of a man of the same weight and who consumes the same amount of alcohol. The BAL of both women and men is lower if alcohol is consumed after a substantial meal than on an empty stomach, as expected. After a meal, women and men can consume three standard drinks at once and remain under the limit.

2.14. Alcohol consumption.

(a) $y(t) = -k_3 t + y_0$, $y(0) = y_0$.

(b) $y(t) = y_0 e^{-(k_3/M)t}$, $y(0) = y_0$.

(c) See Figure 2

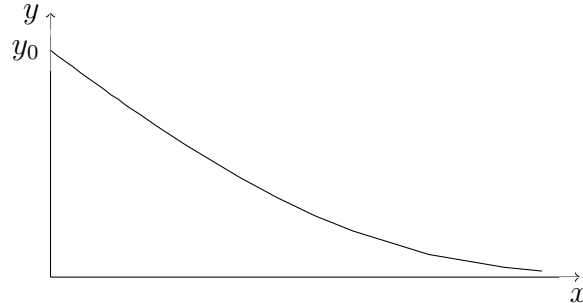


Figure 2: Sketch of the solution of the differential equation in Question 14.

(d) If the observed rate of removal frequently reaches a saturation value.

2.15. Solving differential equations.

The normal form of

$$t \frac{dx}{dt} = x$$

is

$$\frac{dx}{dt} - \frac{x}{t} = 0$$

and its solution is

$$x(t) = \frac{x_0}{t_0} t.$$

A unique solution to the IVP exists on an interval containing t_0 provided $t_0 \neq 0$. The response to the initial data is $\frac{x_0}{t_0} t$. There is no response to the input.

The normal form of

$$y^2 \frac{dx}{dy} + xy = 2y^2 + 1$$

is

$$\frac{dx}{dy} + \frac{x}{y} = 2 + \frac{1}{y^2}.$$

Its solution is

$$x(y) = y + \frac{1}{y} (\ln y + x_0 y_0 - y_0^2 - \ln y_0).$$

A unique solution to the IVP exists on an interval containing y_0 provided $y_0 \neq 0$. The response to the initial data is $\frac{y_0 x_0}{y}$ and the response to the input is $y + \frac{1}{y} (\ln y - y_0^2 - \ln y_0)$.

2.16. Formulating DEs for alcohol case study.

Conservation of mass requires

$$\begin{aligned}\frac{d}{dt}(V_b C_1) &= i - F_1 \times \frac{C_1 V_b}{V_g}, \\ \frac{d(V_b C_2)}{dt} &= \alpha F_1 \times \frac{C_1 V_b}{V_g} - F_2 \times C_2,\end{aligned}$$

So

$$I = \frac{i}{V_b}, \quad k_1 = \frac{F_1}{V_g}, \quad k_2 = \frac{\alpha F_1}{V_g}, \quad k_4 = \frac{F_2}{V_b}.$$

Since C_1 is the concentration in the GI-tract measured in terms of the volume in the blood, the mass of alcohol is therefore $V_b C_1$ and since F_1 measures flow rate as volume of fluid in the GI tract we need to multiply by the fraction $(C_1 V_b)/V_g$ as the appropriate concentration in the first equation.

2.17. Economic model.

(a) From the case study, the ratio of capital to labour, $r(t)$ satisfies

$$\left(\frac{dr}{dt} + nr\right) L_0 e^{nt} = sF(K, L_0 e^{nt})$$

so

$$\frac{dr}{dt} = s \min\left\{\frac{r}{a}, \frac{1}{b}\right\} - nr,$$

For $r/a < 1/b$ we have $r < a/b$ and the relevant equation is

$$\frac{dr}{dt} = \left(\frac{s}{a} - n\right) r.$$

The solution, satisfying $r(t) = r_0$, is

$$r(t) = r_0 e^{(s/a - n)t}.$$

(b) For $s/a - n < 0$ then r is always decreasing. With $r_0 > a/b$, then r decreases towards $s/nb < a/b$. When $r(t)$ reaches a/b it decreases further.

Note that at $r = a/b$, labour supply and capital stock are in balance. Thereafter, the capital-labour ratio decreases with labour becoming redundant, and this redundancy growing.

2.18. Return to scale property.

$$F(aK, aL) = (aL)^\alpha (aK)^{1-\alpha} = aK^\alpha L^{1-\alpha} = aF(K, L).$$

2.19. Stability of equilibrium solution.

The stability of the steady states can be determined by examining the sign of dr/dt for different values of r , as shown in Table 1. If $r < r_e^{(1)}$, dr/dt is positive and r increases towards $r_e^{(1)}$. On the other hand, if $r_e^{(1)} < r < r_e^{(2)}$, r decreases towards $r_e^{(1)}$. Thus $r_e^{(1)}$ is a stable steady state. By similar arguments, we see that $r_e^{(2)}$ is unstable and $r_e^{(3)}$ is stable.

r	Value of $sf(r)$	Sign of dr/dt
$r < r_e^{(1)}$	$sf(r) > nr$	+
$r_e^{(1)} < r < r_e^{(2)}$	$sf(r) < nr$	-
$r_e^{(2)} < r < r_e^{(3)}$	$sf(r) > nr$	+
$r > r_e^{(1)}$	$sf(r) < nr$	-

Table 1: Stability of the steady states of $r(t)$ for Question 19.

3 Chapter 3 Solutions

3.1. Farming fish.

(a)

$$\left\{ \begin{array}{c} \text{rate of change} \\ \text{of number} \\ \text{of fish} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{normal} \\ \text{rate of} \\ \text{deaths} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of} \\ \text{deaths by} \\ \text{harvesting} \end{array} \right\}$$

$$\frac{dX}{dt} = 0.7X - 0.2X - 300, \quad X(0) = X_0.$$

(b) 1 176 000

(c) $X_e = 600$.

3.2. Modelling the spread of technology.

(a) $1 - N/N^* = (N^* - N)/N^*$.

(b) Approximately 12.6.

3.3. Density dependent births.

$$X' = \left\{ \left[\beta - (\beta - \alpha) \delta \frac{X}{K} \right] - \left[\alpha + (\beta - \alpha) (1 - \delta) \frac{X}{K} \right] \right\} X = (\beta - \alpha) X \left(1 - \frac{X}{K} \right),$$

which is in the form of the standard logistic equation with $r = \beta - \alpha$.

3.4. Mouse population model.

$$\left\{ \begin{array}{c} \text{rate of change} \\ \text{of number} \\ \text{of mice} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{normal} \\ \text{rate of} \\ \text{deaths} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of} \\ \text{deaths by} \\ \text{mouse traps} \end{array} \right\}$$

$$M' = 8M - 2M - 80, \quad M(0) = 1000.$$

3.5. Harvesting model.

(a) The two equilibrium values are

$$X_e = \frac{K}{2} \left(1 + \sqrt{1 - \frac{4h}{rK}} \right) \quad \text{and} \quad X_e = \frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right)$$

and with the parameter values in Figure 3.3.2 the larger equilibrium population is

$$X_e = \frac{10}{2} \left(1 + \sqrt{1 - \frac{4 \times 9}{10 \times 1 \times 10}} \right) = 9.$$

(b) Non-zero equilibrium values do not exist when $1 - 4h/(rK) < 0$. The largest value of h for which non-zero equilibrium values exist is $h_c = rK/4$.

(c) Factoring the right-hand side of the differential equation gives

$$X' = -\frac{r}{K} \left[X - \frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right) \right] \left[X - \frac{K}{2} \left(1 + \sqrt{1 - \frac{4h}{rK}} \right) \right].$$

Using this factored form, the behaviour of the function for different initial conditions can be determined and is shown in Table 2. If x_0 is lower than the critical value,

$$x_c = \frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right),$$

the population becomes extinct.

x_0	Sign of X'	Behaviour of X
$x_0 < \frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right)$	$X' < 0$	X tends to zero
$\frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right) < x_0 < \frac{K}{2} \left(1 + \sqrt{1 - \frac{4h}{rK}} \right)$	$X' > 0$	X tends to the larger steady state
$x_0 > \frac{K}{2} \left(1 + \sqrt{1 - \frac{4h}{rK}} \right)$	$X' < 0$	X tends to the larger steady state

Table 2: Behaviour of $X(t)$ for different initial conditions for Question 5.

3.6. Fishing with quotas.

(b) The population will become extinct when $X_e = K(1 - h/r) = 0$. This occurs at when $h = r$.

3.7. Predicting with a model.

The analytic solution is

$$X(t) = \frac{200}{1 + e^{-t/5}}.$$

The population reaches approximately 120 after 2 months.

3.8. Plant biomass.

(a)

$$\left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{plant biomass} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate of growth} \\ \text{due to} \\ \text{nutrient uptake} \end{array} \right\}$$

$$\frac{dx}{dt} = x(x_f - x), \quad x(0) = x_0.$$

(b) The analytic solution is

$$x(t) = \frac{x_0 x_f}{x_0 (1 - e^{-x_f t}) + x_f e^{-x_f t}}.$$

(c) With $x(t) = x_f$ we have, from part (b),

$$x_f = \frac{x_0 x_f}{x_0 (1 - e^{-x_f t}) + x_f e^{-x_f t}}.$$

Algebraic simplification leads us to the equation

$$e^{-x_f t} (x_f - x_0) = 0.$$

We are not interested in the case where $x_f = x_0$ because then $x' = 0$ for all time. However, $e^{-x_f t} \rightarrow 0$ as $t \rightarrow \infty$. The plant biomass cannot reach x_f in a finite time.

3.9. Modelling the population of a country.

(a)

$$\left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{population} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{deaths} \end{array} \right\} + \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{immigration} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{emigration} \end{array} \right\}$$

Let the population at time t be $x(t)$, the per-capita birth rate be β and the per-capita death rate be α . Let the immigration rate be i and the emigration rate be e . Then

$$\frac{dx}{dt} = (\beta - \alpha)x + i - e.$$

(b)

$$\left\{ \begin{array}{c} \text{rate of change} \\ \text{of population} \\ \text{in country } X \end{array} \right\} = \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{deaths} \end{array} \right\} + \left\{ \begin{array}{c} \text{net rate of} \\ \text{movement to} \\ \text{country } X \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{rate of change} \\ \text{of population} \\ \text{in country } Y \end{array} \right\} = \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{of} \\ \text{deaths} \end{array} \right\} + \left\{ \begin{array}{c} \text{net rate of} \\ \text{movement to} \\ \text{country } Y \end{array} \right\}$$

where the net movement into each country incorporates immigration to and emigration from the country.

Let $x(t)$ be the population in country X at time t and $y(t)$ be the population in the neighbouring country Y at time t . Let the per-capita birth rate be β and the per-capita death rate be α . Subscripts x and y denote the rates in countries X and Y , respectively. Let γ be the constant of proportionality associated with movement of people between the two countries. Then

$$\begin{aligned} \frac{dx}{dt} &= \beta_x x - \alpha_x x + \gamma(x - y), \\ \frac{dy}{dt} &= \beta_y y - \alpha_y y - \gamma(x - y). \end{aligned}$$

3.10. Sensitivity to initial conditions.

The solutions are almost indistinguishable for both $r = 1.9$ and $r = 2.5$; the populations move together rapidly. With $r = 2.8$, the solutions move apart as time progresses.

