Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 09

Indian Institute of Technology Guwahati

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Expectation of Function of RV

Example 1: Let the random variable X be a DRV with PMF

$$f_X(x) = \begin{cases} \frac{1}{7} & \text{if } x = -2,\ -1,\ 0,\ 1 \\ \frac{3}{14} & \text{if } x = 2,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^2$. Find the expectation of Y.

Expectation of Function of RV

Theorem: Let X be a DRV with PMF $f_X(\cdot)$ and support S_X. Let $g : \mathbb{R} \to \mathbb{R}$. Then

$$E\left[g(X)\right] = \sum_{x \in S_X} g(x) f_X(x) \quad \text{provided} \quad \sum_{x \in S_X} |g(x)| f_X(x) < \infty.$$

Theorem: Let X be a CRV with PDF $f_X(\cdot)$. Let $g : \mathbb{R} \to \mathbb{R}$. Then

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad \text{provided} \quad \int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty.$$

Expectation of Function of RV

Theorem: Let X be a RV (either DRV or CRV). Then

1. Let $A \subset \mathbb{R}$. Then $E(I_A(X)) = P(X \in A)$.
2. $h_1(x) \leq h_2(x),$ for all $x \in \mathbb{R}$, then $E\left[h_1(X)\right] \leq E\left[h_2(X)\right]$, provided all the expectations exist.
3. $a < b$ are two real numbers such that $S_X \subset [a,\, b]$, then $a \leq E(X) \leq b$, provided the expectation exists.
4. $E(a + bX) = a + bE(X)$, where a and b are two real numbers.
5. Let $h_1(\cdot),\, \ldots,\, h_p(\cdot)$ be real valued function of real numbers such that $E(h_i(X))$ exists for all $i = 1,\, 2, \ldots,\, p$, then

$$E\left(\sum_{i=1}^{p} h_i(X)\right) = \sum_{i=1}^{p} E\left(h_i(X)\right).$$

Remarks

- For $r = 1, 2, \ldots$, $\mu_r = E(X^r)$ is called rth raw moment of X, if the expectation exists.
- $\mu'_r = E\left[(X - E(X))^r\right]$ is called rth central moment of X, if the expectations exist.
- $\mu'_2 = E\left[(X - E(X))^2\right]$ is called variance of X when it exists and is denoted by $Var(X)$.
- $Var(X) = E(X^2) - (E(X))^2$.

Moment Generating Function

Def: The moment generating function of random variable X is defined by

$$M_X(t) = E\left(e^{tX}\right)$$

provided there exists a real number $a > 0$ such that the expectation exists for all $t \in (-a, a)$ (the expectation exists in a neighbourhood of the origin).

Example 2: $X \sim Bin(n, p)$, then $M_X(t) = (1 - p + pe^t)^n$ for all $t \in \mathbb{R}$.

Example 3: $X \sim Exp(\lambda)$, then $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$ for all $t < \lambda$.

Example 4: $X \sim N(\mu, \sigma^2)$, then $M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$ for all $t \in \mathbb{R}$.

Def: X and Y are said to be same in distribution if $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$.

Theorem: Let X and Y be two random variables having MGFs $M_X(\cdot)$ and $M_Y(\cdot)$, respectively. Suppose that there exists a positive real number a such that $M_X(t) = M_Y(t)$ for all $t \in (-a, a)$. Then X and Y are same in distribution.

Example 5: Let $X \sim N(\mu, \sigma^2)$. Find the distribution of $Y = a + bX$.

The above is Technique 3 of Transformation of Random Variable.

Remark: If the MGF $M_X(t)$ exist for $t \in (-a, a)$ for some $a > 0$, the derivatives of all order exist at $t = 0$ and

$$E\left(X^k\right) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}$$

for all positive integer k.