Lie Algebras

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January 21, 2021

These are my notes from NC State's introductory class in Lie Algebra (and some representations), MA 720. Throughout the document we denote an arbitrary field by $\mathbb{F}$.

Contents

1 January 19, 2021 (Tuesday)

1.1 Motivations

The main motivation for the study of Lie algebra comes from the notation of symmetry. Mathematically, we can describe symmetry using groups.

For example, consider the symmetry group S_3 of order 6, which consists of rotations and reflections (on an equilateral triangle).

Sometimes we have an infinite amount of symmetry, consider the group which acts on a circle, this is an example of a **continuous group of symmetry**. (For example, we can take all real numbers in $\mathbb{R}$ mod 2π).

We can also take the linear symmetries on a vector space, and this forms the General Linear group, GL, consisting of all invertible matrices. $GL(n)$ consists of $n \times n$ matrices.

When we express the group actions in terms of coordinates, they are smooth and become differentiable. This is an example of a **Lie group**.

Lie algebras are the tangent spaces to Lie groups. The Lie algebra under $GL(n)$ is denoted by $\mathfrak{gl}(n)$.

The main difference between the algebras and groups in this case is that the Lie algebra is also a vector space. A Lie group is always nonlinear.

1.2 Lie algebras and subalgebras

Consider $\mathfrak{gl}(n)$, the vector space of all $n \times n$ matrices under a field $\mathbb{F}$. We can multiply and add matrices, and this gives the structure of a vector space. As we can multiply matrices, we have an algebra, and it is associative. In general, an algebra over a field $\mathbb{F}$ is associative with a bilinear product.

Consider the **commutator**, which is defined as $[A, B] = AB - BA$. The commutator is skew symmetric, as $[A, B] = -[B, A]$. Note that the commutator is also bilinear. The bracket is also skew symmetric: $[A, B] = -[B, A]$. Note that $[A, A] = 0$, as this is not obvious if the characteristic of $\mathbb{F}$ is 2. The final property of the commutator is known as the **Jacobian identity**:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

We can now state the formal definition of a Lie algebra:

> **Definition 1.1:** A **Lie algebra** is a vector space with a operation $[\cdot, \cdot]$ such that it satisfies bilinearity, skew-symmetry, and the Jacobian identity.

Note that we also have the following (equivalent) form of the Jacobian identity:

$$\sum_{\text{cyc}} [A, [B, C]] = 0.$$

Every associative algebra is a Lie algebra under the commutator. If the product is defined as AB for a general associative algebra, a Lie algebra can be generated under the commutator (preserving the original multiplication), $AB - BA$. Consider the following examples:

> **Example 1.2:** Consider the vector space $\mathfrak{sl}(n) \subset gl(n)$ of all $n \times n$ matrices with trace 0. Note that $\dim \mathfrak{sl}(n) = n^2 - 1$ (codimension 1 in $\mathfrak{gl}(n)$). By preserving the commutator from $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ is also a Lie algebra.

The above is an example of a subalgebra:

Definition 1.3: A **subalgebra** is a subspace of a Lie algebra closed under the bracket.

$sl(n)$ is a very important Lie algebra, and it will reappear many times.

Example 1.4: Consider the vector space $\mathbb{R}^3$. Note that the cross product is skew symmetric

$$a \times b = -(b \times a)$$

, so this is a candidate for a Lie algebra. Define the bracket $[a, b] = a \times b$.

Note that the bracket is *not* fixed for all Lie algebras. (oops)

Remark. To give $[\cdot, \cdot]$ on a vector space L, it suffices to specify $[x_i, x_j]$ for a given basis $\mathcal{B} = \{x_1, x_2, \cdots\}$. Then $[\cdot, \cdot]$ is extended $\forall x, y$ using the property of bilinearity. We can explicitly define the Lie bracket as

$$[x_i, x_j] = \sum_k c_{ij}^k x_k,$$

where c_{ij} are known as the **structure constants**. It suffices to verify that the axioms of a Lie algebra hold on just the basis vectors.

Example 1.5: Consider the vector space $L = \operatorname{span} x, y, z$. Define $[x, y] = z$, $[z, x] = [z, y] = 0$. It suffices to verify the jacobian identity on the basis vectors of L:

$$\sum_{\text{cyc}} [x, [y, z]] = 0 + 0 + 0 = 0.$$

Such a z as above is known as a **central element** ($[z, a] = 0$ for all $a \in L$).

The above Lie algebra is known as the **Heisenberg Lie algebra**. ($[x, y] = z$ where z is central)

Example 1.6: Classify all 1 dimensional Lie algebras.

Note that all elements of a 1 dimensional vector space are scalar multiples of each other, so

$$[\alpha x, \beta x] = \alpha\beta[x, x] = 0.$$

Thus all 1 dimensional Lie algebra have $[\cdot, \cdot] = 0$. Note that this alebra is **abelian**.

