Probability Theory and Random Processes (MA225)

Lecture Slides
Lecture 01

Dr. Palash Ghosh

Indian Institute of Technology Guwahati

July-Nov 2022

Syllabus

- Probability spaces, independence, conditional probability, and basic formulae;
- Random variables, distribution functions, probability mass/density functions, functions of random variables; Standard univariate discrete and continuous distributions and their properties;
- Mathematical expectations, moments, moment generating functions, characteristic functions; Random vectors, multivariate distributions, marginal and conditional distributions, conditional expectations;
- Modes of convergence of sequences of random variables, laws of large numbers, central limit theorem;
- Definition and classification of random processes, discrete-time Markov chains, classification of states, limiting and stationary distributions, Poisson process, continuous-time Markov chains.

Books

- Text Books
  - P. G. Hoel, S. C. Port and C. J. Stone, *Introduction to Probability Theory*, Universal Book Stall, 2000.
  - G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, 3rd Ed., Oxford University Press, 2001.
- Reference Books
  - *Introduction to Probability Models* by *Seldon M. Ross*.
  - *An Introduction to Probability Theory and its Applications* by *W. Feller*.

Grading Policy

- Weights in different examination are as follows:
  - Quiz I: 15%
  - Mid-semester Examination: 30%
  - Quiz II: 15%
  - End-semester Examination: 40%
- An **F** grade will be awarded if you obtain less than 20% of total marks after the end semester examination.

Why should we study Probability and Statistics?

[1] Probability and Statistics by Dr. Arnab Chakraborty

Why should we study Probability and Statistics?

- Lets a play a game!

[1] Probability and Statistics by Dr. Arnab Chakraborty

Why should we study Probability and Statistics?

- Lets a play a game!
- You are familiar with Ludo (board game). Similar to that, consider a game having a die with four outcomes (instead of six in Ludo-die): 1, 2, 3 and 4.

[1] Probability and Statistics by Dr. Arnab Chakraborty

Why should we study Probability and Statistics?

- Lets a play a game!
- You are familiar with Ludo (board game). Similar to that, consider a game having a die with four outcomes (instead of six in Ludo-die): 1, 2, 3 and 4.
- Here, the board is $\mathbb{R}^2$. At the beginning, you are at $(0,0)$. If you are at (x,y), then your (the token) next move is determined by the following Table[1] using the outcome by rolling the die:

[1] Probability and Statistics by Dr. Arnab Chakraborty

Why should we study Probability and Statistics?

- Lets a play a game!
- You are familiar with Ludo (board game). Similar to that, consider a game having a die with four outcomes (instead of six in Ludo-die): 1, 2, 3 and 4.
- Here, the board is $\mathbb{R}^2$. At the beginning, you are at $(0, 0)$. If you are at (x, y), then your (the token) next move is determined by the following Table[1] using the outcome by rolling the die:

Outcome of the die	New position in $\mathbb{R}^2$
1	(0.8x + 0.1, 0.8y + 0.04)
2	(0.5x + 0.25, 0.5y + 0.4)
3	(0.355(x-y) + 0.266, 0.355(x+y) + 0.078)
4	(0.355(x+y) + 0.378, 0.355(y-x) + 0.434)

[1] Probability and Statistics by Dr. Arnab Chakraborty

Statistical Regularity: Random to Deterministic...

- From totally meaningless movement to something meaningful in long run!
- See the final outcome-graph (repeating many times) of the above play in R-software.
- The beauty of Probability and Statistics is to get valuable information from Random Experiment which may seems meaningless at the beginning!
- The notion of getting something meaningful (regularity) from a random phenomena (experiment) is called **Statistical Regularity**.

A World without randomness means

Following people/organization will have no job/value:

- Life Insurance agent
- Weather Forecast
- Stock Market
- Gambling
- Computer Game Industry (most of them)
- Medical Science (You know when you will die!)
- Education (You know your marks all the time!)
- Myself! :)

A World without randomness means

Following people/organization will have no job/value:

- Life Insurance agent
- Weather Forecast
- Stock Market
- Gambling
- Computer Game Industry (most of them)
- Medical Science (You know when you will die!)
- Education (You know your marks all the time!)
- Myself! :)
- Without the notion of randomness the entire world becomes standstill!

Random Experiment

Def: An experiment is called a random experiment if it satisfies the following three properties:

1. All the out comes of the experiment is known in advance.
2. The outcome of a particular performance of an experiment is not known in advance.
3. The experiment can be repeated under identical conditions.

Example 1: Toss a coin.
Example 2: Toss a coin until the first head appears.
Example 3: Measuring the height of a student.
All the above can be can be repeated under identical conditions (by the experimenter: You).

Random Experiment

Def: An experiment is called a random experiment if it satisfies the following three properties:

1. All the out comes of the experiment is known in advance.
2. The outcome of a particular performance of an experiment is not known in advance.
3. The experiment can be repeated under identical conditions.

Example 1: Toss a coin.
Example 2: Toss a coin until the first head appears.
Example 3: Measuring the height of a student.
All the above can be can be repeated under identical conditions (by the experimenter: You).
Example 4: Raining Tomorrow:

Random Experiment

Def: An experiment is called a random experiment if it satisfies the following three properties:

1. All the out comes of the experiment is known in advance.
2. The outcome of a particular performance of an experiment is not known in advance.
3. The experiment can be repeated under identical conditions.

