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(1)

$P_1 \; P_2 \; P_3 \; P_4 \; P_5 \; P_6 \; P_7$

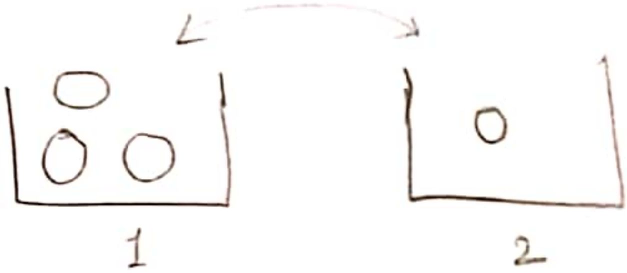
$H_1 \; H_2 \; \underbrace{H_3 \; H_4 \; H_5}$

H_3 or H_4 or H_5

$$P(\text{nobody } H1 \text{ or } H2) = \frac{3^7}{5^7}$$

total # of choices

(2)



$$P = \begin{array}{c c} \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} & \end{array}$$

$$P_{ij} = P(X_1 = j \mid X_0 = i)$$

$P_{00} = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0$ $\quad P_{10} = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$

$P_{01} = \frac{1}{2} \times 1 + \frac{1}{2} \cdot 1 = 1$ $\quad P_{11} = P_{13} = P_{14} = 0$

$P_{12} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

$P_{02} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

$\vdots$

(3)

$$P = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0 & 0.25 \\ 0.75 & 0.25 & 0 & 0 \end{pmatrix}$$

$N(i) = E(\text{number of steps to reach state 2} \mid X_0 = i)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -0.5 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.25 & 1 & -0.25 & 1 \\ -0.75 & -0.25 & 0 & 1 & 1 \end{array}\right)$$

$$N = \begin{bmatrix} 2.4 \\ 0 \\ 2.9 \\ 2.8 \end{bmatrix}$$

$$N(1) = \boxed{2.4}$$

(4) $X \longrightarrow$ uniform over $[0,2]$

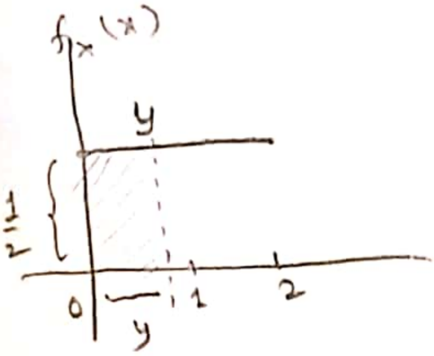
$Y \longrightarrow \quad f_Y(y) = 2y \qquad 0 \le y \le 1$

$f_X(x) = \frac{1}{2}$; let, $z = X - Y \quad X \le Y$

$\Rightarrow Z \le 0$

$$P(X \le Y) = P(z \le 0)$$

$$= \int_{-\infty}^{\infty} P(z \le 0 \mid Y = y)\, f_Y(y)\, dy$$



$$= \int_{-\infty}^{\infty} P(x - y \le 0 \mid Y = y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} P(x \le y) \times f_Y(y)\, dy$$

$$= \int_0^2 \frac{1}{\cancel{2}} \times y \times \cancel{2} y \times dy$$

$$= \left.\frac{y^3}{3}\right|_0^2 = \boxed{\frac{8}{3}}$$

(5)

Consider the time when "C" finds free ATM.

At this point, A or B would have left the ATM and one would still be in service.

Memory less property of exponential $\Bigg\} \Rightarrow$ amount of time other customer (A or B) spends in ATM is exponentially distributed with meen 3 minutes.

$\therefore$ This is similar to starting the service again

$\therefore$ By Symmetry,

$$P(C \text{ Completes before } A) = \boxed{\frac{1}{2}}$$

(6) $E\left[|X_n|\right] \leq \frac{1}{n}$

$$P(X_n \geq n) \leq P(|X_n| \geq n)$$

$$\leq \frac{E(|X_n|)}{n} \quad \left(\text{Markov's inequality}\right)$$

$$\leq \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} P(X_n \geq n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

$\therefore$ Borel cantelli lemma 1

$$\Rightarrow P(X_n \geq n \text{ i.o.}) = 0$$

(7)

$$p_{ii}(n) = P(X_n = i \mid X_0 = i) \geq \frac{1}{n+7}$$

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} \geq \frac{1}{8} + \frac{1}{9} + \cdots$$

$$\geq \underbrace{\sum_{x=1}^{\infty} \frac{1}{x}}_{\text{Harmonic series}} - \sum_{y=1}^{7} \frac{1}{y}$$

$$= \infty$$

$\Rightarrow$ "i" is persistent.

(8)

(a) $E[N_3 | N_2 = k]$

$$= E[N_3 | N_2 = k, H] \times P(H | N_2 = k)$$

$$+ E[N_3 | N_2 = k, T] \times P(T | N_2 = k)$$

$$= [1 + E(N_2)] \times p + (1 + E[N_3 | \cancel{N_2 = k}]) \times (1-p)$$

$$E[N_3 | N_2 = k] = \frac{\left(1 + \frac{1}{p(1-p)}\right) \times p + (1-p)}{p}$$

(b) $$E(N_3) = \sum_{k=2}^{\infty} E[N_3 | N_2 = k] \times P(N_2 = k)$$

$$= \sum_{k=2}^{\infty} \frac{\left(1 + \frac{1}{(1-p)}\right)}{p} \times P(N_2 = k)$$

$$= \sum_{k=2}^{\infty} \frac{\left(1 + \frac{1}{1-p}\right)}{p} \times \frac{1}{p(1-p)}$$