# Indian Institute of Technology Guwahati
(Supplementary Answer Sheet)

| Name of student : | | Roll No. | |
|---|---|---|---|
| Course No. | | Signature of the student : | |

1. $A \cap B^c \cap C^c$, $(A\cap B)\cup(A\cap C)\cup(B\cap C)$, $A\cap B\cap C^c$, $(A^c \cup B^c)\cap(A^c\cup C^c)\cap(B^c \cup C^c)$.

2. (c) NO, $P(\cdot)$ is not a probability. Because if it is then we get,

$$1 = P(S) = P\left(\bigcup_{i=1}^{\infty}\{i\}\right) = \sum_{i=1}^{\infty} P(\{i\}) = 0.$$

3. Since $S \in \mathcal{F}_\alpha \ \forall \alpha$, $S \in \bigcap \mathcal{F}_\alpha$.

If $A \in \bigcap \mathcal{F}_\alpha$, then $A \in \mathcal{F}_\alpha \ \forall \alpha$,

$\Rightarrow A^c \in \mathcal{F}_\alpha \ \forall \alpha$ [$\because \mathcal{F}_\alpha$ is a $\sigma$-algebra]

$\Rightarrow A \in \bigcap \mathcal{F}_\alpha$.

If $\{A_i\}_{i=1}^{\infty} \in \bigcap \mathcal{F}_\alpha$, then $\{A_i\}_{i=1}^{\infty} \in \mathcal{F}_\alpha, \forall \alpha$,

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha \ \forall \alpha$, $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap \mathcal{F}_\alpha$.

4. let $S = \mathbb{R}$, let $\mathcal{F}_1 = \{\phi, \mathbb{R}, (0,2), (0,2)^c\}$

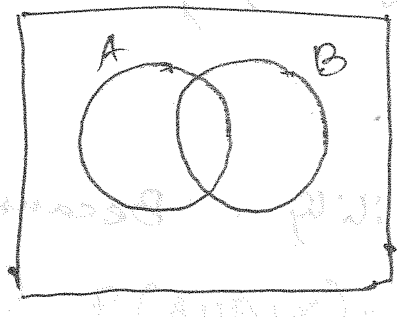
and let $\mathcal{F}_2 = \{\phi, \mathbb{R}, (1,3), (1,3)^c\}$

Then $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \mathbb{R}, (0,2), (1,3), (0,2)^c, (1,3)^c\}$.

$(0,2) \cap (1,3) = (1,2) \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Thus $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a $\sigma$-algebra.

5. (a) $\{\phi, A, A^c, S\}$

(b) 


<u>Step 1</u> Form a partition of $S$ using $A$ & $B$.

$A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c$

<u>Step 2</u> Take all possible unions of the above sets. There will be total 16 sets.

$, A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c, A, (A \cap B^c) \cup (B \cap A^c),$
$B^c, B, (A \cap B) \cup (A^c \cap B^c), A^c, A \cup B, (A \cap B)^c, B^c \cup A, A^c \cup B,$
$S$.

6. We prove by induction.

Basis step: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$$\leq P(A_1) + P(A_2)$$

Induction hypothesis: True for $n-1$.

Inductive step: $P(A_1 \cup A_2 \cup \ldots A_n) \leq P(A_1 \cup \ldots A_{n-1}) + P(A_n)$

$$\leq \sum_{i=1}^{n} P(A_i).$$

7. Define $B_1 = A_1$, $B_2 = A_2 - A_1$, ..., $B_n = A_n - (A_1 \cup \cdots \cup A_{n-1})$.

Thus $B_i$'s are disjoint. $B_i \subset A_i \; \forall i$ and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i .$$

Thus $P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$

$\leq \sum_{i=1}^{\infty} P(A_i)$. [Last inequality by monotonicity property.]

8. We prove by induction.

Basis step: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Induction hypothesis: True for $n-1$.

Inductive Step: $P(A_1 \cup A_2 \cdots A_n) = P(A_1 \cup A_2 \cdots \cup A_{n-1})$

$+ P(A_n) - P\left(\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right)$

$= P(A_1 \cup \cdots \cup A_{n-1}) + P(A_n) - P\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right)$

$= \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i_1=1 \\ i_1 < i_2}}^{n} \sum_{i_2=1}^{n} P(A_{i_1} \cap A_{i_2}) + \cdots (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$

9. ~~Left~~ Right hand side inequality already shown in problem 6.