Example 1.7: Classify all 2 dimensional Lie algebras.

We can represent all such vector spaces as $L = \operatorname{span} x, y$. Then

$$[x, y] = \alpha x + \beta y = z.$$

Then

$$[x, z] = \alpha[x, x] + \beta[x, y] = \beta[x, y] = \beta z.$$

If $\beta = 0$, $[x, y] = \alpha x$. If $\alpha = 0$, then $[x, y] = 0 \implies$ the algebra is abelian. If $\alpha \neq 0$, $[-\frac{1}{\alpha} y, x] = x$. If $\beta \neq 0$, then $[\frac{1}{\beta} x, z] = z$. Thus we have the following classifications: All 2 dimensional Lie algebras are either abelian, or has a basis $\{a, b\}$ such that $[a, b] = b$.

Remark. The identity matrix $I \in \mathfrak{gl}(n)$ is central, as $IA = AI = A \implies [A, I] = 0$ for all A. In fact, any central element of $\mathfrak{gl}(n)$ is a scalar multiple of I.

Example 1.8: Consider $\mathfrak{gl}(2)$, which has a basis

$$\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Note that $E_{11}E_{12} = E_{12}$, and that $E_{12}E_{11} = 0$. This implies that $[E_{11}, E_{12}] = E_{12}$. Thus we have $L = \operatorname{span} E_{11}, E_{12}$ as closed under the bracket, and is a subalgebra of $\mathfrak{gl}(2)$.

The above example is not restricted to the choice of elementary matrices, any choice of basis vectors gives a subalgebra under the space.

Example 1.9: THe set of strictly upper triangular matrices A such that $a_{ij} = 0$ if $\forall i \geq j$ is a subalgebra of $\mathfrak{gl}(n)$, and is denoted by $\mathfrak{n}(n)$.

2 January 21, 2021 (Thursday)

2.1 Ideals

An ideal is similar to a subalgebra, but has a stronger limiting condition.

Definition 2.1: An **ideal** of a Lie Algebra L is a subspace I such that

$$[x, y] \in I \ \forall x \in L \text{ and } y \in I.$$

Any ideal is also a subalgebra of L.

This is the analog to an ideal for a ring.

Example 2.2 (Quotient Algebra)**:** Let L be a Lie algebra, and $I \subset L$ be an ideal. Then the vector space L/I is also a Lie algebra. The vectors are the *cosets*

$$x + I = \{x + y | y \in I\} \subset L.$$

We define the bracket as

$$[x + I, y + I] = [x + y] + I.$$

As all $[x, y] \in I$, this is well defined for cosets. This Lie algebra is called the **quotient algebra**.

Let M and N be subspaces of L, an arbitrary Lie algebra. Then

$$[M, N] = \operatorname{span}\{[x, y] | x \in M, y \in N\} = \operatorname{span}\{\sum_i [x_i, y_i] | x_i \in M, y_i \in N.$$

In particular, if M is a subalgebra, then

$$[M, M] \subseteq M,$$

and if M is an ideal, then

$$[L, M] \subseteq M.$$

Definition 2.3: The **derived algebra** of L is $[L, L]$.

We also have the following:

Lemma

$[L, L] \subset L$ is an ideal, which implies it is a subalgebra, and thus itself is a Lie algebra.

Proof. Let $I = [L, L]$, which is a subspace of L by constraint. Then

$$[L, I] \subseteq [L, L] = I,$$

as $I \subseteq L$, and we are done. $\square$

Lemma

Let $L = \mathfrak{gl}(n)$. Then

$$[L, L] = \mathfrak{sl}(n).$$

Proof. We have $[L, L] \subseteq \mathfrak{sl}(n)$, as the trace of any commutator is equal to 0.

Consider the basis for $\mathfrak{sl}(n)$. The basis for $\mathfrak{gl}(n)$ is all E_{ij} such that entry $(i, j) = 1$ and all others are 0. Then the basis for $\mathfrak{sl}(n)$ is all E_{ij}, along with all $E_{k-1,k-1} - E_{k,k}$ for $1 \leq k \leq n$.

Recall that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Then the bracket is

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}.$$

For $i \neq j$, $[E_{ij}, E_{jj}] = E_{ij}$. Then $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. As all basis vectors can be generated by brackets, we are done. $\square$

Recall that $\mathfrak{n}(n)$ was the Lie algebra of all strictly upper triangular matrices, and this is a subalgebra of $\mathfrak{t}(n)$, the Lie algebra on all triangular matrices. These are both subalgebras of $\mathfrak{gl}(n)$.

Exercise 2.4. Show that $\mathfrak{n}(n) = [\mathfrak{t}(n), \mathfrak{t}(n)]$.

