DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA221: Discrete Mathematics, July - November 2017

Practice Problems: Combinatorics

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1. There are n married couples. How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple.

2. Show that any set of 7 distinct integers includes 2 integers x and y such that either $x+y$ or $x-y$ is divisible by 10.

3. How many among the first $100,000$ positive integers contain exactly one 3, one 4, and one 5 in their decimal representation? [**Ans. 2940**]

4. A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find (a) the total number of passwords that can be created, and (b) the number of passwords in which no digit repeats. [**Ans. 286000, 149760.**]

5. If a set X has $2n+1$ elements, find the number of subsets of X with at most n elements. [**Ans.** 2^{2n}]

6. Find the number of binary sequences of length n that contain an even number of 1 s. [**Ans.** 2^{n-1}]

7. Ten different paintings are to be allocated to n office rooms so that no room gets more than 1 painting. Find the number of ways of accomplishing this, if (a) $n=14$ and (b) $n=6$. [**Ans.** $P(14,10)$, $P(10,6)$**.**]

8. Establish Newton's identity combinatorially: $C(n,r)C(r,k)=C(n,k)C(n-k,r-k)$.

9. Prove that if n is a prime number then $C(n,r)$ is divisible by n for $r=1,2,...,n-1$.

10. Use a combinatorial argument to prove the following:

    (a) $C(n,r)=C(n,n-r)$;

    (b) $\sum\limits_{r=0}^{n} C(n,r)=2^n$, $\sum\limits_{r=0}^{n}(-1)^n C(n,r)=0$, $\sum\limits_{r \text{ even}}^{n} C(n,r)=\sum\limits_{r \text{ odd}}^{n} C(n,r)=2^{n-1}$;

    (c) $C(p+q,r)=\sum\limits_{j=0}^{r} C(p,j)C(q,r-j)$; [Convolution Rule or Vandermonde Identity]

    (d) $C(n+r+1,r)=\sum\limits_{j=0}^{r} C(n+j,j)$;

    (e) $C(m+n,n)=C(m,0)C(n,0)+C(m,1)C(n,1)+\ldots+C(m,n)C(n,n)$;

    (f) $C(n,0)^2+C(n,1)^2+\ldots+C(n,n)^2=C(2n,n)$;

    (g) $C(n,r)=C(r,r)C(n-r,0)+C(r,r-1)C(n-r,1)+\ldots+C(r,0)C(n-r,r)$.

11. Prove that $C(pn,pn-n)$ is a multiple of p in two ways. [Hint. Newton's Identity]

12. Find the number of 5-digit positive integers such that in each of them every digit is greater than the digit to its right. [**Ans.** $C(10,5)$]

13. There are 3 apartments A, B, and C-for rent in a building. Each unit will accept either exactly 3 or exactly 4 occupants. Find the number of ways of renting the apartments to 10 students. [**Ans.** $3C(10,4)C(6,3)$]

14. Find the number of ways of seating m women and n men $(m < n)$ at a round table so that no 2 women sit side by side. [**Ans.** $(n-1)!P(n,m)$]

15. Let $1, 2, 3, ..., n$ be n fixed points on the circumference of a circle. Each of these points is joined to every one of the remaining $n-1$ points by a straight line, and the points are so positioned on the circumference that at most 2 straight lines meet in any interior point of the circle. Find the number of such interior intersection points. [**Ans.** $C(n,4)$]

16. Show that $(m!)^n$ divides $(mn)!$. [Use generalized permutation.]

17. How many arrangements of the alphabates in $MISSISSIPPI$ are there so that no two S are adjacent? [**Ans.** $P(11; 1, 2, 4, 4)$]

18. How many integer solutions are there to the equation $x + y + z = 18$ such that $0 \leq x, y, z \leq 9$? [**Use ordinary generating function**]

19. How many integer solutions are there to the equation $x + y + z + w = 40$ such that $1 \leq x \leq 5, \quad 2 \leq y \leq 7, \quad 3 \leq z \leq 9, \quad 5 \leq w$? [**Use ordinary generating function**]

20. Find the ordinary generating functions for the following sequences:

    (a) $(1, -1, 1, -1, ...)$;

    (b) $(1, 2, 3, 4, ...)$; and

    (c) $(1, -2, 3, -4, ...)$.