We use induction for Left hand inequality.

Basis Step $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Induction hypothesis: True for $n-1$.

Inductive Step: $P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{n-1} A_i) + P(A_n) - P(\bigcup_{i=1}^{n-1} A_i \cap A_n$

$$\geq \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} A_i \cap A_j .$$

10. a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$
$- P(B \cap C) + P(A \cap B \cap C) = 0.6 + 0.5 + \cancel{0.4} - 0.3 - \cancel{0.2}$
$- \cancel{0.2} + 0.1 = 0.9$

$P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = 0.1$

b) $P((A \cup B) \cap C) = P(A \cup B) + P(C) - P(A \cup B \cup C)$
$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cup B \cup C)$

$= 0.6 + 0.5 + 0.4 - 0.3 - 0.9 = 0.3$.

$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C)$
$= 0.6 + 0.2 - 0.1 = 0.7$.

c) $P((A^c \cup B^c) \cap C^c) = P(A^c \cup B^c) + P(C^c) - P(A^c \cup B^c \cup C^c)$
$= 1 - P(A \cap B) + 1 - P(C) - (1 - P(A \cap B \cap C))$

$= 1 - 0.3 + \cancel{1} - 0.4 - \cancel{1} + 0.1 = 0.4$

$((A^c \cap B^c) \cup C^c) = P(A^c \cap B^c) + P(C^c) - P(A^c \cap B^c \cap C^c)$
$= 1 - (P(A) + P(B) - P(A \cap B)) + 1 - P(C) - 0.1 = 1 - 0.6 - 0.5 + 0.3 + 1 - 0.4$
$= 0.7$

d) $0 \leq P(D \cap B \cap C) \leq P(B \cap D) = 0$

$0 \leq P(A \cap C \cap D) \leq P(C \cap D) = 0$

e) $P(A \cup B \cup D) = P(A) + P(B) + P(D) - P(A \cap B) - P(A \cap D)$

$- P(B \cap D) \quad + P(A \cap B \cap D) = 0.6 + 0.5 + 0.2 - 0.3 - 0.1$

$= 0.9$

$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) - P(A \cap B) -$

$P(A \cap C) - P(A \cap B) - P(B \cap C) + P(A \cap B \cap C)$

$= 0.6 + 0.\cancel{5} + 0.4 + 0.\cancel{2} - 0.\cancel{3} - 0.\cancel{2} - \cancel{0.1} - 0.\cancel{2}$

$+ 0.\cancel{1} = 1.0$

f) $P((A \cap B) \cup (C \cap D)) = P(A \cap B) + P(C \cap D) - P(A \cap B \cap C \cap D)$

$= 0.3$

11) $P(A)P(B^c) - P(A \cap B^c)$

$= P(A)(1 - P(B)) - P(A \cap B^c) = P(A) - P(A \cap B^c) - P(A)P(B)$

$= P(A \cap B) - P(A)P(B)$

$P(A^c)P(B) - P(A^c \cap B) = (1 - P(A))P(B) - P(A^c \cap B)$

$= P(B) - P(A^c \cap B) - P(A)P(B)$

$= P(A \cap B) - P(A)P(B)$

$P((A \cup B)^c) - P(A^c)P(B^c)$

$= P((A \cup B)^c) - (1 - P(A))(1 - P(B))$

$= P((A \cup B)^c) - 1 + P(A) + P(B) - P(A)P(B)$

$= P(A) + P(B) - P(A \cup B) - P(A)P(B) = P(A \cap B) - P(A)P(B).$

**Q12**

Total number of ways in which $n$ persons stand in a row is $n!$

The $r$ persons can be choose in $\binom{n-2}{r}$ ways from $(n-2)$ persons [leaving $P_1$ & $P_2$ there are $(n-2)$ persons].

These $r$ persons can be arranged in $r!$ ways between $P_1$ & $P_2$

Now consider the unit from $P_1$ to $P_2$ (including $r$ persons in between) as one ~~unit~~ person, then we need to arrange $(n-2-r+1)$ persons in a row, and it can be performed in $(n-r-1)!$ ways.

Again $P_1$ & $P_2$ can change there position giving 2 ways to ~~arran~~ make the row.