Example 1: Toss a coin.
Example 2: Toss a coin until the first head appears.
Example 3: Measuring the height of a student.
All the above can be can be repeated under identical conditions (by the experimenter: You).
Example 4: Raining Tomorrow: Can it be repeated under identical conditions?

Random Experiment

Def: An experiment is called a random experiment if it satisfies the following three properties:

1. All the out comes of the experiment is known in advance.
2. The outcome of a particular performance of an experiment is not known in advance.
3. The experiment can be repeated under identical conditions.

Example 1: Toss a coin.
Example 2: Toss a coin until the first head appears.
Example 3: Measuring the height of a student.
All the above can be can be repeated under identical conditions (by the experimenter: You).
Example 4: Raining Tomorrow: Can it be repeated under identical conditions? Yes, by the Nature (the experimenter!)

Classical Probability

- S: Set of all possible outcomes of a Random Experiment.
- Def: $P(A) = \frac{\text{Favourable number of cases to } A}{\text{Total number of cases}} = \frac{\#A}{\#S}$.
- Example 1: A die is rolled. What is the probability of getting 3 on upper face?
  ▶ Ans: 1/6.
- Example 2: Consider a target comprising of three concentric circles of radii $1/3$, 1, and $\sqrt{3}$ feet. At random, a shot is aimed by a shooter at the target. What is the probability that the shooter hits inside the inner circle?
  ▶ Both $\#A$ as well as $\#S$ are infinite, the classical probability can not be used here.

Remarks

- The classical definition works in the first example but does not work in the second.
- Need a better definition which works for wider class of models.
- Start with classical definition and take three key properties to give more general definition of probability.
- Define the probability as a **set function**.
- Define the **domain** properly.

Countability and Uncountability

For any positive integer n, let $J_n = \{1, 2, \ldots, n\}$ and $\mathbb{N}$ be the set of all positive integers (natural numbers).

Def: We say that two sets A and B are equivalent if there exists a bijection from A to B. We denote it by $A \sim B$.

Def: For any set A we say:

1. A is finite if $A = \phi$ or $A \sim J_n$ for some $n \in \mathbb{N}$. n is said to be the cardinality of A or number of elements in A.
2. A is countable if $A \sim \mathbb{N}$
3. A is atmost countable if either A is finite or A is countable.
4. A is uncountable if A is not atmost countable.

Example 3: $\mathbb{Z}$ is countable (integer numbers).
Remark: If a set is countable, then it can be written as sequence $\{x_n\}$ of distinct terms.

Summary of Results

Theorem: Every subset of an atmost countable set is again atmost countable.

Theorem: Let $\{E_n\}_{n\geq 1}$ be a sequence of atmost countable sets and put $S = \cup_{n=1}^{\infty} E_n$. Then S is again atmost countable.

Theorem: Let $A_1,\ A_2,\ \ldots,\ A_n$ be atmost countable sets. Then $B = A_1 \times A_2 \times \ldots \times A_n$ is also atmost countable.

Corollary: The set of rationals is countable.

Theorem: The set of all binary sequences is uncountable.

Corollary: $[0,\ 1]$ is uncountable.

Corollary: $\mathbb{R}$ is uncountable.

Corollary: Q^c is uncountable.

Corollary: Any interval is uncountable.

Sample Space

Def: The collection of all possible outcomes of a random experiment is called the sample space of the random experiment. It will be denoted by $\mathcal{S}$.

Example 4: Toss a coin: $\mathcal{S} = \{H, T\}$.
Example 5: Toss a coin until the first head appears: $\mathcal{S} = \{H, TH, TTH, \ldots\}$
Example 6: Measuring the height of a student: $\mathcal{S} = (0, \infty)$
Example 7: Raining Tomorrow: $\mathcal{S} = \{Yes, No\}$.

σ-algebra

Def: A non-empty collection, $\mathcal{F}$, of subsets of $\mathcal{S}$ is called a $\sigma-$algebra (or $\sigma-$field) if

1. $\mathcal{S} \in \mathcal{F}$
2. $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$
3. $A_1,\, A_2,\, \ldots \in \mathcal{F}$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example 8: Toss a coin: $\mathcal{F}_1 = \{\phi,\, \mathcal{S},\, \{H\},\, \{T\}\}$, $\mathcal{F}_2 = \{\phi,\, \mathcal{S}\}$,
$\mathcal{F}_3 = \{\phi,\, \mathcal{S},\, \{H\}\}$
Show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ fields, but $\mathcal{F}_3$ is not a σ-field.
Example 9: $\mathcal{F} = \mathcal{P}(\mathcal{S})$
Example 10: $\mathcal{F} = \{\phi,\, \mathcal{S},\, (4,\, 5),\, (4,\, 5)^c\}$

σ-algebra

Remark: Note that there could be multiple σ-field on subsets of a sample space. Power set of sample space is always a σ-field and it is the largest σ-field. On the other hand $\{\mathcal{S}, \emptyset\}$ is also a σ-field and it is the smallest σ-field.

Def: [Measurable Space] Let $\mathcal{S}$ be a sample space of a random experiment and $\mathcal{F}$ is a σ-field on subsets of $\mathcal{S}$. Then the ordered pair $(\mathcal{S}, \mathcal{F})$ is called a measurable space.