A STUDY ON PARTITION FUNCTIONS

A Project Report Submitted for the Course

MA498 Project II

by

M.Siva Venkata Ranga Reddy (Roll No. 160123053)
Pranav Jangir (Roll No. 160123028)

ABSTRACT

The main aim of the project is to understand what partitions are and explore various properties of partitions. We study what ranks and cranks of partitions are, and various types of other ranks defined to tackle the problems. After that, we study some proofs of Mao's conjectures and some other identites of ranks and cranks.

Previous work

Basics

The ordinary partition function $p(n)$ is the number of representations of the positive integer n as a sum of positive integers, where different orders of the summands are not considered to be distinct.

Examples

For example $p(4) = 5$. Because 4 can be written in 4 different forms which are

4
$3 + 1$
$2 + 2$
$2 + 1 + 1$
$1 + 1 + 1 + 1$

Basics

We use q-series to prove many theorems related to partitions.First we introduce q-series.

Definition

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

Basics

We use q-series to prove many theorems related to partitions.First we introduce q-series.

Definition

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

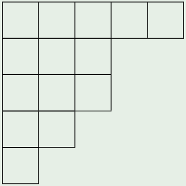
Generating Function

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

Ferrer's Diagram

Examples

Ferrers graph of $14 = 5 + 3 + 3 + 2 + 1$



Definition

The conjugate of a partition of n represented by a ferrers graph is the partition obtained by reading the graph from top to bottom.

There is a bijection between a ferrer's graph and it's conjugate.

Euler's pentagonal theorem

Theorem

$$(q;q)_\infty = \sum_{s=-\infty}^{\infty} (-1)^s q^{(s(s+1)/2)}. \tag{1}$$

Fractional Partitions

So far we have looked at $(a;q)_\infty^k$ where k is an integer.
Now we look at

$$\sum_{n=0}^{\infty} p_k(n)q^n = (q;q)_\infty^k.$$

Where k is a rational number.

Main Theorem

Theorem

Suppose a, b,d$\in Z$, b$\geq$1 and $gcd(a,b)=1$. Let l be a prime divisor of $a+db$ and $0\leq r\leq l$. Suppose d,l and r satisfy any of the following conditions:

1. $d=1$ and $24r+1$ is a quadratic non-residue modulo l.
2. $d=3$ and $8r+1$ is a quadratic non-residue modulo l or
$$8r+1\equiv 0 \pmod{l}.$$
3. $d\in\{4,8,14\}$,l $\equiv$ 5(mod 6) and $24r+d\equiv 0 \pmod{l}$;
4. $d\in\{6,10\}$ and l $\geq$5 and l $\equiv$ 3(mod 4) and 24r+d $\equiv$ 0(mod l).
5. $d=26$, l $\equiv$ 11(mod 12) and 24r + d $\equiv$ 0 (mod l).

Then for n$\geq$0,

$$p_{-a/b}(ln+r)\equiv 0 \pmod{l} \tag{2}$$

Why worry about fractional partitions?

The question then comes why do we need to study fractional partitions?
Is it possible to relate them to the orignal partition function?

It turns out that it's actually easier to prove Ramanujan's Congruences using fractional partitions.

Quick look at how!

We will use the following instance, we get from the previous theorem.

$$p_{-1/2}(7n+r) \equiv 0(mod\ \ 7), r \in \{2,4,5,6\}$$

Theorem

For any integer n$\geq$0,

$$p(7n+5) \equiv 0(mod\ \ 7). \tag{3}$$

Proof

We divide original q-series into q-series containing fractional powers and use congruences we derived for fractional partitions to prove the above congruence.

$$\sum_{n=0}^{\infty} p(n)q^n = (\sum_{n=0}^{\infty} p_{-1/2}(n)q^n)^2, \tag{4}$$

We get

$$p(n) = \sum_{k=0}^{n} p_{-1/2}(k)p_{-1/2}(n-k). \tag{5}$$

For any two integers k and n,one of the following is always true:

1. k is congruent to 2,4,5 or 6.
2. 7n+5-k is congruent to 2,4,5 or 6.

So using this and fractional partition congruence,we get the required

Other congruences

Of course, apart from providing different methods for proving Ramanujan's Congruence, fractional partitions also provide a lot of different congruences, and reveal things that can be very useful. The list of congruences that can be obtained from the previous theorem is endless.

Introduction to rank and crank

Content

- Introduction
- Some advancements and conjectures
- Dyson's Rank
- Dyson's Conjectures
- Andrews-Garvan Crank
- Vector Partitions
- Theorems for Vector Partitions
- Completing the Circle
- Other cranks

Introduction

In 1919, Ramanujan stated and proved the following conjectures

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

He also conjectured that if $\delta = 5^a, 7^b$ or 11^c and $24\lambda \equiv 1 \pmod 8$, then

$$p(\delta n + \lambda) \equiv 0 \pmod \lambda$$

with the exception that

$$p(7^b n + \lambda) \equiv 0 \pmod{7^{[(b+2)/2]}}$$

There have been various proofs of the above conjectures, we are going to look into combinatorial methods.