3.11. Investigating parameter change.

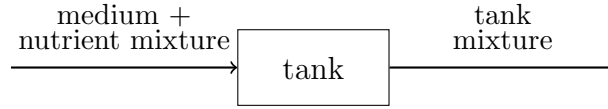
Similar behaviour to the logistic model is exhibited. With small r the population grows exponentially and then levels off as overcrowding becomes significant. With higher r values, the steady state is again reached, although damped oscillations are exhibited. With $r > 7$ and $r > 12$ we see 2-cycle and 4-cycle solutions, respectively. With $r = 15$ chaotic behaviour is observed. A 2-cycle appears when $r \approx 7$ and 4-cycles occur from $r \approx 12$ to $r \approx 13$.

3.12. Modelling insects with a difference equation.

In this system extinction occurs with $r \leq 1$ since only the zero steady state exists. For $r > 1$ there exists a stable non-zero steady state to which the solutions tend. For r near 3 solutions exhibit damped oscillations towards the non-zero steady state.

3.13. Aquatic environments.

(a) The compartment diagram is shown below.



(b) $S'(t) = (F/V) c_{in} - (F/V) S(t)$ where c_{in} is the concentration of nutrient in the inflowing mixture and V is the volume of the tank.

(c) $S'(t) = (F/V) c_{in} - (F/V) S(t) - p(S) x(t)$

(d) $x'(t) = \beta p(S) x(t) - (F/V) x(t)$

(e)

$$p'(S) = \frac{ma}{(a + S)^2} > 0 \quad \forall S \in \mathbb{R}.$$

The function is monotonically increasing and its maximum value is given by $\lim_{x \rightarrow \infty} p(S) = m$. Therefore, the function is bounded from above. The value of $p(a) = m/2$ is half this maximum, m , which is why it is called the half-saturation constant.

3.14. Stability of 2-cycles.

(a)

$$X_{n+1} = X_n + rX_n(1 - X_n)$$

and

$$X_{n+2} = X_{n+1} + rX_{n+1}(1 - X_{n+1}).$$

Substituting the first equation into the second and simplifying gives the required result.

(b) If $S = X_{n+2} = X_n$ then

$$X_{n+2} - X_n = (2r + r^2) S - (2r + 3r^2 + r^3) S^2 + (2r^2 + 2r^3) S^3 - r^3 S^4 = 0.$$

Since $S = 0$ and $S = 1$ satisfy the equation,

$$-rS(S - 1)(r^2 S^2 - (2r + r^2)S + 2 + r) = 0$$

The solutions to the quadratic equation $(r^2 S^2 - (2r + r^2)S + 2 + r) = 0$ are

$$S = \frac{(2 + r) \pm \sqrt{r^2 - 4}}{2r}.$$

3.15. Linear differential-delay equation.

Letting $X(t) = Ce^{mt}$ we obtain

$$mCe^{mt} = Ce^{m(t-1)}$$

and hence m satisfies the equation

$$m = e^{-m}.$$

We can solve this equation graphically to obtain $m \simeq 0.5671$. So $X(t) = Ce^{0.5671t}$. Applying the initial condition $X(0) = 1$ gives $C = 1$ so $X(t) = e^{0.5671t}$.

4 Chapter 4 Solutions

4.1. Round-off errors.

- (a) 0.011
(b) 0. The two answers are different.

4.2. Numerical schemes.

- (a) and (b) See Figure 3.

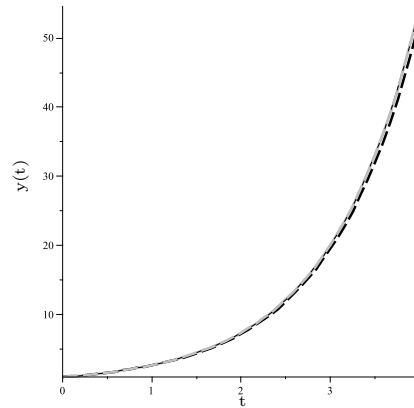


Figure 3: Analytical solution (black solid line) and solutions computed using a forward Euler method (black dashed line), Huen's method (grey solid line) and RK4 (grey dashed line) for the system in Question 2. All numerical solutions use a stepsize of 0.1.

- (c) See Figure 4.

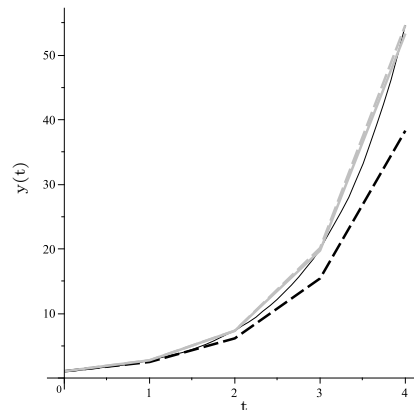


Figure 4: Analytical solution (black solid line) and solutions computed using a forward Euler method (black dashed line), Huen's method (grey solid line) and RK4 (grey dashed line) for the system in Question 2. All numerical solutions use a stepsize of 0.5.

4.3. Numerical and analytical solutions.

See Figures 5 and 6.

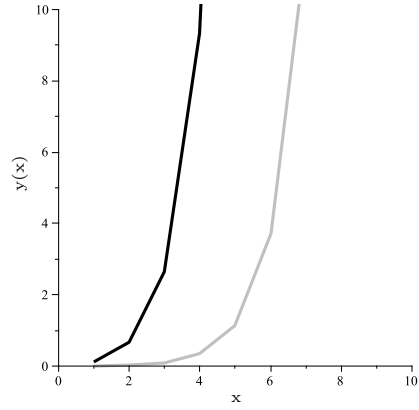


Figure 5: The difference between the solutions obtained for the system in Question 3 by different numerical methods. The black line shows the difference between the forward Euler method and RK4 solutions and the grey line shows the difference between the Huen's method and RK4 solutions. All numerical solutions use a stepsize of 0.1.

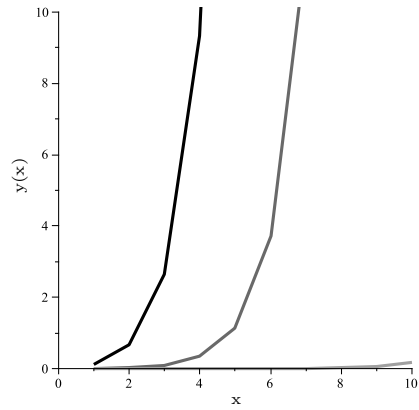


Figure 6: The difference between the analytical solution and the solution obtained using the forward Euler method (black line), Heun's method (dark grey line) and RK4 (light grey line) for the system in Question 3. All numerical solutions use a stepsize of 0.1.

4.4. Discretisation and round-off errors.

When Digits is set to 20, the error decreases as the stepsize increases. With Digits set to 10, however, it appears that the error is larger when a stepsize of 0.005 is used than when a stepsize of 0.01 is used.

5 Chapter 5 Solutions

5.1. Numerical solution of basic epidemic model.

- (a) **Correction:** In the question b should be β and r should be γ .
- (b) Approximately 50.
- (c) Only two susceptibles never become infected.
- (d) The maximum number of infectives appears to double, as expected. The number of susceptibles who never become infected falls to approximately 2.

5.2. Contagious for life.

(a)

$$\begin{aligned}\left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{susceptibles} \end{array} \right\} &= - \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{infectives} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\}\end{aligned}$$

Let $S(t)$ and $I(t)$ be the number of susceptibles and infectives at time t , respectively, and let β be the transmission coefficient. Then

$$S' = -\beta SI, \quad I' = \beta SI.$$

(b) Eventually the entire population is infected.

5.3. Disease with no immunity.

The word-equations are

$$\begin{aligned}\left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{susceptibles} \end{array} \right\} &= - \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} + \left\{ \begin{array}{c} \text{rate infectives} \\ \text{recover} \end{array} \right\}, \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{infectives} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate infectives} \\ \text{recover} \end{array} \right\}.\end{aligned}$$

Hence

$$S' = -\beta SI + \gamma I, \quad I' = \beta SI - \gamma I.$$

5.4. Continuous vaccination.

$$\begin{aligned}\left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{susceptibles} \end{array} \right\} &= - \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{vaccinated} \end{array} \right\}, \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{infectives} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{infectives} \\ \text{removed} \end{array} \right\}\end{aligned}$$

Let v be the vaccination rate. Then

$$S' = -\beta SI - vS, \quad I' = \beta SI - \gamma I.$$

5.5. SEIR model, disease with latent period.

$$\begin{aligned}\left\{ \begin{array}{l} \text{rate of} \\ \text{change in no.} \\ \text{susceptibles} \end{array} \right\} &= - \left\{ \begin{array}{l} \text{rate} \\ \text{susceptibles} \\ \text{exposed} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of} \\ \text{change in no.} \\ \text{exposed} \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate} \\ \text{susceptibles} \\ \text{exposed} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate exposed} \\ \text{become} \\ \text{infected} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of} \\ \text{change in no.} \\ \text{infectives} \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate exposed} \\ \text{become} \\ \text{infected} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate} \\ \text{infectives} \\ \text{removed} \end{array} \right\}\end{aligned}$$

$$S' = -\beta SI, \quad E' = \beta SI - \sigma E, \quad I' = \sigma E - \gamma I,$$

where β is the transmission coefficient, σ^{-1} is the latent period and γ^{-1} is the infectious period.

5.6. Two prey and one predator.

$$\begin{aligned}\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{prey } X \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate of} \\ \text{prey } X \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate prey} \\ X \text{ killed} \\ \text{by predators} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{prey } Y \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate of} \\ \text{prey } Y \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate prey} \\ Y \text{ killed} \\ \text{by predators} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{predators} \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate of} \\ \text{predator} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of} \\ \text{predator} \\ \text{deaths} \end{array} \right\}\end{aligned}$$

Let $X(t)$ and $Y(t)$ be the two prey species and $Z(t)$ be the predator species. We assume that the populations are sufficiently large to neglect random differences between individuals, that only the two prey species and one predator species affect the ecosystem, that the prey populations grow exponentially in the absence of a predator and that the prey do not compete with each other. Then

$$X' = b_1 X - c_1 XZ \quad Y' = b_2 Y - c_2 YZ, \quad Z' = d_1 XZ + d_2 YZ - a_3 Z,$$

where b_1 and b_2 are the per-capita birth rates of the prey species, a_3 is the per-capita death rate of the predators and c_1 , c_2 , d_1 and d_2 are positive constants representing the interaction terms.

5.7. Effect of DDT.

(a) If there are initially more predators than prey, the mean densities of predators and prey are higher than if we initially have more prey than predators.

(d) The presence of DDT decreases the mean predator density but increases the mean prey density. Furthermore, the equilibrium predator population decreases but the equilibrium prey population increases.

5.8. Predator-prey with density dependence.

(c) The populations oscillate around the equilibrium populations and settle down to these values over time.

