Assigment 1 - Solutions

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Problem 1

Find the general solution to

$$t\frac{dy}{dt} - 2ty = t^2 - t$$

Solution. The given equation holds true when,

$$t = 0 \quad or \quad \frac{dy}{dt} - 2y = t - 1$$

The integration factor for the above differential equation $= e^{\int -2dt} = e^{-2t}$

Therefore the solution is given by,

$$\begin{aligned} y(t) &= e^{2t} \int e^{-2t}(t-1)dt \\ &= e^{2t}\left(\frac{te^{-2t}}{-2} - \int \frac{e^{-2t}}{-2} - \frac{e^{-2t}}{-2} + c_1\right) \\ &= e^{2t}\left(-\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{e^{-2t}}{2} + c\right) \\ &= -\frac{t}{2} + \frac{1}{4} + ce^{2t} \end{aligned} \tag{1}$$

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Problem 2

Draw the slope plot for $\frac{dy}{dt} = (1-t)y$ and the trajectory $y(0) = 0$.

Solution. The following table represents values of $\frac{dy}{dt}$ for first few values of y and t.

Slope plot		
t	y	$\frac{dy}{dt} = (1-t)y$
0	0	0
0	1	1
1	0	0
1	1	0
0	-1	-1
-1	0	0
-1	1	2
1	-1	0
-1	-1	-2

