1> Since (X, Y) a bivariate normal.

So $X+Y$ and $X-Y$ are independent if we can show $Cov(X+Y, X-Y) = 0$

$$cov(X+Y, X-Y) = cov(X+Y, X) - cov(X+Y, Y)$$

$$= cov(X, X) + cov(Y, X) - \{cov(X, Y) + cov(Y, Y)\}$$

$$= var(X) + cov(Y, X) - cov(X, Y) - var(Y)$$

$\{\because cov(Y, X) = cov(X, Y)$ and $var(X) = var(Y)$ Given$\}$

$$= 0 \quad \text{proved.}$$

2> a> $E(X) = 0 \quad E(Y) = -1 \quad var(X) = 1 \quad var(Y) = 4$

$$\rho = \frac{cov(X, Y)}{\sigma_X \sigma_Y} \quad \& \quad \rho = -1/2$$

$$cov(X, Y) = -1/2 \times 1 \times 2 = -1$$

$$E(X+Y) = 0 - 1 = -1$$

$$var(X+Y) = var(X) + var(Y) + 2\,cov(X, Y)$$

$$= 1 + 4 + 2 \times -1 = 3$$

$$X+Y \sim \cancel{BVN}\ N(-1, 3)$$

$$P(X+Y > 0) = P\left(\frac{X+Y-(-1)}{\sqrt{3}} > \frac{0-(-1)}{\sqrt{3}}\right)$$

$$= 1 - P(Z \leq 1/\sqrt{3}) = 1 - \Phi(1/\sqrt{3})$$

$$= 0.2819$$

(b) Find a such that $aX+Y$ and $X+2Y$ are independent

$$\text{Cov}(aX+Y,\ X+2Y) = 0$$

$$\text{cov}(aX+Y, X) + \text{cov}(aX+Y, 2Y) = 0$$

$$a\,\text{Cov}(X,X) + \text{cov}(Y,X) + 2a\,\text{cov}(X,Y) + 2\,\text{Cov}(Y,Y) = 0$$

$$a\,\text{var}(X) + \text{cov}(X,Y)(1+2a) + 2\,\text{var}(Y) = 0$$

$$a + -1 \times (1+2a) + 2 \times 4 = 0$$

$$a - 1 - 2a = -8$$

$$-a = -7 \quad \Rightarrow \quad a = 7$$

(c) $P(X+Y>0 \mid 2X-Y=0)$

Let X and Y are jointly (Bivariate) Normal RV with parameters μ_X, σ_X^2, μ_Y, σ_Y^2 and ρ.

Then given $X=x$, Y is normally distribute.

$$Y \mid X=x \sim N\left(\mu_Y + \rho\sigma_Y\left(\frac{x-\mu_X}{\sigma_X}\right),\ (1-\rho^2)\sigma_Y^2\right)$$

Let $P = X+Y$ $\quad Q = 2X-Y$

P and Q are ~~Bivariate~~ (univariate) normals (Separately).

$$E(P) = 0 - 1 = -1 \qquad E(Q) = 2EX - EY = 2\times 0 - (-1) = 1$$

$$\text{var}(P) = \text{var}(X) + \text{var}(Y) + 2\text{Cov}(X,Y) = 1 + 4 + (-2) = 3$$

$$\text{var}(Q) = 4\,\text{var}(X) + \text{var}(Y) - 2\times 2\,\text{Cov}(X,Y) = 4\times 1 + 4 - 4(-1) = \cancel{10}\ 12$$

$$\operatorname{cov}(P,Q) = \operatorname{cov}(X+Y,\ 2X-Y)$$
$$= \operatorname{cov}(X+Y,\ 2X) - \operatorname{cov}(X+Y,\ Y)$$
$$= \operatorname{cov}(X,2X) + \operatorname{cov}(Y,2X) - \{\operatorname{cov}(X,Y) + \operatorname{cov}(Y,Y)\}$$
$$= 2\operatorname{Var}(X) + 2\operatorname{cov}(Y,X) - \operatorname{cov}(X,Y) - \operatorname{Var}(Y)$$
$$= 2\times 1 + \operatorname{cov}(Y,X) - 4$$
$$= -2 + \times(-1) = -3$$

$$\rho(P,Q) = \frac{\operatorname{cov}(P,Q)}{\sigma_P\,\sigma_Q} = \frac{-3}{\sqrt{3\times 12}} = \frac{-1}{2}$$

$$\operatorname{Prob}(P>0 \mid Q=0) = ?$$

$$P \mid Q=q \sim N\left(\mu_P + \rho(P,Q)\left(\frac{q-\mu_Q}{\sigma_Q}\right)\sigma_P,\ (1-\rho^2_{P,Q})\,\sigma_P^2\right)$$