Hence the required probability is

$$\frac{2(n-r-1)!\binom{n-2}{r}r!}{n!} = \frac{2(n-r-1)}{n(n-1)}.$$

$\square$

~~Q13~~ **Q13**

There are $6^3$ ways to choose three numbers from $\{1,2,\ldots 6\}$

Root will be real if $b^2 \geq 4ac$. We here count in how many ways ~~are~~ it can happen

| $(a,c)$ | Possible values of $b \ni b^2 \geq 4ac$ | No. of ways |
|---|---|---|
| (1, 1) | 2, 3, 4, 5, 6 | 5 |
| (2,1) & (1,2) | 3, 4, 5, 6 | 8 |
| (3,1) & (1,3) | 4, 5, 6 | 6 |
| (4,1) & (1,4) | 4, 5, 6 | 6 |
| (5,1) & (1,5) | 5, 6 | 4 |
| (6,1) & (1,6) | 5, 6 | 4 |
| (2,2) | 4, 5, 6 | 3 |
| (3,2) & (2,3) | 5, 6 | 4 |

| $(a, c)$ | Choice of $b \ni b^2 \geq 4ac$ | # ways. |
|---|---|---|
| $(4,2)$ & $(2,4)$ | 6 | 2 |
| $(3,3)$ | 6 | 1 |
| | Total | 43. |

Hence the required probability is $\frac{43}{6^3}$. $\square$

~~Q 15~~

**Q14 (a)** From the set $\{1, 2, \ldots, 50\}$, three number can be chosen in $\binom{50}{3}$ ways.

Three ~~into~~ real numbers will be in AP if

$a + \cancel{c} = 2b$. (Here $a < b < c$)

It means that if we choose $a$ & $c$ such that the sum of $a + c$ is an even integer, then $b$ can be found automatically (Taking $b = \frac{a+c}{2}$).

Now $a + c$ will be even if both $a$ & $c$ are even or both $a$ & $c$ are odd.

There are 25 evens and 25 odd in the set $\{1, 2, \ldots, 50\}$

Hence total no. of ways $a, b, \& c$ can be drawn from the set $\{1, 2, \ldots 50\}$ such that ~~a~~ they are in AP is $\binom{25}{2} + \binom{25}{2}$.

The required probability is $\frac{2\binom{25}{2}}{\binom{50}{3}} = \frac{600}{\binom{50}{3}}$

$\square$

$\boxed{\cancel{Q15(b)}}$

$\boxed{Q14(b)}$ For $a < b < c$ in GP we have $b = ar$ and $c = ar^2$, where $r$ is the common ratio. We consider $r > 1$.

Here $1 \le a < ar < ar^2 \le 50 \Rightarrow 1 < r^2 \le 50 \Rightarrow 1 < r \le \sqrt{50}$.

**Case-I:** $r$ is an integer. For each $r$, $1 \le a \le \frac{50}{r^2}$

$r \in \{2, 3, \ldots, 7\}$.

| $r$ | Possible values of $a$ | # |
|---|---|---|
| 2 | 1, 2, 3, ... 12 | 12 |
| 3 | 1, 2, 3, 4, 5 | 5 |
| 4 | 1, 2, 3 | 3 |
| 5 | 1, 2 | 2 |
| 6 | 1 | 1 |
| 7 | 1 | 1 |
| | Total | 24 |

**Case-II** $r$ is not an integer.

Let $r = \frac{m}{n}$, $m$ & $n$ are co-prime and $m > n > 1$.

$1 \le a \le a\frac{m}{n} \le a\frac{m^2}{n^2} \le 50$.

As $a\frac{m^2}{n^2}$ is an integer $\Rightarrow a$ is a multiple of $n^2$.

For fixed $r = \frac{m}{n}$, we have $1 \le a \le \frac{50n^2}{m^2}$ & $a$ is a multiple of $n^2$

| $r$ | Range of $a$ | Possible values of $a$ | # |
|---|---|---|---|
| $\frac{3}{2}$ | $[1, 22.22]$ | 4, 8, 12, 16, 20 | 5 |
| $5/2$ | $[1, 8]$ | 4, 8 | 2 |
| $7/2$ | $[1, 4.08]$ | 4 | 1 |
| $4/3$ | $[1, 28.125]$ | ~~3, 6, 9, 12, 15, 18, 21, 24~~ 9, 18, 27 | 3 |
| $5/3$ | $[1, 18]$ | 9, 18 | 2 |
| $7/3$ | $[1, 9.18]$ | 9 | 1 |
| $5/4$ | $[1, 32]$ | 16, 32 | 2 |

| $r$ | Range for $a$ | Possible values of $a$ | # |
|---|---|---|---|
| $7/4$ | $[1, 16.33]$ | 16 | 1 |
| $6/5$ | $[1, 34.72]$ | 25 | 1 |
| $7/5$ | $[1, 25.51]$ | 25 | 1 |
| $7/6$ | $[1, 36.73]$ | 36 | 1 |
| | | Total | 20. |

