Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 03

Indian Institute of Technology Guwahati

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Continuity of Probability

Def: A sequence, $\{E_n\}_{n\geq 1}$, of events are said to be increasing if

$$E_n \subseteq E_{n+1} \text{ for all } n = 1,\, 2,\, \ldots .$$

Def: A sequence, $\{E_n\}_{n\geq 1}$, of events are said to be decreasing if

$$E_{n+1} \subseteq E_n \text{ for all } n = 1,\, 2,\, \ldots .$$

Def: For an increasing sequence, $\{E_n\}_{n\geq 1}$, of events, define

$$\lim_{n\to\infty} E_n = \cup_{n=1}^{\infty} E_n.$$

Def: For a decreasing sequence, $\{E_n\}_{n\geq 1}$, of events, define

$$\lim_{n\to\infty} E_n = \cap_{n=1}^{\infty} E_n.$$

Continuity of Probability

Continuity from below:

Theorem: Let $\{E_n\}_{n\geq 1}$ be an increasing sequence of events, then

$$P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Continuity from above:

Theorem: Let $\{E_n\}_{n\geq 1}$ be a decreasing sequence of events, then

$$P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

- Finite additivity, and continuity from below, implies countable additivity.

Conditional Probability

- A die is thrown twice. What is the probability that the sum is 6?

Ans: 5/36

- Now suppose you have observed the outcome of the first throw and it is 4. Now what is the probability that the sum will be 6?

Ans: 1/6.

Once you are given some information or you observe something, the sample space changes. Conditional probability is a probability on the changed sample space.

Def: Let H be an event with $P(H) > 0$. For any arbitrary event A, the conditional probability of A given H is defined by

$$P(A|H) = \frac{P(A \cap H)}{P(H)}.$$

$$P(A \cap B) = \begin{cases} P(A)P(B|A) & \text{if } P(A) > 0 \\ P(B)P(A|B) & \text{if } P(B) > 0 \end{cases}$$

Def: A collection of events $\{E_1, E_2 \ldots\}$ is said to be **mutually exclusive** if $E_i \cap E_j = \phi, \forall i \neq j$. It is said to be **exhaustive** if $\cup_i E_i = \mathcal{S}$.

Theorem of Total Probability:

Theorem: Let $\{E_1, E_2 \ldots\}$ be a collection of mutually exclusive and exhaustive events with $P(E_i) > 0, \forall i$. Then for any event E,

$$P(E) = \sum_i P(E|E_i)P(E_i)\,.$$

Bayes' Theorem:

Theorem: Let $\{E_1, E_2 \ldots\}$ be a collection of mutually exclusive and exhaustive events with $P(E_i) > 0, \forall i$. Let E be any event with $P(E) > 0$. Then

$$P(E_i|E) = \frac{P(E|E_i)P(E_i)}{\sum_j P(E|E_j)P(E_j)} \qquad i = 1, 2 \ldots.$$

Example 1: There are 3 boxes. Box 1 containing 1 white, 4 black balls. Box 2 containing 2 white, 1 black ball. Box 3 containing 3 white, 3 black balls. First you throw a fair die. If the outcomes are 1, 2 or 3 then box 1 is chosen, if the outcome is 4 then box 2 is chosen and if the outcome is 5 or 6 then box 3 is chosen. Finally you draw a ball at random from the chosen box.

a) Given the drawn ball is white what is the (conditional)probability that the ball is from box 1.

b) Given the drawn ball is white what is the (conditional)probability that the ball is from box 2.

Remarks

In the theorem of total probability and Bayes' theorem, we have considered a countable collection of events $\{E_1, E_2, \ldots\}$. However, the theorems hold true even if we have a finite collection of mutually exclusive and exhaustive events *(Why?)*.