1. Let $f(x)$ be the density function of X. First suppose $c > m$.

Now $$E(|X-c|) = \int_{-\infty}^{\infty} |x-c| f(x)\,dx$$

$$= \int_{-\infty}^{c} |x-c| f(x)\,dx + \int_{c}^{\infty} |x-c| f(x)\,dx$$

$$= \int_{-\infty}^{c} (c-x) f(x)\,dx + \int_{c}^{\infty} (x-c) f(x)\,dx$$

$$= \int_{-\infty}^{m} (c-x) f(x)\,dx + \int_{m}^{c} (c-x) f(x)\,dx + \int_{m}^{\infty} (x-c) f(x)\,dx - \int_{m}^{c} (x-c) f(x)\,dx$$

$$= \int_{-\infty}^{m} (m-x+c-m) f(x)\,dx + \int_{m}^{c} (c-x) f(x)\,dx + \int_{m}^{\infty} (x-m+m-c) f(x)\,dx - \int_{m}^{c} (x-c) f(x)\,dx$$

$$= \int_{-\infty}^{m} (m-x) f(x)\,dx + \int_{m}^{\infty} (x-m) f(x)\,dx + (c-m)\left[\int_{-\infty}^{m} f(x)\,dx - \int_{m}^{\infty} f(x)\,dx\right] + 2\int_{m}^{c} (c-x) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-m| f(x)\,dx + (c-m)\left[F(m) - 1 + F(m)\right] + 2\int_{m}^{c} (c-x) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-m| f(x)\,dx + 2\int_{m}^{c} (c-x) f(x)\,dx$$

$$\geq E|X-m|.$$

Sim. for $c < m$, we get,

$$E\,|X - c| = E\,|X - m| + 2\int_{c}^{m}(x-c)f(x)\,dx$$

$$\geq E\,|X - m| \,.$$

2. $F_n(x) = 0$ for $x < 0$

$= \frac{1}{n}$ for $0 \leq x < n$

$= 1$ for $x \geq n$.

Then $F_n(x) \to 0$ for all x.

Thus X_n cannot converge in distribution to any RV X.

3. $$P(|X_n - X| > \epsilon) \leq \frac{E|X_n - X|^r}{\epsilon^r} \quad \text{(By Markov Ineq.)}$$

$$\to 0 \text{ as } n \to \infty.$$

Thus $\{X_n\}$ converges to X in probability.

4. a) Follows from the facts that

$$\left|\int_{-\infty}^{\infty} f(x)\,dx\right| \le \int_{-\infty}^{\infty} |f(x)|\,dx \;\&$$

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n| .$$

b) $$|E(X_n) - E(X)| = |E(X_n - X)| \le E|X_n - X|$$

Hence proved.

c) Let $\{X_n\}$ be a seq of RVs defined on a probability space $(S, \mathcal{F}, P)$ such that,

$$P(X_n = n) = \frac{1}{2n}$$

$$P(X_n = -n) = \frac{1}{2n}$$

$$P(X_n = 0) = 1 - \frac{1}{n}.$$

Then $0 = EX_n \to EX$ where X is the zero RV.

But $E|X_n - X| = E|X_n| = 1 \not\to 0$.

$\boxed{Q5}$ For $t \neq 0$,

$$M_{X_n}(t) = E(e^{tX_n})$$

$$= \frac{1}{2^n} \sum_{k=1}^{2^n} e^{tk/2^n}$$

$$= \frac{1}{2^n} e^{t/2^n} \sum_{k=0}^{2^n-1} e^{tk/2^n}$$

$$= \frac{e^{t/2^n}}{2^n} \times \frac{e^t - 1}{e^{t/2^n} - 1}$$

$$= \frac{e^{t/2^n}(e^t - 1)}{t \times \frac{e^{t/2^n} - 1}{t/2^n}} \longrightarrow \frac{e^t - 1}{t}$$

Now if $X \sim U(0,1)$, $M_X(t) = \frac{e^t - 1}{t}$ if $t \neq 0$

Hence $X_n \longrightarrow X$ in distⁿ. $\square$

$\boxed{Q6}$ Note that $S_n^2 = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right]$.