Some advancements and conjectures

Atkin found in 1969 that

$$p(59^4 13n + 111247) \equiv 0 \pmod{13}$$

$$p(23^3 17n + 2623) \equiv 0 \pmod{17}$$

The generalization of this, in 2000 Ken Ono proved that
For every prime $l \geq 5$ there exists infinitely many pairs (A, B) such that

$$p(An + B) \equiv 0 \pmod{l}$$

Kolberg in 1959

$$p(n) \equiv 0 \pmod{2} \quad \text{for infinitely many } n$$

$$p(n) \equiv 1 \pmod{2} \quad \text{for infinitely many } n$$

Conjecture

$$|n \leq N : p(n) \text{ is even}| \approx \frac{N}{2} \quad as \quad N \to \infty$$

Remark

Serre in 1998 did prove that

$$\lim_{N\to\infty} |n \leq N : p(n) \text{ is even } | \times \frac{1}{\sqrt{N}} = +\infty$$

Conjecture

$$p(n) \equiv 0 \pmod 3 \quad \text{for infinitely many values of n}$$

Dyson's Rank

Dyson rank of a partition is the largest summand of the partition minus the number of summands.

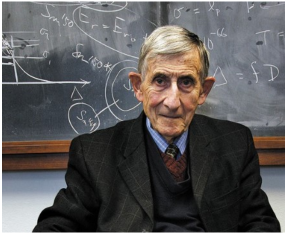


Figure: Dyson

Example:
$\lambda = (7, 7, 6, 5, 3, 3, 1, 1, 1)$
rank of $\lambda = (7 - 9) = -2$

Definition

Let $N(m, n)$ denote the number of partition of n with rank m.

Theorem

$N(m, n) = N(-m, n)$ for all m, n.

Definition

Let $(N(m, t, n)) =$ number of partitions of n with rank $m \pmod{t}$

Theorem

$N(m, t, n) = N(t - m, t, n)$ for all t, k, n.

Dyson's Conjectures

Conjectures

$$N(m,5,5n+4) = \frac{p(5n+4)}{5} \qquad 0 \le m < 5$$

$$N(m,7,7n+5) = \frac{p(7n+5)}{7} \qquad 0 \le m < 7$$

or in other words the residues of the rank modulo 5 (7) divides the partition of $5n+4$ $(7n+5)$ into 5 (7) equal classes.
The dyson conjectures were proved by Atkin and Swinnerton - Dyer in 1954. Their proof is analytic and depends on ellipitic functions and q-series identities.
No combinatorial proof is known.

These conjectures however do not extend to the partitions of $11n + 6$

Examples

Why? For example, to show this we observe that $p(6) = 11$. but $N(4, 11, 6) = 0$ which clearly does not fit in the pattern.

Andrews-Garvan Crank

Figure: Andrews

Figure: Garvan

Dyson (1944) conjectured that there exists some statistic called the "crank" which would explain the last Ramanujan congruence

$$p(11n+6) \equiv 0 \pmod{11}$$

Crank

For a partition λ let $l(\lambda)$ = the largest part ; and let $w(\lambda)$ = number of ones in λ and $\mu(\lambda)$ = number of parts of $\lambda > w(\lambda)$

Then

$$crank(\lambda) = \begin{cases} l(\lambda) & \text{if } w(\lambda) = 0, \\ \mu(\lambda) - w(\lambda) & \text{if } w(\lambda) > 0 \end{cases}$$

Like before, Let $M(m, n)$ = number of partitions of n with crank m.
and
Let $M(k, t, n)$ = number of partitions of n with crank $\equiv k \pmod{t}$

Like before

$$M(-m, n) = M(m, n) \qquad \text{for } n \geq 2$$

Theorems for Crank

Theorems

$$M(k,5,5n+4) = \frac{p(5n+4)}{5} \qquad 0 \leq k < 5$$

$$M(k,7,7n+5) = \frac{p(5n+4)}{5} \qquad 0 \leq k < 7$$

$$M(k,11,11n+6) = \frac{p(11n+6)}{11} \qquad 0 \leq k < 11$$

There are ways of proving these theorems, but a rather very interseting way to prove these is by defining another rank, and proving equivalency,
Enter vector partitions!

Vector Partitions - Introduction

An algebric number α is a complex number that satisfies
$a_n\alpha^n + \cdots + a_1\alpha + a_0 = 0$
some polynomial of degree n where a_j are integers and $a_n \neq 0$

Vector Partitions - Introduction

An algebric number α is a complex number that satisfies
$a_n\alpha^n + \cdots + a_1\alpha + a_0 = 0$
some polynomial of degree n where a_j are integers and $a_n \neq 0$

Examples

$\sqrt{2}$ is algebric since $x^2 - 2 = 0$.
$\kappa = e^{\frac{2\pi i}{5}}$ is algebric since it satisfies $z^5 - 1 = 0$

Vector Partitions - Introduction

An algebric number α is a complex number that satisfies
$a_n\alpha^n + \cdots + a_1\alpha + a_0 = 0$
some polynomial of degree n where a_j are integers and $a_n \neq 0$

Examples

$\sqrt{2}$ is algebric since $x^2 - 2 = 0$.
$\kappa = e^{\frac{2\pi i}{5}}$ is algebric since it satisfies $z^5 - 1 = 0$

Minimal Polynomial

The minimal polynomial of an algebric number is the **unique and irreducible monic polynomial** of smallest degree $p(x)$ with rational coefficients such that $p(\alpha) = 0$.

Theorem

Let p be a prime. The minimal polynomial of $\kappa_p = e^{\frac{2\pi i}{p}}$ is $p(x) = 1 + x + \cdots + x^{p-1}$.