Hence the required probability is $\dfrac{24+20}{\binom{50}{3}} = \dfrac{44}{\binom{50}{3}}$. $\square$

$\boxed{15}$ The number of ways four groups (each of size 4) can be made is $\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}\Big/4!$

The number of ways four groups ~~ea~~ each having one graduate student can be mad is

$$\frac{4!\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!}$$

Hence the required probability is

$$\frac{4!\cdot\dfrac{12!}{3!\,\cancel{9!}}\times\dfrac{\cancel{9!}}{3!\,\cancel{6!}}\times\dfrac{\cancel{6!}}{3!\,3!}\times\dfrac{\cancel{3!}}{\cancel{3!}\,1!}\times\dfrac{1}{\cancel{4!}}}{\dfrac{16!}{4!\,\cancel{12!}}\times\dfrac{\cancel{12!}}{4!\,\cancel{8!}}\times\dfrac{\cancel{8!}}{4!\,4!}\times\dfrac{\cancel{4!}}{\cancel{4!}\,1!}\times\dfrac{1}{\cancel{4!}}}$$

$$= \frac{4!\times 12!\times 4!\,4!\,4!\,4!}{16!\;3!\;3!\;3!\;3!}$$

$$= \frac{4\times3\times2\times\cancel{4^4}\,4^2}{\cancel{16}\times15\times14\times13} = \frac{2\times3\times4^3}{15\times14\times13}. \qquad \square$$

**Q16** Define the event that the $i^{th}$ letter is inserted into the $i^{th}$ envelope by $A_i$. $i = 1, 2, \ldots, n$.

We need to find $P\left[\left(\bigcup_{i=1}^{n} A_i\right)^c\right] = 1 - P\left(\bigcup_{i=1}^{n} A_i\right)$.

Again $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j) + \cdots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)$

Here $P(A_i) = \frac{1}{n}$ $\forall i = 1, 2, \ldots, n$.

For $i < j$, $P(A_i \cap A_j) = \frac{1}{n(n-1)}$.

For $i_1 < i_2 < \cdots < i_m$, $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = \frac{1}{n(n-1)\cdots(n-m+1)}$

Hence the required probability is

$$1 - \left[n \times \frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3} \times \frac{1}{n(n-1)(n-2)} + \cdots + (-1)^n \frac{1}{n!}\right]$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}.$$ $\square$

**Q17** Three ~~same~~ digits can be drawn in $10^3$ ways.

~~Two digits can be drawn in~~

~~Two distinct digits can be drawn in $10 \times 9$ ways.~~

~~Now these two there are $\frac{3!}{2!\,1!} = 3$ permutation can be made~~

~~using these two distinct digits~~

Three distinct digits can be drawn in $10 \times 9 \times 8$ ways.

Three same digits can be drawn in $10$ ways.

Hence exactly two different digits can be drawn in

$10^3 - (10 \times 9 \times 8 + 10)$ ways

$= 10^3 - 730$ "

Hence the required probability is $1 - 0.73 = 0.27$. $\square$

[Q 18] If we arrange $r$ balls in a row and then insert $(n-1)$ bars among the balls, we will get $n$ groups of balls. These groups can be considered as the balls in different cells.

For example in the following picture there are $n=3$ cells and $r=4$ balls

$\cdot\cdot|\cdot|\cdot$ $\quad r_1=2,\ r_2=1,\ r_3=1$

$\cdot\cdot\cdot\cdot||$ $\quad r_1=4,\ r_2=0,\ r_3=0.$

[(a)] To obtain the distinguishable distⁿ, we have $r$ balls and $(n-1)$ bars. Hence if we have $(n-1+r)$ places and if we choose $r$ places to put the balls and rest $(n-1)$ places to put the bars, we will have distinguishable distributions. Clearly it can be done in $\binom{n+r-1}{r}$ ways. $\square$

[(b)] If we arrange the ball in a row, there will be $(r-1)$ placese in between. Now if we put $(n-1)$ bars in ~~these~~ $(n-1)$ places among these $(r-1)$ places then we get no empty cells. Clearly it can be done in $\binom{r-1}{n-1}$. $\square$

[(c)] The required probability is $\begin{cases} \dfrac{\binom{r-1}{n-1}}{\binom{n+r-1}{r}} & \text{for } r \geq n \\ 0 & \text{o.w.} \end{cases}$ $\square$.

