MATH 7241: Probability 1 FALL 2020 MAKEUP TEST 1

Important:

- This Makeup Test will be available at **6pm on Friday November 6. You must start the Test at 6pm**.
- This Test must be **completed within 2 hours** – you will not be able to upload your answer after that time.
- You must **upload your answer as a pdf file**. Photos, jpg files etc will not be accepted. You may wish to install and use a **scanner app on your phone**.
- You must put your **full name and student ID** at the top of your answer.
- Send me an email if you have any questions or encounter any problems.
- You may use any material from the class, including notes, problem sets and recordings. **You may not access material from any other source, and you may not discuss these problems with anyone until they have been submitted**.

$\boxed{\textbf{Questions:}}$

1) Five people play a game, where each person rolls a die once. Find the probability that the number 6 is rolled by at least one person. [Hint: look at the complementary event].

2) Four balls are shared between box #1 and box #2. At each step a biased coin is tossed which comes up Heads with probability p. If the coin comes up Heads and box #1 is not empty, a ball is removed from box #1 and placed in box #2. If the coin comes up Heads and box #1 is empty, no balls are moved. If the coin comes up Tails and box #2 is not empty, a ball is removed from box #2 and placed in box #1. If the coin comes up Tails and box #2 is empty, no balls are moved. Let X_n be the number of balls in box #1 after n steps.

Find the transition matrix for the Markov chain $\{X_n\}$.

3) Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4, 5\}$ in the order presented.

Given that the chain starts in state 1, find the expected number of steps until the first return to state 1.

4) The continuous random variables X and Y are independent. X is uniform on $[2,4]$, and Y is uniform on $[0,1]$. Compute $\mathrm{P}(X \leq Y+2)$. [Hint: first condition on Y to compute $\mathrm{P}(X \leq Y+2 \,|\, Y=y)$, then undo the conditioning on Y].

5) A supply depot has three working computers. The lifetimes of the three computers are independent, and all have exponential distributions with the same mean equal to 1 year. One of the computers fails after year 1, and another computer fails after year 2. Find the probability that the third computer is still working after year 4.

6) Let $A_1, A_2, \ldots$ be a sequence of independent events, and suppose that

$$\sum_{k=1}^{n} \mathrm{P}(A_k) \geq \sqrt{n+4} \qquad \text{for all } n \geq 1.$$

Compute

$$\mathrm{P}(A_n \text{ i.o.})$$

where i.o. means 'infinitely often'. [Hint: use the Borel-Cantelli Lemma]

7) Let $\{X_n\}$ be a Markov chain, and suppose that for state i we have

$$\sum_{k=1}^{n} p_{ii}(k) = \sum_{k=1}^{n} \mathrm{P}(X_k = i \,|\, X_0 = i) = 3 - \frac{9}{\sqrt{n+8}} \quad \text{for all } n \geq 1.$$

Determine whether state i is transient or persistent (explain your reasoning).

8) A biased coin has probability p of coming up Heads. The coin is tossed repeatedly. Let N_3 be the number of tosses until the first occurrence of the sequence (Heads, Tails, Heads), and let N_4 be the number of tosses until the first occurrence of the sequence (Heads, Tails, Heads, Tails).

a) Compute the conditional probability $\mathrm{E}[N_4 \mid N_3 = k]$ for any $k \geq 3$ (your answer should involve k and also $\mathrm{E}[N_4]$).

b) We have

$$\mathrm{E}[N_4] = \sum_{k=3}^{\infty} \mathrm{E}[N_4 \mid N_3 = k] \, \mathrm{P}(N_3 = k)$$

Substitute your answer from part (a) into this formula and compute $\mathrm{E}[N_4]$ (your answer should depend on p, but nothing else). **NOTE:** you should use the result that

$$\mathrm{E}[N_3] = \frac{1}{p} + \frac{1}{p^2(1-p)}.$$