Note Title 11/18/2020

Lecture 30:

§ Rings of continuous functions:

Let $R := C([a,b], \mathbb{R}) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$.

For $\alpha \in [a,b]$, let $M_\alpha = \{ f \in R \mid f(\alpha) = 0 \}$

= set of all continuous functions $f : [a,b] \to \mathbb{R}$ such that $f(\alpha) = 0$.

① M_α is a maximal ideal for every $\alpha \in [a,b]$.

__Proof:__ For $\alpha \in [a,b]$, define $F_\alpha : R \to \mathbb{R}$, where $R = C([a,b], \mathbb{R})$.

$$F_\alpha(f) = f(\alpha).$$

It is easy to prove that $\boxed{F_\alpha \text{ is an onto ring homomorphism}}$

Hence, by 1st isomorphism theorem,

$$R/\ker(F_\alpha) \cong \mathbb{R}.$$

Since $\mathbb{R}$ is a field, so $R/\ker(F_\alpha)$ is also a field, and hence $\ker(F_\alpha)$ is an maximal ideal.

Now, $\ker(F_\alpha) = \{f \in R \mid F_\alpha(f) = 0\} = \{f \in R \mid f(\alpha) = 0\} = M_\alpha$.

$\therefore$ If $\alpha \in [a,b]$, then M_α is a maximal ideal in $R = C([a,b], \mathbb{R})$.

Converse:

(2) Any maximal ideal of $C([a,b], \mathbb{R})$ is of the form M_α for some $\alpha \in [a,b]$.

Proof: Let A be any maximal ideal of $C([a,b],\mathbb{R})$.

Let $Z_A = \{x \in [a,b] : f(x) = 0 \ \ \forall f \in A\}$

$=$ the common zeros of the elements of A.

Claim: $Z_A \neq \phi$. That is, if A is a maximal ideal, then there is a common zero.

Suppose that $Z_A = \phi$. Then, for $x \in [a,b]$, there exists $f_x \in A$ such that $f_x(x) \neq 0$. (that $x \in [a,b]$ is not a zero for at least one function in A).

Since f_x is continuous, there exists a neighborhood U_x of x such that $U_x \subseteq [a, b]$ and f_x has no zero in U_x.

(that is, $f_x(y) \neq 0 \ \forall y \in U_x$)

Clearly, $[a, b] = \bigcup_{x \in [a,b]} U_x$.

Since $[a, b]$ is closed and bounded, so $[a, b] = \underbrace{U_{x_1} \cup U_{x_2} \cup U_{x_3} \cup \dots \cup U_{x_n}}_{\text{finitely many}}$ for some $x_1, x_2, \dots, x_n \in [a, b]$.

Let $f = f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2$. Then, $f(y) \neq 0 \ \forall y \in [a, b]$.

$\Rightarrow$ $1/f$ is well-defined on $[a, b]$ which is again continuous. $\Rightarrow$ f is a unit.

But $f_{x_1}, f_{x_2}, \dots, f_{x_n} \in A$ and hence $f \in A$.

Since f is a unit, so $A = R$. But A is a maximal ideal and therefore, this is a contradiction.

$\therefore$ We must have $Z_A \neq \phi$. Proof of the claim is complete.

Now, let $\alpha \in Z_A$. Then $f(\alpha) = 0 \quad \forall\ f \in A$

$$\Rightarrow \quad A \subseteq M_\alpha$$

Since both are maximal ideals, so $M_\alpha = A$.

Thus, from (1) and (2), we have that the maximal ideals of $C([a,b],\mathbb{R})$ are of the form M_α for $\alpha \in [a,b]$.

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§ <u>Polynomial ring:</u> Let R be a commutative ring with identity $1 \neq 0$. The polynomial ring $R[x]$ in the indeterminate x with coefficients from R is the set of all formal sums $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 0$ and each $a_i \in R$. If $a_n \neq 0$, then the polynomial is of degree n, $a_n x^n$ is the leading term, and a_n is the leading co-efficient. Also, a_0 is called the constant term.

$R[x]$ is a ring with respect to the following operations:

<u>Addition:</u> Addition is "componentwise".

$$\sum_{i=0}^{n} a_i x^i + \sum_{j=0}^{m} b_j x^j = \sum_{k=0}^{l} (a_k + b_k) x^k, \text{ where}$$

$l = \max\{n, m\}$. Also, if $n < m$, we take $a_k = 0$ $\forall k = n+1, n+2, \dots, m$.

If $m < n$, we take $b_k = 0$ $\forall k = m+1, m+2, \dots, n$

<u>Example:</u> $(a_0 + a_1 x) + (b_0 + b_1 x + b_2 x^2)$

$= (a_0 + b_0) + (a_1 + b_1) x + b_2 x^2$. Here, $a_2 = 0$.