$$P \mid Q=0 \sim N\left(-1 + \frac{-1}{2}\left(\frac{0-1}{\sqrt{12}}\right)\times\sqrt{3},\ \left(1-\frac{1}{4}\right)\times 3\right)$$

$$P \mid Q=0 \sim N\left(\frac{-3}{4},\ \frac{9}{4}\right)$$

$$\operatorname{Prob}(P>0 \mid Q=0) = \operatorname{Prob}\left(\frac{P-(-3/4)}{3/2} > \frac{3/4}{3/2} \,\Big|\, Q=0\right)$$
$$= \operatorname{Prob}(Z > 1/2)$$
$$= 1 - \operatorname{Prob}(Z \le 1/2)$$
$$= 1 - \Phi(1/2)$$

OR, $P(X+Y>0 \mid 2X-Y=0)$

$= P(X+Y>0 \mid 2X=Y)$

$= P(X+Y>0 \mid X=Y/2)$

$= P\left(\frac{Y}{2}+Y>0\right)$

$= P\left(\frac{3Y}{2}>0\right) = P(Y>0)$

$= 1-P(Y\leq 0)$

$= 1-P\left(\frac{Y-(-1)}{2} \leq \frac{0-(-1)}{2}\right)$

$= 1-P(Z \leq 1/2)$

$= 1-\Phi(1/2)$

(4) $\mu_x = 5 \quad \sigma_x^2 = 1 \qquad \rho_{X,Y} = \rho$

$\mu_y = 10 \quad \sigma_y^2 = 25$

$Y \mid X=x \sim N\left(\mu_Y + \rho\sigma_Y\left(\frac{x-\mu_X}{\sigma_X}\right), (1-\rho^2)\sigma_Y^2\right)$

$Y \mid X=5 \sim N\left(10+\rho\times 5\frac{(5-5)}{1}, (1-\rho^2)25\right)$ (it will not matter here though!)

$Y \mid X=5 \sim N(10, 25(1-\rho^2))$

$y \in (4,16)$

$P(4 \leq Y \leq 16 \mid X=5) = P(Y \leq 16 \mid X=5) - P(Y < 4 \mid X=5)$

$= \cancel{P\left(\frac{Y-10}{5\sqrt{(1-\rho^2)}} \mid X=5\right)} - 1$

$= P\left(\frac{Y-10}{(\sqrt{1-\rho^2})\times 5} \leq \frac{16-10}{(\sqrt{1-\rho^2})5} \mid X=5\right)$

$- P\left(\frac{Y-10}{(\sqrt{1-\rho^2})\times 5} \leq \frac{4-10}{(\sqrt{1-\rho^2})\times 5} \mid X=5\right)$

$P(4 \le Y \le 16 | X = 5) =$

$$P\left(Z \le \frac{6}{5\sqrt{1-\rho^2}}\right) - P\left(Z \le \frac{-}{5}\right.$$

L

$$= \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \left(\Phi\left(\frac{-6}{5\sqrt{1-\rho^2}}\right)\right)$$

$$= \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \left(1 - \Phi\left(\frac{6}{5\sqrt{1}}\right.\right.$$

