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(1) $E[x_n] = 0 \quad ; \quad E[x_n^2] = 1 \quad \forall n \geq 1$

$$P(x_n \geq n) \leq P(|x_n| \geq n)$$

$$\leq \frac{E(|x|^2)}{n^2} \quad \left(\text{using Markov's inequality}\right)$$

$$= \frac{E(x^2)}{n^2} = \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\therefore P(x_n \geq n) \leq \frac{\pi^2}{6} < \infty$$

$\therefore$ According to Borel-Cantelli Lemma 1,

$$P(x_n \geq n \text{ i.o.}) = 0$$

I suppose you meant X_k is independent on $\{1,2,\dots n\}$

(2) $P(X_K = i) = \frac{1}{n} \quad \forall \quad 1 \leq i \leq n$

$\therefore P(X_K = 5) = \frac{1}{n}$

$\Rightarrow \sum_{n=1}^{\infty} P(X_K = 5) = \sum_{n=1}^{\infty} \frac{1}{n}$

$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\downarrow} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\downarrow} + \dots$

$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

$= \infty$

$\therefore$ According to Borel - Cantelli Lemma 2,

X_K is independent & $\sum P(X_K = 5) = \infty$

$\Rightarrow P(X_K = 5 \text{ i.o.}) = 1.$

(3)

Let,

$$U_n = X_n X_{n-1}^{-1}$$

$$U_{n-1} = X_{n-1} X_{n-2}^{-1}$$

$$\vdots$$

$$U_1 = X_1 X_0^{-1}$$

$$\therefore U_n U_{n-1} \cdots U_1 = X_n X_0^{-1}$$

U_n is uniform on $[0, 1]$

$$X_0 = 1$$

$$\therefore U_n U_{n-1} \cdots U_1 = X_n$$

$$\Rightarrow \sum_{i=1}^{n} \ln(U_i) = \ln(X_n)$$

$$\frac{1}{n} \ln(X_n) = \frac{1}{n} \sum_{i=1}^{n} \ln(U_i)$$

$\therefore$ According to Weak law of large numbers

$$\Rightarrow \lim_{n \to \infty} P\left(\left|\frac{1}{n}\ln(X_n) - \mu\right| > \varepsilon\right) = 0$$

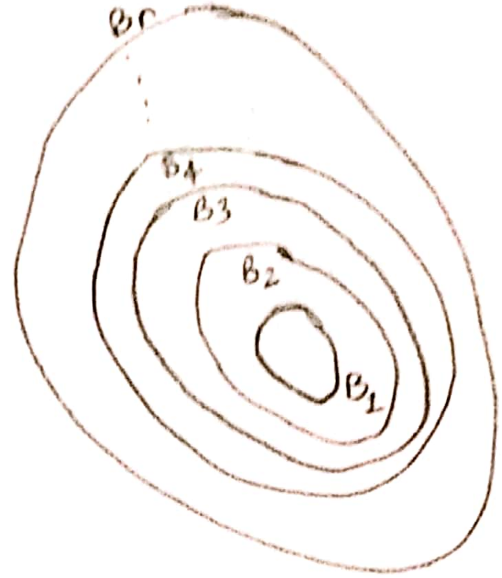
where, $\mu = E[\ln(U)] = \int_0^1 \ln(x) \times 1 \times dx$

$$= -1$$

Comparing with expression in question,

$$c = \mu = -1.$$

④ a)

$C_1 = B_1$ 

$C_2 = B_2 \cap B_1^c$

$C_3 = B_3 \cap B_2^c$

$\vdots$

$C_n = B_n \cap B_{n-1}^c$

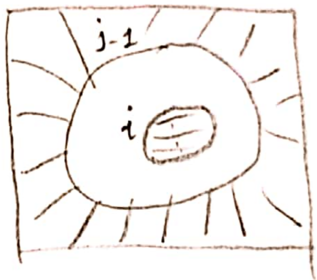
For $i \neq j$, consider,

$$C_i \cap C_j = \left(B_i \cap B_{i-1}^c\right) \cap \left(B_j \cap B_{j-1}^c\right)$$

$$= \left(B_i \cap B_{j-1}^c\right) \cap \left(B_j \cap B_{i-1}^c\right)$$

Without loss of generality,

Assume $i < j$,

$\therefore B_i \cap B_{j-1}^c =$ 

$= \phi$ (As, $j \geq i+1$)

