Lecture 21 16th Sep 2022

Note Title 9/19/2022

Theorem 1: Let $f \in S_n$. Then, either f is a cycle or a product of disjoint cycles.

Ex 1: Let G be a group, and $a, b \in G$. Suppose that $o(a)$ and $o(b)$ are finite and $ab = ba$.

(i) $o(ab) \mid l$, where $l = \text{lcm}(o(a), o(b))$.

(ii) if $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $o(ab) = \text{lcm}(o(a), o(b))$.

Soln: (i) $(ab)^l = a^l b^l = e \Rightarrow o(ab) \mid l$

(ii) Let $o(ab) = m$. Then, $(ab)^m = e \Rightarrow a^m = b^{-m}$.

$\therefore a^m \in \langle a \rangle \cap \langle b \rangle$ and $b^m \in \langle a \rangle \cap \langle b \rangle \Rightarrow a^m = e = b^m$

$\Rightarrow o(a) \mid m$ and $o(b) \mid m$

$\Rightarrow \ell = \mathrm{lcm}(o(a), o(b)) \mid m.$

$\Rightarrow \ell \mid o(ab)$

$\therefore \; o(ab) = \ell.$

<u>Theorem 2</u>: Let $f, g \in S_n$. If f and g are disjoint, then

$$o(fg) = \mathrm{lcm}(o(f), o(g))$$

<u>Proof</u>: Since f and g are disjoint, so $fg = gf$. and $\langle f \rangle \cap \langle g \rangle = \{(1)\}$.

$\therefore \; o(fg) = \mathrm{lcm}(o(f), o(g)).$

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__Corollary__: Let $f \in S_n$ and $f = \alpha_1 \alpha_2 \cdots \alpha_m$ as a product of disjoint cycles. Then, $o(f) = \operatorname{lcm}(o(\alpha_1), \ldots, o(\alpha_m))$.

__Ex 2:__ Let $\beta \in S_8$, and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$

We have $\beta = (2\ 3\ 8\ 4\ 7)(5\ 6) = \alpha_1 \alpha_2$

$\therefore\ o(\beta) = \operatorname{lcm}(o(\alpha_1), o(\alpha_2)) = \operatorname{lcm}(5, 2) = 10.$

__Ex 3:__ Let $f = (1\,2\,5\,6\,7)(2\,3\,8\,9) \in S_{10}$.

To find $o(f)$, we first need to express f as a product of disjoint cycles. We have $f = (1\,2\,3\,8\,9\,5\,6\,7)$ $\therefore\ o(f) = 8$.

Theorem 3: S_n is not abelian if $n \geq 3$.

Proof: Let $f = (1\ 2)$ and $g = (1\ 2\ 3)$. Then, $f, g \in S_n$ if $n \geq 3$.

Now, $fg = (1\ 2)(1\ 2\ 3) = (2\ 3)$

$gf = (1\ 2\ 3)(1\ 2) = (1\ 3)$

$\therefore fg \neq gf \Rightarrow S_n$ is not abelian if $n \geq 3$.

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Ex 4: Let $\beta = (1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} as a product of disjoint cycles.

We have $\beta = (1\ 2\ 3)(1\ 4\ 5) = (1\ 4\ 5\ 2\ 3)$ $\therefore\ o(\beta) = 5$

$$\therefore \beta^{100} = (1) \Rightarrow \beta^{99} = \beta^{-1} = (1\,4\,5\,2\,3)^{-1}$$

$$= (3\,2\,5\,4\,1),$$

which is already a cycle.

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