    [**Ans.** $(1+x)^{-1}, \quad (1-x)^{-2}, \quad (1+x)^{-2}$.]

21. Give the ordinary generating function of the sequence $(n(3+5n))$. [**Ans.** $(8x+2x^2)(1-x)^{-3}$]

22. Find the ordinary generating function $f(x)$ that can be associated with the combinatorial problem of finding the number of solutions in positive integers of the equation $x + y + z + w = r$.

23. Find the ordinary generating function $f(x)$ that can be associated with the combinatorial problem of finding the number of solutions in integers of the inequality $x + y + z \leq r$, where $2 \leq x, y, z \leq 5$.

24. Find the number of ways of distributing 8 apples and 6 oranges to 3 children so that each child can get at least 2 apples and at most 2 oranges. **Ans.** [**6**]

25. Find the ordinary generating functions associated with the sequences (2^r) and $(r2^r)$.

26. Find the exponential generating functions of

(a) $(0, 0, 1, 1, 1, 1, , \ldots)$;

(b) $(1, a, a^2, a^3, \ldots)$;

(c) $(0, 1, 2a, 3a^2, 4a^3, \ldots)$.

[**Ans.** $e^x - x - 1, \quad e^{ax}, \quad xe^{ax}$]

27. If a leading digit of 0 is permitted, find the numbers of r-digit binary numbers that can be formed using (a) an even number of 0's and an even number of 1's; (b) an odd number of 0's and an odd number of 1's. [**Ans.** $\left(\frac{e^x+e^{-x}}{2}\right)^2, \quad \left(\frac{e^x-e^{-x}}{2}\right)^2$]

28. Find the number of r-letter sequences that can be formed using the letters P, Q, R, and S such that in each sequence there are an odd number of P's and an even number of Q's. [**Ans.** $\frac{1}{4}(e^{4x} - 1)$]

29. Let $X = \{A, B, C, D\}$. Using exponential generating functions, obtain

    (a) the number of r-permutations that can be formed using these four letters such that in each permutation there is at least one A, at least one B and at least one C; and

    (b) the number of r-permutations that can be formed using these four letters such that in each permutation there is an even even number of A's and an odd number of B's.

30. There are n lines drawn in a plane such that no 2 lines are parallel and no 3 lines are concurrent. If the plane is thereby divided into a_n regions, find a recurrence relation for (a_n). Also, solve the recurrence relation. [**Ans.** $a_n = a_{n-1} + n, \quad a_1 = 2$]

31. Let a_n be the number of n-letter sequences that can be formed using the letters A, B, and C such that any nonterminal A has to be immediately followed by a B. Find the recurrence relation for (a_n) and the corresponding generating function. Also, solve the recurrence relation. [**Ans.** $a_n = 2a_{n-1} + a_{n-2}, \quad a_1 = 3, a_2 = 7$]

32. Find the ordinary generating functions of the following sequences:

$$(a) \;\; (n(n-1)) \quad (b) \;\; (n^2) \quad (c) \;\; (n^3) \quad (d) \;\; ((n+1)n(n-1)).$$

33. Find the ordinary generating function of the numbers of solutions in integers of $u_1 + u_2 + \ldots + u_{10} = n$, where each variable is at least -2 and at most 2.

34. Find the ordinary generating function of the numbers of solutions in integers of $x+y+z = r(r = 0, 1, 2, ...)$, if (a) each variable is nonnegative and at most 3; (b) each variable is at least 2 and at most 5; and (c) $0 \le x \le 6, 2 \le y \le 7, 5 \le z \le 8, x$ is even and y is odd.