Following is the Python code to draw the slope plot using matplotlib library:

```python

import numpy as np
from matplotlib import pyplot as plt

def diff(t,y):
    return (1-t)*y

t = np.linspace(-4,4,20)
y = np.linspace(-4,4,20)

for j in t:
    for k in y:
        slope = diff(j,k)
        domain = np.linspace(j-0.05,j+0.05,2)
        def fun(t1,y1):
            z = slope*(domain-t1)+y1
            return z
        plt.plot(domain,fun(j,k),solid_capstyle='projecting',solid_joinstyle='bevel')

plt.rcParams["figure.figsize"] = (10,10)
plt.grid(True)
plt.show()
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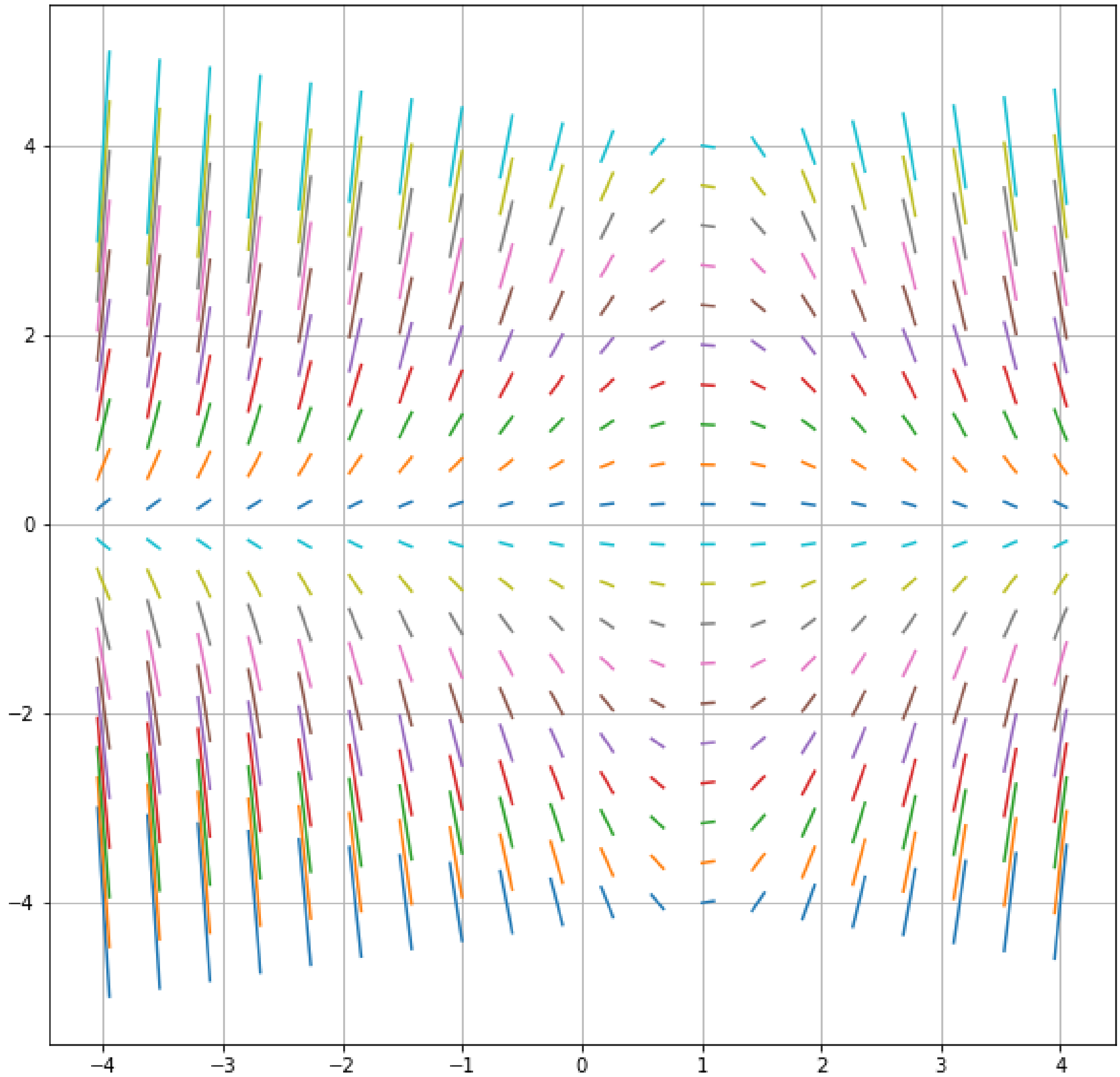
Listing 1: Slope plot code

Figure 1: Slope plot for $\frac{dy}{dt} = (1-t)y$



Solving the initial value problem, $\frac{dy}{dt} = (1-t)y \quad y(0) = 0$ we get,

$$\frac{dy}{y} = (1-t)dt \tag{2}$$

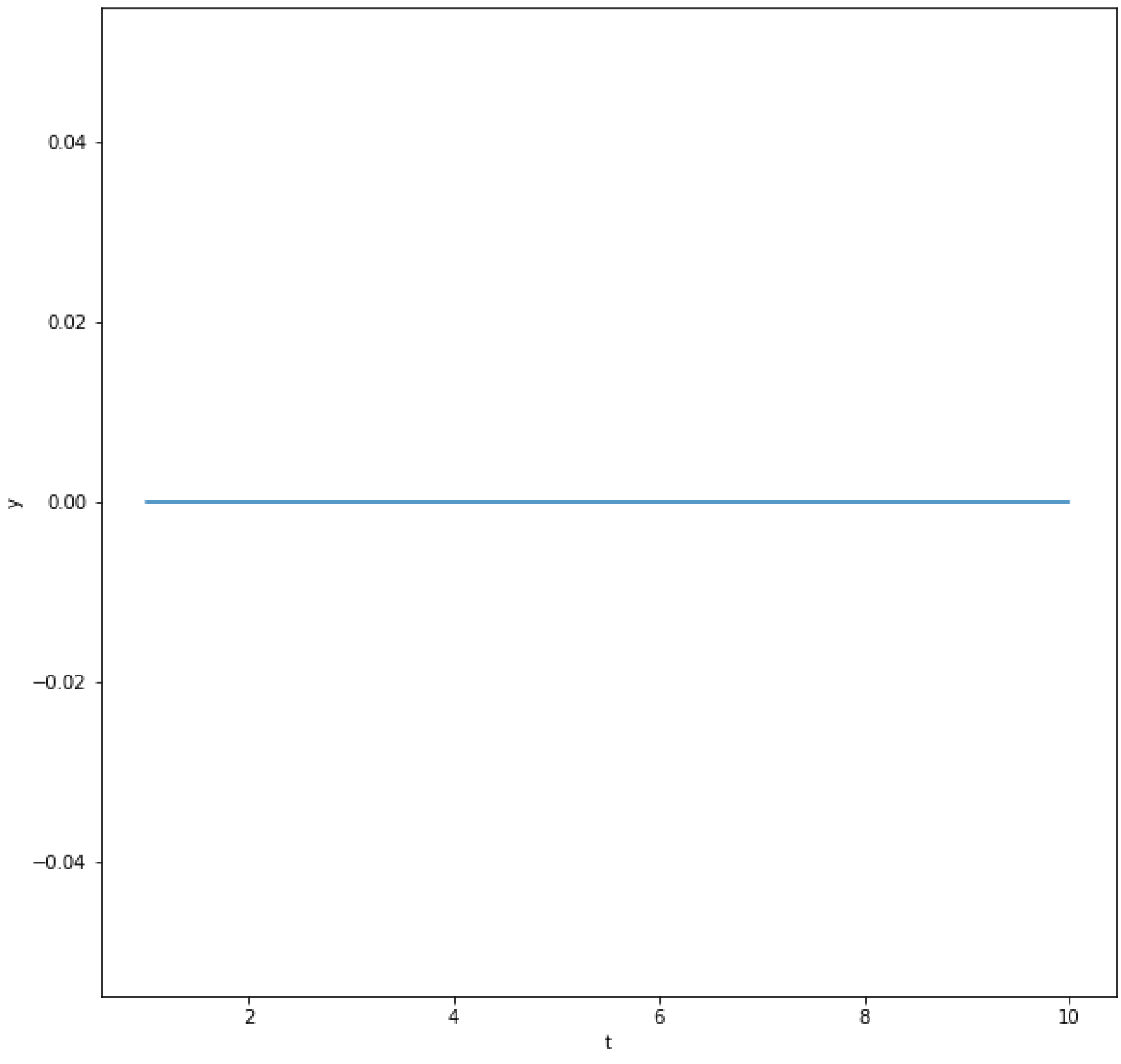
Integrating on both sides,

$$\begin{aligned}\int \frac{dy}{y} &= \int (1-t)dt \\ ln(y) &= t - \frac{t^2}{2} + c_1 \\ y(t) &= ce^{t-\frac{t^2}{2}}\end{aligned} \tag{3}$$

Substituting the initial conditions, we get,

$$y(t) = 0 \tag{4}$$

Figure 2: Solution to I.V.P. $\frac{dy}{dt} = (1-t)y \quad y(0) = 0$



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Problem 3

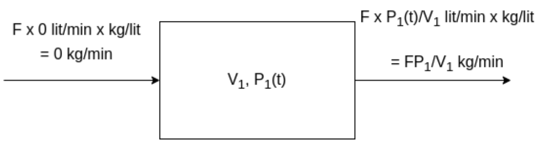
Consider the lake system with Lake A flowing into Lake B. Lake A is contaminated with arsenic due to a coal gasification plant out-flowing near the lake. Clean water enters Lake A at a rate F and all water flows into Lake B. Write a pair of differential equations, one for the concentration of pollutants in Lake A and one for Lake B. Solve the system of differential equations by first solving for Lake A and then for Lake B.

