Math 5110 Applied Linear Algebra -Fall 2020.

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Homework 2.

1. Reading: [Gockenbach], Chapters 2 and 3.

Reminder: Two notations of **column** vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (v_1, v_2, v_3)$.

2. Questions: (You can use Matlab to calculate **rref**)

Question 1. Show that $\{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 13 \end{bmatrix}\} \in \mathbb{R}^3$ is linearly dependent by writing one of the vectors as a linear combination of the others.

Solution: Solve $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$
We can get a solution $x_1 = -7; x_2 = -4; x_3 = 1$.
So, $\vec{v}_3 = 7\vec{v}_1 + 4\vec{v}_2$ □

Question 2. Consider the following vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 17 \\ 85 \\ 56 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 3 \\ 16 \\ 13 \end{bmatrix}$$

(a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ spans $\mathbb{R}^3$.

(b) Find a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ that is a basis for $\mathbb{R}^3$.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 17 & 1 & 3 \\ 5 & 5 & 85 & 5 & 16 \\ 4 & 3 & 56 & 2 & 13 \end{bmatrix}$ Then $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & 12 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ So, $\text{rank}(A) = 3$.
(2) $\vec{v}_1, \vec{v}_2, \vec{v}_5$ form a basis of $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\} = \mathbb{R}^3$ □

Question 3. Consider the following vectors in $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 7 \\ -5 \end{bmatrix}$$

(a) Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent.

(b) Find a vector $\vec{u}_4 \in \mathbb{R}^4$ such that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a basis for $\mathbb{R}^4$.

Solution: (1) Let $A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 9 \\ 5 & 9 & 7 \\ 1 & 0 & -5 \end{bmatrix}$ Then $\textbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ So, $\text{rank}(A) = 3$.

(2) Try $\vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ But we need to verify that $\textbf{rref}([\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{e}_4]) = I_4$. □

Question 4. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Show that $S = \text{Span}\{\vec{u}, \vec{v}\}$ is a plane in $\mathbb{R}^3$ by showing there exist constants $a, b, c \in \mathbb{R}$ such that

$$\text{Span}\{\vec{u}, \vec{v}\} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid ax_1 + bx_2 + cx_3 = 0 \right\}$$

Solution: We want to find a, b, c such that $[a\ b\ c]\vec{u} = 0$ and $[a\ b\ c]\vec{v} = 0$. That is to solve linear system $[A\ \vec{0}]$ with

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

We find that one solution is $a = 2, b = -3, c = -1$.
Denote $T = \left\{ \vec{x} \in \mathbb{R}^3 \mid 2x_1 - 3x_2 - x_3 = 0 \right\} = \ker[2\ -3\ -1]$. So T is a subspace. It is clear that $S \subseteq T$.
Since both S and T has dimension 2, so $S = T$. □

Question 5. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix}; \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -4 \\ 0 \end{bmatrix}; \vec{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ be vectors in $\mathbb{R}^5$.

(1) Show that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is linearly independent.

(2) Extend $\vec{u}_1, \vec{u}_2, \vec{u}_3$ to a basis for $\mathbb{R}^5$.

Solution: (1) Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$. Then $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So. $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is independent.

(2) You can try to add $\vec{e}_4$ and $\vec{e}_5$, but we need to check that $\mathbf{rref}([\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{e}_4\ \vec{e}_5]) = I_5$. An more general method is to use decomposition

$$\mathbb{R}^5 = \mathrm{Row}(A^T) \oplus \ker A^T$$

where

$$A^T = \begin{bmatrix} 1 & 4 & 0 & -5 & 1 \\ 1 & 3 & 0 & -4 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}$$

□

Question 6. Consider the following vectors in $\mathbb{R}^5$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_5 = \begin{bmatrix} 2 \\ 10 \\ 3 \\ 6 \\ 2 \end{bmatrix},$$

Let $S = \mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$. Find a subset of $\mathrm{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$ that is a basis for S.

Solution: Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{u}_4\ \vec{u}_5]$. Then

$$\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns form a basis for S. So $\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}$ is a basis for S.

□

Question 7. Apply the row reduction algorithm to solve each of the following systems of equations. In each case, state whether the system has no solution, exactly one solution, or infinitely many solutions. Also, state the rank and nullity of A, where A is the coefficient matrix of the system, and find a basis for $\ker(A) = Nul(A)$ and a basis for $\mathrm{im}(A) = \mathrm{Col}(A)$, where possible.

$$\begin{aligned} -x_1 - 5x_2 + 10x_3 - x_4 &= \ \ 2 \\ 2x_1 + 11x_2 - 23x_3 + 2x_4 &= -4 \\ -4x_1 - 23x_2 + 49x_3 - 4x_4 &= \ \ 8 \\ x_1 + 2x_2 - x_3 + x_4 &= -2 \end{aligned}$$

Solution: $\mathbf{rref}(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 5 & 1 & -2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The system has infinitely many solutions.

rank$(A) = 2$ and *nulity*$(A) = 2$.