35. Find the ordinary generating function of the sequence defined by $a_{n+2} - 3a_{n+1} - 4a_n = 0$ with $a_1 = 1, a_2 = 3$. [**Ans.** $\frac{x}{1-3x-4x^2}$]

36. Let a_n be the number of n-letter sequences that can be formed using the letters P, Q, R, S, and T, if each sequence must involve an odd number of P's. Find a recurrence relation for (a_n). Also, solve the recurrence relation. [**Ans.** $a_n = 3a_{n-1} + 5^{n-1}, \quad a_1 = 1$]

37. Solve the following recurrence relations:

(a) $a_n = a_{n-2} + 4n$ with $a_0 = 3, a_1 = 2$;

(b) $a_n = 3a_{n-1} - 4a_{n-3}$ with $a_0 = 0, a_1 = 2, a_2 = -1$;

(c) $a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = 1, a_1 = 2$;

(d) $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1, a_1 = 4$;

(e) $a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n$ with $a_0 = 0, a_1 = -11, a_2 = -15$;

(f) $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$ with $a_0 = 3, a_1 = 6, a_2 = 14$;

(g) $a_{n+3} = 4a_{n+2} - 5a_{n+1} + 2a_n$ with $a_0 = 2, a_1 = 4, a_2 = 7$;

(h) $a_n = a_{n-1} + 2a_{n-2} + 4(3^n)$ with $a_0 = 11, a_1 = 28$;

(i) $a_n = 4(a_{n-1} - a_{n-2}) + 2^n$;

(j) $a_n = a_{n-1} + 6a_{n-2} + 3n$ with $a_0 = a_1 = 0$;

(k) $a_n = a_{n-1} + 6a_{n-2} + 2^n$ with $a_0 = a_1 = 0$;

(l) $a_n = a_{n-1} + 2a_{n-2} + 2n$ with $a_0 = a_1 = 0$;

(m) $a_n = a_{n-1} + 2a_{n-2} + 2^n$ with $a_0 = a_1 = 0$;

(n) $a_n = 3a_{n-1} - 4n + 3(2^n)$;

(o) $a_{n+2} = 4a_{n+1} - 3a_n + 16$ with $a_0 = 4, a_1 = 2$.

38. Let a_n be the number of subsets of a set that has n elements. Find a recurrence relation for a_n. Also, solve the recurrence relation.

39. The roots of the characteristic equation of a linear homogeneous recurrence relation with constant coefficients are $1, 2, 2$ and 3. Write down the relation and its general solution.

40. Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = h(n)$, where

$$(a)\ h(n) = 1,\quad (b)\ h(n) = n,\quad (c)\ h(n) = 3^n,\quad (d)\ h(n) = 2^n,\quad (e)\ h(n) = 1 + n + 2^n + 3^n.$$

41. Solve $a_n = 4a_{n-1} + 5(3^n)$.

42. Solve $a_n = 4a_{n-1} + 5(4^n)$.

43. Use generating function to solve the recurrence relations that appear in the previous problems. (As many as you can.)

For the following problems on Fibonacci numbers, define a_n by $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

44. For the Fibonacci numbers a_n, show the following:

(a) $a_{-n} = (-1)^{n-1}a_n$ for all $n \geq 1$.

(b) $a_{n+k} = a_k a_{n+1} + a_{k-1}a_n$ for all non-negative integers n, k.

(c) a_{kn} is a multiple of a_n, that is, $s|t \Rightarrow a_s|a_t$.

(d) $\gcd(a_m, a_n) = a_d$, where $d = \gcd(m, n)$.

(e) If $n > 2$ and a_m is a multiple of a_n, then m is a multiple of n.

45. Looking at $\frac{1}{1-x-x^2} = 1 + (x + x^2) + (x + x^2)^2 \ldots$, prove for $n \geq 1$ that a_n is the number of ways of adding 1's and 2's to get n.

46. Use lattice paths to construct a proof of $\sum_{k=0}^{n} C(n, k)^2 = C(2n, n)$. [Hint: $C(n, k)$ is the number of lattice paths from $(0, 0)$ to $(n - k, k)$ as well as from $(n - k, k)$ to (n, n).]

