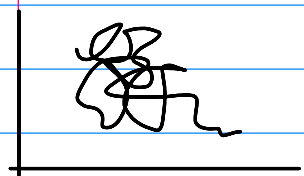
Stochastic Differential Equations (SDE)

Brownian Motion



$\frac{d\beta}{dt} = W(t) \leftarrow$ white noise $\quad E[W(t)] = 0$

$E[W(t)W(s)] = \delta(t-s)\, Q$

MSD $\sim t$

SDE: $\quad \frac{dx}{dt} = F(x,t) + L(x,t)\, W(t)$

$$x(t) = x(t_0) + \int_{t_0}^{t} F(x,t)\,dt + \boxed{\int_{t_0}^{t} L(x,t)\, \underline{W(t)\, dt}}$$

$\int_{t_0}^{t} L(x,t)\, d\beta$

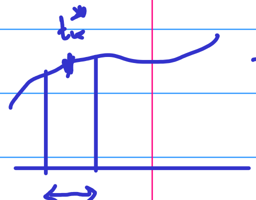
Forward: $\int_0^t L\, d\beta \sim \sum_{k=1}^{n} L_{t_{k-1}} \left[\beta(t_k) - \beta(t_{k-1})\right]$

Backward: $\int_0^t L\, d\beta \sim \sum_{k=1}^{n} L_{t_k} \left[\beta(t_k) - \beta(t_{k-1})\right]$

Cauchy-Schwarz inequality:

$$\sum_{k=1}^{N} \left(L_{t_k} - L_{t_{k-1}}\right)\left(\beta(t_k) - \beta(t_{k-1})\right) \leq \sqrt{\underbrace{\sum_{k=1}^{n} \left(L_{t_k} - L_{t_{k-1}}\right)^2}_{\neq 0} \; \underbrace{\sum_{k=1}^{n} \left(\beta(t_k) - \beta(t_{k-1})\right)^2}_{\sim t}}$$

Riemann Integral: $I = \int_{t_0}^{t} L(x,t)\, d\beta(t) = \lim_{n\to\infty} \sum_k L\left(x(t_k^*), t_k^*\right)\left[\beta(t_{k+1}) - \beta(t_k)\right]$



$t_0 < t_1 < t_2 \cdots < t_n$

$t_k^* \in [t_k, t_{k+1}]$

Ito Integral $\quad t_k^* = t_k \Rightarrow$

$$I = \lim_{n\to\infty} \sum_k L\left(x(t_k), t_k\right)\left[\beta(t_{k+1}) - \beta(t_k)\right]$$

Ito Chain Rule $\quad Y = f(t,x)$

$$dY = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\, dx\, dx$$

$\sigma^2(dx) \sim t$

Black Scholes equation: An exactly solvable example

$dx = \mu x dt + \sigma x dW(t)$

$$\boxed{x(t) = x_0 \exp\left[\mu t - \frac{1}{2}\sigma^2 t + \sigma \eta(t)\right]}$$

$x(0) = x_0$

$\sigma^2(dW(t)) = dt$

$$\eta(t) = \int_0^t dW(t)$$

Consider the growth model:

$$\frac{dx}{dt} = \mu x$$

$x(t) = x_0 \exp[\mu t]$

$\mu > 0 \quad x \to \infty$

$\mu < 0 \quad x \to 0$

$\mu = 0 \quad x = x_0$

In presence of noise $\mu > \sigma^2/2 \quad x \to \infty$

$\mu < \sigma^2/2 \quad x \to 0$

Stochastic Stabilization !!

Euler Maruyama Method

$$dX(t) = a(t,x)dt + b(t,x)dW(t) \qquad X(t=0) = X_0$$

$$\begin{cases} y_0 = X_0 \\ y_{i+1} = y_i + a(t_i, y_i)\Delta t_{i+1} + b(t_i, y_i)\Delta W_{i+1} \end{cases}$$

$\Delta t_{i+1} = t_{i+1} - t_i$

$\Delta W_i = Z_i \sqrt{\Delta t} \qquad Z_i = N(0,1)$

$\sigma^2(Z) = 1$

$\sigma^2(\sqrt{\Delta t}\, Z) = \Delta t$

$\sigma^2(\Delta W) = \Delta t$

For Black Scholes:

$x_0 = X_0$

$x_{i+1} = x_i + \mu x_i \Delta t_i + \sigma x_i \Delta W_{i+1}$

Birth-Death Process

$$DNA \xrightarrow{k_1} mRNA \xrightarrow{k_2} \phi$$

$d \rightarrow$ DNA concentration

$n \rightarrow$ mRNA concentration

Kinetic Rate Equation:

$$\frac{dn}{dt} = k_1 d - k_2 n \quad \circledast$$

$$n(t) = \frac{k_1 d}{k_2}\left[1 - \exp(-k_2 t)\right]$$

$$n_{st} = \frac{k_1}{k_2} \qquad [d] = 1$$

Stochastic Process

P_n = probability of mRNAs

Master Equation: $\displaystyle \frac{dP_n}{dt} = \sum_{n'} \left[W_{nn'} P_{n'} - W_{n'n} P_n\right]$

Transition Rates: $\left.\begin{aligned} W_{n+1,n} &= k_1 \\ W_{n-1,n} &= k_2 n \end{aligned}\right\}$

$$\frac{dP_n}{dt} = \underbrace{k_2(n+1)P_{n+1} + k_1 P_{n-1}}_{\text{gain terms}} \underbrace{- n k_2 P_n - k_1 P_n}_{\text{loss terms}}$$

Mean $\displaystyle \langle n \rangle = \sum_{n=0}^{\infty} n P_n \Rightarrow \frac{d\langle n \rangle}{dt} = k_1 - k_2 \langle n \rangle \Leftarrow \circledast \times P_n$

$$\langle n \rangle_{st} = \frac{k_1}{k_2}$$ mean evolves deterministically.

Steady state solv. $\displaystyle P_n = \frac{1}{n!}\left(\frac{k_1}{k_2}\right)^n \exp\left[-k_1/k_2\right]$

Poisson Process: mean $= k_1/k_2$