Lecture 12:

Monday, 29/8/2022

Note Title 8/29/2022

Let G be a group. G is called abelian or commutative if $a * b = b * a \quad \forall a, b \in G$.

§ **Matrix groups:** $(M_{n \times m}(\mathbb{Z}), +)$, $(M_{n \times m}(\mathbb{Q}), +)$, $(M_{n \times m}(\mathbb{R}), +)$ are groups.

We have $M_{n \times m}(\mathbb{Z}) \leq M_{n \times m}(\mathbb{Q}) \leq M_{n \times m}(\mathbb{R}) \leq M_{n \times m}(\mathbb{C})$.

Matrix groups are abelian under addition.

Matrix groups under multiplication: Let $GL_n(\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}),\ \det(A) \neq 0\}$.

Then, $GL_n(\mathbb{R})$ is a group under matrix multiplication.

We have $GL_n(\mathbb{Q}) \leq GL_n(\mathbb{R})$, and these are non-commutative groups.

We define $SL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$.

Clearly, $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$.

We have $SL_n(\mathbb{Z}) \leq SL_n(\mathbb{Q}) \leq SL_n(\mathbb{R})$.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then, $O(A)$ is infinite.

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, $O(B) = 4$. $C = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $O(C) = 6$.

We have $A = BC$. Thus, B and C both have finite orders, but BC has infinite order.

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Recall that $S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$.

$$\mu_\infty = \bigcup_{n \geq 1} \mu_n$$

Thus, every element of μ_∞ looks like $e^{2\pi i k/n}$ for some $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

$$\therefore \mu_\infty = \{e^{\pi i q} \mid q \in \mathbb{Q}\}.$$

Also, if $x \notin \mathbb{Q}$, then $e^{\pi i x}$ has infinite order in S^1.

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§ Rotations in $\mathbb{R}^2$: Let $\theta \in \mathbb{R}$. Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \cdot \begin{pmatrix} x \\ y \end{pmatrix}$.

Then, T gives a rotation by an angle θ.

We have $|R_\theta| = 1$. So, $R_\theta \in SL_2(\mathbb{R})$.

If $n \geq 1$, then $R_\theta^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = R_{n\theta}$.

Also, $R_\theta^{-1} = R_{-\theta}$ $\therefore$ $R_\theta^{-n} = R_{-n\theta}$ $\forall\, n \geq 1$

Thus, $R_\theta^k = R_{k\theta}$ $\forall\, k \in \mathbb{Z}$

If $\theta = \frac{2\pi}{n}$, then $o(R_\theta) = n$.

Clearly, R_θ has finite order $\Leftrightarrow \theta = \pi \cdot q$, where $q \in \mathbb{Q}$.

We have $\{R_\theta \mid \theta \in \mathbb{R}\}$ is a group under matrix multiplication.

Elements of finite orders in $\{R_\theta \mid \theta \in \mathbb{R}\}$ is the group $\{R_\theta \mid \theta = \pi \cdot q,\ q \in \mathbb{Q}\} = \{R_{\pi \cdot q} \mid q \in \mathbb{Q}\}$.

——— x ———

We define a relation $\sim$ on $\mathbb{R}$ as follows:

For $x, y \in \mathbb{R}$, $x \sim y$ if $x - y \in \mathbb{Z}$.

If $x \in \mathbb{R}$, then the equivalence class containing x is the set $\{y \in \mathbb{R} \mid y \sim x\} = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$

$$= \{x + k \mid k \in \mathbb{Z}\} = x + \mathbb{Z}.$$

Let $\mathbb{R}/\mathbb{Z}$ denote the set of all the equivalence classes.

Then $\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} \mid x \in \mathbb{R}\}$
$= \{r + \mathbb{Z} \mid r \in [0,1)\}$.

- $\mathbb{R}/\mathbb{Z}$ is a group under the operation:
$$(r_1 + \mathbb{Z}) + (r_2 + \mathbb{Z}) = (r_1 + r_2) + \mathbb{Z}.$$

- $\mathbb{Q}/\mathbb{Z} = \{q + \mathbb{Z} \mid q \in \mathbb{Q}\} = \{q + \mathbb{Z} \mid q \in [0,1) \cap \mathbb{Q}\}$
is a subgroup of $\mathbb{R}/\mathbb{Z}$.

Every element of $\mathbb{Q}/\mathbb{Z}$ has finite order. #

Thm, we have

(1) $\mu_\infty \leq S^1$

(2) $\{R_{\pi \cdot q} \mid q \in \mathbb{Q}\} \leq \{R_\theta \mid \theta \in \mathbb{R}\}$

(3) $\mathbb{Q}/\mathbb{Z} \leq \mathbb{R}/\mathbb{Z}$.

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