

Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 29



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Def: For $i \in S$, $N_n(i) = \# \{t : 0 \leq t \leq n, X_t = i\}$ is the number of visits to state i during $\{0, 1, \dots, n\}$.

Def: For $i \in S$, define $L_n(i) = \frac{N_n(i)}{n+1}$. Then $\{L_n(i) : i \in S\}$ is called empirical distribution at time n .

Theorem: Fix the state $i \in S$. Then

- ① i is transient iff $\sum_{k=0}^{\infty} p_{ii}^{(k)} < \infty$.
- ② i is null recurrent iff $\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty$ and $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n p_{ii}^{(k)} = 0$.
- ③ i is positive recurrent iff $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n p_{ii}^{(k)} > 0$.

Example 1: For a simple symmetric random walk, the state 0 is null recurrent.

Stationary Distribution

Def: A vector $\{\pi_i\}_{i \in S}$ is called a stationary distribution for a MC with transition probability matrix $P = ((p_{ij}))$ if $\pi_i \geq 0$ for all $i \in S$, $\sum_{i \in S} \pi_i = 1$ and $\sum_{j \in S} \pi_j p_{ji} = \pi_i$ for all $i \in S$.

Remark: $\sum_{j \in S} \pi_j p_{ji} = \pi_i$ for all $i \in S \implies \underline{\pi}P = \underline{\pi}$. Thus a stationary distribution is a left eigen vector corresponding to eigen value 1 and $\underline{\pi}\underline{1} = 1$.

Remark: Let $\{\pi_i\}_{i \in S}$ be a stationary distribution of a MC $\{X_n\}$ and the initial distribution is same as the stationary distribution. Then, $P(X_n = i) = \pi_i$ if $P(X_0 = i) = \pi_i$ for all $i \in S$.

Remark: If S is finite, then stationary distribution exists.

Theorem: Let $\{X_n\}$ be a MC having stationary distribution π , then $\pi(i) > 0 \implies i$ is positive recurrent.

Theorem: Let $\{X_n\}$ be a MC, then a stationary distribution exists iff there is at least one positive recurrent state.

Corollary: Simple random walk does not admit a stationary distribution.

Corollary: A finite state MC has atleast one positive recurrent state.

Remark: In general, stationary distribution may not exist. If it exist, it may not be unique.