

Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 07

- ✓1. Let X be a continuous random variable. A real number m is said to be median of X if $F_X(m) = 0.5$. Show that, for all $c \in \mathbb{R}$, $E|X - c| \geq E|X - m|$.
- ✓2. Let $\{X_n\}$ be a sequence of random variables with $P(X_n = n) = 1 - \frac{1}{n}$ and $P(X_n = 0) = \frac{1}{n}$. Does X_n converge to some random variable X in distribution? [Note: This example shows that even if a sequence of distribution functions converges, it may not converge to a distribution function.]
- ✓3. Let $X_n \rightarrow X$ in r th mean, for some $r > 0$. Show that $X_n \rightarrow X$ in probability.
- ✓4. (a) Show that $|E(X)| \leq E|X|$.
 (b) Show that if $X_n \rightarrow X$ in 1st mean, then $E(X_n) \rightarrow E(X)$.
 (c) Give an example of a sequence of random variables $\{X_n\}$ such that $E(X_n) \rightarrow E(X)$, but $X_n \not\rightarrow X$ in 1st mean.
- ✓5. Let X_n be a sequence of discrete random variables such that $P(X_n = \frac{k}{2^n}) = \frac{1}{2^n}$ for $k = 1, 2, \dots, 2^n$. Show that $X_n \rightarrow X$ in distribution, where $X \sim U(0, 1)$.
- ✓6. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with finite variance σ^2 . Let $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$. Show that $\{S_n^2\}$ converges to σ^2 almost surely.
- ✓7. Let $\{X_n\}$ be a sequence of identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, where $\sigma > 0$. Also assume that $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$. Show that $\bar{X}_n \rightarrow \mu$ in probability.
- ✓8. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with mean 0 and variance 1. Find the limiting distribution of

$$Z_n = \sqrt{n} \frac{X_1 X_2 + X_3 X_4 + \dots + X_{2n-1} X_{2n}}{X_1^2 + X_2^2 + \dots + X_n^2}.$$
- ✓9. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with mean α and variance σ^2 , and let $\{Y_n\}$ be a sequence of *i.i.d.* random variables with mean $\beta (\neq 0)$. Find the limiting distribution of $Z_n = \frac{\sqrt{n}(\bar{X}_n - \alpha)}{Y_n}$.
- ✓10. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with mean μ and finite variance σ^2 . Show that $\sqrt{n} \frac{\bar{X}_n - \mu}{S_n} \rightarrow Z$ in distribution, where $Z \sim N(0, 1)$.
- ✓11. Let X_i and Y_i , $i = 1, 2, \dots$ are independently and identically distributed $U(0, 1)$ random variables. Let $N_n = \#\{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\}$. Show that $\frac{4N_n}{n}$ converges to π with probability one.
- ✓12. Let X_i , $i = 1, 2, \dots, 50$, be independent random variables each being uniformly distributed over the interval $(0, 1)$. Find the approximate value of $P\left(\sum_{i=1}^{50} X_i > 30\right)$. You may use the fact that $\Phi(\sqrt{6}) = 0.9928$. Ans: 0.0071.
- ✓13. Show that

$$\lim_{n \rightarrow \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!} = \frac{1}{2}.$$