Probability Theory and Random Processes (MA225)

LECTURE SLIDES
Lecture 07

Indian Institute of Technology Guwahati

July-Nov 2022

Properties of PDF

1. $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f_X(x) = 1$.

Theorem: Suppose a real valued function $g : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

1. $g(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} g(x)dx = 1$.

Then $g(\cdot)$ is a probability density function of some continuous random variable.

RV which is neither discrete nor continuous

Consider the random variable X whose distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ x+1 & \text{if} \quad -1 \leq x < -1/2 \\ 1 & \text{if} \quad x \geq -1/2. \end{cases}$$

Observe that $F_X = \frac{1}{2}F_1 + \frac{1}{2}F_2$ where F_1 and F_2 are distribution functions given by

$$F_1(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ 2(x+1) & \text{if} \quad -1 \leq x < -1/2 \\ 1 & \text{if} \quad x \geq -1/2. \end{cases}$$

$$F_2(x) = \begin{cases} 0 & \text{if} \quad x < -1/2 \\ 1 & \text{if} \quad x \geq -1/2. \end{cases}$$

Expectation of DRV

Def: Let X be a discrete RV with PMF $f_X(\cdot)$ and support S_X. The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S_X} x f_X(x) \quad \text{provided} \quad \sum_{x \in S_X} |x| f_X(x) < \infty\,.$$

- E(X) is the weighted average of the values taken by X.
- If $\sum_{x \in S_X} |x| f_X(x) = \infty$ then we say that expectation does not exist.

Example 1: X = outcome of a roll of a fair die. What is $E(X)$?
Example 2: $X \sim Bin(n, p)$. What is $E(X)$?
Example 3: $X \sim Geo(p)$. What is $E(X)$?
Example 4: $X \sim Poi(\lambda)$. What is $E(X)$?
Example 5:

$$f_X(x) = \begin{cases} \frac{c}{x^2}, & x \in \mathbb{N}, \quad \text{where} \quad c = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} \\ 0 & \text{otherwise}. \end{cases}$$

Let X be a DRV having the above PMF, then $E(X)$ does not exist.

Expectation of CRV

Def: Let X be a CRV with PDF $f_X(.)$. The expectation of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \quad \text{provided} \quad \int_{-\infty}^{\infty} |x| f_X(x) dx < \infty \,.$$

Example 6: $X \sim U(a, b)$, what is $E(X)$?
Example 7: $X \sim Exp(\lambda)$, what is $E(X)$?
Example 8: $X \sim N(\mu, \sigma^2)$, what is $E(X)$?
Example 9: Let X be a CRV having PDF $f_X(x) = \frac{1}{\pi(1+x^2)}, \forall x \in \mathbb{R}$. What is $E(X)$?