$\Rightarrow C_i \cap C_j = \phi \Rightarrow C_i, C_j$ are disjoint $\forall\, n \geq 2$

Since, B_N is the biggest set containing all $B_i, \forall\, i < n$

$$B_N = \bigcup_{n=1}^{N} B_n$$

Consider

$$C_i \cup C_{i+1} = \left(B_i \cap B_{i-1}^c\right) \cup \left(B_{i+1} \cap B_i^c\right)$$

$$= \left(B_{i+1} \cap B_{i-1}^c\right)$$

$$(C_i \cup C_{i+1}) \cup C_{i+2} = \left(B_{i+1} \cap B_{i-1}^c\right) \cup \left(B_{i+2} \cap B_{i+1}^c\right)$$

$$= \left(B_{i+2} \cap B_{i-1}^c\right)$$

$$\vdots$$

$$\left(C_i \cup C_{i+1} \cup C_{i+2} \cup \cdots C_N\right)$$

$$= \left(B_N \cap B_{i-1}^c\right)$$

Put $i=2$,

$$C_2 \cup C_3 \cdots C_N = B_N \cap B_1^c = B_1^c \cap B_N$$

$$\Rightarrow \underline{\underline{C_1}} \cup C_2 \cup C_3 \cdots C_N = (\underbrace{\underline{\underline{B_1}} \cup B_1^c}_{\text{Universal set}}) \cap B_N \quad (\because C_1 = B_1)$$

$$\Rightarrow \bigcup_{n=1}^{N} C_n = B_N$$

b) Let, $x \in \bigcup_{n=1}^{\infty} B_n$

$\Rightarrow x \in B_K$ for some $1 \leq K \leq \infty$

$\Rightarrow x \in \bigcup_{n=1}^{K} C_n$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} C_n$

$\Rightarrow \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} C_n \quad -(1)$

Now,

let, $y \in \bigcup_{n=1}^{\infty} C_n$

$\Rightarrow y \in C_K$ for some $1 \leq K \leq \infty$

$\Rightarrow y \in B_K \cap B_{K-1}^{c}$

$\Rightarrow y \in B_K$

$\Rightarrow y \in \bigcup_{n=1}^{\infty} B_n$

$\Rightarrow \bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} B_n \quad -(2)$

From (1) & (2),

we get,

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n$$

c)

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = P\left(\bigcup_{n=1}^{\infty} C_n\right) \quad (\text{from b}))$$

$$= \sum_{n=1}^{\infty} C_n \quad \left(\text{As } C_i \cap C_j = \phi \ \forall\, i \neq j \text{ from a})\right)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} P(C_n)$$

$$= \lim_{N\to\infty} P\left(\bigcup_{n=1}^{N} C_n\right)$$

$$= \lim_{N\to\infty} P(B_N)$$

d)

$A_n \to$ decreasing

$\Rightarrow A_n^C \to$ increasing

$\therefore$ From c), $P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = \lim_{N\to\infty} P(A_N^c)$

Consider, $P\left(\bigcap_{n=1}^{\infty} A_n\right) + P\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right) = 1$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$

$$= 1 - \lim_{N\to\infty} P(A_N^c)$$

$$= \lim_{N\to\infty} P(A_N)$$

e) Let,

$x_n, n \geq 1$ be a decreasing sequence converging to x.

$x_n = \{x + h_1, x + h_2, \ldots\}$

$\therefore A_n = \{X \leq x_n\}$ is also decreasing

Also,

$$\{X \leq x\} = \bigcap_{n=1}^{\infty} A_n$$

$$\therefore P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N \to \infty} P(A_N)$$

$$= \lim_{N \to \infty} P(X \leq x_N)$$

$$= \lim_{N \to \infty} F(x_N) \quad -(1)$$

Also, $\therefore \lim_{N \to \infty} F(x_N) = \lim_{N \to \infty} P(A_N)$

$$= P(X \leq x)$$

$$= F(x) \quad -(2)$$

From (1) & (2),

$$\lim_{N \to \infty} F(x_N) = F(x)$$

$\downarrow$

Choose $x_N \to x + h$ and $h \to 0$

$\therefore \lim_{h \to 0} F(x+h) = F(x)$ as series is decreasing