# Lecture - 3:

Tuesday, 2/8/2022

Note Title 8/1/2022

**1st principle of mathematical induction:** Let $A$ be a non-empty subset of $\mathbb{N}$.

Suppose that (i) $1 \in A$ and (ii) $n \in A \Rightarrow 1+n \in A$.

Then, $A = \mathbb{N}$.

**2nd principle of mathematical induction:** Let $A$ be a non-empty subset of $\mathbb{N}$. Suppose that (i) $1 \in A$ and (ii) $1, 2, \cdots, n \in A \Rightarrow 1+n \in A$.

Then $A = \mathbb{N}$.

**Theorem:** The above two principles are equivalent to the well-ordering principle of $\mathbb{N}$.

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§ Primes: An integer $p > 1$ is called a prime number if there is no divisor $d$ of $p$ satisfying $1 < d < p$ That is, 1 and $p$ are the only positive divisors of $p$

If an integer $a > 1$ is not a prime, then it is called a composite number.

Theorem 1: If $p$ is a prime, and $p \mid ab$, then $p \mid a$ or $p \mid b$.

More generally, if $p \mid a_1 \cdot a_2 \cdots a_n$, then $p$ divides at least one factor $a_i$ of the product.

Proof: Suppose that $p \nmid a$. Then, $\gcd(a, p) = 1$. Since $p \mid ab$, so $p \mid b$.

If $p \mid a_1 a_2 a_3$, then $p \mid a_1 b_1$, where $b_1 = a_2 a_3$. $\Rightarrow p \mid a_1$ or $p \mid b_1$

If $p \mid b_1$, then $p \mid a_2 a_3$ implies $p \mid a_2$ or $p \mid a_3$. Hence, $p \mid a_i$ for some $i = 1, 2, 3$.

In general, we can use mathematical induction.
So, we assume that the result holds whenever $p$ divides a product with fewer than $n$ factors.

Now, if $p \mid a_1 a_2 \cdots a_n$, we write $p \mid a_1 c$ where $c = a_2 a_3 \cdots a_n$.

Then, $p \mid a_1$ or $p \mid c$. If $p \mid a_1$, we are done.
If $p \mid c$, we apply the induction hypothesis to conclude that $p \mid a_i$ for some $i = 2, 3, \ldots, n$.

Hence, $p \mid a_i$ for some $i = 1, 2, \ldots, n$.

This completes the proof. #

**Theorem 2 (Fundamental Theorem of Arithmetic):**
Every integer $n > 1$ can be expressed as a product of primes. Also, the factorization of $n$ into primes is unique apart from the order of the prime factors.

**Proof:** Let $n > 1$. If $n$ is a prime, then the integer itself stands as a product with a single factor. Otherwise, $n = n_1 n_2$, where $1 < n_1 < n$, $1 < n_2 < n$. If $n_1$ is a prime, then $n_1$ remains in the factorization. Otherwise, it will factor into, say, $n_3 n_4$ where $1 < n_3 < n_1$ and $1 < n_4 < n_1$. Similarly, we proceed with $n_2$. This process of writing each composite number that arises as a product of factors must terminate because the factors are smaller than the composite number itself, and each factor is greater than 1.

Thus, we can write $n$ as a product of primes as

$$n = p_1^{a_1} \cdots p_k^{a_k}, \quad \text{where } p_1, \ldots, p_k \text{ are distinct primes, and } a_1, \ldots, a_k \text{ are positive.}$$

We next prove that the factorization of $n$ into product of primes is unique apart from the order of the prime factors.

Suppose that there is an integer $n$ with two different factorizations. Dividing out any primes common to the two representations, we would have an equality of the form

$$p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s \longrightarrow (1)$$

where the factors $p_i$ and $q_j$ are primes, not necessarily all distinct and $p_i \neq q_j$ for any $i$ and $j$.

But this impossible, because $p_1 \mid q_1 q_2 \cdots q_s \Rightarrow p_1 \mid q_j$ for some $j$ (by Theorem 1).

$\Rightarrow p_1 = q_j$ for some $j$.

This completes the proof. #

**Theorem 3 (Euclid):** There are infinitely many primes. That is, there is no end to the sequence of primes

$$2, 3, 5, 7, 11, 13, \ldots$$

**1st proof:** Suppose that there are only finitely many primes, say, $p_1, p_2, \ldots, p_r$. Consider $n = 1 + p_1 p_2 \cdots p_r$.

Since $n > p_i$ for all $i$, so $n$ is not a prime (as $p_1, \ldots, p_r$ are the only primes)

By Fundamental Theorem of Arithmetic, $n$ has a prime divisor $p$.

Since $p_1, \ldots, p_r$ are the only primes, so $p = p_j$ for some $j$.

$\Rightarrow p_j$ divides $1 = n - p_1 p_2 \cdots p_r$, which is a contradiction.

$\therefore$ There are infinitely many primes. #

**2nd proof:** Fermat's numbers are defined as:

$$F_n = 2^{2^n} + 1, \quad n \geq 1$$

$F_1 = 5$, $F_2 = 2^4 + 1 = 17$

$F_3 = 2^8 + 1 = 257, \ldots$

We first prove that $\gcd(F_n, F_{n+k}) = 1$ for all $k \geq 1$

(That is, any two distinct Fermat's numbers are co-prime).

Let $m \mid F_n$ and $m \mid F_{n+k}$, where $k \geq 1$.

Put $x = 2^{2^n}$. Then, $\dfrac{F_{n+k} - 2}{F_n} = \dfrac{2^{2^{n+k}} - 1}{2^{2^n} + 1} = \dfrac{x^{2^k} - 1}{x + 1}$

$= x^{2^k - 1} - x^{2^k - 2} + \ldots - 1$, which is an integer.

$\Rightarrow F_n$ divides $F_{n+k} - 2$

Since $m \mid F_n$ and $m \mid F_{n+k}$, so $m \mid 2$.
But Fermat's numbers are odd, and hence $m = 1$.
$\therefore \gcd(F_n, F_{n+k}) = 1$ for all $k \geq 1$.

Now, consider the prime factorizations of the Fermat's numbers.
Since $\gcd(F_n, F_{n+k}) = 1$, so the prime divisors of $F_n$ and $F_{n+k}$ are all different.

Since there are infinitely many Fermat's number, so there are infinitely many prime numbers.

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