Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 05

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Random Variables

- In most of the practical situations we are interested in numerical characteristic of a random experiment.
- For example, we may be interested in number of heads out of 10 tosses of a coin, number of bug reported for a newly developed software, value of total yield of a crop in different months of a year in Assam, the level of water in Brahmaputra river each day at a particular site, etc.
- Hence, it is helpful to use a function which maps a sample space to $\mathbb{R}$. Such a function is called a random variable.
- Moreover, we have rich mathematical tools on the set of real numbers. These mathematical tools can be used to analyze several properties of probability of the quantity of interest if we can transform any arbitrary sample space to $\mathbb{R}$ or a subset of $\mathbb{R}$.

Random Variables

Def: A function $X : \mathcal{S} \to \mathbb{R}$ is called a random variable.

Example 1: Tossing a fair coin n times. Assume that the tosses are independent. Let $X : \mathcal{S} \to \mathbb{R}$ be defined by the no. of tails.

Example 2: Throwing a fair die twice. Assume the throws are independent. Let $X : \mathcal{S} \to \mathbb{R}$ be defined by the sum of the outcomes.

Example 3: Suppose we are testing the reliability of a battery. Define $X_1 : \mathcal{S} \to \mathbb{R}$ by $X_1(\omega) = \omega$. Here, X_1 denotes the lifetime of the battery. Now suppose we are mainly interested in whether the battery would last more than 2 years or not. Then define $X_2 : \mathcal{S} \to \mathbb{R}$ by $X_2(\omega) = I_{(2,\infty)}(\omega)$.

Example 4: In example 1, take n=2. $P(X=0)=P(X=2)=1/4$, $P(X=1)=1/2$.

Example 5: In example 2, $P(X=2)=1/36, P(X=3)=2/36, P(X=4)=3/36, P(X=5)=4/36, P(X=6)=5/36, P(X=7)=6/36$, $P(X=8)=5/36, P(X=9)=4/36, P(X=10)=3/36$, $P(X=11)=2/36, P(X=12)=1/36$.

Example 6: In example 3, for some interval $I \in (0,\infty), P(I)=\int_I e^{-t}dt$, defines a probability on $\mathcal{B}(0,\infty)$. $P(X_2=1)=e^{-2}, P(X_2=0)=1-e^{-2}$.

Cumulative Distribution Function

Def: The cumulative distribution function (CDF) of a random variable X is a function $F_X : \mathbb{R} \to [0, \infty)$ defined by

$$F_X(x) = P(X \leq x) \,.$$

* Note that CDF is defined for all real numbers. Though in the definition $[0, \infty)$ is written as co-domain for CDF, it clear that CDF lies in the interval $[0, 1]$ for all real numbers.

Example 7: From example 4,

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1/4 & \text{if} \quad 0 \leq x < 1, \\ 3/4 & \text{if} \quad 1 \leq x < 2, \\ 1 & \text{if} \quad x \geq 2 \,. \end{cases}$$

Example 2: From example 5,

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 2, \\ 1/36 & \text{if} \quad 2 \le x < 3, \\ 3/36 & \text{if} \quad 3 \le x < 4, \\ 6/36 & \text{if} \quad 4 \le x < 5, \\ 10/36 & \text{if} \quad 5 \le x < 6, \\ 15/36 & \text{if} \quad 6 \le x < 7, \\ 21/36 & \text{if} \quad 7 \le x < 8, \\ 26/36 & \text{if} \quad 8 \le x < 9, \\ 30/36 & \text{if} \quad 9 \le x < 10, \\ 33/36 & \text{if} \quad 10 \le x < 11, \\ 35/36 & \text{if} \quad 11 \le x < 12, \\ 1 & \text{if} \quad x \ge 12\,. \end{cases}$$

Example 3: From example 6,

$$F_{X_1}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - e^{-x} & \text{if} \quad x \geq 0\,. \end{cases}$$

$$F_{X_2}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - e^{-2} & \text{if} \quad 0 \leq x < 1, \\ 1 & \text{if} \quad x \geq 1\,. \end{cases}$$

Proposition: The CDF of a random variable has the following properties:

(1) $F_X(\cdot)$ is non-decreasing [and hence can have only jump discontinuities.]

(2) $\lim_{x\uparrow\infty} F_X(x) = 1, \lim_{x\downarrow-\infty} F_X(x) = 0$.

(3) $\lim_{h\downarrow 0} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$, thus CDF is right continuous.

(4) $\lim_{h\downarrow 0} F_X(x-h) = F_X(x) - P(X = x), \forall x \in \mathbb{R}$.

Theorem: Let F be a function satisfying properties (1)-(3). Then F is a CDF.

Remarks

- Random variable is just a function and does not depend on the probability. But the distribution of the random variable depends on the probability. So keeping the function same if we change the probability then the random variable will remain the same but its distribution will change. Consider Example 1, but with the probabilities $P(HH) = 9/16, P(TT) = 1/16, P(HT) = P(TH) = 3/16$. What will be the distribution function in this case?
- If $x \in \mathbb{R}$ is such that $P(X = x) > 0$, then x is said to be an atom of the distribution function of X. Thus if the distribution function of a random variable has no atoms then it is continuous.
- $P(a < X \leq b) = F_X(b) - F_X(a)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a-)$.
- $P(a < X < b) = F_X(b-) - F_X(a)$.
- $P(a \leq X < b) = F_X(b-) - F_X(a-)$.