5.9. Competing species with no density dependence.

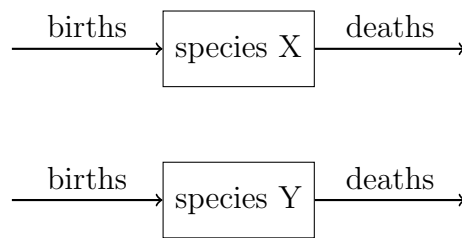
(c) It is not easy to determine which species survives and which becomes extinct.

5.10. Competing species with density dependence.

(a) $K_1 = 1.5$, $K_2 = 1.2$.

(b) $X \rightarrow 0.5$, $Y \rightarrow 1$.

5.11. Symbiosis.



Let $X(t)$ and $Y(t)$ be the density of species X and Y , respectively, at time t . Then

$$X' = c_1XY - \alpha_1X, \quad Y' = c_2XY - \alpha_2Y,$$

where α_1 and α_2 are per-capita death rates and c_1 and c_2 are interaction coefficients.

5.12. Simple age-based model.

$$\begin{aligned} \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{juveniles} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate of} \\ \text{juvenile} \\ \text{births} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate at which} \\ \text{juveniles} \\ \text{reach maturity} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of} \\ \text{juvenile} \\ \text{deaths} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{adults} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate at which} \\ \text{juveniles} \\ \text{reach maturity} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of} \\ \text{adult} \\ \text{deaths} \end{array} \right\} \end{aligned}$$

$$J' = \beta_1A - \gamma J - \alpha_1J, \quad A' = \gamma J - \alpha_2A,$$

where β_1 is the per-capita birth rate of juveniles, α_1 and α_2 are the per-capita death rates of juveniles and adults, respectively, and γ is the rate at which juveniles mature into adults.

5.13. Beetle population model.

There are three compartments, as shown in figure 7. There is a flow from larvae to pupae to adult as each individual ages through the three stages of growth.

Expressed in words the equations describing the inputs and outputs are

$$\begin{aligned} \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{larvae} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate} \\ \text{larvae} \\ \text{born} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{larvae} \\ \text{die} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate larvae} \\ \text{age to} \\ \text{pupae} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{pupae} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate larvae} \\ \text{age to} \\ \text{pupae} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{pupae} \\ \text{die} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate pupae} \\ \text{age to} \\ \text{adults} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{adults} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate pupae} \\ \text{age to} \\ \text{adults} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{adults} \\ \text{die} \end{array} \right\}. \end{aligned}$$

We obtain the differential equations

$$\begin{aligned} \frac{dL}{dt} &= b_1 A - d_1 L - \sigma_1 L, \\ \frac{dP}{dt} &= \sigma_1 L - d_2 P - \sigma_2 P, \\ \frac{dA}{dt} &= \sigma_2 P - d_2 A. \end{aligned}$$

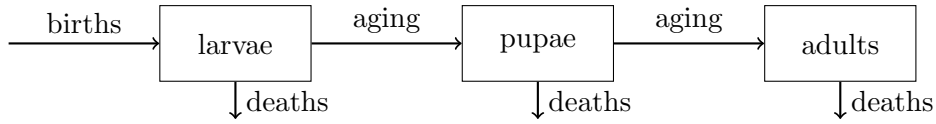


Figure 7: Compartment diagram for Question 13 with larvae, pupae and adult beetles.

5.14. Wine fermentation.

Let the density of yeast cells, the amount of alcohol and the amount of sugar at time t be $Y(t)$, $A(t)$ and $S(t)$, respectively. Then

$$Y' = \beta_1 SY - \alpha_1 AY, \quad A' = \beta_2 YA, \quad S' = -\alpha_3 YS,$$

where $\beta_1 S(t)$ is the per-capita birth rate of yeast, $\beta_2 Y(t)$ is the production rate of alcohol, $\alpha_1 A(t)$ is the per-capita death rate of yeast and $\alpha_3 Y(t)$ is the consumption rate of the sugar.

5.15. Cycles of measles epidemics.

$$\begin{aligned} \left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{susceptibles} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate of} \\ \text{births of} \\ \text{susceptibles} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change in no.} \\ \text{infectives} \end{array} \right\} &= \left\{ \begin{array}{c} \text{rate} \\ \text{susceptibles} \\ \text{infected} \end{array} \right\} - \left\{ \begin{array}{c} \text{rate} \\ \text{infectives} \\ \text{removed} \end{array} \right\} \end{aligned}$$

$$S' = aS - \beta SI, \quad I' = \beta SI - \gamma I,$$

where a is the per-capita birth rate of susceptibles.

5.16. Density dependent contact rate.

With $\epsilon = 0.5$, the number of infectives is at a maximum after approximately 8 days. As ϵ increases, the maximum number of infectives is higher and is reached after only 7 days. Conversely, with $\epsilon = 0$, the maximum number of infectives is lower and is reached after 9 days.

As ϵ decreases from 1 to 0, the spread of infection among the susceptibles takes longer to occur and the number of susceptibles who never become infected increases.

5.17. Battle loss due to disease.

$$R' = -a_1 B - \alpha R, \quad B' = -a_2 R,$$

where α is the per-capita rate of deaths from disease.

5.18. Jungle warfare.

(a)

$$\begin{aligned} \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{red soldiers} \end{array} \right\} &= - \left\{ \begin{array}{c} \text{rate red soldiers} \\ \text{wounded by} \\ \text{blue army} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{blue soldiers} \end{array} \right\} &= - \left\{ \begin{array}{c} \text{rate blue soldiers} \\ \text{wounded by} \\ \text{red army} \end{array} \right\} \end{aligned}$$

(b)

$$R' = -a_1 B, \quad B' = -a_2 RB.$$

(c)

$$R' = -a_1 B + r_1, \quad B' = -a_2 RB + r_2,$$

where r_1 and r_2 are the rates of reinforcements of the red and blue armies, respectively.

5.19. Jungle warfare.

(a) For a bullet fired at random into the area occupied by the hidden blue army, the probability of that bullet striking a red soldier is the ratio of the total area exposed by $B(t)$ soldiers to the total area occupied, $p_b = A_b \times B(t)/A$.

(b) Hence we can now estimate the mortality rates. For the blue army

$$\begin{aligned} \left\{ \begin{array}{c} \text{rate blue soldiers} \\ \text{wounded by} \\ \text{red army} \end{array} \right\} &= \left\{ \begin{array}{c} \text{number} \\ \text{bullets fired} \\ \text{by red soldiers} \end{array} \right\} \times \left\{ \begin{array}{c} \text{probability} \\ \text{each bullet} \\ \text{strikes a blue soldier} \end{array} \right\} = f_r \times R(t) \times \frac{A_b B(t)}{A} \\ &= \frac{f_r A_b}{A} RB. \end{aligned}$$

For the red army soldiers

$$\left\{ \begin{array}{c} \text{rate red soldiers} \\ \text{wounded by} \\ \text{blue army} \end{array} \right\} = f_b \times R(t) \times p_r = f_b p_r R.$$

where $p_r = 1/100$.

(c) The differential equations are

$$B' = -a_1 RB, \quad R' = -a_2 R$$

where $a_2 = f_r p_r = 400 \times 1/100 = 4$ and $a_1 = f_b A_r / A = 400 \times 0.1/10^5 = 4 \times 10^{-4}$.

Using MATLAB, solving this system numerically. we find that after approximately 5 days, (rounding answers to closest integer) that the red army has effectively no soldiers left whereas the blue army still has 150 soldiers.

5.20. Exact solution for battle model.

(a) The relevant equation is

$$\frac{d^2 R}{dt^2} = a_1 a_2 R.$$

(b) Assume $R(t) = C e^{\lambda t}$. Substitution into the differential equation in (a) gives

$$C (\lambda^2 - a_1 a_2) e^{\lambda t} = 0$$

which implies that $\lambda = \pm (a_1 a_2)^{1/2}$. Thus

$$R(t) = c_1 e^{-\alpha t} + c_2 e^{\alpha t} = c_3 \cosh(\alpha t) + c_4 \sinh(\alpha t),$$

where $\alpha = \sqrt{a_1 a_2}$. Then,

$$B(t) = -\frac{1}{a_1} \alpha (c_3 \sinh(\alpha t) + c_4 \cosh(\alpha t)).$$

Solving for c_3 and c_4 , we find that

$$R(t) = r_0 \cosh(\alpha t) - b_0 \nu \sinh(\alpha t),$$

and

$$B(t) = b_0 \cosh(\alpha t) - r_0 \nu^{-1} \sinh(\alpha t),$$

where $\nu = \sqrt{a_1/a_2}$.

5.21. Spread of malaria by mosquitoes.

There are four compartments, susceptible mosquitoes and humans and infected mosquitoes and humans. A compartment diagram is shown in figure 8. Note that there is no recovery for either humans or mosquitoes. In this model we have also ignored any births and deaths of mosquitoes or humans.

Infections can only occur by a mosquito biting a human. The rate that mosquitoes get infected is proportional to both the number of susceptible mosquitoes and the number of infected humans, $\beta_1 S_m I_h$. The rate of infection of humans is proportional to both the number of susceptible humans and the number of infected mosquitoes, $\beta_2 S_h I_m$. Note the transmission coefficients β_1 for human to mosquito infection and β_2 for mosquito to human infection are generally different numbers since the rates the infection becoming successful

from a single bite may be different for humans than mosquitoes. Hence we obtain the differential equations

$$\begin{aligned}\frac{dS_h}{dt} &= -\beta_1 S_h I_m, \\ \frac{dS_m}{dt} &= -\beta_2 S_m I_h, \\ \frac{dI_h}{dt} &= \beta_1 S_h I_m, \\ \frac{dI_m}{dt} &= \beta_2 S_m I_h.\end{aligned}$$

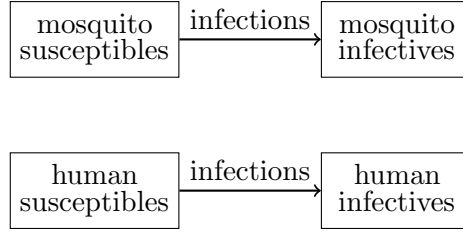


Figure 8: Compartment diagram for Question 21.

5.22. Spread of a religion. In the simplest model, unconverted become converted and converted become missionaries. A compartment diagram is shown below.



Since conversion of unbelievers to converts is due to contacts with missionaries we assume that the rate of conversion is bMU , which is independent of the current number of converted. The system of differential equations is

$$U' = -bMU, \quad C' = bMU - aC, \quad M' = aC.$$

Defining $N(t) = U(t) + C(t) + M(t)$ as the total population, we see that

$$\frac{dN}{dt} = \frac{dU}{dt} + \frac{dC}{dt} + \frac{dM}{dt} = -bMU + bMU - aC + aC = 0$$

so the total population is constant.

5.23. Predator-prey with protection of young prey.

The equations are

$$X_1' = bX_2 - aX_1 - cX_1, \quad X_2' = cX_1 - dX_2 - eX_2Y, \quad Y' = -fY + gX_2Y.$$

where b and a are the birth and death rates of juveniles, respectively and c is the rate at which juveniles become adults. The natural death rate of adult prey is d and the rate of death by predation is eY . The per-capita birth and death rates of predators are gX_2 and f , respectively.