Now we are done with the prerequisites, we can move on to the actual definition.

Vector Partitions

Definition

A vector partition τ is a triple

$$\tau = (\pi_1, \pi_2, \pi_3)$$

π_1 is partition with distinct parts
π_2, π_3 are unrestricted partitions
We say that τ is a vector partition of n if

$$n = |\pi_1| + |\pi_2| + |\pi_3|$$

Vector Partitions

Definition

A vector partition τ is a triple

$$\tau = (\pi_1, \pi_2, \pi_3)$$

π_1 is partition with distinct parts
π_2, π_3 are unrestricted partitions
We say that τ is a vector partition of n if

$$n = |\pi_1| + |\pi_2| + |\pi_3|$$

Examples

$$\tau = [(5,3,2),(2,2),(5,1,1)]$$

is a vector partition of 21.

Weight and Crank for Vector Partitions

For a vector partition $\tau = (\pi_1, \pi_2, \pi_3)$ we define the weight $w(\tau)$ and $crank(\tau)$ by
$w(\tau) = (-1)^{\#\pi_1}$
$crank(\tau) = \#\pi_2 - \#\pi_3$

Now let V be a complete set of vector partitions.

Definition

Let $N_v(m, n) =$ the number of vector partitions of n with crank m counted according to their weights i.e.

$$N_v(m, n) = \sum_{\tau \in V, |\tau| = n} w(\tau)$$

Examples

For $n = 2$ we have: (there are total 8 vector partitions of 2)

$$N_v(2,2) = 1$$

$$N_v(1,2) = 1 - 1 = 0$$

$$N_v(0,2) = 1 - 1 = 0$$

$$N_v(-1,2) = 1 - 1 = 0$$

$$N_v(-2,2) = 1$$

Generating Function

Let $|q| < 1$ and $|q| < |z| < \frac{1}{|q|}$

$$\sum_{n=0}^{\infty} \sum_{m} N_v(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

Corollaries

Nv(m,n) = Nv(-m, n)
for all m,n

$$N_v(k,t,m) = N_v(t-k,t,m)$$

for all k, n, t where $t \geq 1$
where

$$N_v(k,t,n) = \sum_{crank(\tau)\equiv k \pmod{t}, |\tau|=n} w(\tau)$$

Proofs are the same as we did before.

Relation to partitions

$$p(n) = \sum_m N_v(m, n)$$

Proof follows simply by letting $z = 1$ in the generating function. Now taking the newly defined N_v as our combinatorial object, we observe the following theorems.

Theorems

$$N_v(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \qquad 0 \leq k < 5$$

$$N_v(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \qquad 0 \leq k < 7$$

$$N_v(k, 11, 11n+6) = \frac{p(11n+6)}{11}, \qquad 0 \leq k < 11$$

Proving these are comparitively simpler using the generating functions.

Proof

We first require the following facts

$$(q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$$

and

Proof

We first require the following facts

$$(q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$$

and

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-n} \frac{z^{2n+1}-1}{z-1} \qquad \text{for } |q|<1$$

Proof

We first require the following facts

$$(q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$$

and

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-n} \frac{z^{2n+1} - 1}{z - 1} \qquad \text{for } |q| < 1$$

Let $\zeta = e^{\frac{2\pi i}{5}}$ so that

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$$

Proof

We first require the following facts

$$(q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$$

and

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-n} \frac{z^{2n+1}-1}{z-1} \qquad \text{for } |q|<1$$

Let $\zeta = e^{\frac{2\pi i}{5}}$ so that

$$1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$$

Recall that

$$1-z^5 = (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z)$$

$$
\begin{aligned}
(q)_\infty(\zeta q)_\infty(\zeta^2 q)_\infty(\zeta^3 q)_\infty(\zeta^4 q)_\infty &= \prod_{n=1}^{\infty}\prod_{k=0}^{4}(1-\zeta^k q^n) \\
&= \prod_{n=1}^{\infty}(1-q^{5n}) = (q^5;q^5)_\infty
\end{aligned} \tag{6}
$$

next we let $z = \zeta$ in

$$
\sum_{n=0}^{\infty}\sum_{m} N_v(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}
$$

$$= \frac{\sum_n (-1)^n q^{n(3n-1)/2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-2n} \frac{z^{2(2n+1)}-1}{z^2-1}}{(q^5;q^5)_\infty}$$

Now whats left is to show that the coefficient of $q^{5n+4} = 0$. We simplify:

$$\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} \equiv 4 \pmod 5$$

We get:

$$i \equiv 1 \pmod 5 \textit{ and } j \equiv 2 \pmod 5$$

But

$$j \equiv 2 \pmod 5 \implies (1 - \zeta^{2(2j+1)} = 0)$$

It follows that the coefficient of q^{5n+4} is equal to zero.

