Probability Theory and Random Processes (MA225)

Lecture Slides
Lecture 08

Indian Institute of Technology Guwahati

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Transformation of RV

1. If X is a random variable then $Y = g(X)$ is a random variable where $g : \mathbb{R} \to \mathbb{R}$.
2. Our aim is to find the distibution (CDF/PMF/PDF) of $Y = g(X)$ for a known distribution of X.
3. There are mainly three techniques.

Technique 1

Find the CDF of $Y = g(X)$ using the definition of CDF. That means that we will try to find $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$ for all $y \in \mathbb{R}$. Note that CDF exists for all type of RVs. Therefore, this technique can be used for any type of RV. This technique is best understood by examples.

Example 1: Let the random variable X has the following PDF:

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = [X]$.

Example 2: Let the random variable X has the following PDF:

$$f(x) = \begin{cases} \frac{|x|}{2} & \text{if } -1 < x < 1 \\ \frac{x}{3} & \text{if } 1 \leq x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X^2$.

Technique 2 for DRV

Find PMF (if Y is DRV) or PDF (if Y is CRV) of Y directly without finding its' CDF. Obviously, first we need to understand whether Y is DRV or CRV. This technique is mainly based on two theorems.
The first theorem consider the case when X is DRV. We will see that if X is DRV, then Y is also a DRV. The second theorem addresses the case when X is CRV.
We will see the under some condition, Y is a CRV if X is CRV. With examples, we will illustrate that if the conditions do not hold, then Y can be DRV as well as CRV. Hence, those conditions are important. Let us start with an example.

Example 3: Let the random variable X has the following PMF:

$$f(x) = \begin{cases} \frac{1}{7} & \text{if } x = -2,\ -1,\ 0,\ 1 \\ \frac{3}{14} & \text{if } x = 2,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X^2$.

Theorem: Let X be a DRV with PMF $f_X(\cdot)$ and support S_X. Let $g : \mathbb{R} \to \mathbb{R}$ and $Y = g(X)$. Then Y is a DRV with support $S_Y = \{g(x) : x \in S_X\}$ and PMF

$$f_Y(y) = \begin{cases} \sum_{x \in A_y} f_X(x) & \text{if } y \in S_Y \\ 0 & \text{otherwise,} \end{cases}$$

where $A_y = \{x \in S_X : g(x) = y\}$.

Example 4: $X \sim Bin(n, p)$. Find the distribution of $Y = n - X$.

Technique 2 for CRV

Theorem: Let X be a CRV with PDF $f_X(\cdot)$ and support S_X, which is an interval. Let $g : S_X \to \mathbb{R}$ be a differentiable function and either $g'(x) < 0$ for all $x \in S_X$ or $g'(x) > 0$ for all $x \in S_X$. Then the RV $Y = g(X)$ is a CRV with PDF

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{for } y \in g(S_X) \\ 0 & \text{otherwise.} \end{cases}$$

Example 5: Let $X \sim U(0,\, 1)$, then $Y = -\ln X \sim Exp(1)$.

Example 6: Let $X \sim Exp(1)$, then find the distribution of $Y = X^2$.

Example 7: Let $X \sim N(0,\, 1)$, then find the distribution of $Y = X^2$.