Probability Theory and Random Processes (MA225)

Lecture Slides
Lecture 14

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Functions of RVs: Technique 2 for CRV

Theorem: Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a CRV with JPDF $f_{\boldsymbol{X}}$.

1. Let $y_i = g_i(\boldsymbol{x})$, $i = 1, 2, \ldots, n$ be $\mathbb{R}^n \to \mathbb{R}$ functions such that $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ is one-to-one. That means that there exists the inverse tranformation $x_i = h_i(\boldsymbol{y})$, $i = 1, 2, \ldots, n$ defined on the range of the transformation.
2. Assume that both the mapping and its' inverse are continuous.
3. Assume that partial derivatives $\frac{\partial x_i}{\partial y_j}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, exist and are continuous.
4. Assume that the Jacobian of the inverse transformation

$$J \doteq \det \left(\frac{\partial x_i}{\partial y_j} \right)_{i,j=1,2,\ldots,n} \neq 0$$

   on the range of the transformation.

Then $\boldsymbol{Y} = (g_1(\boldsymbol{X}), \ldots, g_n(\boldsymbol{X}))$ is a CRV with JPDF

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(h_1(\boldsymbol{y}), \ldots, h_n(\boldsymbol{y}))|J|.$$

Functions of RVs: Technique 2 for CRV

Example 1: Let X_1 and X_2 be *i.i.d.* $U(0, 1)$ random variables. Find the JPDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

Example 2: Let X_1 and X_2 be *i.i.d.* $N(0, 1)$ random variables. Find the PDF of $Y_1 = X_1/X_2$.

Remark: If X and Y are independent, then $g(X)$ and $h(Y)$ are also independent.

Moment Generating Function

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a RV. The moment generating function (MGF) of $\boldsymbol{X}$ at $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ is defined by

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = E\left(\exp\left(\sum_{i=1}^{n} t_i X_i\right)\right),$$

provided the expectation exists in a neighborhood of origin $\mathbf{0} = (0,\, 0,\, \ldots,\, 0)$.

Remark: $E(X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}) = \frac{\partial^{r_1+r_2+\ldots+r_n}}{\partial t_1^{r_1} \partial t_2^{r_2} \ldots \partial t_n^{r_n}} M_{\boldsymbol{X}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0}$.

Def: Two RVs $\boldsymbol{X}$ and $\boldsymbol{Y}$ are said to have the same distribution, denoted by $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}$, if $F_{\boldsymbol{X}}(\cdot) = F_{\boldsymbol{Y}}(\cdot)$.

Theorem: Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two RVs. Let $M_{\boldsymbol{X}}(\boldsymbol{t}) = M_{\boldsymbol{Y}}(\boldsymbol{t})$ for all $\boldsymbol{t}$ in a neighborhood around 0, then $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}$.

Theorem: X and Y are independent iff $M_{X,Y}(t_1, t_2) = M_X(t_1)M_Y(t_2)$.

Example 3: Let X_i, $i = 1, 2, \ldots, k$ be independent $Bin(n_i, p)$ RVs. Then $\sum X_i \sim Bin(\sum n_i, p)$.

Example 4: Let X_i, $i = 1, 2, \ldots, k$ be iid $Exp(\lambda)$ RVs. Then $\sum X_i \sim Gamma(k, \lambda)$.

Example 5: Let X_i, $i = 1, 2, \ldots, k$ be independent $N(\mu_i, \sigma_i^2)$ RVs. Then $\sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$.

Expectation and Variance of a Random Vector

Expectation of a random vector is given by

$$E(\boldsymbol{X}) = (EX_1, EX_2, \ldots, EX_n) = \boldsymbol{\mu}\,.$$

The variance-covariance matrix of a n-dimensional random vector, denoted by Σ, is defined by

$$\Sigma = [Cov(X_i, X_j)]_{i,j=1}^{n} = E(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^t\,.$$