47. Give a bijection between 'the solution set of $x_0 + x_1 + x_2 + \ldots + x_k = n$ in non-negative integers' and 'the number of lattice paths from $(0, 0)$ to (n, k)'. [Hint: For a string of length $n + k$ with n many R's and k many U's, let x_0 = number of R's before the first U and for $i \geq 1$, let x_i = number of R's between the $(i - 1)$-th and i-th U's.]

48. Let the non-negative integers $\lambda_1, \ldots, \lambda_m$ be given. Find the number of integer solutions of $x_1 + x_2 + \ldots + x_m = n$ such that $x_i \geq \lambda_i$ for each i. [Hint: Take $x_i = \lambda_i + y_i$ and $\lambda = \lambda_1 + \ldots + \lambda_m$.]

49. Use Newtons generalized binomial series $(1 - y)^{-r} = \sum_{n=0}^{\infty} \frac{r(r-1)\ldots(r-n+1)}{n!} y^n$ in the OGF $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ to prove that $C_n = \frac{C(2n,n)}{n+1}$.

50. $C_{n+1} = \frac{2(2n+1)}{n+2} C_n$ for $n \geq 0$. [Easily obtained from $C_n = \frac{C(2n,n)}{n+1}$]

51. Prove that the EGF of $\{s(n, k)\}_{n\geq 0}$ is $\frac{(-1)^k}{k!}(\log(1 - x))^k$. [Use expansion of $(1 - x)^{-m}$ and $[m]^n$.]

52. Prove by induction that $s(n + k + 1, k) = \sum_{r=0}^{k} (n + r)s(n + r, r)$.

53. Determine the number of ways of putting n distinguishable/distinct balls into r indistinguishable boxes with the restriction that no box is empty. [Hint: Stirling Number]

54. Use the formula $S(n + 1, r) = S(n, r - 1) + rS(n, r)$ and $r^n = \sum_{k=1}^{\ell} C(r, k)k!S(n, k)$ to calculate $S(n, r)$ for $1 \leq n, r \leq 6$.

55. Determine the number of ways of

(a) selecting r distinguishable objects from n distinguishable objects, when $n \geq r$.

(b) distributing 20 distinct toys among 4 children if each children gets 5 toys.

(c) placing r distinguishable balls into n indistinguishable boxes if no box is empty

(d) placing r distinguishable balls into n indistinguishable boxes.

56. Suppose 13 people get on the lift at level o. If all the people get down at some level, say 1, 2, 3, 4 and 5 then, calculate the number of ways of getting down if at least one person gets down at each level.

57. Let $\mathcal{B}$ be the set of all mappings $f : J_r \to J_{n+1}$. Compute $|\mathcal{B}|$ in two ways to prove $(n+1)^r = \sum_{k=0}^{r} C(r,k)n^k$.

58. Prove that $S(n,2) = 2^{n-1} - 1$ and $S(n,n-1) = C(n,2) = |s(n,n-1)|$.

59. Find the number of functions from J_n into J_k such that the ranges of these functions each have exactly r elements.

60. Prove by induction that

    (a) $S(n+1,k+1) = \sum_{r=0}^{n} (k+1)^{n-r} S(r,k)$.

    (b) $S(n+k+1,k) = \sum_{r=0}^{k} rS(n+r,r)$.

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Optional:

1. If 5 points are chosen at random in the interior of an equilateral triangle each side of which is 2 units long, show that at least 1 pair of points has a separation of less than 1 unit.

2. The total number of games played by a team in a 15-day season was 20. The rules required the team to play at least 1 game daily. Show that there was a period of consecutive days during which exactly 9 games were played.

3. Prove that any set of 3 distinct integers includes 2 integers x and y such that $F(x,y) = x^3y - xy^3$ is divisible by 10.

4. Show that any sequence of n^2+1 distinct real numbers contains a subsequence of at least $n+1$ terms that is either an increasing sequence or a decreasing sequence. In particular, every sequence of n distinct numbers has a monotone subsequence of length at least $\sqrt{n}$.