Hence, $\sum_{k=0}^{4} N_v(k,5,5n+4)\zeta^k = 0$
and ζ is the root of the polynomial

$$p(x) = \sum_{k=0}^{4} N_v(k,5,5n+4)x^k = 0 \in \mathbb{Z}[x]$$

But the minimal polynomial of ζ is

$$p(x) = \sum_{k=0}^{4} x^k = 0 \in \mathbb{Z}[x]$$

It follows that

$$N_v(i,5,5n+4) = N_v(j,5,5n+4) \qquad \text{for all } 0 \leq i,j < 5$$

since we also know that

$$\sum^{4} N_v(k,5,5n+4) = p(5n+4)$$

It follows that

$$N_v(i, 5, 5n+4) = \frac{p(5n+4)}{5} \qquad \text{for all } 0 \leq i < 5$$

Theorem

$$M(m,n) = N_v(m,n)$$

for all $n \geq 2$ and for all m.

$$\sum_{n=0}^{\infty}\sum_{m} N_v(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}$$

$$= \frac{(1-q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty}$$

Now if we look at the j^{th} term in the summation we see

$$\frac{q^j z^{-j}}{(1-q^2)(1-q^3)\dots(1-q^j)(1-zq^{j+1})(1-zq^{j+2}\dots)}$$

This is clearly the Generating function for partitions with $w(\pi) = j$ (for $j \geq 1$) and with the coefficient of z giving us the value of
$\mu(\pi) - w(\pi)$
and the first term

$$\frac{(1-q)}{(1-zq)(1-zq^2)\dots}$$

gives us the generating function where coefficient of $q^i z^j$ is the number of partitions of i without any 1 and the largest part equal to j.
Hence, in the the RHS the coefficient of $z^m q^n$, $n \geq 1$ is the number of partitions of n in which $c(\pi) = m$.
Hence proved.

Corrolaries

(1) $M(-m,n) = M(m,n)$ for $n \geq 2$.
(2) $M(k,5,5n+4) = \frac{p(5n+4)}{5}$
(3) $M(k,7,7n+5) = \frac{p(7n+5)}{7}$
(4) $M(k,11,11n+6) = \frac{p(11n+6)}{11}$
for appropriate values of k in all cases as before.

Other cranks

(t-residue diagram)

Given the ferrers diagram of a partition we label node (i, j) by least non negative residue $(j - i) \pmod t$.
The resulting diagram is called the t-residue diagram.

Examples

$\pi = (11, 7, 3, 3)$ and $t = 5$

0 1 2 3 4 0 1 2 3 4 0
4 0 1 2 3 4 0
3 4 0
2 3 4

For each $0 \leq i < t$ let

$$r_i = \text{ number of nodes labelled } i \text{ in the t-residue diagram of } \pi$$

Define t-core-crank as

Definition

$$t - core - crank(\pi) = \sum_{j=0}^{t-1} (j - \frac{t-1}{2})^{t-3} (r_j - r_{j+1})$$

where $r_t = r_0$

Theorem

Let $(t, \delta) = (5, 4) or (7, 5) or (11, 6)$
Then the t-core-crank $\pmod{t}$ divides the partitions of $tn + \delta$ into 't' equal classes.

Proof of some of Mao's conjectures

Mao's work

Using works of Atkin and Swinnerton-Dyer and works of Lovejoy and Osburn, Mao proved several identities between rank differences for unrestricted partitions, and M_2 rank differences for partitions with distinct odd parts.

He left some as conjectures. They are mainly for **modulo 10 and 6.**

We now study some proofs of Mao's conjectures given by **Alwaise and Swisher**.

M_2-rank

- First defined by Lovejoy and Osburn for partitions with distinct odd parts.
- Let λ be a partition with distinct odd parts. Then
$$M_2 - rank(\lambda) = \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda).$$
- *Let* $N_2(s, m, n)$ be the number of partitions of n with rank $\equiv s \pmod{m}$.
- Lovejoy and Osburn got generating functions for rank differences of the form
$$N_2(s, l, ln + b) - N_2(t, l, ln + b).$$
for $l = 3$ and $l = 5$.

Mao's conjectures

Conjecture Mathematical simulations suggest that

$$N(0,10,5n)+N(1,10,5n)>N(4,10,5n)+N(5,10,5n) \tag{7}$$

$$N(1,10,5n)+N(2,10,5n)\geq N(3,10,5n)+N(4,10,5n) \tag{8}$$

$$N_2(0,10,5n)+N_2(1,10,5n)>N_2(4,10,5n)+N_2(5,10,5n) \tag{9}$$

$$N_2(0,10,5n+4)+N_2(1,10,5n+4)>N_2(4,10,5n+4)+N_2(5,10,5n+4) \tag{10}$$

Cont..

$$N_2(1,10,5n)+N_2(2,10,5n) > N_2(3,10,5n)+N_2(4,10,5n) \tag{11}$$

$$N_2(1,10,5n+2)+N_2(2,10,5n+2) > N_2(3,10,5n+2)+N_2(4,10,5n+2) \tag{12}$$

$$N_2(0,6,3n+2)+N_2(1,6,3n+2) > N_2(2,6,3n+2)+N_2(3,6,3n+2) \tag{13}$$

Conjectures 8, 11 and 12 are for $n \geq 1$, rest for $n \geq 0$.

Basic idea to prove Mao's conjectures.

Take for example (7). We first get generating function of

$$(N(0,10,5n)+N(1,10,5n)-N(4,10,5n)-N(5,10,5n)) \text{ i.e.,}$$

$$\sum_{n=0}^{\infty}(N(0,10,5n)+N(1,10,5n)-N(4,10,5n)-N(5,10,5n))q^n.$$

Then we prove that this series has positive coefficients for all q^n.

Remark

All the required generating functions were already given by Mao. We only need to prove their coefficients are positive.

Proof of (7)

We use the following representation used by Mao.