[Q19] There are $n!$ ways to arrange $n$ keys. The right key will be found in $k^{th}$ trail if the right key is in the $k^{th}$ position. There are $(n-1)!$ ways to arrange $n$ keys ~~such~~ such that right key is in the $k^{th}$ position. Hence the required prob is $\frac{1}{n}$ $\square$.

$\boxed{\text{Q20}}$ The no. of ways in which two pairs can be ~~joint~~ joined is $(2n-1)(2n-3)\cdots 1 = \dfrac{(2n)!}{2^n\, n!}$.

Hence the required probability is $\dfrac{2^n\, n!}{(2n)!}$. $\square$

$\boxed{\text{Q21}}$

(a) $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{4}$, $P(A\cap B) = \frac{1}{16}$. Hence $A$ & $B$ are indep. $\square$

(b) $P(A) = \frac{1}{4}$, $P(B) = \frac{4}{16} = \frac{1}{4}$, $P(A\cap B) = \frac{1}{16}$. Hence $A$ & $B$ are indep. $\square$

(c) $P(A) = \frac{3}{16}$, $P(B) = \frac{4}{16} = \frac{1}{4}$, $P(A\cap B) = \frac{1}{16}$. Hence $A$ & $B$ are not indep. $\square$

$\boxed{\text{Q22}}$ $A\cap B\cap C = (\frac{1}{4}, \frac{1}{2}) \Rightarrow P(A\cap B\cap C) = \frac{1}{4}$.

$P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{2}{3}$. Hence $P(A\cap B\cap C) = P(A)P(B)P(C)$. $\square$

$A\cap B = (\frac{1}{4}, \frac{1}{2}) \Rightarrow P(A\cap B) = \frac{1}{4} \neq \frac{1}{2}\times\frac{3}{4} = P(A)P(B)$.

$\boxed{\text{Q23}}$ Trivial.

$\boxed{\text{Q24}}$ Trivial.

$\boxed{\text{Q25}}$

(a) $P(A\cap B|C) = \dfrac{P(A\cap B\cap C)}{P(B\cap C)} \times \dfrac{P(B\cap C)}{P(C)} = P(A|B\cap C)\,P(B|C)$. $\square$

Hence proved.

(b) This statement can be disproved using the problem 23. $\square$

$\boxed{\text{Q26}}$

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c) \quad \text{as } A_1, A_2, \ldots, A_n \text{ are independent.}$$

$$= \prod_{i=1}^{n} (1-P(A_i))$$

$$\leq \prod_{i=1}^{n} e^{-P(A_i)} = e^{-\sum_{i=1}^{n} P(A_i)}.$$ $\square$

**Q27**

(a) Consider ~~the tosses~~ two independent tosses of a fair coin.

Let $A = \{HT, HH\}$, $B = \{TT\}$, $C = \{TH, HH\}$.

Then $P(A \cap B) = 0$, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$. Hence $A$ & $B$ are (-ve)ly associated.

$P(B \cap C) = 0$, $P(C) = \frac{1}{2} \Rightarrow$ $B$ & $C$ are (-ve)ly associated.

$P(A \cap C) = \frac{1}{4} = P(A)P(C) \Rightarrow$ $A$ & $C$ are not (-ve)ly associated. $\square$

(b) ~~Let $A = \{HH, TH, HT\}$~~

Let $A = \{HT, HH\}$ $B = \{HH, TH, HT\}$, $C = \{TH, HH\}$.

$P(A) = \frac{1}{2} = P(C)$ and $P(B) = \frac{3}{4}$.

$P(A \cap B) = \frac{1}{2}$, $P(B \cap C) = \frac{1}{2}$

$P(A \cap B) > P(A)P(B)$ & $P(B \cap C) > P(B)P(C)$.

Hence $A$ & $B$ are (+ve)ly associated and $B$ & $C$ are (+ve)ly associated.

Now $P(A \cap C) = \frac{1}{4} = P(A)P(C)$. $\square$

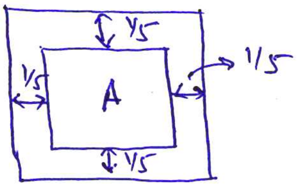
$\Rightarrow$ $A$ & $C$ are not positively associated.

