Lecture 17: Friday, 9th Sep. 2022

Let H be a subgroup of a group G.

- For $a, b \in G$, $aH = bH \Leftrightarrow a \sim_L b \Leftrightarrow b^{-1}a \in H \Leftrightarrow a^{-1}b \in H$

  In particular, $aH = H \Leftrightarrow a \in H$.

- For $a, b \in G$, $Ha = Hb \Leftrightarrow a \sim_R b \Leftrightarrow ab^{-1} \in H \Leftrightarrow ba^{-1} \in H$.

  In particular, $Ha = H \Leftrightarrow a \in H$.

§ Normal subgroup: Let $H \leq G$. H is called a normal subgroup of G if $gH = Hg$ $\forall g \in G$.

Notation: We write $H \trianglelefteq G$ to mean that H is a normal subgroup of G.

- In a group G, $\{e\}$ and G are always normal subgroups of G.
- Every subgroup of an abelian group is normal.
- Let $H \leq G$. For $g \in G$, let

$$gHg^{-1} = \{ghg^{-1} : h \in H\}.$$

- Easy to check that $gHg^{-1} \leq G \ \forall g \in G$. (Prove this fact)

<u>Theorem 1:</u> Let H be a subgroup of G. Then, the following are equivalent:

(1) $H \trianglelefteq G$ (2) $gHg^{-1} = H \ \forall g \in G$ (3) $ghg^{-1} \in H \ \forall g \in G, \forall h \in H$.

<u>Proof</u>: (1) $\Rightarrow$ (2): Let $H \trianglelefteq G$. Then, $gH = Hg \ \forall g \in G$.

<u>claim</u>: $gHg^{-1} = H$

Let $x \in gHg^{-1}$. Then, $x = ghg^{-1}$ for some $h \in H$.

Since, $gH = Hg$, so $gh = h_1 g$ for some $h_1 \in H$

$\therefore \ x = ghg^{-1} = (h_1 g) g^{-1} = h_1 \in H \Rightarrow gHg^{-1} \subseteq H. \rightarrow (1)$

Again, if $h \in H$, then $hg \in Hg = gH$

$\Rightarrow hg = gh_2$ for some $h_2 \in H \Rightarrow h = gh_2 g^{-1} \in gHg^{-1}$

$\therefore \ H \subseteq gHg^{-1} \rightarrow (2)$. From (1) & (2), we have $gHg^{-1} = H$ if $H \trianglelefteq G$.

<u>(2) ⇒ (3)</u>: $gHg^{-1} = H \quad \forall g \in G$

$$\Rightarrow ghg^{-1} \in H \quad \forall g \in G \text{ and } \forall h \in H.$$

<u>(3) ⇒ (1)</u>: Let $ghg^{-1} \in H$ $\forall g \in G$ and $\forall h \in H$.

<u>Claim</u>: $H \trianglelefteq G$, that is, $gH = Hg$ $\forall g \in G$.

Let $g \in G$ and $x \in gH$. Then, $x = gH$ for some $h \in G$.

Since $ghg^{-1} \in H$, so $ghg^{-1} = h_1$ for some $h_1 \in H$

$$\Rightarrow gh = h_1 g \Rightarrow x = gh = h_1 g \in Hg$$

$\therefore gH \subseteq Hg$. Similarly, we can prove that $Hg \subseteq gH$.

This completes the proof. #

Thm 2: Let $H \leq G$ be such that $[G:H] = 2$.

Then, $H \trianglelefteq G$.

Proof: Let $g \in G$.

Case I: $g \in H$. Then, $gH = H = Hg$.

Case II: $g \notin H$. Then, $H \neq gH$ and $H \neq Hg$.

Since $[G:H] = 2$, so $G = H \cup gH$ and $G = H \cup Hg$

$\Rightarrow gH = Hg$.

Hence $gH = Hg \ \forall g \in G \Rightarrow H \trianglelefteq G$. #

Ex: $SL_2(\mathbb{R})$ is a normal subgroup of $GL_2(\mathbb{R})$.