Definition

For integers i, j with $i < j$, define

$$J_j := (q^j; q^j)_\infty$$

$$J_{i,j} := (q^i, q^{j-i}, q^j; q^j)_\infty$$

Remarks

J_i is a series in q^i.
$J_{i,j}$ is a series in $q^{gcd(i,j)}$.

Examples

J_3 is a series in q^3.
$J_{15,20}$ is a series in $q^{gcd(15,20)} = q^5$.

Generating function proved by Mao:

Theorem

We have that

$$\sum_{n=0}^{\infty}(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n))q^n$$
$$=\left(\frac{J_{25}J_{50}^2J_{20,50}^2}{J_{10,50}^4J_{15,50}^3}+\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}\right)$$
$$+q\left(\frac{J_{25}J_{50}^2}{J_{5,50}J_{10,50}^2J_{15,50}^2}\right)+q^2\left(\frac{J_{25}J_{50}^5}{J_{5,50}^2J_{15,50}J_{20,50}^2}\right) \tag{14}$$
$$+q^3\left(\frac{J_{25}J_{50}^5J_{10,50}^2}{J_{5,50}^3J_{20,50}^4}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+10}}\right)$$
$$+q^4\left(\frac{2J_{50}^6}{J_{25}J_{5,50}J_{15,50}J_{20,50}}\right)$$

Cont..

Now, we can write the above equation as

$$\sum_{n=0}^{\infty}(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n))q^n$$
$$=L_0+qL_1+q^2L_2+q^3L_3+q^4L_4$$

where each L_i is a series in q^5.

Cont..

So taking only those powers of q which are $\equiv 0 \pmod 5$, we have

$$\sum_{n=0}^{\infty}(N(0,10,5n)+N(1,10,5n)-N(4,10,5n)-N(5,10,5n))q^{5n}=L_0.$$

So, replacing q^5 by q in the above equation we have,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(N(0,10,5n)+N(1,10,5n)-N(4,10,5n)-N(5,10,5n))q^{n}\\
&=\frac{J_5J_{10}^2J_{4,10}^2}{J_{2,10}^4J_{3,10}^3}+\frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{15n(n+1)/2+1}}{1+q^{5n+1}}\\
&=\frac{1}{J_5}\left(\frac{J_5^2J_{10}^2J_{4,10}^2}{J_{2,10}^4J_{3,10}^3}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{15n(n+1)/2+1}}{1+q^{5n+1}}\right)
\end{aligned}$$

The summation term

Rearranging and reindexing the summation term gives

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2+1}}{1+q^{5n+1}} = S - T_1 - T_2 - T_3 - T_4$$

where

$$S := \sum_{n=0}^{\infty} \frac{q^{30n^2+15n+1}}{1-q^{20n+2}} + \sum_{n=0}^{\infty} \frac{q^{30n^2+55n+22}}{1-q^{20n+12}} + \sum_{n=1}^{\infty} \frac{q^{30n^2-5n}}{1-q^{20n-2}} + \sum_{n=1}^{\infty} \frac{q^{30n^2-25n+4}}{1-q^{20n-12}}$$

Cont..

$$T_1 = \sum_{n=0}^{\infty} a_1(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+25n+2}}{1-q^{20n+2}},$$

$$T_2 = \sum_{n=0}^{\infty} a_2(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+45n+16}}{1-q^{20n+12}}$$

$$T_3 = \sum_{n=0}^{\infty} a_3(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+5n-1}}{1-q^{20n-2}}$$

$$T_4 = \sum_{n=0}^{\infty} a_4(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2-35n+10}}{1-q^{20n-12}}$$

Cont..

So we have

$$\frac{J_5^2 J_{10}^2 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2+1}}{1+q^{5n+1}}$$
$$= \frac{J_5^2 J_{10}^2 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} + S - T_1 - T_2 - T_3 - T_4$$

Note that S, T_1, T_2, T_3, T_4 **have positive coefficients.**

Cont..

So it is sufficient to prove that

$$\frac{J_5^2 J_{10}^2 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} - T_1 - T_2 - T_3 - T_4$$

has positive coefficients.

Cont..

So it is sufficient to prove that

$$\frac{J_5^2 J_{10}^2 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} - T_1 - T_2 - T_3 - T_4$$

has positive coefficients.
Suppose

$$T_1 + T_2 + T_3 + T_4 = \sum_{n=1}^{\infty} a(n) q^n$$

and

$$\frac{J_5^2 J_{10}^2 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} = 1 + \sum_{n=1}^{\infty} b(n) q^n.$$

So our goal is to show that $b(n) - a(n) > 0$ *for* $n \geq 1$.

Cont..

Expanding the denominator of T_1 gives

$$T_1 = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} q^{30n^2+(20l+25)n+2l+2}.$$

Cont..

Expanding the denominator of T_1 gives

$$T_1 = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} q^{30n^2+(20l+25)n+2l+2}.$$

From above equation, we can see that the coefficient of $q^N, N \geq 0$ in T_1 is nothing but the number of non-negative integer solutions (n, l) of the diophantine equation

$$N = 30n^2 + (20l + 25)n + 2l + 2. \tag{15}$$

We can get an upper bound on $a_1(N)$ for $N \geq 1$.

Cont..

Clearly, for any integer $n \geq 0$, there is atmost one integer $l \geq 0$ such that (15) is satisfied.

Cont..

Clearly, for any integer $n \geq 0$, there is atmost one integer $l \geq 0$ such that (15) is satisfied.

