Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 04

Indian Institute of Technology Guwahati

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Independence

Observe that $P(B_1|W) = 9/34 < 1/2 = P(B_1)$, whereas $P(B_2|W) = 5/17 > 1/6 = P(B_2)$. Thus the "occurrence of one event is making the occurrence of a second event more or less likely".

Def: Let A and B be two events. They are said to be

a) negatively associated if $P(A \cap B) < P(A)P(B)$,

b) positively associated if $P(A \cap B) > P(A)P(B)$,

c) independent if $P(A \cap B) = P(A)P(B)$.

Theorem: If A and B are independent, so are A and B^c, A^c and B, A^c and B^c.

- If $P(B) = 0$ then A and B are independent.
- If $P(B) = 1$ then A and B are independent.
- In particular any event A is independent of $\mathcal{S}$ and ϕ.

Def: A countable collection of events $E_1, E_2, \ldots$ are said to be pairwise independent if E_i and E_j are independent for $i \neq j$.

Def: A finite collection of events $E_1, E_2, \ldots, E_n$ are said to be independent (or mutually independent) if for any sub-collection $E_{n_1}, \ldots, E_{n_k}$ of $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcap_{i=1}^{k} E_{n_i}\right) = \prod_{i=1}^{k} P(E_{n_i})\,.$$

Def: A countable collection of events $E_1, E_2, \ldots$ are said to be independent if any finite sub-collection are independent.

Remarks

- To verify the independence of $E_1, E_2, \ldots, E_n$ we must check $2^n - n - 1$ conditions. For example, for $n = 3$, the conditions that need to be checked are $P(E_1 \cap E_2) = P(E_1)P(E_2), P(E_1 \cap E_3) = P(E_1)P(E_3), P(E_2 \cap E_3) = P(E_2)P(E_3), P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$.
- Independence implies pairwise independence.
- Pairwise independence does not imply independence in general.

Example 1: Let $S = \{HH, HT, TH, TT\}$. Suppose all elementary events are equally likely. Let $E_1 = \{HH, HT\}$, $E_2 = \{HH, TH\}$ and $E_3 = \{HH, TT\}$. Then E_1, E_2, E_3 are pairwise independent but not independent because

$$1/4 = P(E_1 \cap E_2 \cap E_3) \neq P(E_1)P(E_2)P(E_3) = 1/8\,.$$

- $P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$ is also not sufficient.

Example 2: Let $S = \{(i, j) : i = 1, \ldots, 6, j = 1, \ldots, 6\}$. Suppose all elementary events are equally likely. Define $E_1 = \{\text{1st roll is } 1, 2 \text{ or } 3\}$, $E_2 = \{\text{1st roll is } 3, 4 \text{ or } 5\}$ and $E_3 = \{\text{Sum of the rolls is } 9\}$.

Def: Given an event C two events A and B are said to be conditionally independent if $P(A \cap B|C) = P(A|C)P(B|C)$.

Example 3: A box contains two coins: a fair coin and one fake two-headed coin (P(H)=1). You choose a coin at random and toss it twice. Define the following events.
A= First coin toss results in a H. B= Second coin toss results in a H. C= Coin 1 (regular) has been selected.
Then A and B are conditionally independent given C. Are A and B independent?