Probability Theory: Finite Markov Chains

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$\boxed{\textbf{Exercise 1}}$ A transition matrix is doubly stochastic if each column sum is 1. Find a stationary distribution for a doubly stochastic chain with M states.

$\boxed{\textbf{Exercise 2}}$ Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

$\boxed{\textbf{Exercise 3}}$ Let $\{X_n\}$ be a regular finite state Markov chain with transition matrix P and stationary distribution w. Define the process $\{Y_n\}$ by $Y_n = (X_{n-1}, X_n)$. Show that $\{Y_n\}$ is a Markov chain, and compute

$$\lim_{n\to\infty} P(Y_n = (i,j)) \tag{1}$$

$\boxed{\textbf{Exercise 4}}$ Determine the equivalence classes of the chain (the transient states, and each closed irreducible class):

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \tag{2}$$

$\boxed{\textbf{Exercise 5}}$ Consider the following transition probability matrix for a Markov chain on 5 states:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0 & 0 & 0.2 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0.4 & 0.2 & 0 \\ 0.3 & 0 & 0.2 & 0 & 0.5 \\ 0.5 & 0.2 & 0 & 0 & 0.3 \end{pmatrix}$$

Number the states $\{1,2,3,4,5\}$ in the order presented.

a). Find and classify the equivalence classes of the states.

b). Find a stationary distribution for the chain. Is it unique?

c). Compute the expected number of steps needed to first return to state 1, conditioned on starting in state 1.

d). Compute the expected number of steps needed to first reach any of the states $\{1,2,5\}$, conditioned on starting in state 3.

Exercise 6 Rework the drunkard's walk on state space $\{0,1,2,3,4\}$, assuming that a step to the right has probability $1/3$ and a step to the left has probability $2/3$.

Exercise 7 [Snell and Grinstead] A city is divided into three areas $1,2,3$. It is estimated that amounts u_1, u_2, u_3 of pollution are emitted each day from these three areas. A fraction q_{ij} of the pollution from region i ends up the next day at region j. A fraction $q_i = 1 - \sum_j q_{ij} > 0$ escapes into the atmosphere. Let $w_i^{(n)}$ be the amount of pollution in area i after n days.
(a) Show that $w^{(n)} = u + uQ + \cdots + uQ^{n-1}$.
(b) Show that $w^{(n)} \to w$.
(c) Show how to determine the levels of pollution u which would result in a prescribed level w.

Exercise 8 [The gambler's ruin] At each play a gambler has probability p of winning one unit and probability $q = 1-p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0? [Hint: define P_i to be the probability that the gambler's fortune reaches N before reaching 0 conditioned on starting in state i. By conditioning on the first step derive a recursion relation between P_i, P_{i+1} and P_{i-1}.]

Exercise 9 Consider a Markov chain on a finite graph. The states are the vertices, and jumps are made along edges connecting vertices. If the chain is at a vertex with n edges, then at the next step it jumps along an edge with probability $1/n$. Argue that the chain is reversible, and find the stationary distribution.

Exercise 10 Consider a set S consisting of $2M$ equally spaced points on a circle, where M is an integer. Two random walkers jump between neighboring points of S; at each time unit they either take a step clockwise with probability p or counterclockwise with probability $1-p$. Successive steps of each walker are independent, and the walkers are independent of each other. Assuming that the walkers start at the same point, what is the expected time until they first meet again?