Probability Theory and Random Processes (MA225)

Lecture Slides
Lecture 06

Indian Institute of Technology Guwahati

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Discrete Random Variable

Def: [Discrete RV] A random variable is said to have discrete distribution if there exists an atmost countable set $S_X \subset \mathbb{R}$ such that $P(X = x) > 0$ for all $x \in S_X$ and $\sum_{x \in S_X} P(X = x) = 1$. S_X is called the support of X.

Def: Define a function $f_X : \mathbb{R} \to [0, 1]$ by

$$f_X(x) = \begin{cases} P(X = x) & \text{if } x \in S_X \\ 0 & \text{otherwise}. \end{cases}$$

The function f_X is called the probability mass function (PMF) of X.

Show that in Example 1, X is discrete, in Example 2, X is discrete, in Example 3, X_1 is not discrete, but X_2 is discrete.

Remarks

- For a discrete random variable X,

$$f_X(x) = F_X(x) - F_X(x-).$$

- For a discrete random variable X,

$$F_X(x) = \sum_{\substack{y \in S_X \\ y \leq x}} f_X(y).$$

Properties of PMF

1. $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\sum_{x \in S_X} f_X(x) = 1$.

Theorem: Suppose a real valued function $h : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

1. $h(x) \geq 0$ for all $x \in \mathbb{R}$. $D = \{x : h(x) > 0\}$ is atmost countable.
2. $\sum_{x \in D} h(x) = 1$.

Then $h(\cdot)$ is a probability mass function (PMF) of some discrete random variable.

Example

1. (Bernoulli Distribution) $X \sim Bernoulli(p)$: $S_X = \{0, 1\}$, $f_X(0) = 1 - p$, $f_X(1) = p$.
2. (Binomial Distribution) $X \sim Bin(n\,, p)$: $S_X = \{0,\ 1,\ \ldots,\ n\}$, $f_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$.
3. (Geometric Distribution) $X \sim Geo(p)$: $S_X = \{0,\ 1,\ \ldots\}$, $f_X(k) = p(1-p)^k$.
4. (Poisson Distribution) $X \sim Poi(\lambda)$ $(\lambda > 0)$: $S_X = \{0,\ 1,\ \ldots\}$, $f_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$.

Application

Example 1: Suppose that an airplane engine will fail, when in flight, with probability $1 - p$ independently from engine to engine. The airplane will make a successful flight if at least 50 percent of its engines remain operating. For what values of p is a four engine plane preferable to a two engine plane? (Ans: $p > 2/3$)

Continuous Random Variable

Def: A random variable is said to have a continuous distribution if there exists a non-negative integrable function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

for all $x \in \mathbb{R}$. The function f_X is called the probability density function (PDF). The set $S_X = \{x \in \mathbb{R} : f_X(x) > 0\}$ is called support of X.

Example 2: (Exponential Distribution: $Exp(\lambda)$)

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3: (Uniform Distribution: $U(a, b)$)

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

Example 4: (Normal Distribution: $N(\mu, \sigma^2)$)

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \quad \text{if } -\infty < x < \infty.$$

Remarks on CRV

- For a continuous random variable X, $P(X = a) = 0$ for all $a \in \mathbb{R}$.
- CDF of a continuous random variable is continuous.
- PDF is not unique.
- Support of a continuous random variable is not unique.
- $P(a \leq X \leq b) = \int_a^b f_X(t)dt$.
- $f_X(x)$ is not $P(X = x)$.