Power Method : Dominant Eigenvalue of a matrix.

$\{V_k\} \rightarrow$ set of linearly independent vectors

$$AV_k = \lambda_k V_k \qquad \lambda_1 > \lambda_2 > \lambda_3 \cdots > \lambda_N$$

Any vector by superposition:

$$X_0 = C_1 V_1 + C_2 V_2 + C_3 V_3 + \cdots + C_N V_N$$

$$AX_0 = C_1 \lambda_1 V_1 + C_2 \lambda_2 V_2 + \cdots + C_N \lambda_N V_N$$

$$A^2 X_0 = C_1 \lambda_1^2 V_1 + C_2 \lambda_2^2 V_2 + \cdots + C_N \lambda_N^2 V_N$$

$$\vdots$$

$$A^m X_0 = C_1 \lambda_1^m V_1 + C_2 \lambda_2^m V_2 + \cdots + C_N \lambda_N^m V_N$$

$$\boxed{\frac{A^m X_0}{\lambda_1^m} = C_1 V_1} + \underbrace{C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^m + \cdots + C_N \left(\frac{\lambda_N}{\lambda_1}\right)^m V_N}_{m \to \infty \quad \sim 0}$$

$$\left.\begin{aligned} \frac{A^m X_0}{\lambda_1^m} \cdot Y &= C_1 V_1 Y \\ \frac{A^{m+1} X_0}{\lambda_1^{m+1}} \cdot Y &= C_1 V_1 Y \end{aligned}\right\} \qquad \lambda_1^{(m)} = \frac{A^{m+1} X \cdot Y}{A^m X \cdot Y}$$

$$\Psi_1 = A^m X_0$$

* Works for non-degenerate matrices

- Convergence depends on λ_2 / λ_1 $\qquad \lim_{m \to \infty} \lambda_1^{(m)} \rightarrow \lambda$
- check $A\Psi_1 = \lambda_1 \Psi_1 \longrightarrow \{\lambda_1, \Psi_1\}$ eigenpair
- scaling $\qquad \{A^m X_0\} \Rightarrow \{a_1, a_2 \cdots a_N\} \times \frac{1}{\max\{a_i\}}$

Method of Deflation : finding the non-dominant eigenvalue

$A \rightarrow \{\lambda_1, \psi_1\}$ is known by power method

Define $\psi_1 \equiv \dfrac{\psi_1}{\|\psi_1\|}$ and construct $\hat{A} = A - \lambda_1 \psi_1 \psi_1^T$

$$A = \{\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_N\} \qquad \hat{A} = \{0, \lambda_2, \lambda_3 \cdots \lambda_N\}$$

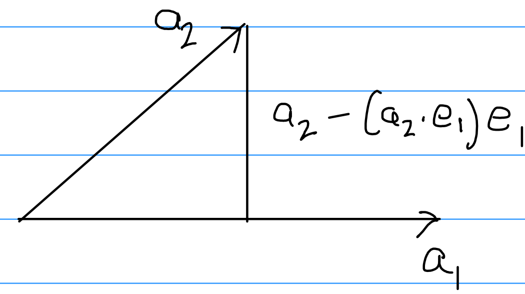
$$\{\psi_1, \psi_2, \psi_3 \cdots \psi_N\} \qquad \{\psi_1, \hat{\psi}_2, \hat{\psi}_3 \cdots \hat{\psi}_N\}$$

\# $\hat{A}\psi_1 = A\psi_1 - \lambda_1 \psi_1 \psi_1^T \psi_1 = 0$ '0' $\rightarrow$ eigenvalue

\## $\hat{A}\psi_k = A\psi_k$ $\quad \because (\psi_1, \psi_k) = 0$ orthonormal set

$= \lambda_k \psi_k$

Gram - Schmidt Process



$$A = [\, a_1 \,|\, a_2 \,|\, \cdots \,|\, a_N \,]$$

$$u_1 = a_1 \qquad e_1 = \frac{u_1}{\| u_1 \|}$$

$$u_2 = a_2 - (a_2 \cdot e_1) e_1 \qquad e_2 = \frac{u_2}{\| u_2 \|}$$

$$\vdots$$

$$u_k = a_k - \sum_{j=1}^{k-1} (a_k \cdot e_j) e_j \qquad e_k = \frac{u_k}{\| u_k \|}$$

$$a_1 = (e_1, a_1) e_1$$

$$a_2 = (e_1, a_2) e_1 + (e_2, a_2) e_2$$

$$\vdots$$

$$a_k = \sum_{j=1}^{k} (e_j, a_k) e_j$$

In matrix form

$$A = [\, a_1 \,|\, a_2 \,|\, \cdots \,|\, a_N \,] = \underbrace{[\, e_1 \,|\, e_2 \,|\, \cdots \,|\, e_N \,]}_{Q} \underbrace{\begin{bmatrix} (e_1, a_1) & (e_1, a_2) & (e_1, a_3) & \cdots \\ 0 & (e_2, a_2) & (e_2, a_3) & \cdots \\ 0 & 0 & (e_3, a_3) & \cdots \\ 0 & 0 & & \ddots \end{bmatrix}}_{R}$$

$$A = Q \cdot R \rightarrow \underline{QR \text{ decomposition}}$$

QR Algorithm : Eigenvalues of a matrix.