5. Show that in any group of 10 people there is always (a) a subgroup of 3 mutual strangers or a subgroup of 4 mutual acquaintances, and (b) a subgroup of 3 mutual acquaintances or a subgroup of 4 mutual strangers.

6. Show that if $n+1$ integers are chosen from the set J_{2n}, then there are always two which differ by 1.

7. Show that if $n+1$ integers are chosen from the set J_{3n}, then there are always two which dffer by at most 2.

8. There 100 people at a party. Each person has an even number (possibly zero) of acquaintances in the group. Prove that there are three people at the party with the same number of acquaintances.

9. Show that if 6 integers are chosen from the set J_{10}, then there are always two whose sum is odd.

10. Show that if more than half of the subsets of J_n are selected, then some two of the selected subsets have the property that one is a subset of the other.

11. There are 15 married couples in a party. Find the number of ways of choosing a woman and a man from the party such that the two are (a) married to each other, (b) not married to each other. [**Ans.** (a) **15,** (b) **210.**]

12. There are n married couples at a party. Each person shakes hands with every person other than her or his spouse. Find the total number of handshakes. [**Ans.** $2n(n-1)$]

13. The n members of the board of directors include the president and 2 vice presidents. Find the number of ways of seating the board at a round table so that the vice presidents are on either side of the president. [**Ans.** $2(n-3)!$]

14. How many solutions in positive integers are there to the equation $x+y+z=7$. $[C(4,2)]$

15. Find the number of solutions in non-negative integers of $a+b+c+d+e<11$. $[C(15,5)]$

16. Find the number of allocations of r identical objects to n distinct locations so that location i gets at least $p_i \geq 0$ objects for $i=1,2,\ldots,n$. $[C(r-p+n-1,n-1), \quad p=\sum p_i]$

17. How many integers between 0 and 99,999 (inclusive) have among their digits each of 2, 5 and 8? $[10^5 - 3\times 9^5 + 3\times 8^5 - 7^5]$

18. At a party there are n men and n women. In how many ways can the n women choose male partners for the first dance? How many ways are there for the second dance if everyone have to change partners? $[n!, \quad D_n]$

19. Suppose the n men and n women at the party check their hats before the dance. At the end of the party their hats are returned randomly. In how many ways can they be returned if each man gets a male hat and each woman gets a female hat, but no one gets the hat he or she checked? $[D_n^2]$

20. Find the number of integers between 1 and 1000 inclusive that are not divisible by 4, 5 or 6. [**Use Inclusion-Exclusion Principle**]

21. Find the number of integers between 1 and 1000 inclusive that are not divisible by 4, 6, 7 or 10. [**Use Inclusion-Exclusion Principle**]

22. Find the number of integers between 1 and 1000 inclusive that are neither perfect squares nor perfect cubes. [**Use Inclusion-Exclusion Principle**]

23. A bakery sells chocolate, cinnamon and plain dough nuts and at a particular time has 6 chocolate, 6 cinnamon and 3 plain. If a box contains 12 dough nuts, how many different options are there for a box of dough nuts? [**Use Inclusion-Exclusion Principle**]

24. Determine the number of permutations of J_8 in which exactly four integers are in their natural position. [**Use Inclusion-Exclusion Principle**]

25. Determine the number of permutations of J_9 in which at least one odd integer is in its natural position. [**Use Inclusion-Exclusion Principle**]

26. Find the number of permutations of the digits 1 through 9 in which (a) none of the blocks 23, 45, and 678 appears; (b) none of the blocks, 34, 45 and 738 appears. [**Use Inclusion-Exclusion Principle**]

27. Each of the n children in a class is given a book by the teacher; the books are all distinct. The students are required to return the books after 1 week. The same n books are again distributed for another week. In how many distributions does nobody get the same book twice? $[n!D_n]$

