# Lecture 18

10th Sep 2022, Saturday.

Note Title 9/10/2022

## § Group homomorphism:

Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups. A map $f: G_1 \to G_2$ is called a group homomorphism if

$$f(x *_1 y) = f(x) *_2 f(y) \quad \forall x, y \in G_1.$$

Example: ① $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = 0 \quad \forall n \in \mathbb{Z}$

② $f: G_1 \to G_2$, $f(x) = e_2 \quad \forall x \in G_1$. Here, $e_2$ is the identity elt. of $G_2$.

③ Fix an integer $k$. Then, $f_k: \mathbb{Z} \to \mathbb{Z}$ defined by

$$f_k(x) = kx \quad \forall x \in \mathbb{Z}.$$

(4) $f : (\mathbb{Z}, +) \to (\mathbb{R}^*, \cdot)$, $\quad f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$

(5) $f : GL_n(\mathbb{R}) \to \mathbb{R}^*$, $\quad f(A) = \det(A)$.

(6) $f : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^*, \cdot)$, $\quad f(x) = e^x$.

Definition: (I) If $f : G_1 \to G_2$ is a group homomorphism, then the kernel of $f$ is defined as $\ker(f) = \{x \in G_1 \mid f(x) = e_2\}$.

(II) A group homomorphism $f : G_1 \to G_2$ is called an isomorphism if $f$ is one-to-one and onto.

(III) An isomorphism $f : G \to G$ is called an automorphism of $G$.

Theorem 1: Let $f: G_1 \to G_2$ be a group homomorphism.

(1) $f(e_1) = e_2$

(2) $f(x^k) = (f(x))^k \quad \forall\, k \in \mathbb{Z}$ and $\forall\, x \in G_1$

(3) $x \in G_1$ and $o(x)$ is finite $\Rightarrow o(f(x)) \mid o(x)$.

(4) $\mathrm{Ker}(f) \trianglelefteq G$ and $f$ is one-to-one $\Leftrightarrow \mathrm{Ker}(f) = \{e_1\}$.

(5) $\mathrm{Im}(f) \leq G$

(6) If $f$ is an isomorphism, then $o(x) = o(f(x)) \quad \forall\, x \in G_1$.

**Theorem 2 (1st Isomorphism Theorem):** If $f: G_1 \to G_2$ is a group homomorphism, then $G_1/\ker(f) \cong \mathrm{Im}(f)$.

**Proof:** Let $N = \ker(f)$. Define $\psi: G_1/N \longrightarrow \mathrm{Im}(f)$

$$gN \longmapsto f(g).$$

(i) $\psi$ is well-defined: Let $aN = bN$, $a, b \in G_1$

Then, $b^{-1}a \in N \iff f(b^{-1}a) = e_2 \iff f(a) = f(b)$

$$\iff \psi(aN) = \psi(bN).$$

(ii) $\psi$ is one-to-one: Reverse the steps used to prove that $\psi$ is well-defined.

(iii) Clearly, $\psi$ is onto.

(4) $\psi$ is a homomorphism:

$$\psi(aN \cdot bN) = \psi(abN) = f(ab)$$

$$= f(a)\,f(b) \qquad [\because f \text{ is a homomorphism}]$$

$$= \psi(aN)\,\psi(bN).$$

$\therefore$ $\psi$ is an isomorphism.

This completes the proof. #

**Corollary:** If $f$ is onto, then $\mathrm{Im}(f) = G_2$ and hence $G_1/\ker(f) \cong G_2$.

**Ex 1:** $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*$

Here, $f : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$ defined by $f(A) = \det(A)$ is an onto homomorphism.

Also, $\ker(f) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$

$= SL_n(\mathbb{R})$.

By 1st theorem of isomorphism, we have

$$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*.$$

__Ex 2:__ $\mathbb{R}/\mathbb{Z} \cong S^1$.

We consider the map $f : \mathbb{R} \longrightarrow S^1$
$x \longmapsto e^{2\pi i x}$

Clearly, $f$ is an onto homomorphism.

We have, $\ker(f) = \{x \in \mathbb{R} \mid e^{2\pi i x} = 1\} = \mathbb{Z}$.

Hence, $\mathbb{R}/\mathbb{Z} \cong S^1$.

__Ex 3:__ $\mathbb{Q}/\mathbb{Z} \cong M_\infty$

We consider the map $f : \mathbb{Q} \longrightarrow M_\infty$, $f(q) = e^{2\pi i q}$, $q \in \mathbb{Q}$

Then, $\ker(f) = \mathbb{Z}$ and $f$ is onto. $\therefore$ $\mathbb{Q}/\mathbb{Z} \cong M_\infty$.

__Theorem 3:__ Every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. If $G$ is a finite cyclic group of order $n$, then $G$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

__Proof:__ (1) Let $G = \langle a \rangle$ be an infinite cyclic group. Then, $o(a) = \infty$.

Define $f : \mathbb{Z} \longrightarrow G$ by $f(k) = a^k$.

Clearly, $f$ is an onto homomorphism.

Also, $\ker(f) = \{k \in \mathbb{Z} \mid a^k = e\} = \{0\}$, since $o(a) = \infty$.

$\therefore$ $f$ is one-to-one

This proves that $f$ is an isomorphism.

(2) Let $G$ be a finite cyclic group of order $n$. Then,

$$G = \{e, a, a^2, \ldots, a^{n-1}\}.$$

Define $f : \mathbb{Z} \longrightarrow G$ by $f(k) = a^k$.

Then, $f$ is an onto homomorphism.

Also, $\operatorname{Ker}(f) = \{k \in \mathbb{Z} \mid a^k = e\} = n\mathbb{Z}$ since $o(a) = n$.

By the 1st isomorphism theorem,

$$\mathbb{Z}/n\mathbb{Z} \cong G.$$

This completes the proof.

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