$$= 2\,\Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1$$

It is given $P(\cancel{x}4 \le Y \le 16 | X = 5) = .954$

so $2\,\Phi\left(\frac{6}{5(\sqrt{1-\rho^2})}\right) - 1 = .954$

$$\Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = 0.977$$

We know $P(Z \le 2) = .97725$

so $\dfrac{6}{5\sqrt{1-\rho^2}} = 2$

$$\left(\frac{6}{10}\right)^2 = 1 - \rho^2$$

$$\Rightarrow \quad \rho^2 = 1 - (.36)$$

$$\rho = .8 \quad \text{Ans.}$$

(5) $\mu_x = 0, \quad \sigma_x^2 = 1$

$\mu_y = 0 \quad \sigma_y^2 = 1 \qquad \rho = 0$

$$P(-c < X < c,\ -c < Y < c) = 0.95$$

$$\Rightarrow P(-c < X < c)\, P(-c < Y < c) = 0.95$$

Bec. of $\rho = 0 \Rightarrow X$ & Y are independent

$X \sim N(0, 1)$

$Y \sim N(0, 1)$

$$\Rightarrow [P(-c < X < c)]^2 = 0.95$$

$$P(X < c) - P(X < -c) = \sqrt{0.95}$$

$$\Rightarrow 2\Phi(c) - 1 = \sqrt{0.95}$$

$$\Phi(c) = \frac{1 + \sqrt{0.95}}{2}$$

$$\Phi(c) = \frac{1 + 0.97467}{2}$$

$$\Phi(c) = \frac{1.97467}{2} = .987335$$

Given $\cancel{\Phi(c) =}$ $\Phi(2.24) = 0.987335$

so $c = 2.24$

(6) $V_x, V_y, V_z \overset{iid}{\sim} N\left(0, \frac{KT}{m}\right)$

$V = (V_x^2 + V_y^2 + V_z^2)^{1/2} \qquad f_V(v) = ?$

$$V^2 = V_x^2 + V_y^2 + V_z^2$$

$$E(V^2) = E(V_x^2) + E(V_y^2) + E(V_z^2)$$

$$= \frac{3KT}{m}$$

We know $X \sim N(0,1)$

$E(X^2) = 1 \qquad E(X^3) = 0$

$E(X^4) = 3$

Let $W = X^2 \qquad E(X^4) = E(W^2)$

$$E(W^2) = Var(W) + (E(W))^2$$

$$(E(W))^2 = (E(X^2))^2 = (var(X) + (E(X))^2)^2$$

$$= (1 + 0^2)^2 = 1$$

$W \sim \chi^2_{(1)} \qquad E(W) = 1 \qquad Var(W) = 2$

$$E(X^4) = E(W^2) = Var(W) + [E(W)]^2$$

$$= 2 + 1^2 = 3$$

$\cancel{v^2 \sim N(}$

Also $\quad X \sim N(0, \sigma^2) \qquad \frac{X}{\sigma} \sim N(0,1) \qquad \frac{X^2}{\sigma^2} \sim \chi^2_{(1)}$

So $\quad V_x \sim N(0, \frac{KT}{m}) \qquad \frac{V_x^2}{\frac{KT}{m}} \sim \chi^2_{(1)}$

$V_y \sim N(0, \frac{KT}{m}) \qquad \frac{V_y^2}{KT/m} \sim \chi^2_{(1)}$ independent.

$V_z \sim N(0, \frac{KT}{m}) \qquad \frac{V_z^2}{KT/m} \sim \chi^2_{(1)}$

$$\frac{V_x^2 + V_y^2 + V_z^2}{KT/m} \sim \chi^2_{(3)} \Rightarrow V^2 \sim \frac{KT}{m}\chi^2_{(3)}$$

$$V^2 \sim Gamma\left(\frac{3}{2}, \frac{2KT}{m}\right)$$

Let $x = v^2$

$$f_X(x) = \begin{cases} \dfrac{e^{-\frac{2KT}{m}x} \, x^{3/2 - 1} \dfrac{(2KT)^{3/2}}{m^{3/2}}}{\Gamma 3/2} & 0 < x < \infty \\ 0 & \text{ow} \end{cases}$$

Let $U = \sqrt{X}$

$X = U^2 \qquad \dfrac{dx}{\partial U} = 2U \qquad |J| = 2|U|$

$$f_U(u) = \begin{cases} \dfrac{e^{-\frac{2KT}{m}u^2} \, (u^2)^{1/2} \dfrac{(2KT)^{3/2}}{m^{3/2}}}{\Gamma 3/2} \times 2u, & 0 < u < \infty \\ 0 & \text{ow} \end{cases}$$

$$f_V(v) = \begin{cases} \dfrac{2^{5/2} e^{-2v^2\frac{KT}{m}} \, v^2 \dfrac{(KT)^{3/2}}{m^{3/2}}}{\Gamma 3/2} & 0 < v < \infty \\ 0 & \text{ow} \end{cases}$$

(7)

Let W : be a RV representing Wives height
H : be a RV representing Husband's height

$P(W > H) = ?$

i.e. $P(W - H > 0) = ?$

$$E(W-H) = EW - EH$$
$$= 66.8 - 70 = -3.2$$

$$Var(W-H) = Var(W) + Var(H) - 2\,cov(W, H)$$
$$= 4 + 4 - 2\rho\sigma_W\sigma_H$$
$$= 8 - 2 \times .68 \times 2 \times 2$$
$$= 8 - 5.44 = 2.56$$

$$P\left(\frac{(W-H) - (-3.2)}{\sqrt{2.56}} > \frac{0 - (-3.2)}{\sqrt{2.56}}\right)$$

$$= P\left(Z > \frac{3.2}{1.6}\right)$$
$$= 1 - \Phi(2) = 1 - 0.977 = 0.023 \text{ Ans.}$$

(8) Some related concepts:-

In order to estimate the value of an unobserved RV Y given that we have observed $X = x$.