28. Given the sequence $X=(x_1,x_2,...,x_{2n})$, find the number of derangements of X such that the first n elements of each derangement are (a) the first n elements of X, and (b) the last n elements of X. $[D_n^2, \quad (n!)^2]$

29. Find the coefficients of x^{27} in (a) $(x^4+x^5+x^6+\ldots)^5$ and (b) $(x^4+2x^5+3x^6+\ldots)^5$. **[Ans. C(11,4), C(16,9)]**

30. Use the generating-function method to count the distinct binary solutions of $x_1+x_2+\ldots+x_n=r$. $[(1+x)^n=\sum_{r=0}^{n} C(n,r)x^r]$

31. Find the number of ways of forming a committee of 9 people drawn from 3 different parties so that no party has an absolute majority in the committee. [**Ans. 10**]

32. The sum of four positive integers in nondecreasing order is r and a_r is the number of ways of choosing these four integers. Find the ordinary generating function associated with the sequence (a_r). $[(x^4+x^8+\ldots)(1+x^3+x^6+\ldots)(1+x^2+x^4+\ldots)(1+x+x^2+\ldots)]$

33. Find the number of ways of distributing 10 distinguishable books among 4 distinguishable shelves so that each shelf gets at least 2 and at most 7 books. [**Ans.** $\frac{10!}{16}$]

34. Use generating function to prove that $p(2n,n)=p(n)$, where $p(2n,n)=$ and $p(n)=$.

35. For the Fibonacci numbers a_n, show the following:

    (a) $a_{2n}+(-1)^n=(a_{n+2}+a_n)\,a_{n-1}$ and $a_{2n}-(-1)^n=(a_n+a_{n-2})\,a_{n+1}$.

    (b) $a_{2n+1}+(-1)^n=(a_{n+1}+a_{n-1})\,a_{n+1}$ and $a_{2n+1}-(-1)^n=(a_{n+2}+a_n)\,a_n$.

    (c) $a_n^2+1=\begin{cases} a_{n-2}a_{n+2} & \text{if } n \text{ is odd,} \\ a_{n-1}a_{n+1} & \text{if } n \text{ is even.}\end{cases}$

    (d) $a_n^2-1=\begin{cases} a_{n-1}a_{n+1} & \text{if } n \text{ is odd,} \\ a_{n-2}a_{n+2} & \text{if } n \text{ is even.}\end{cases}$

36. Find all positive integer n such that either a_n+1 or a_n-1 is a prime number.

37. Find all positive integer n such that either a_n^2+1 or a_n^2-1 is a prime number.

38. Prove the identity $\sum_{k=0}^{n}\frac{1}{a_{2^k}}=3-\frac{a_{2^n-1}}{a_{2^n}}$ for $n\geq 1$. What is $\sum_{k=0}^{n}\frac{1}{a_{3.2^k}}$?

39. Use lattice paths to construct a proof of $\sum_{k=0}^{n} C(n,k)=2^n$.

40. Show that the number of monotonic increasing functions f from J_n to J_n such that $f(i)\leq i$ for all $i\in J_n$ is C_n.

    [Hint: Such function will have a graph that can be embedded in a unique way in a lattice path from $(0,0)$ to (n,n) that does not rise above the line $y=x$.]

41. Show that $\sum_{k=0}^{n} s(n,k)=n!$, $s(n,2)=(n-1)!H_{n-1}$ and $s(n,3)=\frac{1}{2}(n-1)!\left[(H_{n-1})^2-H_{n-1}^{(2)}\right]$.

42. Show that $|s(n,k)|\geq S(n,k)$ for all n,k. [Hint: View $s(n,k)$ as number of cyclic decomposition.]

43. Use $[x]_r=(-1)^r[-x]^r$ to show that $x^n=\sum_{k=1}^{n}(-1)^{n-k}S(n,k)[x]^k$.

44. Show that $\sum_{k=0}^{\infty}(-1)^{m-k}S(n,k)n(k,m)=\delta_{nm}$, where δ_{nm} is the Kronecker delta.

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