Now $E(X_i^2) = \sigma^2 + \mu^2$, where $\mu = E(X_1)$.

Using ~~a~~ SLLN,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \longrightarrow \sigma^2 + \mu^2 \quad \text{almost surely.}$$

$$\bar{X} \longrightarrow \mu \quad \text{almost surely}$$

$$\Rightarrow \bar{X}^2 \longrightarrow \mu^2 \quad \text{almost surely } [\text{as } g(x) = x^2 \text{ is continuous}]$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \longrightarrow \sigma^2 \quad \text{almost surely.}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \longrightarrow \sigma^2 \quad \text{almost surely}$$

Now $S_n = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right] \longrightarrow \sigma^2$ almost surely $\square$

as $\frac{n}{n-1} \longrightarrow 1$ as $n \longrightarrow \infty$.

$\boxed{Q7}$ $E(\bar{X}_n) = \mu \quad \forall n$,

$Var(\bar{X}_n) = \sigma^2/n \quad \forall n$.

Using Chebyshev's inequality

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

As prob ≥ 0, $\lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$

$\Rightarrow \bar{X}_n \longrightarrow \mu$ in probability. $\square$

$\boxed{Q8}$ Define $Y_n = X_{2n-1} X_{2n}$.

Then $E(Y_n) = E(X_{2n-1}) E(X_{2n}) = 0. \quad \forall n$

$$Var(Y_n) = E(Y_n^2) = E(X_{2n-1}^2) E(X_{2n}^2) = 1. \quad \forall n$$

As $\{X_n\}$ are i.i.d., $\{Y_n\}$ are also i.i.d.

Using CLT, $\left\{ \sqrt{n} \frac{\overline{Y}_n - 0}{\sqrt{1}} = \frac{X_1 X_2 + X_3 X_4 + \cdots + X_{2n-1} X_n}{\sqrt{n}} \right\}$

converges in dist to Z, where $Z \sim N(0,1)$. ①

Now define $U_n = X_n^2 \quad \forall n$.

Then $E(U_n) = E(X_n^2) = 1$.

Using SLLN, $\overline{U}_n = \frac{X_1^2 + \cdots + X_n^2}{n} \longrightarrow 1$ ~~in~~ almost surely. ②

Using ① and ②,

$$\sqrt{n} \frac{X_1 X_2 + \cdots + X_{2n-1} X_{2n}}{X_1^2 + \cdots + X_n^2} \longrightarrow Z \text{ in distribution} \quad \square$$

$\boxed{Q9}$ Using CLT, $\frac{\sqrt{n}(\overline{X}_n - \alpha)}{\sigma} \longrightarrow Z$ in dist^n, where

$Z \sim N(0,1)$.

Using SLLN, $\overline{Y}_n \longrightarrow \beta$ ~~in~~ almost surely.

$$\Rightarrow \frac{\overline{Y}_n}{\beta} \longrightarrow 1 \text{ almost surely. (as } \beta \neq 0).$$

Hence $\frac{\beta}{\sigma} \times \frac{\sqrt{n}(\overline{X}_n - \alpha)}{\overline{Y}_n} \longrightarrow Z$ in dist^n

$$\Rightarrow \frac{\sqrt{n}(\overline{X}_n - \alpha)}{\overline{Y}_n} \longrightarrow Z_1 \text{ in dist}^n \text{; where}$$

$$Z_1 \sim N\left(0, \frac{\sigma^2}{\beta^2}\right).$$

Q10 $\sqrt{n}\,\dfrac{\overline{X}_n - \mu}{\sigma} \longrightarrow Z \sim N(0,1)$ in dist^n.

Using problem 6, $S_n^2 \longrightarrow \sigma^2$ almost surely.

Take $f(x) = \sqrt{x}$ for $x > 0$.

$f(\cdot)$ is a continuous function.