**Q28**

Given that $P(A \cap B) > P(A)P(B)$. $\square$

Now $P(A \cap B^c) = P(A) - P(A \cap B) < P(A)P(B^c)$.

**Q29**

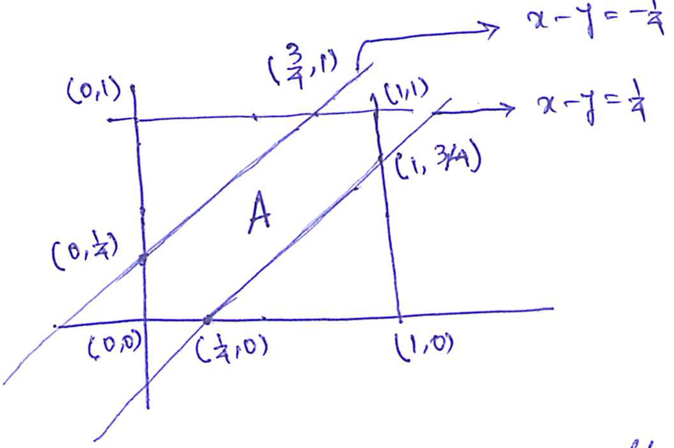


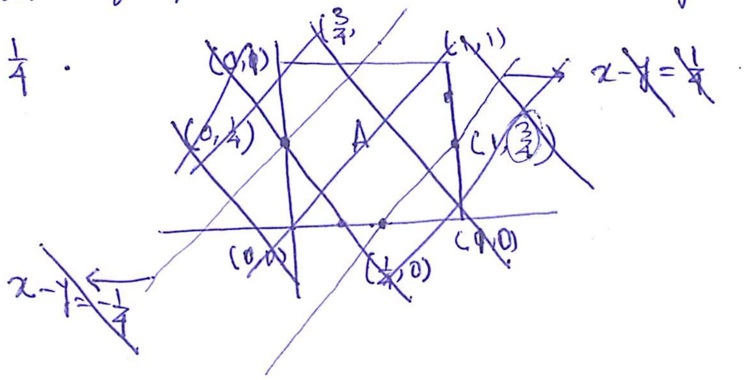

With respect to the diagram, the required probability is

$$\frac{\text{Area of } (S - A)}{\text{Area of } S} = \frac{1 - \left(\frac{3}{5}\right)^2}{1} = \frac{16}{25}. \quad \square$$

**Q30**

Let, $x$ & $y$ be the arrival time of person 1 & 2, respectively, where $0 \le x \le 1$ and $0 \le y \le 1$.
They will meet if $|x-y| \le \frac{1}{4}$.




Using the figure above the required prob is $\dfrac{\text{Area } A}{\text{Area of } [0,1]^2}$

$$= 1 - 2 \times \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4}$$

$$= \frac{7}{16}. \quad \square$$

**Q31**

Either he wins in first game or he looses the $1^{st}$ and wins the $2^{nd}$.
Hence the required prob is $0.5 + (0.5)^2 = \frac{3}{4}$.

**Q33**

Let $E$ be the event that sum total is at least 4.
Let $A_i$ ~~& B~~ be the events that the $1^{st}$ roll results in $\{i\}$ ~~2} and~~ $i = 1, 2, \ldots, 4$.
~~{3,4,5,6}, respectively.~~

The required prob is

$$\cancel{P(E) = P(E|A)P(A) + P(E|B)P(B)}$$

$$P(E) = \sum_{i=1}^{4} P(E|A_i)P(A_i)$$

$$\cancel{= \frac{4}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} + 0 \times \frac{1}{6} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6}}$$

$$= \frac{2}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4}$$

$$= \frac{9}{16}. \quad \square$$

$\boxed{Q34}$

Let $E_i$ be the event that the student is upto date after the week $i$, $i = 1,2,3$.

Let $E_0$ be the event that the student is upto date ~~atd~~ at the begining

Then $P(E_0) = 1$.

Now $P(E_1) = P(E_1|E_0)P(E_0) = 0.8$.

$$P(E_2) = P(E_2|E_1)P(E_1) + P(E_2|E_1^c)P(E_1)$$

$$= 0.8 \times 0.8 + 0.4 \times 0.2$$

$$= 0.24$$

$$P(E_3) = P(E_3|E_2)P(E_2) + P(E_3|E_2^c)P(E_2^c).$$

$$= 0.8 \times 0.24 + 0.4 \times 0.76$$

$$= 0.496.$$

□.

