Complex Analysis

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1 January 19, 2021 (Tuesday)

1.1 The Set of Complex Numbers

We can define $\mathbb{C}$ as follows:

$$\mathbb{C} = \{z = ix + y | (x, y) \in \mathbb{R}^2\}.$$

Note that $\mathbb{C}$ is a commutative field, under standard addition and multiplication. Formally, addition is defined as

Exercise 1.1. Verify that $\mathbb{C}$ satisfies the definition of a commutative field.

1.2 Motivations: The Fundamental Theorem of Algebra

Theorem 1.2 (Fundamental Theorem of Algebra)
A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_i \in \mathbb{C}$ for all i and $a_n \neq 0$, is a product of n linear factors; There exist $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that

$$p(z) = a_n \prod_{k=1}^{n} z - r_k.$$

This is a fundamental result, and it can finally be proved using complex-analytic techniques.

Corollary
If $p(z)$ has all real coefficients, it factors into a product of linear and irreducible over $\mathbb{R}$ quadratic factors.

As complex roots come in complex pairs, their product $(z \cdot \overline{z})$, we obtain a quadratic factor that is irreducible over $\mathbb{R}$.

1.3 Complex Plane

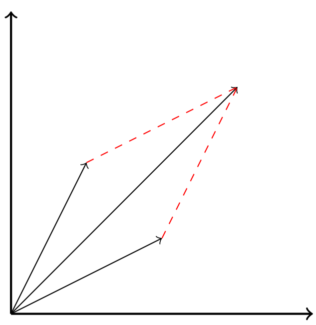
$\mathbb{C}$ is *not* an ordered set! For $w, z \in \mathbb{C}$, we cannot write $z < w$ or $w < z$. Recall that addition in the complex plane follows the **parallelogram law**. This leads to the following:

Proposition 1.3 (Triangle Inequality in Complex Plane)
For any $w, z \in \mathbb{C}$, we have

$$|w + z| \leq |w| + |z|.$$

Proof. We can construct a triangle using side lengths equal to the moduli, and the result follows. □

Visually, we have the following:



We also have the following corollary:

Corollary 1.4

For a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, where $a_i \in \mathbb{C}$ for all i, and $a_n \neq 0$, there exists an $R \in \mathbb{R}$ such that

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}$$

for all z, such that $|z| > R$.

Proof. Let

$$\omega = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}.$$

Then $\omega z^n = a_0 + a_1 z + \cdots a_{n-1} z^{n-1} = p(z) - a_n z^n$.
Note that

$$|\omega||z|^n = |\omega z^n| \leq |a_0| + |a_1||z| + \cdots |a_{n-1}||z|^{n-1}$$

implies that

$$|\omega| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}.$$

There exists an $R \in \mathbb{R}$ such that

$$\max_{i=0,\ldots,n-1} \left\{\frac{|a_i|}{R^{n-i}}\right\} < \frac{|a_n|}{2n}.$$

This implies that $|\omega| < \frac{|a_n|}{2}$, for all z such that $|z| > R$. As $p(z) = (a_n + \omega)z^n$ for $z \neq 0$, we have

$$|p(z)| = |a_n + \omega||z|^n \geq ||a_n| - |\omega|||z|^n.$$

Then for all z such that $|z| > R$,

$$|p(z)| \geq \frac{a_n}{2} R^n,$$

as desired. $\square$

2 January 21, 2021 (Thursday)

2.1 Exponential Form of Complex Numbers

We define the **argument** of a complex number z as the set of θ such that

$$\arg z = \{\theta + 2\pi n | n \in \mathbb{Z}\},$$

where θ is the **principal argument** of z.
Recall the following:

Theorem 2.1 (Euler's Formula)

For any $z = x + iy = r(\cos\theta + i\sin\theta)$, we have $z = re^{i\theta}$ such that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Then all laws of exponents apply to operations with complex numbers.

We also have the following corollary:

Corollary

For any $z \in \mathbb{C}$, $\arg zw = \arg z + \arg w$.

It is better to understand this as for $\theta_1 \in \arg z$ and $\theta_2 \in \arg w$, $\theta_1 + \theta_2 \in \arg zw$.
From this it follows that $\arg z^n = n \arg z$ and that $\arg \frac{z}{w} = \arg z - \arg w$.

2.2 Roots of Complex Numbers

We qcan derive a general form for the nth root of a complex number z:

Proposition 2.2

For all $z \in \mathbb{C}$, we have

$$z^{\frac{1}{n}} = \{c_k = \sqrt[n]{r_0} e^{i\varphi_k} | \varphi_k = \frac{\theta + 2\pi k}{n}, k = 0, \ldots, n-1\}$$

If θ is the principal argument of z ($\theta \in (-\pi, \pi]$), then

$$c_0 = \sqrt[n]{r_0} e^{i\frac{\theta}{n}}$$

is called the **principal nth root** of z.

It is obvious that the other roots of z are obtained by rotating the principal root by a factor of $\frac{2\pi}{n}$ degrees (as this is analogous to multiplying the principal root by $e^{i\frac{2\pi}{n}}$ each time).

