Indian Institute of Technology Guwahati
Probability Theory and Stochastic Processes (MA225)
Problem Set 04

1. Check weather the following functions are CDFs of 2-dim random vector or not.

$$\text{(a) } F(x,\, y) = \begin{cases} 1 & \text{if } x + 2y \geq 1 \\ 0 & \text{if } x + 2y < 1. \end{cases}$$

$$\text{(b) } F(x,\, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1 & \text{otherwise.} \end{cases}$$

2. Let $F(\cdot, \cdot)$ be the CDFs of a two-dimensional random vector $(X,\, Y)$, and let $F_1(\cdot)$ and $F_2(\cdot)$, respectively, be the marginal CDFs of X and Y. Define $U(x,\, y) = \min\{F_1(x),\, F_2(y)\}$ and $L(x,\, y) = \max\{F_1(x) + F_2(y) - 1,\, 0\}$. Prove the followings.
   (a) $L(x,\, y) \leq F(x,\, y) \leq U(x,\, y)$.
   (b) $L(x,\, y)$ and $U(x,\, y)$ are CDFs of 2-dimensional random vector.
   (c) The marginal distributions of $L(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are same as that of $F(\cdot, \cdot)$.

3. Let the random variable X have CDF $F_1(\cdot)$ and let $Y = g(X)$ have distribution function $F_2(\cdot)$, where $g(\cdot)$ is some function. Prove that
   (a) If $g(\cdot)$ is increasing, $F_{X,\, Y}(x,\, y) = \min\{F_1(x),\, F_2(y)\}$.
   (b) If $g(\cdot)$ is decreasing, $F_{X,\, Y}(x,\, y) = \max\{F_1(x) + F_2(y) - 1,\, 0\}$.

4. Consider the following joint PMF of the random vector $(X,\, Y)$.

x \ y	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

   (a) Find the probabilities $P\,(X + Y < 8)$, $P\,(X + Y > 7)$, $P\,(XY \leq 14)$.
   (b) Find the $Corr\,(X, Y)$

5. For the bivariate negative binomial distribution, the PMF is given by

$$f_{X,\, Y}(x,\, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x \theta_2^y (1 - \theta_1 - \theta_2)^k & \text{if } x \in \{0,\, 1,\, 2,\, \ldots\},\, y \in \{0,\, 1,\, 2,\, \ldots\} \\ 0 & \text{otherwise,} \end{cases}$$

   k is a positive integer, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the marginal distributions.

6. Three balls are randomly placed in three empty boxes B_1, B_2, and B_3. Let N denote the total number of boxes which are occupied and let X_i denote the number of balls in the box B_i, $i = 1,\, 2,\, 3$.
   (a) Find the joint PMF of $(N,\, X_1)$.
   (b) Find the joint PMF of $(X_1,\, X_2)$.
   (c) Find the marginal distributions of N and X_2.
   (d) Find the marginal PMF of X_1 from the joint PMF of $(X_1,\, X_2)$.

7. Let $X_1,\, \ldots,\, X_n$ be *i.i.d.* random variables with mean μ and variance σ^2. Then $E\left(\overline{X}\right) = \mu$, $Var\left(\overline{X}\right) = \frac{\sigma^2}{n}$, and $Cov\left(\overline{X},\, X_i - \overline{X}\right) = 0$ for all $i = 1,\, 2,\, \ldots,\, n$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

8. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables such that $P(X_i = 0) = 1 - p = 1 - P(X_i = 1)$, $i = 1, \ldots, n$, for some $p \in (0, 1)$. Let X be the number of $X_1, \ldots, X_n$ that are as large as X_1. Find the PMF of X.

9. For the bivariate beta random vector (X, Y) having PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} y^{\theta_2-1}(1-x-y)^{\theta_3-1} & \text{if } x > 0,\, y > 0,\, x + y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_i > 0$, $i = 1, 2, 3$. Find both the marginal PDFs.

10. The joint PDF of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} 4xy & \text{if } 0 < x < 1,\, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PDFs of X and Y.

(b) Verify whether X and Y are independent.

(c) Find $P(\{0 < X < 0.5,\, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$.

11. Let $X = (X_1, X_2, X_3)$ be a random vector with joint PDF

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)} \left(1 + x_1x_2x_3 e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}\right) \qquad \text{if } (x_1, x_2, x_3) \in \mathbb{R}^3$$

(a) Are X_1, X_2, and X_3 independent?

(b) Are X_1, X_2, and X_3 pairwise independent?

12. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$, and $Corr(X, Y) = 1/3$. Find $Corr\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)$.

13. Suppose that the random vector (X, Y) is uniformly distributed over the region $A = \{(x, y) : 0 < x < y < 1\}$. Find $Cov(X, Y)$.