QR factorize : $A = A_1 = Q_1 R_1$ $\xrightarrow{\text{k steps}}$ $A_k = Q_k R_k$

Define : $A_2 = R_1 Q_1$ $\qquad$ $A_{k+1} = R_k Q_k$

for $k \gg 1$ $\quad A_k$ converges to a triangular matrix $\quad \text{diag}(A_k) \approx \{\lambda_k\}$

★ $A_1 = Q_1 R_1$

$$\Rightarrow Q_1^* A_1 Q_1 = Q_1^* Q_1 R_1 Q_1$$

$$= R_1 Q_1$$

$$\left.\begin{array}{l} Q_1^* A_1 Q_1 = A_2 \longrightarrow A_1, A_2 \rightarrow \text{similar} \\ \vdots \\ Q_k^* A_k Q_k = A_{k+1} \longrightarrow A_k, A_{k+1} \rightarrow \text{similar} \end{array}\right\} A, A_k \rightarrow \text{similar}$$

★ Proof of A_k converges to a triangular matrix see notes below

Useful Identities

$A_{k+1} = Q_k^* A_k Q_k$

$$= Q_k^* Q_{k-1}^* A_{k-1} Q_{k-1} Q_k$$

$$= Q_k^* \cdots Q_1 A_1 Q_1 \cdots Q_k$$

$$= P_k^* A_1 P_k \quad (1)$$

Let, $P_k = Q_1 Q_2 \cdots Q_k$

$T_k = R_k R_{k-1} \cdots R_1$

$P_k T_k = P_{k-1} Q_k R_k T_{k-1}$

$$= P_{k-1} A_k T_{k-1}$$

$$= A_1 P_{k-1} T_{k-1}$$

$$\vdots$$

$$= (A_1)^k \quad (2)$$

$A_{k+1} = P_k^* A_1 P_k$ from (1)

$P_k A_{k+1} = A_1 P_k$

$\leftarrow \; P_{k-1} A_k = A_1 P_{k-1}$

Proof of convergence of A_k to a triangular matrix:

Let, $A = X \Lambda X^{-1}$ $\quad \Lambda = \{\lambda_1, \lambda_2 \ldots \lambda_n\}$

$X = Q_x R_x$ $\quad Q_x$ - orthogonal $\quad R_x$ - upper triangular

$X^{-1} = L_y R_y$ $\quad L_y$ - lower triangular $\quad R_y$ - upper triangular

(2) $\Rightarrow$

$$
\begin{aligned}
P_k T_k = A^k &= X \Lambda^k X^{-1} \\
&= Q_x R_x \Lambda^k L_y R_y \\
&= Q_x R_x (\Lambda^k L_y \Lambda^{-k})(\Lambda^k R_y) \\
&= Q_x R_x (I + E_k) \Lambda^k R_y \qquad \text{From (3)} \\
&= Q_x [I + R_x E_k R_x^{-1}] R_x \Lambda^k R_y \\
&= Q_x [I + G_k] R_x \Lambda^k R_y \\
&= Q_x \underbrace{[I + Z_k][I + W_k]}_{\text{QR factorization of } G_k} R_x \Lambda^k R_y
\end{aligned}
$$

QR factorization is unique $\quad P_k \equiv Q_x [I + Z_k]$

Limit $k \to \infty \Rightarrow E_k \to 0 \Rightarrow G_k \to 0 \Rightarrow Z_k \to 0 \Rightarrow P_k \to Q_x$

$$
\begin{aligned}
A_k = P_k^{-1} A P_k \to Q_x^{-1} A Q_x &= Q_x^{-1} X \Lambda X^{-1} Q_x \\
&= Q_x^{-1} Q_x R_x \Lambda R_x^{-1} \\
&= R_x \Lambda R_x^{-1} \to \text{upper triangular}
\end{aligned}
$$

$\square$

$$
(\Lambda^k L \Lambda^{-k})_{ij} = \begin{cases} l_{ij} \left(\dfrac{\lambda_i}{\lambda_j}\right)^k & i > j \\ 1 & i = j \\ 0 & i < j \end{cases}
$$

$\lim_{k \to \infty} \Lambda^k L \Lambda^{-k} \to I$

$\Lambda^k L \Lambda^{-k} = I + E_k$ (3)