Solution: For $A \in GL_2(\mathbb{R})$ and $B \in SL_2(\mathbb{R})$,

$\det(ABA^{-1}) = 1$. Hence, $ABA^{-1} \in SL_2(\mathbb{R})$ $\forall A \in GL_2(\mathbb{R})$ $\forall B \in SL_2(\mathbb{R})$.

$\therefore SL_2(\mathbb{R}) \trianglelefteq GL_2(\mathbb{R})$.

§ Quotient groups: Let $H \trianglelefteq G$. Let G/H denote the set of all the left cosets of H in G (or set of all the right cosets of H in G), that is, $G/N = \{gH \mid g \in G\} = \{Hg \mid g \in G\}$. Then, the operation

$$aH \cdot bH = abH$$ is well-defined.

__Proof__: Let $aH = cH$ and $bH = dH$.

__Claim__: $abH = cdH$.

Let $x \in abH$. Then, $x = abh$

$= c\underline{h_1 d} h_2 h$

Since $H \trianglelefteq G$, so $h_1 d = d h_3$ for some $h_3 \in H$

$\therefore x = abh = c\underline{h_1 d} h_2 h = c d h_3 h_2 h$

$\in cdH$

$\therefore abH \subseteq cdH$.

Similarly, we can prove that $cdH \subseteq abH$.

$\therefore abH = cdH$. #

$a \in aH$
$\Rightarrow a = ch_1$
$b \in bH$
$\Rightarrow b = dh_2$
for some $h_1, h_2 \in H$

Theorem 3: Let $H \trianglelefteq G$. Then $G/H = \{gH \mid g \in G\}$ is a group under the operation $aH \cdot bH = abH$, $a, b \in G$.

Proof: (i) We have prove that the binary map

$$G/H \times G/H \longrightarrow G/H$$

$$(aH, bH) \longmapsto abH$$ is well-defined.

(ii) H is the identity

(iii) $(aH)^{-1} = a^{-1}H$

(iv) $(aH \cdot bH) \cdot cH = abH \cdot cH$
$= ((ab)c)H = (a(bc))H$
$= aH \cdot (bH \cdot cH)$.

$\therefore G/H$ is a group.

Definition: Let $H \trianglelefteq G$. Then, the group G/H is called the quotient group of H in G.

Ex: $$GL_2(\mathbb{R}) / SL_2(\mathbb{R}) = \{ A \cdot SL_2(\mathbb{R}) \mid A \in GL_2(\mathbb{R}) \}.$$

$$= \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot SL_2(\mathbb{R}) \;\middle|\; \gamma \in \mathbb{R},\ \gamma \neq 0 \right\}.$$

- $\begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix} \cdot SL_2(\mathbb{R}) = \begin{pmatrix} \gamma_2 & 0 \\ 0 & 1 \end{pmatrix} \cdot SL_2(\mathbb{R}) \Leftrightarrow \gamma_1 = \gamma_2$

Proof: $\begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{R}) = \begin{pmatrix} \gamma_2 & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{R}) \Leftrightarrow \begin{pmatrix} \gamma_2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$

$\Leftrightarrow \dfrac{\gamma_1}{\gamma_2} = 1 \Leftrightarrow \gamma_1 = \gamma_2$

- For $A, B \in GL_2(\mathbb{R})$, $A \cdot SL_2(\mathbb{R}) = B \cdot SL_2(\mathbb{R})$

$$\iff |A| = |B|$$

<u>Proof</u>: $|A| = |B| \implies |A^{-1}B| = 1$

$$\implies A^{-1}B \in SL_2(\mathbb{R}) \implies A \cdot SL_2(\mathbb{R}) = B \cdot SL_2(\mathbb{R})$$

Conversely, $A \cdot SL_2(\mathbb{R}) = B \cdot SL_2(\mathbb{R})$

$$\implies A^{-1}B \in SL_2(\mathbb{R}) \implies |A^{-1}B| = 1 \implies |A| = |B|.$$

$\therefore A \cdot SL_2(\mathbb{R}) = B \cdot SL_2(\mathbb{R}) \iff \det(A) = \det(B)$.

In particular, if $\det(A) = r$, $r \neq 0$, then
$A \cdot SL_2(\mathbb{R}) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \cdot SL_2(\mathbb{R})$.

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