A basis of $\ker(A)$ is $\{\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$

A basis of $\mathrm{im}(A)$ is $\{\begin{bmatrix} -1 \\ 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 11 \\ -23 \\ 2 \end{bmatrix}\}$

□

Question 8. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the following conditions:

(a) L is linear; (b) $L(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; (c) $L(\vec{e}_2) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$; (d) $L(\vec{e}_3) = \begin{bmatrix} 7 \\ -3 \\ 9 \end{bmatrix}$;

Here $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Prove that there is a 3×3 matrix A such that $L(x) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. What is the matrix A?

Solution: Proof in lecture notes. The matrix of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $A = [L(\vec{e}_1)\ L(\vec{e}_2)\ L(\vec{e}_3)]$. So, $A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & -3 \\ 3 & 1 & 9 \end{bmatrix}$ □

Question 9. Find bases of the **kernel** and the **image** of the linear map $L : \mathbb{R}^4 \to \mathbb{R}^3$ described by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 0 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

(with respect to the standard bases). Is L injective or surjective? (We already have $\mathbf{rref}(A)$ in homework1.)

Solution:
$\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$

□

Question 10. Define $M : \mathbb{R}^4 \to \mathbb{R}^3$ by $M(x) = \begin{bmatrix} x_1 + 3x_2 - x_3 - x_4 \\ 2x_1 + 7x_2 - 2x_3 - 3x_4 \\ 3x_1 + 8x_2 - 3x_3 - 16x_4 \end{bmatrix}$.

Find the rank and nullity of M.

Solution: The matrix of the transformation is $A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix}$. Then $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. So, $\text{rank}(M) = 3$ and nullity of M is 1.

□

Question 11. Consider the 4×5 matrix

$$A := \begin{bmatrix} -1 & -2 & -1 & 1 & -1 \\ 2 & 4 & 5 & 1 & 2 \\ 1 & 2 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

over a field $\mathbb{F} = \mathbb{R}$.

(a) Find the row reduced echelon form for A.

(b) Find the rank of A.

(c) Find a basis of $\text{im}(f_A)$, where the linear mapping $f_A : \mathbb{F}^5 \to \mathbb{F}^4$ is defined by $f_A(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{F}^5$.

(d) Find a basis of the solution set of $f_A(\vec{x}) = 0$, with f_A as in part (c).

(e*) Solve problems (a)(b)(c)(d) for the case: $\mathbb{F} = \mathbb{Z}_3$. (Optional)

Solution:

(a) $\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\text{rank}(A) = 3$

(c) A basis for $\text{im}(A)$ is $\{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 2 \end{bmatrix}\}$

(d) From $\mathbf{rref}(A)$, $\begin{cases} x_1 = -2x_2 - 2x_5 \\ x_3 = -x_2 + \frac{1}{2}x_5 \\ x_4 = -\frac{1}{2}x_5 \end{cases}$

So,

$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix}$. So, a basis for $\ker(A)$ is $\{\begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix}\}$.

(e) Over $\mathbb{Z}_3$, $\mathbf{rref}(A; \mathbb{Z}_3) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So, $\text{rank}(A; \mathbb{Z}_3) = 2$. A basis for $\text{im}(A; \mathbb{Z}_3) = \{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 2 \end{bmatrix}\}$.

A basis for $\ker(A; \mathbb{Z}_3)$ is $\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\}$.

□

Matlab Input

```
A=[-1  -2  -1  1  -1;
2  4  5  1  2;
1  2  4  4  2;
0  0  0  2  1]

rrefA3 = rrefgf(A,3)
```

Question 12. Let A be an $m \times n$ matrix with real entries, and suppose $n > m$. Prove the linear transformation defined by A is not injective. (That is, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$.)

Solution: There are several different arguments for this question.
$\text{rank}(A) \le m < n$. So $Nullity(A) = n - \text{rank}(A) = n - m > 0$. So the null space has dimension ≥ 1. So, $A\vec{x} = \vec{0}$ has a nontrivial solution $x \in \mathbb{R}^n$. □

Question 13. Find matrix of each linear operator: (Hint: using theorem on matrix of linear transformation.)