Hence $f(S_n^2) \longrightarrow f(\sigma^2)$ almost surely.

$$\Rightarrow \quad S_n \longrightarrow \sigma \quad \text{almost surely.}$$

$$\Rightarrow \quad \frac{S_n}{\sigma} \longrightarrow 1 \quad \text{almost surely (Taking } g(x) = \tfrac{x}{\sigma}\text{)}$$

Hence $\sqrt{n}\,\dfrac{\overline{X}_n - \mu}{S_n} \longrightarrow Z \sim N(0,1)$ in distribution. $\square$

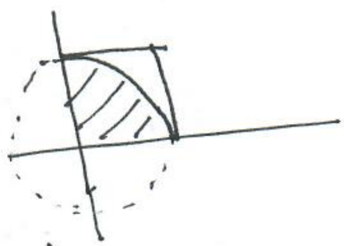
Q11 Define for $i = 1, 2, \ldots$

$$Z_n = \begin{cases} 1 & \text{if } X_n^2 + Y_n^2 \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Clearly $N_n = \sum_{i=1}^{n} Z_i$. Moreover $\{Z_i\}$ is a sequence of ~~RVs~~ i.i.d. RVs. with

$E(Z_n) = P(X_n^2 + Y_n^2 \le 1)$ ~~= Area of a circle of unit radius~~

$= \pi/4$



Using SLLN,

$$\overline{Z}_n \longrightarrow \frac{\pi}{4} \quad \text{almost surely}$$

$$\Rightarrow \quad \frac{4N_n}{n} \longrightarrow \pi \quad \text{almost surely.} \quad \square$$

$\boxed{\text{Q12}}$ Let ~~X_1, X_2~~ $\{X_n\}$ be a seq of ~~R~~ i.i.d RV with $U(0,1)$ distribution. Then

$$\sqrt{n}\,\frac{\bar{X}_n - \frac{1}{2}}{\sqrt{1/12}} \longrightarrow Z \sim N(0,1) \quad \text{in distribution}$$

$$\Rightarrow \sqrt{12n}\,(\bar{X}_n - \tfrac{1}{2}) \longrightarrow Z \sim N(0,1) \quad \text{in distribution.}$$

$$P\left(\sum_{i=1}^{50} X_i > 30\right) = P\left(\bar{X}_{50} > \tfrac{3}{5}\right)$$

$$= P\left(\sqrt{12\times 50}\,(\bar{X}_{50} - \tfrac{1}{2}) > \sqrt{12\times 50}\,(\tfrac{3}{5} - \tfrac{1}{2})\right)$$

$$= P\left(\sqrt{12\times 50}\,(\bar{X}_{50} - \tfrac{1}{2}) > \sqrt{6}\right)$$

$$\simeq 1 - \Phi(\sqrt{6}) = 0.0072. \qquad \square$$

$\boxed{\text{Q13}}$ Consider a sequence of i.i.d RVs $\{X_n\}$ where $X_n \sim P(1)$.

~~Then using CLT,~~

$$\cancel{\lim_{n\to\infty} P(\sqrt{n}(\bar{X}_n - 1) \le 0) = \Phi(0) = \tfrac{1}{2}}.$$

Now, $\sum_{i=1}^{n} X_i \sim P(n) \quad \forall n.$

~~Again~~ $\cancel{P(\sqrt{n}(\bar{X}_n - 1) \le 0)}$

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i \le n\right)$$

$$= \lim_{n\to\infty} P\left(\bar{X}_n \le 1\right)$$

$$= \lim_{n\to\infty} P\left((\bar{X}_n - 1) \le 0 \right)$$

$$= \lim_{n\to\infty} P\left(\sqrt{n}(\bar{X}_n - 1) \le 0 \right)$$

$$= \frac{1}{2}$$ ~~as Et using CLT~~

As using CLT, $\sqrt{n}(\bar{X}_n - 1) \longrightarrow Z \sim N(0,1)$ in $dist^n$. $\square$