Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 09

1. In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. Find the probability that the grandson of a man from Harvard went to Harvard. Ans: 0.7.

2. A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit)? Ans: $p^2 + q^2$

3. Smith is in jail and has 3 dollars; he can get out on bail if he has 4 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6.
   (a) Find the probability that he wins 4 dollars before losing all of his money if he bets 1 dollar each time (timid strategy). Ans: 0.58
   (b) Find the probability that he wins 4 dollars before losing all of his money if he bets, each time, as much as possible but not more than necessary to bring his fortune up to 4 dollars (bold strategy). Ans: 0.64
   (c) Which strategy gives Smith the better chance of getting out of jail? Ans: Bold strategy

4. A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears. Ans: 10

5. Consider an experiment of mating rabbits. We watch the evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid–the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a gene from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2. Start with a rabbit of given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.
   (a) Write down the transition probabilities of the Markov chain thus defined.
   (b) Assume that we start with a hybrid rabbit. Let ν_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\nu_n(GG)$, $\nu_n(Gg)$, $\nu_n(gg)$ are the probabilities that the n-th generation rabbit is GG, Gg, or gg, respectively. Compute ν_1, ν_2, ν_3.

6. An urns contains two balls. Ball colors are red and blue. At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the same color and with probability 0.2 with opposite color, as the ball it replaces. If initially both balls are red, find the probability that the 5th ball selected is red.

7. Find the communicating classes for Markov chains having following one-step transition probabilities matrices:

   (a) $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$ (c) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (d) $P = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{bmatrix}$ (e) $P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$.