Multiplication: Multiplication is performed by first defining $(ax^i)\cdot(bx^j) = ab\,x^{i+j}$, and then we define

$$\left(\sum_{i=0}^{n} a_i x^i\right)\cdot\left(\sum_{i=0}^{m} b_i x^i\right) = \sum_{k=0}^{n+m}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k$$

Example: $(a_0 + a_1 x)\cdot(b_0 + b_1 x + b_2 x^2)$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,x + (a_0 b_2 + a_1 b_1)\,x^2 + a_1 b_2 x^3.$$

Under this two operations, $R[x]$ becomes a commutative ring. The constant polynomial 1 plays the role of identity of $R[x]$.

Note that $a_0 + a_1 x + \cdots + a_n x^n$ is the zero polynomial in $R[x]$ if and only if $a_0 = a_1 = \cdots = a_n = 0$.
Also, two polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$ are equal if $a_i = b_i \ \forall i$.

For a non-zero polynomial f, $\deg(f)$ denotes the degree of f.

__Theorem:__ Let R be an integral domain. Then,

(i) $R[x]$ is an integral domain.

(ii) $\deg(f\cdot g) = \deg(f) + \deg(g) \quad \forall f, g \in R[x]$.

__Proof:__ Let $f, g \in R[x]$ and both f and g are non-zero.

Then, $\exists\, a_n \neq 0$ and $b_m \neq 0$ such that

$f(x) = a_0 + a_1 x + \dots + a_n x^n$ and $g(x) = b_0 + b_1 x + \dots + b_m x^m$.

But then, the leading term of $f(x)\cdot g(x)$ is

$a_n b_m x^{n+m}$. Since, $a_n \neq 0$, $b_m \neq 0$ and R is an integral domain, so $a_n b_m \neq 0$

and hence $f(x) \cdot g(x) \neq 0$. This proves that $R[x]$ has no zero divisors. Since R is an integral domain, so R is commutative with 1 and hence $R(x)$ is also commutative with 1.

This proves that $R[x]$ is an integral domain.

<u>Part(ii)</u>: Let $deg(f) = n$ and $deg(g) = m$.

Then, $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m$

where $a_n \neq 0$, $b_m \neq 0$.

Then, the leading term $a_n b_m x^{n+m}$ of $f(x) \cdot g(x)$ is also non-zero. Hence $\deg(fg) = n+m = \deg(f) + \deg(g)$.

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Ex: Let $f(x) = 1 + 2x$ and $g(x) = 2x^2$ be two polynomials of $\mathbb{Z}_4[x]$, $\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

Then, $\deg(f) = 1$ & $\deg(g) = 2$.

But $fg = 2x^2 + 4x^3 = 2x^2$ ($\because 4 = 0$ in $\mathbb{Z}_4$).

and hence $\deg(fg) = 2 < 1+2$.

Hence, $\deg(fg) = \deg(f) + \deg(g)$ need not be true if R is not an integral domain

Theorem: Let F be a field. Then, $F[x]$ is a PID.

Proof: Since F is a field, so F is an integral domain. Hence, we need to prove that every ideal of $F[x]$ is principal. Clearly, zero ideal is principal.

Let I be a non-zero ideal of $F[x]$.

Then, $\exists\, g \in I$ such that g is non-zero. $\longrightarrow$ (*)

Let $D = \{ \deg(f) : f \neq 0,\ f \in I \}$, and due to (*), $D \neq \phi$.

By well-ordering principle, D has a least element.

Let $f \in I$ be a polynomial of least degree.

__Claim:__ $I = (f)$, the principal ideal generated by f.

To prove the claim, we need division algorithm for $F[x]$

__Theorem:__ Let $p(x), g(x) \in F[x]$ with $p(x) \neq 0$.

Then, $\exists$ $q(x)$ and $r(x)$ such that

$$g(x) = q(x)\,p(x) + r(x), \text{ where either}$$

$$r(x) = 0 \quad \text{or} \quad r(x) \neq 0 \text{ with } \deg(r) < \deg(p).$$

We will discuss about this algorithm in the next class.

Proof of the claim: Recall that $f \in I$ with least degree. Let $g \in I$. Since $f \neq 0$, so by using division algorithm, $\exists\, q(x)$ and $r(x)$ such that

$$g(x) = f(x).q(x) + r(x), \text{ with either } r(x) = 0$$
$$\text{or } r(x) \neq 0,\ \deg(r) < \deg(f).$$

$$\Rightarrow r(x) = \underset{\substack{\cap \\ I}}{g(x)} - \underbrace{f(x).q(x)}_{\substack{\cap \\ I}} \in I.$$

If $r(x) \neq 0$, then $\deg(r) < \deg(f)$ is a contradiction, since f has least degree in I.

$\therefore r(x) = 0 \Rightarrow g(x) = f(x).q(x)$

This proves that $g(x) \in (f)$.

$\therefore\ I \subseteq (f)$.

Since, $f \in I$, so $(f) \subseteq I$.

This implies that $I = (f)$.

$\therefore\ F[x]$ is a PID.