Probability Theory and Random Processes (MA225)

LECTURE SLIDES

Lecture 19

Indian Institute of Technology Guwahati

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Relation between Modes of Convergence

$$
\begin{tikzcd}
\textit{a.s} \arrow[dr] & & \textit{mean} \arrow[dl] \\
& \textit{Prob} \arrow[d] & \\
& \textit{Dist} &
\end{tikzcd}
$$

Counter Examples

Example 1: Let $\mathcal{S} = [0,1], \mathcal{F} = \mathcal{B}([0,1])$ and P be the uniform measure. Define $X_n = n1_{[0,\frac{1}{n}]}$. X_n converges to 0 in probability and almost surely but not in rth mean for any $r \geq 1$.

Example 2: Let $X_{1,1} = 1_{[0,1/2]}, X_{2,1} = 1_{[1/2,1]}$
$X_{1,2} = 1_{[0,1/4]}, X_{2,2} = 1_{[1/4,1/2]}, X_{3,2} = 1_{[1/2,3/4]}, X_{4,2} = 1_{[3/4,1]} \ldots$.
Then $X_{m,n}$ converges (as $n \to \infty$) in rth mean and in probability but not almost surely.

Example 3: Let X be a $N(0,1)$ RV defined on some probability space $(\mathcal{S}, \mathcal{F}, P)$. Define $X_n = X$ for all n. Then X_n converges in distribution to $-X$ but not in probability.

Theorem: Suppose $\{X_n\}$ is a sequence of RVs defined on a single probability space and X_n converges in distribution to some constant c, then X_n also converges in probability to c.

Theorem: Let $\{X_n\}$ be a sequence of random variables with moment generating functions $M_n(t)$. Let X be a random variable with moment generating function $M(t)$. If $M_n(t) \to M(t)$ for all t in an open interval containing zero, then X_n converges to X in distribution.

Example 4: Let $X_n \sim Bin(n, p_n)$, where $p_n \to 0$ and $np_n = \lambda(> 0)$. Let $X \sim Poi(\lambda)$. Then X_n converges to X in distribution.

Theorem: Let $\{X_n\}$ be a sequence of DRVs with PMF $f_n(\cdot)$. Let X be a DRV with PMF $f(\cdot)$. If $f_n(x) \to f(x)$ for all x, then X_n converges to X in distribution.

Example 5: Prove the claim of the previous example using the above Theorem.

Theorem: Let $\{X_n\}$ be a sequence of CRVs with PDF $f_n(\cdot)$. Let X be a CRV with PDF $f(\cdot)$. If $f_n(x) \to f(x)$ for all x, then X_n converges to X in distribution.

Example 6: Let $X_n \sim U(0, 1 + 1/n)$ and $X \sim U(0, 1)$. Then X_n converges to X in distribution.