$\boxed{Q32}$ The required probability is

$$\binom{3}{1}\left(\frac{1}{2}\right)^3 + \binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{4}.$$

□

35. Let $C_i$, $i=1,2,3$, denote the events that the car is behind door $i$.

Let $X_i$, $i=1,2,3$, denote the events that you choose door $i$ in the beginning.

Let $D_i$, $i=1,2,3$, denote the events that the host opens door $i$.

Then
$$P(D_3|C_1,X_1)=\frac{1}{2} \qquad P(D_2|C_1,X_1)=\frac{1}{2}$$
$$P(D_3|C_2,X_1)=1 \qquad P(D_2|C_2,X_1)=0$$
$$P(D_3|C_3,X_1)=0 \qquad P(D_2|C_3,X_1)=1$$

$$P(C_i)=\frac{1}{3}, \quad P(D_3|X_1)=\frac{1}{2}, \quad P(D_2|X_1)=\frac{1}{2}$$

Now
$$P(C_1|D_3,X_1)=\frac{P(C_1\cap D_3\cap X_1)}{P(D_3\cap X_1)}=\frac{P(D_3|C_1,X_1)\,\underbrace{P(C_1,X_1)}}{P(D_3\cap X_1)}$$
$$=\frac{\frac{1}{2}\cdot P(C_1)\,\cancel{P(X_1)}}{P(D_3|X_1)\,\cancel{P(X_1)}}=\frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{1}{3}.$$

$$P(C_2|D_3,X_1)=\frac{P(C_2\cap D_3\cap X_1)}{P(D_3\cap X_1)}=\frac{P(D_3|C_2,X_1)\,P(C_2,X_1)}{P(D_3\cap X_1)}$$
$$=\frac{P(C_2)\,\cancel{P(X_1)}}{P(D_3|X_1)\,\cancel{P(X_1)}}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}.$$

Sim. $P(C_1|D_2,X_1)=\frac{1}{3}$, $P(C_3|D_2,X_1)=\frac{2}{3}$.

Thus winning probability increases with switching.

36. The required event will happen, if in the next three times, 2 black balls are placed and 1 white ball is placed. Thus the required probability is $\binom{3}{2}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{3} = \cancel{3}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{\cancel{3}} = \frac{4}{9}$.

37. Let $A_{ij}$ denote the event that $j$th machine produces code $i$, where $i = 0, 1$, $j = 1, 2, 3, 4$.

Using Bayes Theorem, the required probability is

$$P(A_{01} \mid A_{14}) = \frac{P(A_{14} \mid A_{01})\,P(A_{01})}{P(A_{14} \mid A_{01})\,P(A_{01}) + P(A_{14} \mid A_{11})\,P(A_{11})}.$$

Now each code, gets changed maximum 3 times. If $M_1$ produces code 0 and $M_4$ produces code 1, then the code has changed odd number of times. Sim., if both $M_1$ & $M_4$ produce code 1 then the code has changed even number of times. Thus

$$P(A_{14} \mid A_{01}) = \binom{3}{1}\left(\frac{3}{4}\right)\cdot\left(\frac{1}{4}\right)^2 + \binom{3}{3}\left(\frac{3}{4}\right)^3$$

$$P(A_{14} \mid A_{11}) = \binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \binom{3}{0}\left(\frac{1}{4}\right)^3.$$

Thus $P(A_{01} \mid A_{14}) = \frac{3}{10}$.

38. a) $P(B \cap C \cap P \cap M) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5} = \frac{1}{120}$.

b) $P(B^c \cap C^c \cap P^c \cap M^c) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)$

$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5} = \frac{24}{120} = \frac{1}{5}$.

c) $P(B\cap C^c\cap P^c\cap M^c) + P(B^c\cap C\cap P^c\cap M^c) + P(B^c\cap C^c\cap P\cap M^c)$

$+P(B^c\cap C^c\cap P^c\cap M) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}$

$+ \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5} = \frac{24+12+8+6}{120}$

$= \frac{5\cancel{0}}{12\cancel{0}} = \frac{5}{12}.$

d) $P(B\cap C\cap P^c\cap M^c) + P(B\cap C^c\cap P\cap M^c) + P(B\cap C^c\cap P^c\cap M) +$
$P(B^c\cap C\cap P\cap M^c) + P(B^c\cap C\cap P^c\cap M) + P(B^c\cap C^c\cap P\cap M)$