Solution. Given, rate of inflow into Lake A = $F\ lit/min$
Let us assume,

1. The volumes of Lake A, Lake B be V_1 and V_2 ($V_1 \neq V_2$) respectively.
2. The amount of Arsenic at a time t in Lake A, Lake B be $P_1(t)$ and $P_2(t)$ respectively.
3. The initial amount of Arsenic in Lake A be $P_1(t=0) = P_0$
4. The initial amount of Arsenic in Lake B be 0.

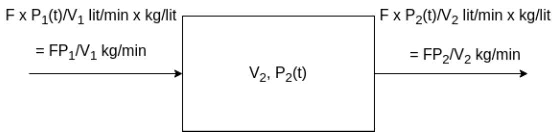
Applying compartment model for two lakes with the above information, we get,

Figure 3: Compartment Model for Lake A



$$\begin{aligned}\frac{dP_1(t)}{dt} &= 0 - \frac{F \times P_1(t)}{V_1}\\ \frac{dP_1(t)}{dt} &= -\frac{FP_1}{V_1}\\ P_1(t) &= P_0 e^{-\frac{Ft}{V_1}}\end{aligned} \tag{5}$$

Figure 4: Compartment Model for Lake B



$$\begin{aligned}\frac{dP_2(t)}{dt} &= \frac{F \times P_1(t)}{V_1} - \frac{F \times P_2(t)}{V_2}\\ \frac{dP_2}{dt} + \frac{FP_2}{V_2} &= \frac{FP_1}{V_1}\end{aligned} \tag{6}$$

Substituting equation (5) into (6), we get,

$$\frac{dP_2}{dt} + \frac{FP_2}{V_2} = \frac{FP_0}{V_1} e^{-\frac{Ft}{V_1}} \tag{7}$$

Using, integrating factor (I.F.) = $e^{\int \frac{F}{V_2} dt} = e^{\frac{Ft}{V_2}}$
And initial condition $P_2(0) = 0$
The solution is,

$$\begin{aligned}P_2(t) &= e^{\frac{-Ft}{V_2}} \int e^{\frac{Ft}{V_2}} \left(\frac{F}{V_1} P_0 e^{\frac{-Ft}{V_1}} \right) dt\\ P_2(t) &= \frac{P_0 V_2}{V_2 - V_1} \left(e^{\frac{-Ft}{V_2}} - e^{\frac{-Ft}{V_1}} \right)\end{aligned} \tag{8}$$

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Problem 4

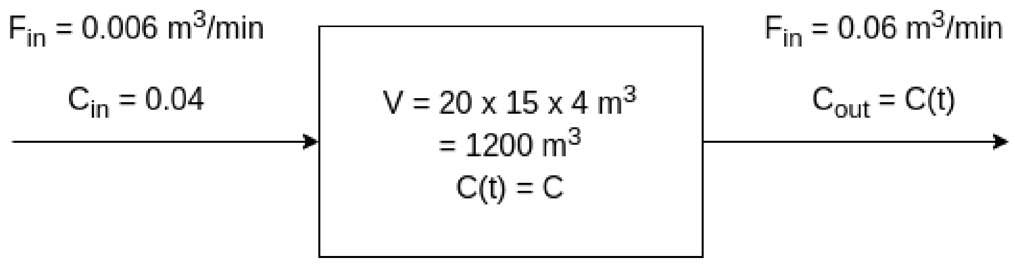