5.24. Diseases with carriers.

$$\begin{aligned}\frac{dS}{dt} &= -(\beta SI + \epsilon\beta SC), \\ \frac{dI}{dt} &= (\beta SI + \epsilon\beta SC) - \gamma I \\ \frac{dC}{dt} &= q\gamma I \\ \frac{dR}{dt} &= (1 - q)\gamma I.\end{aligned}$$

(c) Typhoid is a classic example. There are others.

5.25. Basic reproduction number.

(a) R_0 would be infinite because $\gamma = 0$.

(b) For the SEIR model R_0 is the same as the SIR model, $R_0 = \beta s_0 \times 1/\gamma$ because the rate of new infections, in a completely susceptible population, is still $\beta s_0 I$ and the time someone is infectious for (can actually transmit the disease) is still γ^{-1} .

Note: the SEIR model is

$$S' = -\beta SI - \sigma S, \quad E' = \beta SI + \sigma S, \quad I' = \sigma S - \gamma I.$$

(c) The revised SEIR model model, including natural deaths, is now

$$S' = -\beta SI - \sigma S - aS, \quad E' = \beta SI + \sigma S - aE, \quad I' = \sigma S - \gamma I - aI$$

where a is the per-capita death rate from natural causes.

(c) Now, the mean residence time in the I compartment is $(\gamma + d)^{-1}$, so

$$R_0 = \frac{\beta s_0}{\gamma + d}.$$

The value of R_0 is smaller than for the SIR model because this model accounts for some people dying of natural causes before they recover for the disease.

5.26. Farmers, bandits and soldiers.

(b) The period between the cycles increases.

(c) The effect of increasing c could be due to better training of soldiers that causes each individual soldier to become more efficient at killing bandits. However, because the rate $cBR/(d + B)$ saturates for large B , once the bandit population grows sufficiently large, then the efficiency of the soldiers at killing bandits will be limited and a period of anarchy will still occur. However, it will take longer for it to happen.

6 Chapter 6 Solutions

6.1. Simple example.

$$(x, y) = (0, 0)$$

6.2. Finding equilibrium points.

- (a) $(X, Y) = (0, 0), (1, 3/2)$
- (b) $(X, Y) = (0, 0), (1, 2)$
- (c) $(X, Y) = (X, 0), (1/2, 1/2)$

6.3. Using the chain rule.

$$Y(X) = 2 \ln(X) + K, \quad K = y_0 - 2 \ln(x_0), \quad \text{where } Y(x_0) = y_0.$$

6.4. Contagious for life.

- (a) $I = N - S$ where N is the total population, i.e., $N = S + I$.
- (b) The only nullclines in this system are $S = 0$ and $I = 0$. With $S, I > 0$, $S' < 0$ and $I' > 0$. A phase-plane trajectory is shown in Figure 9.

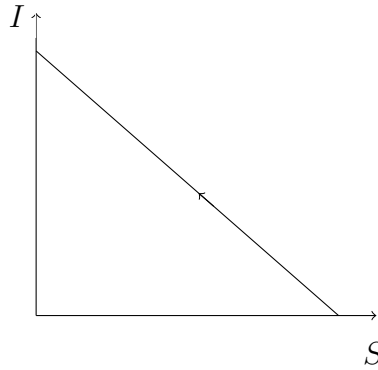


Figure 9: A typical trajectory direction for Question 4.

- (c) Yes.

6.5. Disease with reinfection.

- (a) $I = N - S$ where $N = S + I$.
- (b) The nullclines are $I = 0$ and $S = \gamma/b$. With $S < \gamma/b$, $S' > 0$ and $I' < 0$. Conversely, with $S > \gamma/b$, $S' < 0$ and $I' > 0$. These directions are shown in Figure 10.
- (c) It is not possible to cross the nullcline since, as we approach $S = \gamma/b$, $S' \rightarrow 0$ and $I' \rightarrow 0$. Therefore, if we initially have a small number of susceptibles, the number of infectives decreases to $N - \gamma/b$ while the number of susceptibles increases to γ/b . On the other hand, if the initial number of susceptibles is large, the number of infectives increases to $N - \gamma/b$ and the number of susceptibles decreases to γ/b .

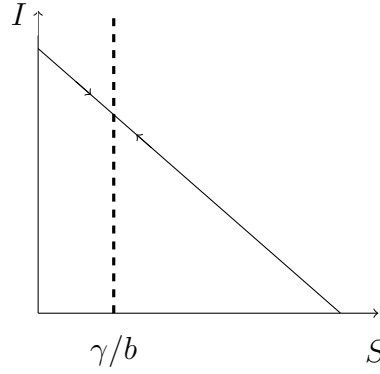


Figure 10: A typical trajectory direction for Question 5.

6.6. Predator-prey with density dependent growth of prey.

(a)

$$(X, Y) = (0, 0), (K, 0), \left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1} \left(1 - \frac{\alpha_2}{c_2 K} \right) \right).$$

(b) The solutions appear to be spirals which tend to the steady state

$$(X, Y) = \left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1} \left(1 - \frac{\alpha_2}{c_2 K} \right) \right).$$

6.7. Predator-prey with DDT.

(a) $(X, Y) = (0, 0), ((\alpha_2 + p_2)/c_2, (\beta_1 - p_1)/c_1)$

(b) The predator population is lower but the prey population is higher. This is in accordance with the results of Figure 5.3.3 in Section 5.3.

(c) If p_1 and p_2 increase, the predator fraction of the total average prey population decreases.

6.8. Predator-prey with density dependence and DDT.

$$(X, Y) = (0, 0), \left(K \left(1 - \frac{p_1}{\beta_1} \right), 0 \right), \left(\frac{\alpha_2 + p_2}{c_2}, \frac{\beta_1}{c_1} \left(1 - \frac{\alpha_2 + p_2}{K c_2} \right) - \frac{p_1}{c_1} \right)$$

6.9. One prey and two predators.

(a)

$$(X, Y, Z) = (0, 0, 0), \left(\frac{b_3}{a_3}, 0, \frac{a_1}{c_1} \right), \left(\frac{b_2}{a_2}, \frac{a_1}{b_1}, 0 \right)$$

It is not possible for three populations to coexist in equilibrium.

(b) Introducing an extra predator causes either this new predator or the existing predator to become extinct.

6.10. Competing species without density dependence.

The equilibrium solutions are $(X, Y) = (0, 0), (\beta_2/c_2, \beta_1/c_1)$. The nullclines are $X = 0$, $X = \beta_2/c_2$, $Y = 0$ and $Y = \beta_1/c_1$. The phase-plane diagram is shown in Figure 11

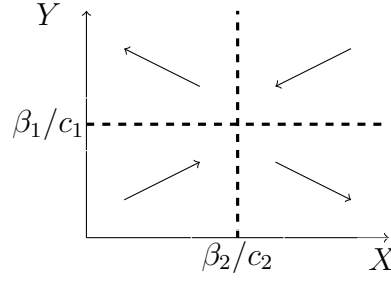


Figure 11: Regions in the phase-plane illustrating the trajectory directions for Question 10. The dashed lines are the non-zero nullclines.

6.11. Rabbits and foxes.

(a)

$$Y = -\frac{b}{a}X + \frac{c}{a}\ln(X) + K,$$

where, if the initial rabbit population is x_0 and the initial population of foxes is y_0 ,

$$K = y_0 + \frac{b}{a}x_0 - \frac{c}{a}\ln(x_0).$$

(b) The nullclines of this system are at $X = 0$, $Y = 0$ and $X = c/b$. With $X < c/b$, $Y' < 0$ and with $X > c/b$, $Y' > 0$. Also, $X' < 0$ for all $X, Y > 0$. A typical phase-plane trajectory is shown in Figure 12.

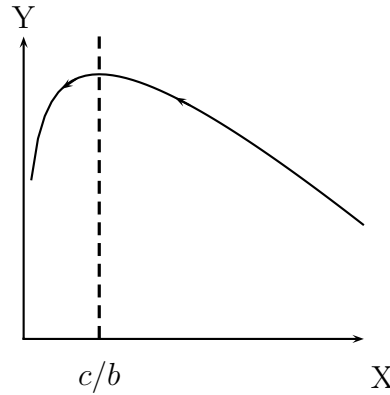


Figure 12: A typical trajectory for Question 11. The dashed line is the non-zero nullcline.

(c) Yes. From Figure 12 it is clear that intersection with the Y axis is possible. This corresponds to the rabbits completely dying out.

6.12. Microorganisms and toxins.

(a)

$$P(T) = \frac{r}{q}T - \frac{\gamma}{2q}T^2 + p_0.$$

(b) The nullclines are at $P = 0$ and $T = r/\gamma$. With $T < r/\gamma$, $P' > 0$ and with $T > r/\gamma$, $P' < 0$. For all $P > 0$, $T' > 0$. A possible phase-plane trajectory is shown in figure 13. The amount of toxin always increases.

The number of microorganisms initially increases but then decreases as more toxin is produced.

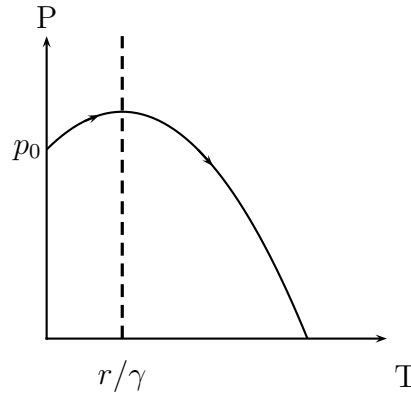


Figure 13: Regions in the phase-plane illustrating the trajectory directions for Question 12. The dashed line is the non-zero nullcline.

6.13. Fatal disease.

(a) After substitution of $S = N - I$ into the equations we obtain

$$\frac{dN}{dt} - \frac{dI}{dt} = -\lambda I \left(1 - \frac{I}{N}\right), \quad \frac{dI}{dt} = \lambda I \left(1 - \frac{I}{N}\right) - \alpha I.$$

Substitution of the second equation into the first and simplifying gives the required result.

(c) If we assume that, initially, $\lambda/N = b$, the maximum number of infectives is higher than in the basic epidemic model. This is what we would expect intuitively. As infectives die, the population size decreases and λ/N increases. This higher transmission rate causes more susceptibles to become infected.

6.14. Battle model with desertion.

(a)

$$R^2(B) = \frac{a_1}{a_2}B^2 + \frac{2c}{a_2}B + K, \quad K = r_0^2 - \frac{a_1}{a_2}b_0^2 - \frac{2c}{a_2}b_0.$$

(b) Since the number of soldiers can only decrease, the direction of the trajectories is as shown in Figure 14

6.15. Jungle warfare.

(a)

$$R = \frac{c_1}{2a_2}B^2 + K, \quad K = r_0 - \frac{c_1}{2a_2}b_0^2.$$

Since both R and B can only decrease, possible trajectories are shown in Figure 15.

(b) There will be 500 red soldiers left.

