Lecture 22: 26th Sep 2022

Note Title 9/26/2022

<u>Theorem 1</u>: Every cycle $(i_1, i_2 \cdots i_r)$ can be expressed as a product of 2-cycles.

<u>Proof</u>: Clearly, if $r=1$, then $(1) = (1\ 2)(1\ 2)$.

Let $r > 1$. Then, we have

$$(i_1\ i_2 \cdots i_r) = (i_1\ i_r)(i_1\ i_{r-1}) \cdots (i_1\ i_2).$$

<u>Remark</u>: (1) expressions into product of 2-cycles need not be unique.

(2) the 2-cycles need not be disjoint.

Theorem 2: Every $f \in S_n$ is a product of 2-cycles.

Proof: We know that every $f \in S_n$ can be expressed as a product of (disjoint) cycles. By Theorem 1, every cycle can be expressed as a product of 2-cycles.

$\therefore$ Every $f \in S_n$ can be expressed as a product of 2-cycles.

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§ **Signum function:** Let $\sigma \in S_n$.

We define
$$\operatorname{sign}(\sigma) = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}.$$

Then, $\mathrm{sign}(\sigma) = \pm 1 \quad \forall \sigma \in S_n$.

<u>Theorem 3:</u> $\mathrm{sign}: S_n \longrightarrow \{1, -1\}$ is a group homomorphism.

That is, $\mathrm{sign}(f \circ g) = \mathrm{sign}(f) \cdot \mathrm{sign}(g)$.

<u>Proof:</u> We have

$$\mathrm{sign}(f \circ g) = \prod_{1 \leq i < j \leq n} \frac{(f \circ g)(i) - (f \circ g)(j)}{i - j}$$

$$= \prod_{1 \leq i < j \leq n} \frac{f(g(i)) - f(g(j))}{g(i) - g(j)} \times \prod_{1 \leq i < j \leq n} \frac{g(i) - g(j)}{i - j}$$

$$= \mathrm{sign}(f) \cdot \mathrm{sign}(g).$$

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Theorem 4: If $f \in S_n$ is a 2-cycle, then $\operatorname{sign}(f) = -1$.

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Let $f \in S_n$, and $f = \alpha_1 \alpha_2 \cdots \alpha_m$, where each α_i is a 2-cycle. Then,

$$\begin{aligned}\operatorname{sign}(f) &= \operatorname{sign}(\alpha_1 \cdot \alpha_2 \cdots \alpha_m) \\ &= \operatorname{sign}(\alpha_1)\operatorname{sign}(\alpha_2) \cdots \operatorname{sign}(\alpha_m) \\ &= (-1)\cdot(-1) \cdots (-1) \quad [\text{By using Theorem 4}] \\ &= (-1)^m.\end{aligned}$$

Theorem 5: Let $(1) = \beta_1 \beta_2 \cdots \beta_k$, where $\beta_1, \beta_2, \ldots, \beta_k$ are 2-cycles.

Proof: We have $1 = \operatorname{sign}(\beta_1 \cdots \beta_k) = (-1)^k \Rightarrow k$ is even.

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Theorem 6: Let $f \in S_n$. Suppose that $f = \alpha_1 \circ \alpha_2 \cdots \circ \alpha_m = \beta_1 \cdot \beta_2 \cdots \beta_k$, where $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_k$ are 2-cycles. Then, both m and k are even or both are odd, that is, $m \equiv k \pmod 2$.

Proof: Since $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_m = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k$, so

$$\operatorname{sign}(\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_m) = \operatorname{sign}(\beta_1 \cdot \beta_2 \cdots \beta_k)$$

$$\Rightarrow (-1)^m = (-1)^k$$

$\therefore$ Both m and k are even or both are odd.

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Definition (odd and even permutation):

$f \in S_n$ is called an even permutation if f can be expressed as a product of even number of 2-cycles.

$f \in S_n$ is called an odd permutation if f can be expressed as a product of odd number of 2-cycles.

Ex:
- Any 2-cycle is an odd permutation.
- $(1\,2\,3)$ is an even permutation since $(1\,2\,3) = (1\,3)(1\,2)$.
- An r-cycle is an even permutation $\iff$ r is odd.

This is because, $(i_1\, i_2 \,\ldots\, i_r) = \underbrace{(i_1\, i_r)(i_1\, i_{r-1}) \cdots (i_1\, i_2)}_{r-1 \text{ no. of 2-cycles}}$.

Let A_n = set of all the even permutations of S_n

B_n = " " " " odd " " ".

we have

(1) $A_n \cap B_n = \phi$ (2) $A_n \cup B_n = S_n$

(3) $|A_n| = |B_n|$. <u>Proof</u>: $\psi : A_n \longrightarrow B_n$

$$f \longmapsto (1\ 2) f$$

ψ is a bijection.

Hence, $|A_n| = |B_n| = \frac{|S_n|}{2} = \frac{n!}{2}$.

(4) kernel of sign $= \{ f \in S_n \mid \text{sign}(f) = 1 \} = A_n$

$\therefore$ A_n is a normal subgroup of S_n.

(5) B_n is <u>not</u> a subgroup of S_n. B_n is not closed, $(1) \notin B_n$.

__Ex:__ For $n \geq 3$, $Z(S_n) = \{(1)\}$.

__Solution:__ Let $f \neq (1)$. __Claim:__ $f \notin Z(S_n)$, that is, $\exists\, g \in S_n$ s.t. $f \circ g \neq g \circ f$.

__Proof of the claim:__ Since $f \neq (1)$, so $\exists\, i \neq j$ such that

$$f(i) = j.$$

Since $n \geq 3$, so $\exists\, k$ s.t. $k \neq i$ and $k \neq j$.

Let $g = (k\ j)$. Then,

$(f \circ g)(i) = f(g(i)) = f(i) = j$ and $(g \circ f)(i) = g(f(i)) = g(j) = k$.

$\therefore f \circ g \neq g \circ f \Rightarrow f \notin Z(S_n)$. #