Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 02

Indian Institute of Technology Guwahati

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Events

Def: A set $E \in \mathcal{F}$ is said to be an event. We will say "the event E occurs" if the outcome of a performance of the random experiment is in E.

Example 1: In measuring height of a student, it turns out to be 4.5 feet. We will say the event (4, 5) has occured.

Axiomatic Definition of Probability

Def: A set function $P : \mathcal{F} \to \mathbb{R}$ is called a probability if

1. $P(E) \geq 0$ for all $E \in \mathcal{F}$
2. $P(\mathcal{S}) = 1$
3. (Countable Additivity) Let $E_1, E_2, \ldots \in \mathcal{F}$ be a sequence of disjoint events then

$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

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- Notice that, the domain of P is $\mathcal{F}$, not $\mathcal{S}$.

Def: [**Probability Space**] Let $\mathcal{S}$ be a sample space and $\mathcal{F}$ be a σ-field on the subsets of $\mathcal{S}$. Let P be a probability defined on $\mathcal{F}$. The triplet $(\mathcal{S}, \mathcal{F}, P)$ is called a probability space.

Examples of Probability

Example 2: Toss a coin: $\mathcal{S} = \{H, T\}$ and $\mathcal{F} = \mathcal{P}(\mathcal{S})$;

Consider a function $P : \mathcal{F} \to \mathbb{R}$ defined by

$P(\phi) = 0$, $P(H) = 0.6$, $P(T) = 0.4$ and $P(\mathcal{S}) = 1$.

Check that P is a probability.

Example 3: For a throw of a die, $\mathcal{S} = \{1,\ 2,\ \ldots,\ 6\}$, $\mathcal{F} = \mathcal{P}(\mathcal{S})$.

1. Scenario 1: define, $P(\phi) = 0$, $P(i) = 1/6$ for $i \in \mathcal{S}$.
2. Scenario 2: define, $P(\phi) = 0$, $P(i) = i/21$ for $i \in \mathcal{S}$.

Note that in above two scenarios, the function $P(\cdot)$ have not defined for all the members in $\mathcal{F}$. However, if we assume that $P(\cdot)$ is a probability defined on the σ-field $\mathcal{F}$, we can uniquely extend $P(\cdot)$ for all other members of $\mathcal{F}$.

Remarks

- Choice of $\mathcal{F}$ is an important issue.

Example 4: Let $\mathcal{S} = \{1, 2, \dots, 60\}$ and $\mathcal{F} = \mathcal{P}(\mathcal{S})$. Define $P(E) = \frac{\#E}{\#S}$ for all $E \in \mathcal{F}$.

Example 5: Now consider the changed problem where $\mathcal{S} = \mathbb{N}$. Let us see if we can use the above definition of P to get a probability for each and every subset of $\mathcal{S}$. The natural extension is

$$P(E) = \limsup_{n\to\infty} \frac{N_n(E)}{n}$$

for $E \in \mathcal{F} = \mathcal{P}(\mathbb{N})$, where $N_n(E)$ is the number of times E occurs in the first n natural numbers.

Remarks

Let $A = \{\omega \in \mathbb{N} : \omega \text{ is a multiple of } 3\}$. Then

$$\frac{N_n(A)}{n} = \begin{cases} \frac{m}{3m} & \text{if } n = 3m \\ \frac{m}{3m+1} & \text{if } n = 3m+1 \\ \frac{m}{3m+2} & \text{if } n = 3m+2. \end{cases}$$

Hence for all $n \in \mathbb{N}$, $\dfrac{1}{3 + \frac{6}{n-2}} \leq \dfrac{N_n(A)}{n} \leq \dfrac{1}{3} \Rightarrow P(A) = \dfrac{1}{3}$.

Similarly, $P(B) = \dfrac{1}{4}$ for $B = \{\omega \in \mathbb{N} : \omega \text{ is a multiple of } 4\}$.

Now assume that $C = \{2\}$. Then

$$\frac{N_n(C)}{n} = \begin{cases} 0 & \text{if } n = 1 \\ \frac{1}{n} & \text{if } n \geq 2. \end{cases}$$

Hence $P(C) = 0$.

Remarks

Similarly, $P(D) = 0$ for any singleton set D.

However, $\mathcal{S} = \mathbb{N} = \cup_{i \in \mathbb{N}} \{i\}$. Hence if P satisfies the 3rd axiom then $P(\mathcal{S}) = \sum_{i=1}^{\infty} P(\{i\}) = 0 \neq 1$, which contradicts the 2nd axiom.

- This P defined on the power set of $\mathcal{S}$ does not satisfy all the three axioms but this P gives meaningful probabilities for sets like A and B.

- This example suggests, depending on our objective we may need to choose from the set of all subsets of $\mathcal{S}$, certain subsets (not all) of $\mathcal{S}$ on which to define a probability P.

Remarks

Note that we can always define a probability on the power set of a sample space. For example, let $\omega_0 \in \mathcal{S}$ be a fixed element. Define $P : \mathcal{P}(\mathcal{S}) \to \mathbb{R}$ by

$$P(A) = \begin{cases} 1 & \text{if } \omega_0 \in A \\ 0 & \text{if } \omega_0 \notin A. \end{cases}$$

▶ It is easy to see that $P(\cdot)$ is a probability.

▶ However, in practice, a probability is used to model a practical situation, where the probability may need to satisfy extra conditions other then three conditions mentioned in the definition of probability.

Properties of Probability

- $P(\phi) = 0$.
- If $E_1, E_2, \dots, E_n$ are n disjoint events, then $P\left(\cup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$.
- P is monotone, *i.e.*, for $E_1, E_2 \in \mathcal{F}$ and $E_1 \subseteq E_2$, $P(E_1) \leq P(E_2)$.
- P is subtractive, *i.e.*, for $E_1, E_2 \in \mathcal{F}$ and $E_1 \subseteq E_2$, $P(E_2 - E_1) = P(E_2) - P(E_1)$.
- $0 \leq P(E) \leq 1$.
- If $E_1, E_2 \in \mathcal{F}$, then $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.
- If $E_1, E_2 \in \mathcal{F}$, then $P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$.
- If $E \in \mathcal{F}$, then $P(E^c) = 1 - P(E)$.

Remarks

- A single-ton event is called an elementary event.
- If $\mathcal{S}$ is finite, and $\mathcal{F} = \mathcal{P}(\mathcal{S})$, it is sufficient to assign probability to each elementary event. Then for any $E \in \mathcal{F}$, $P(E) = \sum_{\omega \in E} P(\{\omega\})$. If the elementary events are equally likely, then we get the classical definition of probability.
- If $\mathcal{S}$ is countably infinite, and $\mathcal{F} = \mathcal{P}(\mathcal{S})$, it is still sufficient to assign probability to each elementary event. Then for any $E \in \mathcal{F}$, $P(E) = \sum_{\omega \in E} P(\{\omega\})$. However, in this case we can not assign equal probability to each elementary event.
- If $\mathcal{S}$ is uncountable, and $\mathcal{F} = \mathcal{P}(\mathcal{S})$, one can not make an equally likely assignment of probabilities. Indeed, one can not assign positive probability to each elementary event without violating the axiom $P(\mathcal{S}) = 1$.