(c) The red army must be hidden because the number of wounded red soldiers depends on both the number of red soldiers and the number of blue soldiers. This is consistent with the wounding that occurs when the enemy is forced to use random fire.

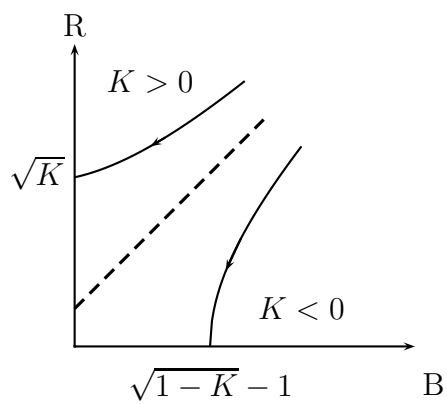


Figure 14: Regions in the phase-plane illustrating the trajectory directions for Question 14.

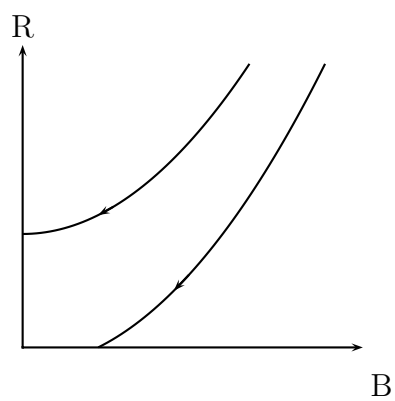


Figure 15: Regions in the phase-plane illustrating the trajectory directions for Question 15.

6.16. Battle with long range weapons.

(a)

$$R = \frac{c_1}{c_2}B + K, \quad K = r_0 - \frac{c_1}{c_2}b_0.$$

(b) The numbers of red and blue soldiers can only decrease so the direction of trajectories is as shown in Figure 16.

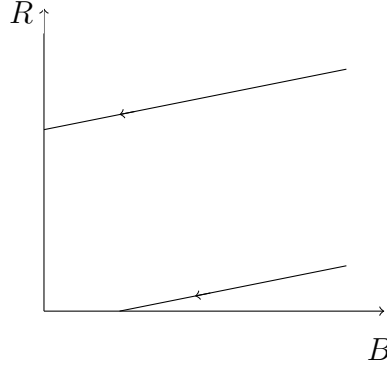


Figure 16: Regions in the phase-plane illustrating the trajectory directions for Question 16.

(c) Referring to Question 19 in Chapter 5, we see that

$$c_1 = \frac{f_b A_r}{A}$$

where f_b is the firing rate of a blue soldier and A_r is the exposed area of a red soldier.

6.17. SIR model, estimating the transmission coefficient. Using the equation for the phase-plane trajectories for the standard SIR model satisfy

$$I = -S + \frac{\gamma}{\beta} \log(S) + K.$$

Let $S = s_0$ and $I = i_0$ initially, which gives

$$I = -S + (s_0 + i_0) + \frac{\gamma}{\beta} \log(S/s_0).$$

At the end of the epidemic, suppose there are s_f susceptibles left and no infectives. Then, we can deduce

$$\frac{\beta n_0}{\gamma} = \frac{\log\left(\frac{s_0/n_0}{s_f/n_0}\right)}{1 - s_f/n_0}.$$

where $n_0 = s_0 + i_0$ is the total population (constant). From the data given, $s_f/n_0 = 0.513$ and $s_0/n_0 = 0.911$. Given $\gamma^{-1} = 1/3$ then we estimate

$$\frac{\beta n_0}{\gamma} = 1.18, \quad \beta N = 0.39, \quad R_0 = \frac{\beta s_0}{\gamma} = \frac{\beta n_0}{\gamma} \frac{s_0}{n_0} = 1.07.$$

6.18. Bacteria in gut.

Easiest to use Maple to find the equilibrium points.

```
eq1 := beta[x]*X +(Xin - X)*F/V - alpha[x]*X;
eq2 := beta[y]+(Yin-Y)*F/V - c[2]*X*Y;
solve([eq1, eq2], [X,Y]);
```

This gives

$$X = \frac{(F/V)X_{\text{in}}}{\beta_x + \alpha_x + F/V}$$

and

$$Y = \frac{-\beta_x\beta_y + \beta_y F/V + \beta_y\alpha_x - Y_{\text{in}}(\beta_x F/V + (F/V)^2 + \alpha_x F/V)}{F/V(F/V - \beta_x + \alpha_x + c_2 X_{\text{in}})}$$

6.19. .

Correction: last equation should be $\frac{dR}{dt} = \gamma I - aR$.

(a) It is easiest to use Maple to find all the equilibrium points. This gives

$$(S, I) = (N, 0), \quad (S, I) = \left(\frac{\gamma + a}{\beta}, \frac{a(\beta N - (\gamma + a))}{\beta(\gamma + a)} \right).$$

The corresponding values for R are $R = 0$ and $R = \gamma I/a$.

(b) The endemic equilibrium ($I \neq 0$) can be written as

$$(S, I) = \left(\frac{N}{R_0}, \frac{a}{\beta}(R_0 - 1) \right)$$

When $R_0 < 1$ the value of I becomes negative, which is unphysical.

This equilibrium is never realised when $R_0 < 1$, instead the solution tends to the disease-free equilibrium. (But when $R_0 > 1$ the opposite occurs.)

7 Chapter 7 Solutions

7.1. Constructing a phase-plane.

- (a) $(x, y) = (0, 0)$ is an unstable focus.
(c) See Figure 17

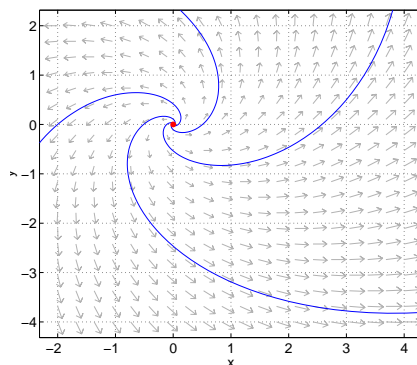


Figure 17: Phase-plane for the system in Question 1.

7.2. Defining the trajectories.

- (a) $(x, y) = (0, 0)$ is a centre.
(b) Implicit differentiation of $x^2 - 2xy + 5y^2 = K$ with respect to t and substitution of $y' = x - y$ gives $x' = x - 5y$. A similar procedure can be used to obtain $y' = x - y$.

7.3. Linearisation.

- (a) $(x, y) = (-1, -1), (1, 1)$.
(b) At an equilibrium point $(x, y) = (x_e, y_e)$ we have the linearised system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -y_e & -x_e \end{bmatrix} \begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix}.$$

The point $(-1, -1)$ is an unstable focus and $(1, 1)$ is a saddle point.

- (d) See Figure 18

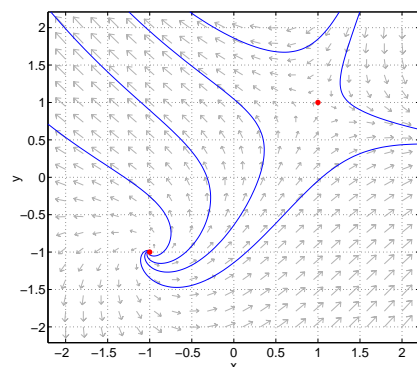


Figure 18: Phase-plane for the system in Question 3.

7.4. Linearisation.

(a) At an equilibrium point $(x, y) = (x_e, y_e)$ the linearised system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -y_e & -x_e \end{bmatrix} \begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix}.$$

(b) $(x, y) = (-4, -1)$ is an unstable node and $(x, y) = (1, 4)$ is a saddle point. See Figure 19 for a typical phase-plane diagram.

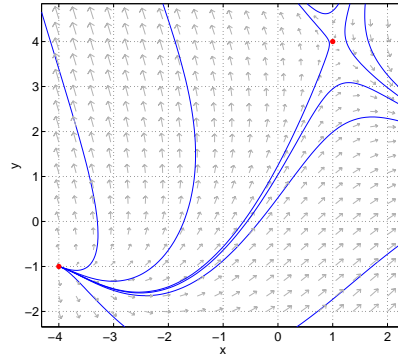


Figure 19: Phase-plane for the system in Question 4.

(c)

$$J = \begin{bmatrix} q & -q \\ -y & -x \end{bmatrix}.$$

The two eigenvalues of J must have different signs at one or more equilibrium points for a saddle point to exist. The determinant of J , evaluated at the equilibrium point, is the product of the eigenvalues and so we must show that the determinant is negative. Now $\det(J) = -q(x_e + y_e)$, where (x_e, y_e) is the equilibrium point. We must have, therefore that $q > 0$ and $x_e + y_e > 0$ or $q < 0$ and $x_e + y_e < 0$.

The two equilibrium points are

$$(x_e, y_e) = \left(-\frac{3}{2q} \pm \frac{1}{2} \sqrt{\left(\frac{3}{q}\right)^2 + 16}, \frac{3}{2q} \pm \frac{1}{2} \sqrt{\left(\frac{3}{q}\right)^2 + 16} \right)$$

and so

$$x_e + y_e = \pm \sqrt{\left(\frac{3}{q}\right)^2 + 16}.$$

Regardless of the value of q the determinant will be negative at one of the equilibrium points and so one of the equilibrium points will be a saddle point.

7.5. Linearisation of competition model.

(a) At an equilibrium point $(X, Y) = (x_e, y_e)$, the linearised system is

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \beta_1 - 2d_1x_e - c_1y_e & -c_1x_e \\ -c_2y_e & \beta_2 - 2d_2y_e - c_2x_e \end{bmatrix} \begin{bmatrix} X - x_e \\ Y - y_e \end{bmatrix}.$$

(b) The eigenvalues of the Jacobian at each steady state are shown in Table 3. $(0, 0)$ is an unstable node, $(0, 1.25)$ and $(1.071, 0)$ are saddle points and $(1.111, 0.278)$ is a stable node. The species will coexist. This is verified by the phase-plane in Figure 20.

Equilibrium point	λ_1	λ_2
$(0, 0)$	3	2.5
$(0, 1.25)$	0.5	-2.5
$(1.071, 0)$	-3	1.429
$(1.111, 0.278)$	-0.433	-2.567

Table 3: Eigenvalues of the system in Question 5 with $\beta_1 = 3$, $\beta_2 = 2.5$, $d_1 = 2.8$, $d_2 = 2$, $c_1 = 2$ and $c_2 = 1$.

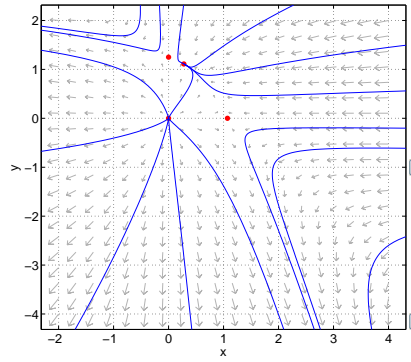


Figure 20: Phase-plane for the system in Question 5 with $\beta_1 = 3$, $\beta_2 = 2.5$, $d_1 = 2.8$, $d_2 = 2$, $c_1 = 2$ and $c_2 = 1$.

