Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 25

Indian Institute of Technology Guwahati

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Examples

Example 1: Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather condition. Suppose that if it is raining today, then it will rain tomorrow with probability $\alpha = 0.7$. If it is not raining today, then it will rain tomorrow with probability $\beta = 0.4$. Calculate the probability that it will rain four days from today given that it is raining today.

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \qquad P^{(4)} = P^4 = \begin{bmatrix} 0.57 & 0.42 \\ 0.57 & 0.43 \end{bmatrix}$$

Examples

Example 2: Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 occupied urns after 9 balls have been distributed?

$$P = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{(4)} = P^4 = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0 & 0.0039 & 0.0952 & 0.9009 \\ 0 & 0 & 0.0198 & 0.9802 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples

Example 3: In a sequence of independent flips of a fair coin, let N denote the number of flips until there is a run of three consecutive heads. Find $P(N \leq 8)$ and $P(N = 8)$.

Fact: Like a random variable is probabilistically specified by its distribution, a stochastic process is specified by its finite dimensional distributions, *i.e.*, by $P\left(X_0 = i_0,\, X_1 = i_1,\, \ldots,\, X_n = i_n\right)$ for $n \geq 0$ and all $i_0,\, i_1,\, \ldots,\, i_n \in S$, where S is the state space.

Remark: Let $P(X_0 = i) = \mu_i$ for $i \in S$. Then

$$P\left(X_0 = i_0,\, X_1 = i_1,\, \ldots,\, X_n = i_n\right) = \left(\prod_{k=0}^{n-1} p_{i_k i_{k+1}}\right) \mu_{i_0}$$

Thus a MC is probabilistically specified by its initial distribution $\{\mu_i\}$ and one step transition probability matrix.

Accessibility

Def: State j is said to be accessible from state i if there exists $n \geq 0$ such that $p_{ij}^{(n)} > 0$, where

$$p_{ij}^{(0)} = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Remark: If j is not accessible from i, then

$$P\left(\text{Ever be in } j | \text{starting from } i\right) = 0.$$

Communication

Def: Two states i and j are said to communicate if i and j are accessible from each other, *i.e.*, there exist $m \geq 0$ and $n \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.

Notation: $i \to j$: j is accessible from i.
$i \leftrightarrow j$: i and j communicate.

Remark:

1. (Reflexivity) $i \leftrightarrow i$.
2. (Symmetry) $i \leftrightarrow j \iff j \leftrightarrow i$.
3. (Transitivity) $i \leftrightarrow k$ and $k \leftrightarrow j \implies i \leftrightarrow j$.

Remark: Thus communication is an equivalence relation. Hence it partitions the state space into equivalence classes.