Note that $(20l + 25)n + 2l + 2$ is positive for $n \geq 0, l \geq 0$. That means that if $n \geq \sqrt{\frac{N}{30}}$, that is, $N - 30n^2 < 0$, there is no solution to (15).

Cont..

Clearly, for any integer $n \geq 0$, there is atmost one integer $l \geq 0$ such that (15) is satisfied.

Note that $(20l + 25)n + 2l + 2$ is positive for $n \geq 0, l \geq 0$. That means that if $n \geq \sqrt{\frac{N}{30}}$, that is, $N - 30n^2 < 0$, there is no solution to (15).

As for a given $n \geq 0$, atmost one solution to (15) is possible, this means that

$$a_1(N) \leq \left\lfloor \sqrt{\frac{N}{30}} \right\rfloor + 1 \text{ for all } N \geq 1.$$

Cont..

Using similar strategies, we bound $a_2(N), a_3(N)$ and $a_4(N)$ as follows:

$$a_2(N) \leq \left\lfloor \sqrt{\frac{N}{30}} \right\rfloor + 1$$
$$a_3(N) \leq \left\lfloor \sqrt{\frac{N}{30}} \right\rfloor$$
$$a_4(N) \leq \left\lfloor \sqrt{\frac{N}{20}} \right\rfloor + 1.$$

All of the above bounds are for $N \geq 1$.

Cont..

Using similar strategies, we bound $a_2(N), a_3(N)$ and $a_4(N)$ as follows:

$$a_2(N) \leq \left\lfloor \sqrt{\frac{N}{30}} \right\rfloor + 1$$
$$a_3(N) \leq \left\lfloor \sqrt{\frac{N}{30}} \right\rfloor$$
$$a_4(N) \leq \left\lfloor \sqrt{\frac{N}{20}} \right\rfloor + 1.$$

All of the above bounds are for $N \geq 1$. As

$$a(l) = a_1(l) + a_2(l) + a_3(l) + a_4(l)$$

we have

$$a(n) \leq 3\left\lfloor \sqrt{\tfrac{n}{30}} \right\rfloor + \left\lfloor \sqrt{\tfrac{n}{20}} \right\rfloor + 3 \text{ for all } n \geq 1.$$

Cont..

Similarly we can bound $b(n)$ as follows:

$$b(n) \geq \left\lfloor \frac{n}{6} \right\rfloor \text{ for } n \geq 1.$$

Cont..

Similarly we can bound $b(n)$ as follows:

$$b(n) \geq \left\lfloor \frac{n}{6} \right\rfloor \text{ for } n \geq 1.$$

We saw that our goal is to prove that $b(n) - a(n) > 0$ for $n \geq 1$.

Cont..

Now

$$a(n) \leq 3\left\lfloor\sqrt{\frac{n}{30}}\right\rfloor + \left\lfloor\sqrt{\frac{n}{20}}\right\rfloor + 3 \; for \; n \geq 1.$$

$$\implies -a(n) \geq -3\left\lfloor\sqrt{\frac{n}{30}}\right\rfloor - \left\lfloor\sqrt{\frac{n}{20}}\right\rfloor - 3 \text{ for } n \geq 1.$$

Cont..

Now

$$a(n) \leq 3\left\lfloor\sqrt{\frac{n}{30}}\right\rfloor + \left\lfloor\sqrt{\frac{n}{20}}\right\rfloor + 3 \; for \; n \geq 1.$$

$$\implies -a(n) \geq -3\left\lfloor\sqrt{\frac{n}{30}}\right\rfloor - \left\lfloor\sqrt{\frac{n}{20}}\right\rfloor - 3 \text{ for } n \geq 1.$$

We also have

$$b(n) \geq \left\lfloor\frac{n}{6}\right\rfloor \text{ for } n \geq 1.$$

Adding bounds for $b(n)$ and $-a(n)$ gives us

$$b(n) - a(n) \geq \left\lfloor\frac{n}{6}\right\rfloor - 3\left\lfloor\sqrt{\frac{n}{30}}\right\rfloor - \left\lfloor\sqrt{\frac{n}{20}}\right\rfloor - 3 \text{ for all } n \geq 1.$$

Using

$$z - 1 \le \lfloor z \rfloor \le z \text{ for any number } z,$$

we get

$$b(n) - a(n) \ge \tfrac{n}{6} - 3\sqrt{\tfrac{n}{30}} - \sqrt{\tfrac{n}{20}} - 4 \text{ for all } n \ge 1.$$

Now,

$$\tfrac{n}{6} - 3\sqrt{\tfrac{n}{30}} - \sqrt{\tfrac{n}{20}} - 4 > 0$$

for a number n will imply $b(n) - a(n) > 0$ for that n.

Using $n = x^2$, it can be shown that

$$\tfrac{n}{6} - 3\sqrt{\tfrac{n}{30}} - \sqrt{\tfrac{n}{20}} - 4 > 0$$

is true for $n \ge 60$. This implies that $b(n) - a(n) > 0$ for $n \ge 60$.

For $1 \leq n \leq 59$, we can manually verify $b(n) - a(n) > 0$. This finishes the proof of conjecture (7).

Remark

The next three conjectures have similar proofs.

Mao's conjecture (13) can be proved if we assume certain conditions. The following theorem was given by Barman and Sachdeva.