(c) It can be shown, by examining the eigenvalues of the general system, that the steady state $(0, \beta_2/d_2)$ will be a stable node provided

$$\frac{\beta_2}{\beta_1} < 1 + \frac{c_1}{d_1}.$$

This requirement is met if $\beta_1 = 2.5$ and $\beta_2 = 3$ with the other parameter values unchanged. From Table 4, we see that the only stable steady state is $(0, \beta_2/d_2)$ and so X will die out. Note that the steady state representing coexistence is not feasible.

Equilibrium point	λ_1	λ_2
$(0, 0)$	2.5	3
$(0, 1.5)$	-3	-0.5
$(0.893, 0)$	-2.5	2.107

Table 4: Eigenvalues of the system in Question 5 for $\beta_1 = 2.5$, $\beta_2 = 3$, $d_1 = 2.8$, $d_2 = 2$, $c_1 = 2$ and $c_2 = 1$.

7.6. Comparison of nonlinear and linearised models.

(a) The steady states are

$$(0, 0), \quad \left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1} \right)$$

and, at an equilibrium point $(X, Y) = (x_e, y_e)$, the linearised system is

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \beta_1 - c_1 y_e & -c_1 x_e \\ c_2 y_e & -\alpha_2 + c_2 x_e \end{bmatrix} \begin{bmatrix} X - x_e \\ Y - y_e \end{bmatrix}.$$

Time-dependent plots for the nonlinear and linearised systems are shown in Figure 21. For initial values close to the non-zero equilibrium solution of $(100, 130)$, the solution obtained using the linearised system is similar to the full solution.

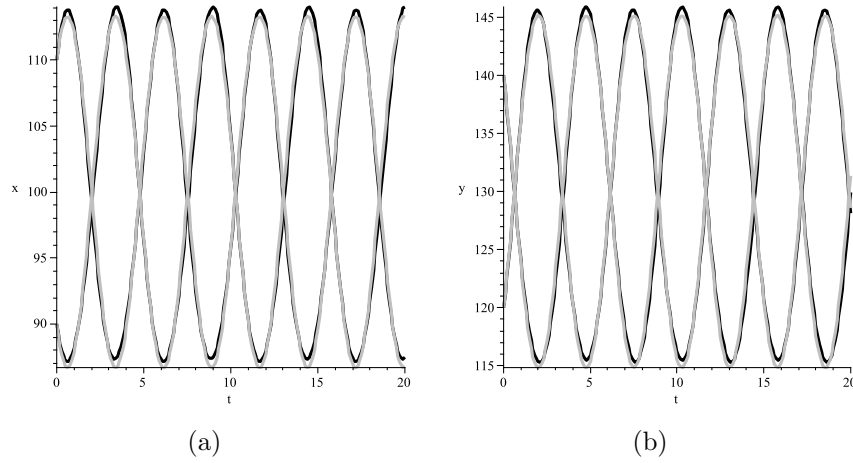


Figure 21: Prey (a) and predator (b) populations over time computed using the linearised system (grey line) and the full system (black line) for Question 6. The initial prey and predator populations are $(110, 120)$ and $(90, 140)$.

(b) A number of trajectories in the phase plane are compared in Figure 22.

7.7. Linearisation of an epidemic model.

(a) **Correction:** In the question there are both r and b should be γ and β .

(b) At the equilibrium point $(S, I) = (\gamma/\beta, \mu/\gamma)$, the linearised system is

$$\begin{bmatrix} S' \\ I' \end{bmatrix} = \begin{bmatrix} -\beta\mu/\gamma & -\gamma \\ \beta\mu/\gamma & 0 \end{bmatrix} \begin{bmatrix} S - s_e \\ I - i_e \end{bmatrix}.$$

(b) $p = -\beta\mu/\gamma$ and $\Delta = (\beta\mu/\gamma)^2 - 4\beta\mu$. Since $p < 0$, the equilibrium point is stable. If $\gamma^2 < \beta\mu/4$ the equilibrium point is a stable node and if $\gamma^2 > \beta\mu/4$ it is a stable spiral.

(c) If the removal rate is lower than $\sqrt{\beta\mu/4}$ solution will approach the steady state in a smooth manner. If the removal rate is higher than this critical value, the solution will oscillate, although it will still tend to the equilibrium point.

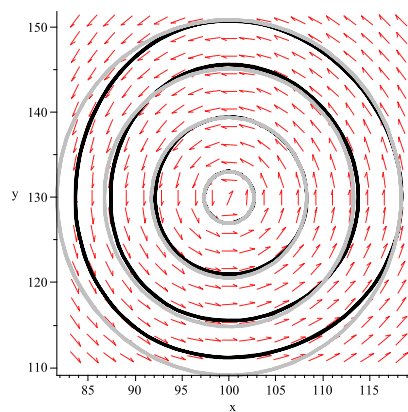


Figure 22: Phase portrait for Question 6 using the nonlinear (black) and linearised (grey) systems.

7.8. Linearisation of jungle warfare model.

The only equilibrium point is at $(R, B) = (0, b)$. The linearised system at this point is

$$\begin{bmatrix} R' \\ B' \end{bmatrix} = \begin{bmatrix} -a_1 b & 0 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} R - 0 \\ B - b \end{bmatrix}.$$

The eigenvalues of the system are $\lambda_1 = 0$ and $\lambda_2 = -a_1 b$. We do not have an equilibrium ‘point’ but a line of equilibrium points along $R = 0$. Since $b > 0$, we have that $\lambda_2 < 0$ and the trajectories move towards this line of steady states. The trajectories are parallel in the vicinity of this line.

7.9. Bacteria in the gut.

Solution not done yet.

8 Chapter 8 Solutions

8.1. Species diversity.

(b)

$$(0, 0), \left(\frac{\epsilon_1}{\alpha_{11}}, 0 \right), \left(0, \frac{\epsilon_2}{\alpha_{22}} \right), \left(\frac{\alpha_{22}\epsilon_1 - \alpha_{12}\epsilon_2}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \frac{\alpha_{11}\epsilon_2 - \alpha_{21}\epsilon_1}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \right).$$

If $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} < 0$ either one or both species must die out.

(c)

$$(0, 0, 0), \left(\frac{\epsilon_1}{\alpha_{11}}, 0, 0 \right), \left(0, \frac{\epsilon_2}{\alpha_{22}}, 0 \right), \left(\frac{\alpha_{22}\epsilon_1 - \alpha_{12}\epsilon_2}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \frac{\alpha_{11}\epsilon_2 - \alpha_{21}\epsilon_1}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, 0 \right), \\ \left(0, \frac{\epsilon_3}{\alpha_{32}}, \frac{\alpha_{32}\epsilon_2 - \alpha_{22}\epsilon_3}{\alpha_{23}\alpha_{32}} \right), \left(\frac{\epsilon_3}{\alpha_{31}}, 0, \frac{\alpha_{31}\epsilon_1 - \alpha_{11}\epsilon_3}{\alpha_{13}\alpha_{31}} \right), \left(\frac{\lambda}{\beta}, \frac{\tau}{\beta}, \frac{\chi}{\beta} \right),$$

where

$$\begin{aligned} \delta &= \alpha_{23}\epsilon_1 - \alpha_{13}\epsilon_2, \\ \lambda &= \epsilon_3 (\alpha_{13}\alpha_{22} - \alpha_{12}\alpha_{23}) + \alpha_{32}\delta, \\ \tau &= \epsilon_3 (\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) - \alpha_{31}\delta, \\ \chi &= \epsilon_1 (\alpha_{22}\alpha_{31} - \alpha_{21}\alpha_{32}) + \epsilon_2 (\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31}) + \epsilon_3 (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}), \\ \beta &= \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}. \end{aligned}$$

8.3. Scaling the extended predator-prey model.

Substitution of

$$\frac{dX}{dt} = r_2 K \frac{dx}{d\tau} \text{ and } \frac{dY}{dt} = r_2 c_3 K \frac{dy}{d\tau}$$

in the equations and simplification leads to the required result.

8.4. Extended predator-prey model.

Correction: λ_3 needs to be increased (from 0.015 to 0.2) in order to find a collapse to a stable focus. Currently the question says to decrease λ_3

As λ_3 is increased from 0.015, the shape of the periodic phase plane solution changes slightly, becoming less elongated and further from the x -axis. This corresponds to both the peak and minimum number of prey decreasing and the peak number of predators increases slightly. Further increases to λ_3 cause the periodic solution to shrink. Correspondingly, the maximum and minimum populations to both tend to 0.17. The transition to a stable focus occurs when $\lambda_3 \approx 0.22$.

The transition from periodic behaviour to a stable solution for different values of λ_2 is shown in Figure 23.

For the last part of the question, where it asked to plot the boundary between periodic and stable behaviour, I can't find the transition when $\lambda_2 < 3$. The solutions are almost periodic for very small λ_3 but they aren't quite over long time scales.

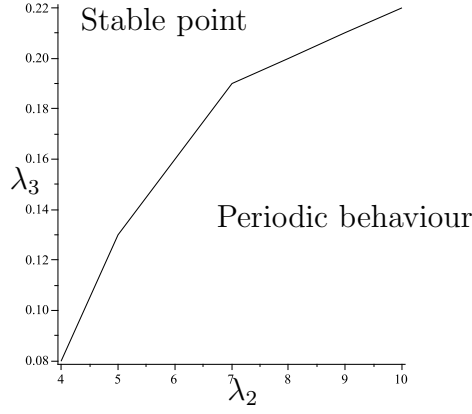


Figure 23: Division of the λ_2 - λ_3 parameter plane into a region of periodic behaviour and a region in which all solutions approach a stable point for the extended predator-prey model (Question 4) with $\lambda_1 = 10$.

8.5. Lemmings.

(a) Over the first 4-8 years it appears that a four-year cycle is followed. After this, oscillations with a yearly period are observed. The phase plane, in which the trajectory is a closed loop, is further evidence of the simple oscillatory behaviour observed over long time periods.

(b)

$$(0, 0), \left(\frac{a_1}{c_1}, 0 \right), \left(\frac{a_2}{b_2}, \frac{a_1 b_2 - a_2 c_1}{b_1 b_2} \right)$$

(c) At an equilibrium point $(n_1, n_2) = (n_{1e}, n_{2e})$ the linearised system is

$$\begin{bmatrix} n'_1 \\ n'_2 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 n_{2e} - 2c_1 n_{1e} & -b_1 n_{1e} \\ b_2 n_{2e} & -a_2 + b_2 n_{1e} \end{bmatrix} \begin{bmatrix} n_1 - n_{1e} \\ n_2 - n_{2e} \end{bmatrix}$$

The $(0, 0)$ steady state is a saddle point and $\left(\frac{a_1}{c_1}, 0 \right)$ is a stable node if $\gamma = a_1 b_2 - a_2 c_1 < 0$ and a saddle point otherwise. The point $(a_2/b_2, \gamma/(b_1 b_2))$ is a stable node if $\frac{4b_2 \gamma}{a_2 c_1^2} < 1$ and is otherwise a stable spiral.