(1.) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** of angle θ about the origin (a positive θ indicates a counterclockwise rotation). Find the matrix A such that $R(x) = Ax$ for all $x \in \mathbb{R}^2$.

(2.) Consider the linear operator mapping $\mathbb{R}^2$ into itself that sends each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to its **projection** onto the x-axis, namely, $\begin{bmatrix} x \\ 0 \end{bmatrix}$. Find the matrix representing this linear operator.

(3.) A (horizontal) **shear** acting on the plane maps a **point** $\begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\begin{bmatrix} x + ry \\ y \end{bmatrix}$, where r is a real number. Find the matrix representing this operator.

(4.) A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L(x) = r\vec{x}$ is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$. What is the matrix of L?

Solution: The matrix of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ ...\ L(\vec{e}_n)]$

(1) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$

(4) $\begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix}$ $\square$

Question 14. Consider the following geometrically defined linear maps of $\mathbb{R}^3$ to itself. Describe each of them by a matrix with respect to the canonical basis of $\mathbb{R}^3$. (Hint: using theorem on matrix of linear transformation.)

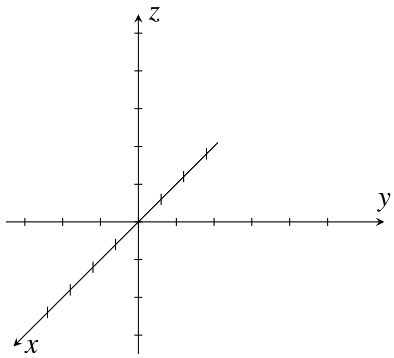
(a) Orthogonal projection onto the xz-plane.

(b) Counterclockwise rotation by 45°about the x-axis.

(c) The map (rotation) of part (b) then followed by the map(projection) of part (a).

(d) Rotation by 120° about the main diagonal in space (spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, taken counterclockwise as you look towards the origin.

Solution: The matrix of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ L(\vec{e}_3)]$, where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, $\vec{e}_3 = (0, 0, 1)$.



(a). $L(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $L(\vec{e}_2) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $L(\vec{e}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$. So, the matrix is $A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$

(b) $L(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $L(\vec{e}_2) = \begin{bmatrix}0\\\cos(\theta)\\\sin(\theta)\end{bmatrix}$ $L(\vec{e}_3) = \begin{bmatrix}0\\-\sin(\theta)\\\cos(\theta)\end{bmatrix}$. So, the matrix is $B = \begin{bmatrix}1 & 0 & 0\\0 & \sqrt{2}/2 & -\sqrt{2}/2\\0 & \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}$

(c) The matrix is $C = AB$.

(d) The plane perpendicular to the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and passing vector $\vec{e}_1$ is $x + y + z = 1$. In fact, this plane also pass $\vec{e}_2$ and $\vec{e}_3$. Draw the triangle and from the geometry, we can see that, $L(\vec{e}_1) = e_2$, $L(\vec{e}_2) = e_3$, $L(\vec{e}_3) = e_1$. So, the matrix is $D = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$

□

Question 15. Let $x \in R^N$ be denoted as $x = (x_1, x_2, ..., x_N)$. Given $\vec{x}, \vec{y} \in \mathbb{R}^N$, the **convolution** of $\vec{x}$ and $\vec{y}$ is the vector $\vec{x} * \vec{y} \in \mathbb{R}^N$ defined by

$$(\vec{x} * \vec{y})_n = \sum_{m=1}^{N} x_m y_{n-m}, \quad \text{for } n = 1, 2, ..., N.$$

In this formula, $\vec{y}$ is regarded as defining a periodic vector of period N; therefore, if $n - m \leq 0$, we take $y_{n-m} = y_{N+n-m}$. For instance, $y_0 = y_N$, $y_{-1} = y_{N-1}$, $y_{-2} = y_{N-2}$, and so forth.

(1) Prove that if $y \in \mathbb{R}^N$ is fixed, then the mapping

$$L : \vec{x} \to \vec{x} * \vec{y}$$

is linear. (2) Find the matrix representing this operator L.