Theorem

Mao's conjecture (13) is true when $3 \nmid (n+1)$*. Particularly, the following inequalities are satisfied for* $n \geq 0$*:*

$$N_2(0,6,9n+2) + N_2(1,6,9n+2) > N_2(2,6,9n+2) + N_2(3,6,9n+2)$$
$$N_2(0,6,9n+5) + N_2(1,6,9n+5) > N_2(2,6,9n+5) + N_2(3,6,9n+5)$$

Proving the above theorem

Similar to previous proofs, we use theta functions and q-series identities to get generating function of

$$(N_2(0,6,3n+2)+N_2(1,6,3n+2)-N_2(2,6,3n+2)-N_2(3,6,3n+2).)$$

The condition $3 \nmid (n+1)$ will be helpful at a certain point in the proof in completely ignoring a few complicated terms.

To give an idea how, consider the following series:

$$f(q) = \sum_{n=0}^{\infty} a_n q^{3n} + \sum_{n=0}^{\infty} b_n q^{3n+1} + \sum_{n=0}^{\infty} c_n q^{3n+2}$$

Cont..

Suppose we want to know coefficient of q^{n_0+1} where $n_0 \in \mathbb{W}$, such that $3 \nmid (n_0 + 1)$, that is, $n_0 + 1 \neq 3k$.

Cont..

Suppose we want to know coefficient of q^{n_0+1} where $n_0 \in \mathbb{W}$, such that $3 \nmid (n_0+1)$, that is, $n_0+1 \neq 3k$.

Clearly, coefficient of q^{n_0+1} in $f(q)$ is same as coefficient of q^{n_0+1} in

$$\sum_{n=0}^{\infty} b_n q^{3n+1} + \sum_{n=0}^{\infty} c_n q^{3n+2}$$

Complexity of deriving coefficient of q^{n_0+1} in $f(n)$ is reduced drastically.

Cont..

Similarly, the assumption that $3 \nmid (n+1)$ simplifies the work needed to be done in the proof of the theorem considerably, and we get our result.

Some identites of ranks and cranks

Introduction

Ranks and cranks are closely related to each other, even though they have very different definitions.

Lewis and Santa-Gadea proved several equalities between ranks and cranks of partitions modulo 4 and 8.

We will study their proofs given by Mortenson.

Some identities proved by Lewis and Santa-Gadea

Identities

$$N(2,4,2n) = M(1,4,2n) \tag{16}$$

$$N(3,8,4n) = M(2,8,4n) \tag{17}$$

$$N(3,8,4n+1) = M(2,8,4n+1) \tag{18}$$

$$M(1,8,4n) = M(3,8,4n) = N(2,8,4n) = N(4,8,4n) \tag{19}$$

Focus on $N(2,4,2n) = M(1,4,2n)$

Note that to prove

$$N(2,4,2n) = M(1,4,2n),$$

it is sufficient to prove

$$N(2,4,2n) - p(2n)/4 = M(1,4,2n) - p(2n)/4$$

for which it is sufficient to prove

$$\sum_{n=0}^{\infty}(N(2,4,2n) - p(2n)/4)q^{2n} = \sum_{n=0}^{\infty}(M(1,4,2n) - p(2n)/4)q^{2n}$$

. Using the above motivations, we define something called deviations of ranks and cranks.

What are deviations?

Definition

Define $D(a, m)$ and $D_C(a, m)$ as follows:

$$D(a, m) := \sum_{n=0}^{\infty}(N(a, m, n) - p(n)/m)q^n$$

$$D_C(a, m) := \sum_{n=0}^{\infty}(M(a, m, n) - p(n)/m)q^n$$

Main idea

Our interest is in proving

$$\sum_{n=0}^{\infty}(N(2,4,2n)-p(2n)/4)q^{2n}=\sum_{n=0}^{\infty}(M(1,4,2n)-p(2n)/4)q^{2n}$$

. Call LHS as A, RHS as B.
Consider

$$\begin{aligned}
D(2,4) &= \sum_{n=0}^{\infty}(N(2,4,n)-p(n)/4)q^{n}\\
&= \sum_{n=0}^{\infty}(N(2,4,2n)-p(2n)/4)q^{2n} \qquad (20)\\
&+ \sum_{n=0}^{\infty}(N(2,4,2n+1)-p(2n+1)/4)q^{2n+1}
\end{aligned}$$

Similarly, consider

$$\begin{aligned} D_C(1,4) &= \sum_{n=0}^{\infty}(M(1,4,n) - p(n)/4)q^n \\ &= \sum_{n=0}^{\infty}(M(1,4,2n) - p(2n)/4)q^{2n} \\ &+ \sum_{n=0}^{\infty}(M(1,4,2n+1) - p(2n+1)/4)q^{2n+1} \end{aligned} \tag{21}$$

Main idea

For proving A=B, it is sufficient to prove that **subseries of** $D(2,4)$ **having even powers of q** is equal to the **subseries of** $D_C(1,4)$ **having even powers of q**.

Further..

- We use generating functions of $D(2,4)$ and $D_C(1,4)$ to proceed.
- We do not go into their details.
- Proofs of other identities have similar ideas.