With $a_1 = 1$, $a_2 = 0.140$ and all other parameter values as given in Figure 8.4.1, $(100, 0)$ is a saddle point. The only stable steady state is the focus representing coexistence. This analysis does not support the author's claim of a four-year cycle or of a one-year oscillation. However, this steady state seems to change to a stable centre upon the inclusion of time dependent parameters.

8.6. Prickly-pears.

(a) The growth in plant biomass and growth in the moth population are logistic, while the reduction in plant biomass is approximately exponential for small V . The functional response is

$$F(V) = c_1 \left(\frac{V}{V + D} \right),$$

and the numerical response is

$$N(V, H) = r_2 \left(1 - \frac{JH}{V} \right).$$

(b) If the plant density V is large compared with D , this term is approximately $c_1 H$, corresponding to an exponential decline of herbivores independent of the plant density V . Alternatively, if V is small compared with D , the rate of decline follows a different rate of decay that is dependent on V , proportional to the product HV .

8.7. Rabbits, hares and geese.

(a) The intrinsic reproduction rate, or the reproduction rate in the absence of overcrowding effects, is r and K is the carrying capacity of the plants. The parameter c_{max} represents the maximum rate at which a single herbivore can consume plants and a is a grazing efficiency term at low plant density. e_{max} is the maximum birth rate of herbivores in the absence of the plant-density effects, b is a coefficient related to the reduction of herbivore births at high plant densities and d is the per-capita death rate.

(b) Both functional response functions are similar at small plant densities. However, while the original function increases and then levels off at c_{max} the modified functional response term has a local maximum at $\left((a^2 + 4a/b)^{1/2} - a \right) / 2$ and tends to 0 at high plant densities. The numerical response functions are the same for both models.

The second model assumes that fear of predators reduces plant consumption by herbivores in high plant-density areas. This reduction is observed in the modified functional response term.

(c) The herbivores die out if $K < 5$ or $K > 25$. For $10 < K < 25$ the rabbits can die out for certain initial conditions.

8.8. Bovine tuberculosis.

(b)

$$J = \begin{bmatrix} b - a - \beta I & \beta(1 - S) \\ \beta I & \beta S - (\alpha_d + a) \end{bmatrix}$$

At the non-zero equilibrium point,

$$\text{trace}(J) = \frac{-\beta(\beta - \alpha)}{\alpha_d - (\beta - \alpha)}$$

and so $p < 0$ provided $\alpha_d > \beta - \alpha$. Since

$$\det(J) = (\beta - \alpha)(\alpha_d - \alpha)$$

$\Delta < 0$ when

$$\frac{4(\alpha_d + \alpha)(\alpha_d + \alpha - \beta)^2}{\beta^2(\beta - \alpha)} > 1,$$

in which case the equilibrium point is a focus. Otherwise, provided $p < \alpha - \beta$ (which must occur unless C and S are to increase without bound), it is a node.

8.9. Bovine tuberculosis and a vaccination program.

(a)

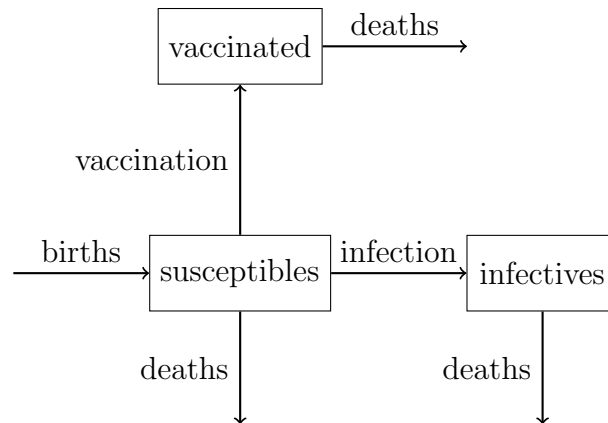


Figure 24: Compartment diagram for Question 9.

8.10. Density dependent contact rate.

Culling is the most effective model. See original reference.

9 Chapter 9 Solutions

9.1. Simple heat calculations.

(a) $108\,000 \text{ J m}^{-2}$

(b) The density of copper is 8290 kg m^{-3} and so

$$Q = (4/3) \pi (0.01)^3 \times 8290 \times 383 \times (25 - 5) \approx 266 \text{ J}$$

(c) $U = (20 / (hS) + 5) ^\circ\text{C}$

9.2. Including an internal heat source.

$$\left\{ \begin{array}{c} \text{rate of} \\ \text{change of} \\ \text{heat in section} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate of flow} \\ \text{of heat} \\ \text{in at } x \end{array} \right\} - \left\{ \begin{array}{c} \text{rate of flow} \\ \text{of heat} \\ \text{out at } x + \Delta x \end{array} \right\} + \left\{ \begin{array}{c} \text{rate heat} \\ \text{generated} \\ \text{in section} \end{array} \right\}$$

$\beta = Q_0/k$ where k is the thermal conductivity.

9.3. Formulating model extensions.

(a) Yes

(b) No. The temperature U depends on x and so the conductivity also depends on x . The appropriate differential equation is

$$\frac{d}{dx} \left(k(U) \frac{dU}{dx} \right) = 0.$$

9.4. Model formulation with circular cross-section.

$$\alpha = \frac{2h}{ak}, \quad \beta = \frac{2h}{ak} u_a.$$

9.5. Nuclear radiation.

$$\frac{d^2 U}{dx^2} + \frac{q(x)}{k} = 0.$$

9.6. Including an internal heat source.

(a)

$$\left\{ \begin{array}{c} \text{rate of change} \\ \text{of heat in shell} \\ \text{in time } \Delta t \end{array} \right\} = \left\{ \begin{array}{c} \text{rate heat} \\ \text{conducted in} \\ \text{at } r \text{ in } \Delta t \end{array} \right\} - \left\{ \begin{array}{c} \text{rate heat} \\ \text{conducted out} \\ \text{at } r + \Delta r \text{ in } \Delta t \end{array} \right\} + \left\{ \begin{array}{c} \text{rate heat} \\ \text{produced in} \\ \text{shell in } \Delta t \end{array} \right\}$$

(b)

$$V = \pi [(r + \Delta r)^2 - r^2] \ell,$$

$$\left\{ \begin{array}{c} \text{rate heat} \\ \text{produced in} \\ \text{shell in } \Delta t \end{array} \right\} = Q_0 \pi \ell \Delta r (2r + \Delta r) \Delta t.$$

(c)

$$\frac{d}{dr} \left(r \frac{dU}{dr} \right) + \frac{Q_0 r}{k} = 0.$$

9.7. Equilibrium temperature inside a sphere.

$$\frac{d}{dr} \left(r^2 \frac{dU}{dr} \right) = 0.$$

9.8. Cylindrical heat fin.

$$\left\{ \begin{array}{c} \text{rate heat} \\ \text{lost to} \\ \text{surroundings} \end{array} \right\} = 2h \left[\pi (r + \Delta r)^2 - \pi r^2 \right] (U(r^*) - u_s),$$

where $r < r^* < r + \Delta r$.

$$\frac{d}{dr} \left(r \frac{dU}{dr} \right) = \frac{2rh}{kl} (U - u_s).$$

10 Chapter 10 Solutions

10.1. Coffee cooling.

(b) $\lambda \approx 0.011515$, $t \approx 26.13$ minutes.

10.2. Beer warming.

Using Newton's law of cooling, 24.1 minutes.

10.4. The hot water heater.

(a) Approximately 33.50°C

(b) Approximately 3.0 hours

(c) Approximately 2.3 hours

(d) Approximately 1.3 hours

(e) The 8 hour cooling method requires three hours of heating for each 11 hour heating and cooling cycle. The 50° method requires 3.9 hours of heating over a 10.8 hour period (corresponding to 3 heating and cooling cycles). It appears that the 8 hour method is more efficient.

(f) No. After 8 hours the temperature of the water is 30.56°C and 3.54 hours of heating are required to raise it to 60°C . On the other hand, it takes 2.0 hours for the temperature to fall to 50° and 1.36 hours of heating to increase it to the original temperature. It is still more economical to turn the heater off for 8 hours and then heat the water.

10.5. Insulating the hot water heater.

$$cm \frac{dU}{dt} = q$$

and so $U(t) = 0.00343t + u_0$. Thus $t \approx 3.6$ hours.

10.6. Horse temperatures.

(b)

$$\frac{U(6) - u_h}{u_0 - u_h} = \frac{U(3) - u_h}{u_0 - u_h} e^{3\lambda}$$

and so

$$\frac{U(6) - u_h}{U(3) - u_h} = e^{3\lambda}$$

(c) $u_h = 36.6$, $\lambda = -0.231049$

10.7. Modelling house temperatures.

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of temperature} \\ \text{in room 1} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate heat} \\ \text{produced by} \\ \text{heater} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate heat} \\ \text{lost to} \\ \text{surroundings} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate heat} \\ \text{lost to} \\ \text{room 2} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of temperature} \\ \text{in room 2} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate heat} \\ \text{lost to} \\ \text{surroundings} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate heat} \\ \text{gained from} \\ \text{room 1} \end{array} \right\}$$

$$M_1 \frac{dU_1}{dt} = q_0 + \frac{S_{10}}{R_{10}} (u_s - U_1) + \frac{S_{12}}{R_{12}} (U_2 - U_1)$$

$$M_2 \frac{dU_2}{dt} = \frac{S_{20}}{R_{20}} (u_s - U_2) + \frac{S_{12}}{R_{12}} (U_1 - U_2)$$

10.8. Stirred chemical reactor.

(a) The straight line represents heat loss and the curve is heat production.

For the solution $\theta^{(1)}$, consider when $\theta_0 < \theta^{(1)}$. Here $d\theta/dt > 0$ (since heat production is greater than heat loss). For $\theta_0 > \theta^{(1)}$ $d\theta < 0$. Hence $\theta^{(1)}$ is a stable equilibrium solution.

Using similar arguments, the equilibrium solution $\theta^{(2)}$ is unstable and the equilibrium solution $\theta^{(3)}$ is stable.

(b) If θ_0 is small, then $\theta \rightarrow \theta^{(1)}$ and if θ_0 is sufficiently large then $\theta(t) \rightarrow \theta^{(3)}$.

10.9. Spontaneous ignition and critical values.

The following Maple code can be used.

Listing 1: Maple code —

```
restart;
lambda := 5; theta_a := 0.1; with(plots):
eq:= lambda*exp(-1/theta) - (theta-theta_a);
fsolve(eq=0, theta, theta = 0..1);
thend := 2;
plot1:= plot(exp(-1/theta), theta=0..thend):
plot2:= plot(1/lambda*(theta-theta_a), theta=0..thend);
display(plot1, plot2);
```

(a) Only one equilibrium solution, $\theta^{(1)} = 1.12$.

(b) Three equilibrium solutions, $\theta^{(1)} = 0.1$, $\theta^{(2)} = 0.32$, $\theta^{(3)} = 4.00$.

10.10. Critical ambient temperature.

(b) $\theta_c \approx 0.38, 0.67$ and so the critical ambient temperature is $\theta_{ac} \approx 0.22$.