So our estimate will be a function of x

$$\hat{y} = g(x)$$

Error in estimate is

$$\tilde{y} = Y - \hat{y}$$
$$= Y - g(x)$$

Mean Square Error :-

$$E[(Y-\hat{y})^2 \mid X=x] = E[(Y-g(x))^2 \mid X=x]$$

min mean square Error is

$$\underset{g(x)}{\arg\min}\ E[(Y-g(x))^2 \mid X=x]$$

Min. MSE is $E[(Y-E(Y|X))^2 \mid X=x]$

Let $\hat{Y} = g(X)$ be an estimator of the RV ~~X~~ Y given that we have observed the RV X
MSE of this estimator is defined as

$$\cancel{E[(X-X} \quad E[(Y-\hat{Y})^2] = E[(Y-g(X))^2]$$

The MMSE estimator of Y

$$\hat{Y}_M = E[Y|X]$$

has the lowest MSE among all possible estimators

MMSE $\quad E[(Y-\hat{Y})^2] = E[(Y-E[Y|X])^2]$

NOW. Given $\mu_x = -1 \quad \sigma_x = 2 \quad E[X^2] = 4+1 = 5$

$$\left.\begin{matrix}\text{Best estimate} \\ \text{of } Y \text{ Given } X\end{matrix}\right\} = E[Y|X] = 3X+7$$

$$\text{MMSE} = E[(Y-E[Y|X])^2] = 28$$

SO $\quad E[E[Y|X]] = E[3X+7] = 3\times -1 + 7 = 4$

$E[Y] = 4 \qquad \mu_y = 4$

$$var[Y] = var[E(Y|X)] + E[var(Y|X)]$$

$$= var[3X+7] + E[E([Y - E[Y|X]]^2|X)]$$

$$= 9Var[X] + \cancel{E[E[28]]}\ E[(Y - E[Y|X])^2]$$

$$= 9 \times 4 + 28 = 64$$

$\sigma_Y = 8$

$\boxed{\mu_y = 4, \quad \sigma_Y = 8}$ $E[y^2] = 64 + 16 = 80$

We know $Y|X=x \sim N\left(\mu_Y + \rho\sigma_Y\left(\frac{x-\mu_x}{\sigma_x}\right), (1-\rho^2)\sigma_Y^2\right)$

$$Y|X \sim N\left(\mu_Y + \rho\sigma_Y\left(\frac{X-\mu_X}{\sigma_X}\right), (1-\rho^2)\sigma_Y^2\right)$$

$$var(Y|X) = (1-\rho^2)\sigma_Y^2$$

OR. Take MMSE.

$$E[(Y - E[Y|X])^2] = 28$$

$$E[(Y - (3X+7))^2] = 28$$

$$E[Y^2 + (3X+7)^2 - 2Y(3X+7)] = 28$$

$$E[y^2] + 9E[X^2] + 42E[X] + 49 - 6E[YX] - 14E[Y] = 28$$

$$\Rightarrow 80 + 9 \times 5 + 42 \times (-1) + 49 - 6E[YX] - 14 \times 4 = 28$$

We know $\rho = \dfrac{E[YX] - EYEX}{\sigma_X \sigma_Y}$

$$16\rho = E[YX] - 4 \times (-1)$$

$$E[YX] = 16\rho - 4$$

so

$$125 + 7 - 6(16\rho - 4) - 56 = 28$$

$$\Rightarrow \quad 132 - 96\rho + 24 - 56 = 28$$

$$156 - 96\rho = 84$$

$$\rho = \frac{72}{96} = \frac{3}{4}$$

$$\boxed{\mu_y = 4 \quad \sigma_y = 8 \quad \rho = 3/4}$$ done.

(3) Given $\mu_x = 0 \quad \sigma_x^2 = 1$

$\mu_y = 0 \quad \sigma_y^2 = 1 \qquad \rho(x, y) = \rho$

(x, y) bivariate normal so X and Y marginally pdf follow normal.

with $\begin{cases} \text{mean } \mu_X & \text{var } \sigma_X^2 \\ \text{mean } \mu_Y & \text{var } \sigma_Y^2 \end{cases}$ respectively.

$$E(X) = E[E(X|Y)]$$

$$E(x^2y^2) = E(E(x^2y^2|Y))$$

$$= E(Y^2 E(X^2|Y))$$

We know $E(X^2|Y) = var(X|Y) - [E(X|Y)]^2$

$$X|Y \sim N\left(\mu_X + \rho\sigma_X\left(\frac{Y - \mu_Y}{\sigma_Y}\right), \ (1-\rho^2)\sigma_X^2\right)$$

$$E(X|Y) = 0 + \rho \times 1\left(\frac{Y-0}{1}\right) = \rho Y$$