Proving the remaining Mao's Conjectures

Content

- Introduction
- Motivation and some Asymptotics
- Proof of the rest of Mao's conjectures

Introduction

We have so far defined the rank and the term $N(m, n)$ which means number of partitions with rank $= m$. This was define with the purpose to prove using a combinatorial argument Ramanujan's congruences.
Of special interest are partitions $p_2(n)$ that count the number of partitions of n that dont have duplicate odd parts.
Common generating functions which are used are:

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n$$

$$P_2(q) = \sum_{n=0}^{\infty} p_2(n)q^n$$

$$R(\zeta; q) = \sum^{\infty} \sum^{\infty} N(m, n)\zeta^m q^n$$

Motivation and some Asymptotics

Hardy and Ramanujan gave the following asymptotic for $p(n)$

$$p(n) = \frac{1}{\sqrt{2}\pi}\sum_{n=1}^{\sqrt{n}} A_k(n)k^{1/2}\frac{d}{dn}\left(\frac{\exp\frac{\pi}{k}\sqrt{2(n-1/24)/3}}{2\sqrt{n-1/24}}\right) + O(n^{\frac{-1}{4}})$$

Where

$$A_k(n) = \sum_{0\leq h,k;(h,k)=1} \omega_{h,k} exp(-\frac{2\pi inh}{k})$$

here, $\omega_{h,k}$ are certain 24^{th} roots of unity.
Rademacher improved this to

$$p(n) = \frac{1}{\sqrt{2}\pi}\sum_{n=1}^{\infty} A_k(n)k^{1/2}\frac{d}{dn}\left(\frac{\exp\frac{\pi}{k}\sqrt{2(n-1/24)/3}}{2\sqrt{n-1/24}}\right)$$

Proof of the rest of Mao's conjectures

R_2 is the Generating Function for N_2 analogous to the way we defined R.

Definition

Suppose a and c are integers with $c > 0$ and c does not divides $2a$. Then with $A(\frac{a}{c}; n)$ defined by

$$R_2(e^{2\pi ia/c}; q) = \sum_{n=0}^{\infty} A(\frac{a}{c}; n)q^n.$$

Now here, some of the values of $A(\frac{a}{c}; n)$ are approximately estimated as

$$A(\frac{1}{6}; n) \approx \frac{2\sqrt{2}\cosh(\pi\sqrt{8n-1}/12)}{\sqrt{8n-1}} \tag{22}$$

$$A(\frac{1}{10}; n) \approx \frac{\sqrt{2}(\sqrt{5}-1)\cosh(3\pi\sqrt{8n-1}/20)}{\sqrt{8n-1}} \tag{23}$$

Also some identities that follow from $N_2(r, m, n) = N_2(m - r, m, n)$ (Shaffer, Reihill, 2019) are as follows:

$$R_2(\zeta_6^a, q) = \sum_{n=0}^{\infty} (N_2(0, 6, n) + N_2(1, 6, n) - N_2(2, 6, n) - N_2(3, 6, n))q^n \tag{24}$$

$$\begin{aligned} R_2(\zeta_{10}^a, q) = \sum_{n=0}^{\infty} &(N_2(0, 10, n) + N_2(1, 10, n) - N_2(2, 10, n) - N_2(5, 10, n \\ &+(\zeta_{10}^{2a} - \zeta_{10}^{3a})(N_2(1, 10, n) + N_2(2, 10, n) - N_2(3, 10, n) - N_2(4, 10, n \end{aligned} \tag{25}$$

Theorem

$$N_2(0,6,n)+N_2(1,6,n)-N_2(2,6,n)-N_2(3,6,n)>0$$

for $n \geq 0$.

The case when numbers are of form $3n+2$ where $3 \nmid (n+1)$ were proved by Barman and Pal Singh Sachdevain.

We only have to show that (22) > 0 for $n \geq 0$

$$A(\frac{1}{6};n)=\frac{2\sqrt{2}\cosh(\pi\sqrt{8n-1}/12)}{\sqrt{8n-1}}+E$$

where E has the bound

$$|E| \leq \frac{112(\sqrt{n}+1)^{3/2}\cosh(\frac{\pi}{24}\sqrt{8n-1})}{3\sqrt{8n-1}}+10^{16}\sqrt{n}$$

E is smaller than (22) for all $n \geq 3823$ and it can be verified easily computationally that the inequality also holds for initial values of n.

Theorem

$$N_2(1, 10, n) + N_2(2, 10, n) - N_2(3, 10, n) - N_2(4, 10, n) > 0$$

for $n \geq 0$.

From 25

$$\frac{R_2(\zeta_{10}; q) - R_2(\zeta_{10}^3; q)}{4\cos(2\pi/5)+1} = \sum_{n\geq 0}(N_2(1,10,n) + N_2(2,10,n) - N_2(3,10,n) - N_2(4,10,n))q^n \tag{26}$$

We have to prove that $A(\frac{1}{10}; n) > A(\frac{3}{10}; n)$ for $n \geq 3$

$$R_2(\zeta_{10}; q) - R_2(\zeta_{10}^3; q) = \frac{\sqrt{2}(\sqrt{5}-1)\cosh(\frac{3\pi\sqrt{8n-1}}{20})}{\sqrt{8n-1}} + E$$

Where E is bounded as

$$|E| \leq \frac{112(\sqrt{n}+1)^{3/2}\left(\cosh(\frac{\pi}{20}\sqrt{8n-1})+\cosh(\frac{\pi}{40}\sqrt{8n-1})\right)}{3\sqrt{8n-1}} + 2.4\times 10$$

The inequality holds true for all $n \geq 1190$ and it can be verified computationally that the same is true for smaller values of n as well.