A bar opens at 6 PM and allows smoking. Smoke contains 4% carbon monoxide and enters the room at a constant rate of .006 m^3/min. Given that the bar's floor area is $20m$ by $15m$ by $4m$ and the bar's ventilation system removes the smoke-air mixture at a 10 times the rate smoke is produced, set up a initial value problem for the concentration of smoke in the bar. Prolonged expose to a concentration of more than 0.012% carbon monoxide can be fatal. At what time will the lethal limit be reached?

Solution. Given,

1. inflow of carbon monoxide, $F_{in} = .006\ m^3/min$.
2. outflow of carbon monoxide, $F_{out} = 10 \times F_{in} = .06\ m^3/min$.
3. volume of bar, $V = 20 \times 15 \times 4\ m^3 = 1200\ m^3$.
4. input concentration of carbon monoxide into bar, $C_{in} = 4\%$.
5. intial concentration of carbon monoxide in the bar, $C(0) = 0$.

Let us assume,
Concentration of carbon monoxide in bar at time $t = C(t) = C$
Applying compartment model for the bar with the above information, we get,

Figure 5: Compartment Model for bar



$$\frac{dC}{dt} = \frac{F_{in}}{V} C_{in} - \frac{F_{out}}{V} C_{out} \tag{9}$$

The solution to the above equation is,

$$C(t) = \frac{F_{in} C_{in}}{F_{out}} \left(1 - e^{-\frac{F_{out}}{V} t}\right) \tag{10}$$

The lethal limit for carbon monoxide is 0.012%. Let, t_h be the time when lethal limit is reached. Therefore, $C(t_h) = 0.012\%$.

Substiting this in equation (10), we get,

$$
\begin{aligned}
0.012 &= \frac{0.006 \times 4}{0.06}\left(1 - e^{-\frac{0.06}{1200}t_h}\right) \\
\implies t_h &= 609.184\; min. \qquad (11) \\
&\approx \boxed{10\; hr.}
\end{aligned}
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