CS6360: ATML

A Primer on Group Theory

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Paper Presentation 1

Homomorphism Autoencoder: Learning Group Structured Representations from Observed Transitions: Hamza Keurti, Hsiao-Ru Pan, Michael Besserve, Benjamin Grewe, Bernhard Schälkopf, *ICML 2023*

The paper attempts to model the effect of interventions as transformations in representation space.
They assert that this problem can be formulated as a problem of learning a homomorphism between the interventional structure of the world and the model's representations of it. This should allow it to be able to reverse-engineer the effects of potential interventions (transformations) through the knowledge of how its representations change.

The Learning Problem

- W is the latent space from which observations are generated, through the process g.
- O is the space of observations.
- Z is the space of representations, mapped to from O through the *inference process* h.

$$\begin{tikzcd}
W \arrow[r, "{(G,\cdot w)}"] \arrow[d, "b"] \arrow[dd, bend right, "f"'] & W \arrow[d, "b"] \arrow[dd, bend left, "f"] \\
O \arrow[d, "h"] & O \arrow[d, "h"] \\
Z \arrow[r, "{(G,\cdot z)}"] & Z
\end{tikzcd}$$

Figure: The proposed group structure of the learning problem

HAE and Two Losses

Definition ($N-$step Prediction Loss)

$$\mathcal{L}^N_{\text{pred}}(\rho, h) = \sum_t \sum_{j=1}^N ||h(o_{t+j}) - (\prod_{i=0}^{j-1} \rho(g_{t+i}))h(o_t)||$$

Definition ($N-$step Reconstruction Loss)

$$\mathcal{L}^N_{\text{rec}}(\rho, h, d) = \sum_t \sum_{j=1}^N ||o_{t+j} - d(\prod_{i=0}^{j-1} \rho(g_{t+i}))h(o_t)||$$

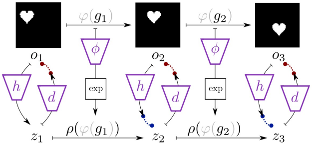

Figure: HAE's Architecture

A weighted sum of both losses, $\mathcal{L}(\rho, h, d) = \mathcal{L}_{\text{rec}^N}(\rho, h, d) + \gamma\mathcal{L}^N_{\text{pred}}(\rho, h)$, is optimized for.

Restrictions on the Representation Class

Theorem

If (ρ, h, d) are continuous and minimize the expectation of $\mathcal{L}^2_{pred}(\rho, h) + \gamma\mathcal{L}^k_{rec}(\rho, h, d)$, for $k \geq 0$, then ρ is a non-trivial group representation and (ρ, h) is a symmetry-based representation.

Informally, a symmetry-based representation is one that satisfies the following:

$$\rho(g_1, g_2, ..., g_n)(z_1 \oplus ... \oplus z_n) = \rho_1(g_1)(z_1) \oplus ... \oplus \rho_n(g_n)(z_n) \text{ where } z_i = h(o_i).$$

Assignment 2

The Basics of Group Theory

- **Goal**: Understand *just* enough group theory to follow along with the proposition's full statement.
- Questions to be addressed:
  1. What is a group? What is a subgroup?
  2. What are homomorphisms and isomorphisms?
  3. What is a product group?
  4. What is a group action?
  5. What are diffeomorphisms and homeomorphisms?
  6. What precisely is a manifold? When is it smooth?
  7. What are lie groups and lie algebras?

  For a more complete treatment of group theory and group actions, refer to [Art98] and [GQ20].

Preliminary Calculus

Definition (Convergence of a Series)

Given a normed vector space $(\mathcal{V}, ||\cdot||)$, a series $\sum_{i=0}^{\infty} v_i$ is said to converge if the sequence $\{S_n = \sum_{i=0}^{n} v_i\}$ converges to some value $v \in \mathcal{V}$.

Definition (Cauchy Sequence)

Given a normed vector space $(\mathcal{V}, ||\cdot||)$, a sequence $\{v_n\}$ is called a Cauchy sequence iff for every $\epsilon > 0$, $\exists N > 0$ such that $\forall m, n > N$,

$$||v_n - v_m|| < \epsilon.$$

Definition (Banach Space)

A normed vector space is said to be complete, or a *Banach* space, if every Cauchy sequence within the space converges.

Groups

A group is a set G equipped with a binary operation $\cdot : G \times G \to G$ such that:

- The law of composition is associative:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c \ \forall a, b, c \in G$$

- There exists an identity element $e \in G$ such that:

$$e \cdot a = a \cdot e = a \ \forall a \in G$$

- Every element $a \in G$ has an inverse $a^{-1} \in G$ such that:

$$a \cdot a^{-1} = a^{-1} \cdot a = e$$

Note: If the law of composition is commutative, the group is called *abelian*.

Groups

Examples

- The set of integers $\mathbb{Z}^+$ under addition is a group.
- The set of even integers under addition is a group.
- The set of non-zero real numbers $\mathbb{R}^\times$ under multiplication is a group.
- The set of odd integers under addition is *not* a group.

Subgroups

A subgroup H of a group G is a subset of G that is itself a group under the same law of composition as G.

Note: The identity element of H is the same as that of G.

Groups as sets of transformations

Groups are often used to formalize and understand sets of transformations. For example:

- $\mathrm{GL}(n, \mathbb{R})$: The *General Linear Group* is the set of all invertible $n \times n$ matrices under matrix multiplication.
- $\mathrm{SL}(n, \mathbb{R})$: The *Special Linear Group* is the set of all $n \times n$ matrices with determinant 1 under matrix multiplication.
- $\mathrm{O}(n)$: The *Orthogonal Group* is the set of all $n \times n$ orthogonal matrices under matrix multiplication.
- $\mathrm{SO}(n)$: The *Special Orthogonal Group* is the set of all $n \times n$ orthogonal matrices with determinant 1 under matrix multiplication.

