Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 21

Indian Institute of Technology Guwahati

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Bivariate normal

Def: A two dimensional random vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is said to have a bivariate normal distribution if $aX_1 + bX_2$ is a univariate normal for all $(a,\, b) \in \mathbb{R}^2 \setminus (0, 0)$.

Theorem: If $\boldsymbol{X}$ has bivariate normal distribution, then each of X_1 and X_2 is univariate normal. Hence, $E(X_1)$, $E(X_2)$, $Var(X_1)$, $Var(X_2)$, and $Cov(X_1,\, X_2)$ exist.

Let us denote $\boldsymbol{\mu} = E(\boldsymbol{X}) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\Sigma = Var(\boldsymbol{X}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, where $\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$, $\sigma_{11} = Var(X_1)$, $\sigma_{22} = Var(X_2)$, and $\sigma_{12} = \sigma_{21} = Cov(X_1,\, X_2)$.

Bivariate normal

Theorem: Let $\boldsymbol{X}$ be a bivariate normal random vector. If $\boldsymbol{\mu} = E(\boldsymbol{X})$ and $\Sigma = Var(\boldsymbol{X})$, then for any fixed $\boldsymbol{u} = (a,\, b) \in \mathbb{R}^2 \setminus (0,0)$,

$$\boldsymbol{u}'\boldsymbol{X} \sim N(\boldsymbol{u}'\boldsymbol{\mu}, \boldsymbol{u}'\Sigma\boldsymbol{u}).$$

Theorem: Let $\boldsymbol{X}$ be a bivariate normal random vector, then $M_{\boldsymbol{X}}(\boldsymbol{t}) = e^{\boldsymbol{t}'\mu + \frac{1}{2}\boldsymbol{t}'\Sigma\boldsymbol{t}}$ for all $\boldsymbol{t} \in \mathbb{R}^2$.

Remark: Thus the bivariate normal distribution is completely specified by the mean vector $\boldsymbol{\mu}$ and the variance-covariance matrix Σ. We may therefore denote a bivariate normal distribution by $N_2(\boldsymbol{\mu},\, \Sigma)$.

Def: A two dimensional random vector $\boldsymbol{X}$ is said to have a bivariate normal distribution if it can be expressed in the form $\boldsymbol{X} = \boldsymbol{\mu} + A\boldsymbol{Y}$, where A is a 2×2 matrix of real numbers, $\boldsymbol{Y} = (Y_1,\, Y_2)$ and Y_1 and Y_2 are i.i.d $N(0,\, 1)$. In this case $E(\boldsymbol{X}) = \boldsymbol{\mu}$ and $\Sigma = AA'$.

Theorem: If $\boldsymbol{X} \sim N_2(\boldsymbol{\mu},\, \Sigma)$, then $X_1 \sim N(\mu_1,\, \sigma_{11})$ and $X_2 \sim N(\mu_2,\, \sigma_{22})$.

Remark: The converse of the above theorem is not true.

Remark: If $\boldsymbol{X} \sim N_2(\boldsymbol{\mu},\, \Sigma)$ and $Cov(X_1, X_2) = 0$, then X_1 and X_2 are independent.

Theorem: Let $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \Sigma)$ be such that Σ is invertible, then, for all $\boldsymbol{x} \in \mathbb{R}^2$, $\boldsymbol{X}$ has a joint PDF given by

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\ &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{A(x,\, y,\, \mu_x,\, \mu_y,\, \sigma_x,\, \sigma_y,\, \rho)}, \end{aligned}$$

where

$$A = -\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}.$$