Solution: (1) (i) Check $L(c\vec{x}) = cL(\vec{x})$.

$$[L(c\vec{x})]_n = [c\vec{x} * \vec{y}]_n = \sum_{m=1}^{N} cx_m y_{n-m} = c\sum_{m=1}^{N} x_m y_{n-m} = [cL(\vec{x})]_n$$

(ii) Check $L(\vec{x} + \vec{z}) = L(\vec{x}) + L(\vec{z})$.

$$[L(\vec{x}+\vec{y})]_n = [(\vec{x}+\vec{z})*\vec{y}]_n = \sum_{m=1}^{N}(x_m+z_m)y_{n-m} = \sum_{m=1}^{N} x_m y_{n-m}+z_m y_{n-m} = \sum_{m=1}^{N} x_m y_{n-m}+\sum_{m=1}^{N} z_m y_{n-m} = L(\vec{x})+L(\vec{z})$$

□

Solution: (2) The matrix of the linear transformation $L : \mathbb{R}^N \to \mathbb{R}^N$ is given by $[L(\vec{e}_1)\ L(\vec{e}_2)\ ...\ L(\vec{e}_N)]$

$$[L(\vec{e}_1)]_n = [\vec{e}_1 * \vec{y}]_n = \sum_{m=1}^{N}(\vec{e}_1)_m y_{n-m} = (\vec{e}_1)_1 y_{n-1} = y_{n-1}$$

So, $L(\vec{e}_1) = \begin{bmatrix} y_N \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix}$

$$[L(\vec{e}_2)]_n = [\vec{e}_2 * \vec{y}]_n = \sum_{m=1}^{N}(\vec{e}_2)_m y_{n-m} = (\vec{e}_2)_2 y_{n-1} = y_{n-2}$$

So, $L(\vec{e}_2) = \begin{bmatrix} y_{N-1} \\ y_N \\ y_1 \\ \vdots \\ y_{N-2} \end{bmatrix}$. Similarly, $L(\vec{e}_3) = \begin{bmatrix} y_{N-2} \\ y_{N-1} \\ y_N \\ \vdots \\ y_{N-3} \end{bmatrix}$... and ..., $L(\vec{e}_N) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix}$ So, the matrix of the transformation is

$$\begin{bmatrix} y_N & y_{N-1} & y_{N-2} & \cdots & y_1 \\ y_1 & y_N & y_{N-1} & \cdots & y_2 \\ y_2 & y_1 & y_N & \cdots & y_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{N-1} & y_{N-2} & y_{N-3} & \cdots & y_N \end{bmatrix}$$

□

Question 16. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, $\vec{b} \in \mathbb{R}^3$ is given, and $\vec{u} = (1, 0, 1)$, $\vec{v} = (1, 1, -1)$ are two solutions to $L(x) = b$. Find two more solutions to $L(\vec{x}) = \vec{b}$.

Solution: Since L is linear, $L(\vec{u}) = \vec{b}$ and $L(\vec{v}) = \vec{b}$, hence, $L(2\vec{u} - \vec{v}) = 2L(\vec{u}) - L(\vec{v}) = 2\vec{b} - \vec{b} = \vec{b}$.
So, $2\vec{u} - \vec{v} = (1, -1, 1)$ is another solution.
Similar, we can find infinitely many solutions $\vec{u} + t(\vec{u} - \vec{v})$. □

Question 17. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ has kernel $\ker(T) = \text{Span}\{\vec{u} = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \vec{v} = \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\}$. Suppose further that $T(\vec{y}) = \vec{b}$, where $\vec{y} = \begin{bmatrix}1\\2\\-1\\1\end{bmatrix}, \vec{b} = \begin{bmatrix}3\\1\\-2\\-1\end{bmatrix}$. (1) Find all possible solutions to $T(\vec{x}) = b$. Explain the reason.

(2) Is $\vec{z} = \begin{bmatrix}0\\4\\0\\1\end{bmatrix}$ a solution of $T(\vec{x}) = \vec{b}$?

Solution: (1) All possible solutions we can find are $\vec{x} = \vec{y} + \ker(T) = \vec{y} + s\vec{u} + t\vec{v}$ where s, t are any real numbers.
$T(\vec{x}) = T(\vec{y} + s\vec{u} + t\vec{v}) = T(\vec{y}) + sT(\vec{u}) + tT(\vec{v}) = \vec{b} + s\vec{0} + t\vec{0} = \vec{b}$.
(2) $\vec{z}$ is a solution of $T(\vec{x}) = \vec{b}$ if and only if $T(\vec{z}) = \vec{b}$; if and only if $T(\vec{z} - \vec{y}) = \vec{0}$; if and only if $\vec{z} - \vec{y} = \begin{bmatrix}-1\\2\\1\\0\end{bmatrix} \in \ker(T) = \text{Span}\{\vec{u}, \vec{v}\}$.
Check **rref**(A) where $A = [\vec{u}\ \vec{v}\ \vec{z} - \vec{y}]$, we can see that there is no solution. Hence, $\vec{z}$ is NOT a solution of $T(\vec{x}) = \vec{b}$

□

Question 18. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy $\ker(L) = \text{Span}\{(1, 1, 1)\}$ and $L(\vec{u}) = \vec{v}$, where $\vec{u} = (1, 1, 0)$ and $\vec{v} = (2, -1, 2)$. Which of the following vectors is a solution of $L(\vec{x}) = \vec{v}$?