The Cyclic Group

If we're given a group G, then the cyclic subgroup generated by an element $a \in G$ is the smallest group containing a:

$$\langle a \rangle = \{a^n | n \in \mathbb{Z}\} = \{...a^{-2}, a^{-1}, 1, a, a^2, ...\}.$$

If, for any $k \in \mathbb{Z}$, $a^k = e$, then the order of a is the smallest such k, since after the k^{th} composition, the elements loop back around on themselves.

Homomorphisms

Definition (Homomorphism)

Let $(G, \cdot)$ and $(H, \circ)$ be groups. A function $\phi : G \to H$ is called a homomorphism if:

$$\phi(a \cdot b) = \phi(a) \circ \phi(b) \ \forall a, b \in G$$

Examples

- The determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^\times$.
- The exponential function $\exp : \mathbb{R}^+ \to \mathbb{R}^\times$.
- The absolute value function $|\cdot| : \mathbb{C}^\times \to \mathbb{R}^\times$.

Definition (The Trivial Homomorphism)

The homomorphism $\varphi : G \to G'$ such that $\varphi(g) = e_{G'}$ for all $g \in G$ is called the trivial homomorphism.

The Image and the Kernel

Let $\varphi : G \to G'$ be a homomorphism.

Definition (The Image of a Homomorphism)

The image of φ, denoted $\mathrm{im}(\varphi)$, is the set of all elements in G' that are the image of some element in G under φ:

$$\mathrm{im}\varphi = \{x \in G' | x = \varphi(a) \text{ for some } a \text{ in } G\}.$$

Definition (The Kernel of a Homomorphism)

The kernel of φ, denoted by $\ker(\varphi)$, is the set of all elements in G that are mapped to the identity element of G':

$$\ker\varphi = \{a \in G | \varphi(a) = e_{G'}\}.$$

Isomorphisms

An isomorphism is a bijective homomorphism.

Definition (Isomorphic Groups)

Two groups G and G' are said to be isomorphic if there exists an isomorphism $\varphi : G \to G'$. This is denoted by $G \cong G'$ or by $G \approx G'$.

Lemma

If $\varphi : G \to G'$ is an isomorphism, then $\varphi^{-1} : G' \to G$ exists, and is also an isomorphism.

Product Groups

If we have two groups G and G', then the product group is the set $G \times G'$, with each group's law of composition applied component-wise:

$$
\begin{aligned}
& (a, a'), (b, b') \in G \times G' && (1) \\
\implies & (a, a') \cdot (b, b') = (a \cdot b, a' \cdot b') && (2)
\end{aligned}
$$

Note: A isomorphism from a group to itself is called an automorphism.

Product Groups

The product group can be understood in terms of its factors through the following four homomorphisms:

- $i(x) = (x, 1), i'(x') = (1, x')$
- $p(x, x') = x, p'(x, x') = x'$

The injective homomorphisms i and i' map the groups G and G' to their images, $G \times 1$ and $1 \times G'$ respectively. The surjective homomorphisms p and p' map their kernels, $1 \times G'$ and $G \times 1$ respectively, to the identity elements of G and G'.

$$
\begin{tikzcd}
G \arrow[dr, "i"] & & G \\
& G\times G' \arrow[ur, "p"] \arrow[dr, "p'"] & \\
G' \arrow[ur, "i'"] & & G'
\end{tikzcd}
$$

Example

The group $\mathcal{T}_3$ consisting of translations of an object in 3D space can be thought of as the group product of the translation groups $\mathcal{X}, \mathcal{Y},$ and $\mathcal{Z}$ along the X, Y and Z axes, i.e.

$$\mathcal{T}_3 \cong \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}.$$

Group Action

Definition (Group Action)

Given a set X and a group G, an action of G on X is a function $\varphi : G \times X \to X$, such that:

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x) \ \forall g, h \in G, x \in X \tag{3}$$

$$\varphi(e, x) = x \ \forall x \in X \tag{4}$$

Normally, we write $\varphi(g, x)$ as $g \cdot x$ or gx, making the above rules

$$g \cdot (h \cdot x) = (gh) \cdot x \ \forall g, h \in G, x \in X \tag{5}$$

$$e \cdot x = x \ \forall x \in X \tag{6}$$

Note: If the set X is such that the group action $\varphi : G \times X \to X$ exists, then X is called a $G-$set.

Group Action

While φ is a function that ranges across $g \in G$ and $x \in X$, we can construct a new function for each $g \in G$, where g is treated as a parameter:

$$\varphi_g : X \to X, \varphi_g(x) = gx.$$

Note that φ_g has an inverse, namely $\varphi_{g^{-1}}$. This means that φ_g is a bijection from X to X, making it a permutation of X.
If we define, by $\mathcal{G}_X$, the group of permutations of X, then the map $g \mapsto \varphi_g$ is a homomorphism from G to $\mathcal{G}_X$.

Equivariant Maps

Definition (Equivariant Map)

Given two $G-$sets X and Y, a map $f : X \to Y$ is called equivariant iff $\forall x \in X$ and $\forall g \in G$,

$$f(g \cdot x) = g \cdot f(x).$$

$$\begin{tikzcd}
X \arrow[r, "\varphi_g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "\psi_g"'] & Y.
\end{tikzcd}$$

References

Michael Artin, *Algebra*, Birkhäuser, 1998.

J. Gallier and J. Quaintance, *Differential geometry and lie groups: A computational perspective*, Geometry and Computing, Springer International Publishing, 2020.