10.11. Liquid explosive.

$R = 1.987 \text{ cal K}^{-1}\text{mol}^{-1}$, $V \approx 196.4 \text{ cm}^3$, $S \approx 187.0 \text{ cm}^{-2}$ and $\rho \approx 1.3188 \text{ g/cm}^2$ and so $\lambda \approx 1.47 \times 10^{11}$.

By solving the equation $\theta_c^2 = \lambda e^{-1/\theta_c}$ we find that $\theta_c = 0.031$ or $\theta_c = 3.8 \times 10^5$. The critical ambient temperature is given by $\theta_{ac} = \theta_c - \theta_c^2$ and so $\theta_{ac} = 0.030$. Thus $T_{ac} = 671^\circ\text{C}$.

11 Chapter 11 Solutions

11.1. Solving a boundary value problem.

- (a) $U(x) = \frac{1}{2}x^2 + Ax + B.$
- (b) $U(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1.$
- (c) $U(x) = \frac{1}{2}x^2 - 2x + 1.$
- (d) $U(x) = \frac{1}{2}x^2 + x - 1.$

Correction: for last part of (d), $x = 1$ should be $x = 2$. There is no solution, in this case, as the two boundary conditions give inconsistent values of the constant A .

11.2. Formulating a boundary value problem.

- (a) $U(\ell) = 30$ and $J(0) = 300/A$, A is the wall area.
- (b) $J(0) = h(10 - U(0))$, $J(\ell) = h(U(\ell) - 25)$
- (c) $J(0) = h(u_0 - U(0))$, $U(\ell) = 80$

11.3. Heat flux through a wall.

- (a) $U(x) = \frac{u_2 - u_1}{L}x + u_1$ (b) $J(x) = -k\frac{u_2 - u_1}{L}$

11.4. Heat flux through a window.

- (a) $U(x) = Ax + B$, where A and B are arbitrary constants.
- (b) $J(0) = h_i(u_i - U(0))$, $J(\ell) = h_o(U(\ell) - u_o)$
- (c) $U(x) = u_i + \frac{u_o - u_i}{k/h_i + k/h_o + \ell} \left(x + \frac{k}{h_i} \right)$
- (e) In general, $J = (u_i - u_o)/R$ and so $R = 1/h_i + 1/h_o + \ell/k$. The total resistance is the sum of the resistance to conduction through the wall (ℓ/k) and the surface resistances at $x = 0$ and $x = \ell$ (which are $1/h_i$ and $1/h_o$, respectively).

11.5. Heat flux and thermal resistance.

- (a)

$$U_1(x) = u_i + \frac{k_2(u_o - u_i)}{k_1d_2 + k_2d_1}(x + d_1), \quad U_2(x) = u_o + \frac{k_1(u_o - u_i)}{k_1d_2 + k_2d_1}(x - d_2)$$

- (c) In general, $J = (u_i - u_o)/R$ and so $R = d_1/k_1 + d_2/k_2$. The total resistance is the sum of the thermal resistances of each material.

11.6. Internal floor heating.

$$U(x) = \frac{q}{2k}x(l - x) + u_0$$

11.7. Nuclear reactor.

- (a) See Figure 25
- (b) $U(x) = -\frac{q_0}{a^2k}e^{-ax} + C_1x + C_2$

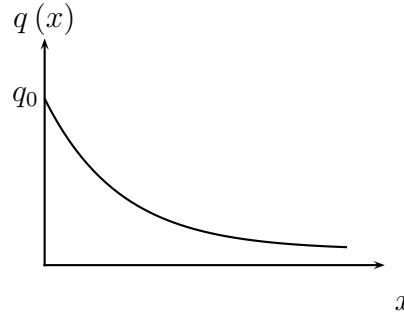


Figure 25: Internal heat generation function for Question 7.

(c)

$$U(x) = \frac{q_0}{a^2 k} \left(1 - e^{-ax} - \frac{x}{\ell} (1 - e^{-a\ell}) \right) + \frac{u_2 - u_1}{\ell} x + u_1.$$

11.8. Combining heat resistances.

One approach is to solve

$$\frac{u_1 - u_a}{R_g} = \frac{u_a - u_b}{R_a}$$

for u_a and substitute into

$$\frac{u_a - u_b}{R_a} = \frac{u_b - u_2}{R_g}$$

to obtain u_b . Substitution into

$$J = \frac{u_b - u_2}{R_g}$$

gives the required result.

11.9. Double glazing.

For the Δ_0 model $J \approx 117.6 \text{ Wm}^{-2}$, for the Δ_1 model $J \approx 54.1 \text{ Wm}^{-2}$ and for the Δ_2 model $J \approx 35.1 \text{ Wm}^{-2}$. With a single pane of glass $J = 2000 \text{ Wm}^{-2}$.

11.10. Comparing glazing models.

For the parameter values in Section 11.3, we find that Δ_2 is the least efficient. In general, Δ_0 , the least realistic model, is the most efficient, although when $r \ll 1$, Δ_1 has a higher efficiency.

11.11. Cooling lizards.

(a)

$$\left\{ \begin{array}{c} \text{rate heat} \\ \text{lost from} \\ \text{cylindrical shell} \end{array} \right\} = \left\{ \begin{array}{c} \text{rate heat} \\ \text{conducted in} \\ \text{at } r \end{array} \right\} - \left\{ \begin{array}{c} \text{rate heat} \\ \text{conducted out} \\ \text{at } r + \Delta r \end{array} \right\} + \left\{ \begin{array}{c} \text{rate heat} \\ \text{produced in} \\ \text{cylindrical shell} \end{array} \right\}$$

and from this we find that

$$0 = J(r) A(r) - J(r + \Delta r) A(r + \Delta r) + q\rho\pi\Delta r (2r + \Delta r) \ell$$

(b)

$$U(r) = -\frac{\rho q r^2}{4k} + C_1 \ln(r) + C_2$$

(c) As $r \rightarrow 0$, $\ln(r) \rightarrow -\infty$ and so $C_1 = 0$.

(d)

$$U(r) = u_0 + \frac{\rho q}{4k} \left(a^2 + \frac{k}{\pi \ell h} - r^2 \right)$$

The maximum temperature occurs at $r = 0$ and is

$$U_{max} = u_0 + \frac{\rho q}{4k} \left(a^2 + \frac{k}{\pi \ell h} \right).$$

11.12. Equilibrium temperature in a spherical shell.

(a) $U(r) = -C_1 r^{-1} + C_2$

(b)

$$U(r) = \frac{ab(u_2 - u_1)}{a - b} r^{-1} + \frac{u_1 a - u_2 b}{a - b}$$

11.13. Hyperbolic functions.

$$U(x) = \cosh(\sqrt{\beta}x)$$

11.14. Cooling an engine.

The general solution is given in question 13.

$$U(x) = u_1 (\cosh(\sqrt{\beta}x) - \tanh(\sqrt{\beta}\ell) \sinh(\sqrt{\beta}x))$$

11.15. Rectangular heat fin.

(a) $U(0) = u_1$, $J(\ell) = hU(\ell)$

(b)

$$U = u_1 \left(\cosh(ax) - \frac{\cosh(a\ell) + ka/h \sinh(a\ell)}{ka/h \cosh(a\ell) + \sinh(a\ell)} \sinh(ax) \right)$$

11.16. Formulating boundary conditions.

(a) $J(a) = 10$ and $U(b) = u_2$

(b) $U(0) = u_2$, $U(L) = u_1$

(c) $U(0) = 100$ and $J(\ell) = 0$

(d) $J(0) = h(u_2 - U(0))$ and $U(\ell) = u_1$

(e) $U(r_1) = u_1$ and $J(r_2) = h(U(r_2) - 3u_1)$

12 Chapter 12 Solutions

12.1. Non-equilibrium temperature model.

$$\alpha = k / (c\rho) \text{ and } \beta = Q_0 / (c\rho)$$

12.2. Non-equilibrium temperature for a cylinder.

$$\frac{\partial U}{\partial t} = \frac{k}{c\rho r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{Q_0}{c\rho}$$

12.4. Using Euler's identity.

$$\lim_{x \rightarrow \infty} e^{bx} = \lim_{x \rightarrow \infty} e^{ax} (\cos(ax) + i \sin(ax)) = \infty$$

since \cos and \sin are bounded between -1 and 1 . Similarly

$$\lim_{x \rightarrow \infty} e^{-bx} = \lim_{x \rightarrow \infty} e^{-ax} (\cos(ax) + i \sin(ax)) = 0$$

for the same reason.

12.5. Oscillating surface temperatures.

$$U(x, t) = u_1 e^{-\sqrt{\omega/(2\alpha)}x} \left(\cos \left(\sqrt{\omega/(2\alpha)}x - \omega t \right) \right)$$

12.6. Amplitude of oscillating temperature.

$$|\hat{U}| = \frac{q}{k} \sqrt{\frac{\alpha}{\omega}} e^{-\sqrt{\omega/(2\alpha)}x}$$

12.7. Land mines.

(a)

$$U_1(x, t) = \frac{q}{k_1 b_1} \frac{(1-r)e^{b_1 x} + (1+r)e^{-b_1 x}}{(1+r) - (1-r)e^{-2b_1 d}} e^{-b_1 d} e^{i\omega t}$$

$$U_2(x, t) = \frac{q}{k_1 b_1} \frac{2e^{-b_2 x}}{(1+r) - (1-r)e^{-2b_1 d}} e^{-b_1 d} e^{i\omega t}$$

Substituting $x = -d$ into the equation for U_1 gives the required result.

12.8. Depth of buried land mine.

From Figure 26 we see that the soil surface temperature changes with the depth of the land mine only when the mine is less than 2 mm below the soil surface. If the mine is buried deeper, the soil surface temperature may be different than if no land mine were buried but this change is independent of the land mine depth.

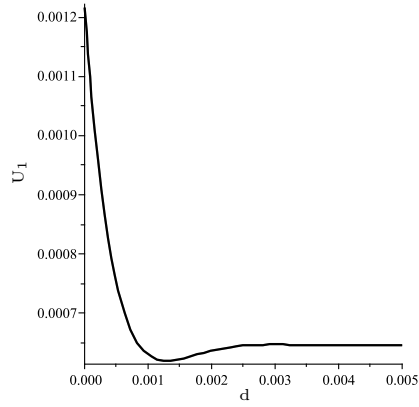


Figure 26: Amplitude of the soil surface temperature versus depth of land mine for Question 8. The temperature is plotted at times $\omega t = 0$ (black), $\omega t = \pi/4$ (thin black) and $\omega t = \pi$ (grey) radians, with $\omega = 1$.

12.9. Lake pollution in steady state.

Since a steady state has been reached we can assume, at any fixed location in the lake, there is no change in pollution concentration over time. Thus

$$\frac{\partial C}{\partial t} = 0$$

for each value of x . Now

$$\frac{F}{A} \frac{\partial C}{\partial x} = 0 = \frac{dC}{dx}.$$

Since C is independent of t in this case, we can convert to an ordinary differential equation, which says that the pollution level is constant throughout the lake. That is $C(x) = K$, where K is an arbitrary constant.