Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 23

Indian Institute of Technology Guwahati

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Multinomial Distribution

Def: Consider n independent trials, each of which results in one of the outcomes 1, 2, $\ldots$, r, with respective probabilities $p_1,\, p_2,\, \ldots,\, p_r$, where $\sum_{i=1}^{r} p_i = 1$. Let N_i be the number of trials that result in outcome i. Then $(N_1,\, \ldots,\, N_r)$ is said to have a multinomial distribution.

Theorem: The joint PMF of $(N_1,\, N_2,\, \ldots,\, N_r)$ is given by

$$f(n_1,\, n_2,\, \ldots,\, n_r) = \begin{cases} \binom{n}{n_1,\, n_2,\, \ldots,\, n_r} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} & \\ \qquad \text{for } n_1 \geq 0,\, \ldots,\, n_r \geq 0,\, \sum_{i=1}^{r} n_i = n & \\ 0 \quad \text{otherwise,} & \end{cases}$$

where $\binom{n}{n_1,\, n_2,\, \ldots,\, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$.

Remark: Notation: $Mult(n,\, p_1,\, p_2,\, \ldots,\, p_r)$.

Theorem: $N_i \sim Bin(n, p_i)$ for all $i = 1, 2, \ldots, r$.

Theorem: Let $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, r\}$. Then the JPMF of $(N_{i_1}, \ldots, N_{i_k})$ is given by

$$f(n_{i_1}, \ldots, n_{i_k}) = \begin{cases} \frac{n!}{w! n_{i_1}! \ldots n_{i_k}!}(1 - \sum_{s=1}^k p_{i_s})^w p_{i_1}^{n_{i_1}} \ldots p_{i_k}^{n_{i_k}} \\ \qquad \text{if } n_{i_1} \geq 0, \ldots, n_{i_k} \geq 0, \sum_{s=1}^k n_{i_s} \leq n \\ 0 \quad \text{otherwise,} \end{cases}$$

where $w = n - \sum_{s=1}^k n_{i_s}$.

Theorem: Let k and l be natural numbers such that $k + l = r$. Let $A = \{i_1, \ldots, i_k\}$ and $B = \{j_1, \ldots, j_l\}$ be a partition of $\{1, 2, \ldots, r\}$. Then the conditional distribution of $(N_{i_1}, \ldots, N_{i_k})$ given $N_{j_1} = n_1, \ldots, N_{j_l} = n_l$ is

$$Mult\left(n - \sum_{j=1}^{l} n_j, \frac{p_{i_1}}{1 - \sum_{s=1}^l p_{j_s}}, \ldots, \frac{p_{i_k}}{1 - \sum_{s=1}^l p_{j_s}}\right).$$

Theorem: $Cov(N_i, N_j) = -np_ip_j$.