2.2 Centers

Recall the central element from the last lecture. This can be generalized:

Definition 2.5: The **center** of a Lie algebra L, $Z(L)$, is defined such that

$$Z(L) = \{z \in L | [z, x] = 0 \ \forall x \in L\}.$$

We have the following:

Lemma

$Z(L) \subset_I L$.

Proof. Let $z_1, z_2 \in Z(L)$. Then we have

$$[z_1 + z_2, x] = [z_1, x] + [z_2, x] = 0.$$

This implies closure under addition. Next, let $\lambda \in \mathbb{F}$. We have

$$[\lambda z, x] = \lambda[z, x] = 0,$$

which implies closure under scalar multiplication. Lastly, consider the bracket. For all $z \in Z(L)$, $x \in L$, we have

$$[x, z] = 0 \in Z(L),$$

so $Z(L)$ is also closed under the bracket, as desired. $\square$

Exercise 2.6. Show that $Z(\mathfrak{gl}(n)) = \operatorname{span} I$.

Example 2.7: Find $Z(\mathfrak{sl}(n))$.

Solution. Assuming that $\operatorname{char}\mathbb{F} \nmid n$, we can write

$$\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{F}I.$$

This implies that $Z(\mathfrak{sl}(n)) = \{0\}$. Then $A = \frac{tr(A)}{n} I \in \mathfrak{sl}(n)$.
If $n = 0$ in $\mathbb{F}$, then we have

$$Z(\mathfrak{sl}(n)) = \mathbb{F}I.$$

□

2.3 Simple Lie Algebras

Definition 2.8: A Lie algebra is called **simple** if it is nonabelian has no nonzero proper ideals.

We require that L is nonabelian as if it were, it would have $[\cdot, \cdot] = 0$ for all possible brackets.

Remark. if L is simple, then we have

$$Z(L) = \{0\}, [L, L] = L.$$

Consider the following:

Proposition 2.9
Asume $\operatorname{char}\mathbb{F} \nmid n$. Then the Lie algebra $\mathfrak{sl}(n, \mathbb{F})$ is simple.

Proof. Let $L = \mathfrak{sl}(n, \mathbb{F})$. Let $I \subset L$ be an ideal, $I \neq \{0\}$. It suffices to show that $I = L$.
Let

$$A = (a_{kl}) \in I = \sum a_{kl} E_{kl}.$$

We have that

$$[E_{ij}, A] \in I \ \forall i, j = \sum_{k,l} a_{kl} [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}.$$

This sum is nonzero only when $k = j$, so we only need to sum over l:

$$\sum_l a_{jl} E_{il} - \sum_k a_{ki} E_{kj} \in I.$$

Consider

$$[E_{ij}, [E_{ij}, A]] = \sum_l a_{jl} [E_{ij}, E_{il}] - \sum_k a_{ki} [E_{ij}, E_{kj}].$$

We can rewrite this as

$$-\sum_l a_{jl} \delta_{li} E_{ij} - \sum_k a_{ki} \delta_{jk} E_{ij}.$$

For nonzero values, we need $l = i$ and $j = k$, so we have the above equal to

$$a_{ji} E_{ij} - a_{ji} E_{ij} = -2a_{ji} E_{ij} \in I.$$

If $A = (a_{(ij)}) \in I$, then $E_{ij} \in I$ whenever $i \neq j$, $a_{ij} \neq 0$.
This implies 2 cases:

- Case 1: I contains some non diagonal A. There exists some $i \neq j$ such that $E_{ij} \in I$, so we have

$$[E_{ki}, E_{ij}] = E_{kj} \in I \ \forall k \neq j.$$

  We also have

$$[E_{jl}, E_{kkj}] = -E_{jk} \in I \ \forall x \neq l \implies E_{kl} \in I \ \forall k \neq l.$$

This implies every offdiagonal matrices are in the ideal. Verifying the diagonal matrices is simple, as we know that

$$[E_{kl}, E_{lk}] = E_{kk} - E_{ll} \in I.$$

Thus $I = \mathfrak{sl}(n)$.

- Case 2: I contains some nonzero diagonal matrix $A = \sum_k a_{kk} E_{kk}$, where $\sum a_{kk} = 0$ (from trace). We have

$$[E_{ij}, A] = (a_{jj} - a_{ii}) E_{ij} \in I,$$

for all $i \neq j$. Since $\operatorname{char} \mathbb{F} \nmid n$, we have that $A \neq \lambda \cdot I$ for some $\lambda \in \mathbb{F}$. Then $\exists i \neq j$ such that $a_{ii} \neq a_{jj}$. THen $E_{ij} \in I$, and we have reduced this to Case 1.

As we have proved Case 1, we are done. $\square$