(a) $\vec{x} = (1, 2, 1)$

(b) $\vec{x} = (3, 3, 2)$

(c) $\vec{x} = (-3, -3, -2)$

Solution: All solutions are $\vec{u} + \ker(L) = \vec{u} + s\begin{bmatrix}1\\1\\1\end{bmatrix}$ where $s \in \mathbb{R}$.
$\vec{x}$ is a solution of $L(\vec{x}) = \vec{v}$ if and only if $\vec{x} - \vec{u} \in \ker(L)$
(1) No. $\vec{x} - \vec{u} = (0, 1, 1)$ is not in $\ker(T)$.
(2)Yes. $\vec{x} - \vec{u} = (2, 2, 2) = 2(1, 1, 1) \in ker(T)$.
(3) $\vec{x} - \vec{u} = (-4, -4, -2)$ is not in $\ker(T)$.

□

Question 19. Consider the linear subspaces U and W of $\mathbb{R}^4$ spanned by $\vec{u}_1 := \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 := \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -3 \end{bmatrix}$

and $\vec{w}_1 := \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{w}_2 := \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \vec{w}_3 := \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \vec{w}_4 := \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ respectively.

Find the **dimensions** of the sum $U + W$, the intersection $U \cap W$, and the quotient spaces $\mathbb{R}^4/U$ and $\mathbb{R}^4/W$.

Solution: Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$, $B = [\vec{w}_1\ \vec{w}_2\ \vec{w}_3\ \vec{w}_4]$, $C = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{w}_1\ \vec{w}_2\ \vec{w}_3\ \vec{w}_4]$
Calculate $\operatorname{rank}(A) = 3$, $\operatorname{rank}(A) = 3$, $\operatorname{rank}(A) = 4$. So, $\dim U = 3$, $\dim W = 3$ and $\dim(U + W) = 4$.
By $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$. $\dim U \cap W = 2$.
$\mathbb{R}^4/U \oplus U = \mathbb{R}^4$. So, $\dim \mathbb{R}^4/U = 1$.
Similarly, $\dim \mathbb{R}^4/W = 1$. □

Question 20. Let V be a vector space over a field $\mathbb{K}$, and let $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors of V such that any $n - 1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent. Show:

(a) There exist scalars $\alpha_1, \ldots, \alpha_n$ in $\mathbb{K}$, **all** nonzero, such that $\sum_{j=1}^n \alpha_j \vec{v}_j = \vec{0}$.

(b) If $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are two sets of nonzero scalars in $\mathbb{K}$ such both $\sum_{j=1}^n \alpha_j \vec{v}_j = 0$ and $\sum_{j=1}^n \beta_j \vec{v}_j = 0$ then there exists a nonzero scalar γ in K such that $\beta_j = \gamma \alpha_j$ for each $j = 1, ..., n$.

Solution: (a) Since $\vec{v}_1, ..., \vec{v}_n$ be n linearly dependent vectors, there exist non-trivial solution $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$
such that $\sum_{j=1}^n c_j \vec{v}_j = \vec{0}$. Claim: all c_j are non-zero. Suppose one of c_j is zero, for example $c_i \neq 0$, then $\vec{v}_1, ..., \widehat{\vec{v}_i}, ..., \vec{v}_n$ is dependent, which contradict the assumption that any $n - 1$ of the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent.
(b) Suppose $\beta_1 = \gamma\alpha_1$. Then

$$\gamma \sum_{j=1}^n \alpha_j \vec{v}_j - \sum_{j=1}^n \beta_j \vec{v}_j = \vec{0}$$

Hence $\sum_{j=1}^n (\gamma\alpha_j - \beta_j)\vec{v}_j = \sum_{j=2}^n (\gamma\alpha_j - \beta_j)\vec{v}_j = \vec{0}$. Since $n - 1$ of the vectors $\vec{v}_2, \vec{v}_3, ..., \vec{v}_n$ are linearly independent, we have $\gamma\alpha_j - \beta_j = 0$ for $j = 2, 3, ..., n$. □