2.3 Roots of Unity

We can apply the above derivation to 1, as it is simply e^{0i}. As $\arg 1 = 2\pi k$ for some $k \in \mathbb{Z}$, the principal argument of 1 is 0. It follows quickly that $r = 1$. We then have n distinct roots of 1, being

$$\{c_k = e^{\frac{2\pi k}{n} i} | k = 0, \ldots, n-1\}.$$

The above set $1^{\frac{1}{n}}$ has a group structure of $\mathbb{Z}_n$ with respect to multiplication, as $c_k \cdot c_l = c_m$ for some k and l where $m = k + l \pmod{n}$.
Then the primitive nth root $\omega_n = e^{\frac{2\pi}{n} i}$ generates the group of nth roots of unity:

$$1^{\frac{1}{n}} = \{\omega_n^k | k = 0, \ldots, n-1\}.$$

Geometrically these are important as they generate regular polygons when plotted in the complex plane.

3 January 25, 2021 (Tuesday)

3.1 Topology in $\mathbb{C}$

Analogous to $\mathbb{R}$, we can define the concept of a neighborhood in $\mathbb{C}$.

Definition 3.1: An **ε-neighborhood** $V_\varepsilon(z_0) \in \mathbb{C}$ is an open disk such that

$$V_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}.$$

The **deleted ε-neighborhood** $V_\varepsilon^\circ(z_0)$ of z_0 is a *punctured* disk such that $0 < |z - z_0| < \varepsilon$.

Again analogous to $\mathbb{R}$, we can define the major types of points and sets in $\mathbb{C}$.
For a set $S \subseteq \mathbb{C}$ we say that

- $z_0 \in \mathbb{C}$ is an **interior point** of S if there exists an $\varepsilon \geq 0$ such that $V_\varepsilon(z_0)$ belongs to S.
- $z_0 \in \mathbb{C}$ is an **exterior point** of S if there exists $\varepsilon > 0$ such that $V_\varepsilon(z_0)$ of z_0 does not belong to S. item $z_0 \in \mathbb{C}$ is a **boundary point** of S if for all $\varepsilon > 0$, $V_\varepsilon(z_0)$ contains at least one point in S and one point in S^c.
- A set S is **closed** if it contains all its boundary points, and is **open** if it does not contain *any* of its boundary points.

- The **closure** of S is the union of S with the set of its boundary points.
- An open set S is **connected** if for any $w, z \in S$ there exists a **polynomial line** that starts at z and ends at w and belongs to S.
- A non-empty connected open subset of S is called a **domain**.
- A union of a domain with a subset of its boundary is called a **region**.
- A set S is **bounded** if it is contained inside a circle:

$$\exists z_0 \in \mathbb{C}, R \in \mathbb{R}, \text{ such that } S \subset \{z : |z - z_0| < R\}.$$

- A point $z_0 \in \mathbb{C}$ is called an **accumulation point** of a set S if for all $\varepsilon > 0$ $V_\varepsilon^\circ(z_0)$ contains at least 1 point of S.

We finish with the following proposition:

Proposition 3.2
A set S is closed if and only if S contains all of its accumulation points.

4 January 27, 2021 (Thursday)

4.1 Functions and Mappings

Definition 4.1: A **function** $f : S \subset \mathbb{C} \to \mathbb{C}$ is a mapping that assigns for each $z \in S$ a complex number w. We write $w = f(z)$ and call S the **domain of** f.

Note that the domain of the function f need not be a domain in the sense of the topological definiton, that is it need not be a connected open subset of $\mathbb{C}$.
A complex function $w = f(z)$ defines two bivariate functions on $\mathbb{R}$:

$$\Re w = u(x, y), \Im w = v(x, y).$$

The **image** of f is the set

$$\operatorname{Im} f = \{w \in \mathbb{C} | \exists z \in S, w = f(z)\}.$$

This has many names, such as the **codomain** of f, or the **image** of S.

Definition 4.2: For $w \in \mathbb{C}$, the *set*

$$\{z \in S | f(z) = w\} = f^{-1}(w)$$

is called the **preimage** of w.

The above is also known as the **inverse image** of w.
Similarly to $\mathbb{R}$, a function f is injective if $\forall w \in \operatorname{Im} f$, we have that $|\{f^{-1}(w)\}| = 1$.

4.2 Multivalued functions

Definition 4.3: A **multivalued function** $f : S \to \mathbb{C}$ is one that maps *more than one* value to a point $z \in S$.

By choose a single value from the set $\{f(z)\}$, we define a **single value branch** of f.

Example 4.4: Consider $f(z) = z^{\frac{1}{n}}$. This maps n values to each $z \neq 0 \in \mathbb{C}$.

Another function that should be obviously multivalued is the function $f(z) = \sqrt[n]{r} e^{\frac{i\pi\theta}{n}}$.

Example 4.5: Consider the function $f(z) = z^2$. We may write this as follows: $f(z) = w = u + iv$, so we have

$$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases} \quad \mathbb{R}^2 \to \mathbb{R}^2.$$

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5 February 2, 2021 (Tuesday)