$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{5}$

$+ \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{5} = \frac{12+8+6+4+3+2}{120} = \frac{35}{120}.$

e) $P(B\cup C\cup P\cup M) = 1 - \frac{1}{5} = \frac{4}{5}$

39. $P(H_j \mid F_j) = \dfrac{P(H_j\cap F_j)}{P(F_j)} = \dfrac{P(F_j\mid H_j)P(H_j)}{P(F_j\mid H_j)P(H_j) + P(F_j\mid H_j^c)P(H_j^c)}$

$= \dfrac{\gamma_j p_j}{\gamma_j p_j + (1-p_j)} < P(H_j) = p_j$

$\left[\begin{array}{l} \gamma_j(1-p_j) < (1-p_j) \\ \Rightarrow \gamma_j < \gamma_j p_j + (1-p_j) \\ \Rightarrow \dfrac{\gamma_j}{\gamma_j p_j + (1-p_j)} < 1 \end{array}\right]$

$P(H_i \mid F_j) = \dfrac{p_i}{\gamma_j p_j + (1-p_j)}$

$> p_i = P(H_i).$

40. Let $A_n$ be the event that a 6 does not show up after n rolls. Then $P(A^c) \leq P(A_n) = \left(\frac{5}{6}\right)^n$

Thus $P(A^c) = 0$.

41. $$P(\text{Disease} \mid \text{positive}) = \frac{P(\text{positive} \mid \text{disease})\, P(\text{disease})}{P(\text{positive} \mid \text{disease})\, P(\text{disease}) + P(\text{positive} \mid \text{no disease}) \times P(\text{no. disease})}$$

$$= \frac{\frac{95}{100} \cdot \frac{5}{1000}}{\frac{95}{100} \cdot \frac{5}{1000} + \frac{1}{100} \cdot \frac{995}{1000}} = \frac{475}{1470} = \frac{95}{294}.$$

42. $$P(\text{Aircraft} \mid \text{alarm}) = \frac{P(\text{alarm} \mid \text{aircraft})\, P(\text{aircraft})}{P(\text{alarm} \mid \text{aircraft})\, P(\text{aircraft}) + P(\text{alarm} \mid \text{no aircraft}) \times P(\text{no aircraft})}$$

$$= \frac{\frac{99}{100} \cdot \frac{5}{100}}{\frac{99}{100} \cdot \frac{5}{100} + \frac{10}{100} \times \frac{95}{100}} = \frac{495}{1445} = \frac{99}{229}$$

$$P(\text{no aircraft} \cap \text{alarm}) = P(\text{alarm} \mid \text{no aircraft})\, P(\text{no. aircraft})$$

$$= \frac{\cancel{10}}{100} \times \frac{\overset{19}{\cancel{95}}}{\underset{2}{\cancel{100}}} = \frac{19}{200}.$$

$$P(\text{aircraft} \cap \text{no alarm}) = P(\text{no. alarm} \mid \text{aircraft})\, P(\text{aircraft})$$

$$= \frac{1}{100} \cdot \frac{\cancel{5}}{\underset{20}{\cancel{100}}} = \frac{1}{2000}.$$

43: $$P(\text{Disease} \mid \text{positive}) = \frac{P(\text{positive} \mid \text{disease})\,P(\text{disease})}{P(\text{positive})}$$

$$= \frac{\frac{95}{100}\cdot\frac{1}{1000}}{\frac{95}{100}\cdot\frac{1}{1000} + \frac{5}{100}\cdot\frac{999}{1000}} = \frac{\cancel{95}\ 19}{\cancel{5090}\ 1018}$$

$$= \frac{19}{1018}.$$

44. a) $1 - \prod_{i=1}^{n}\left(1 - p_i(t)\right)$

b) $\prod_{i=1}^{n} p_i(t)$

c) $\sum_{i=k}^{n} \binom{n}{i}\left(p(t)\right)^{i}\left(1 - p(t)\right)^{n-i}$

45. Required probability is $(0.75)(0.95)$
$+ (0.90)(0.80)(0.90) + (0.90)(0.95)(0.85) - (0.75)(0.95)$
$(0.90)(0.80)(0.90) - (0.75)(0.95)(0.90)(0.95)(0.85)$
$-(0.90)(0.80)(0.90)(0.95)(0.85) + (0.75)(0.95)(0.90)$
$(0.80)(0.90)(0.95)(0.85) = 0.957$