Assignment 1

1. **State whether the following statement is True or False: “It is possible to have an instance of the Stable Matching problem in which two women have the same best valid partner.”**

   Ans.
   False.

   Counterexample:
   Consider 3 men m, m’, m’’ and 3 women w, w’, w’’
   Preference list for men:
   m = [w, w’, w’’]
   m’ = [w’, w’’, w]
   m’’ = [w’’, w, w’]

   Preference list for women:
   w = [m, m’, m’’]
   w’ = [m’, m, m’’]
   w’’ = [m’, m’’, m]

   If m proposes to w, w accepts as m is her preferred choice.
   Here, w’ and w’’ are unmatched and both prefer m', but m’ would propose to w’ as she is higher on the preference list than w’’.
   w’ would accept it leaving w’’ with m’’ over her preferred option of m’.
   Final pairs that we get are: (m, w), (m’, w’), (m’’, w’’).

   Here, w’’ could not match which her preferred man m’. Hence, the statement “It is possible to have an instance of the Stable Matching problem in which two women have the same best valid partner.” Is proven false through this counterexample.

2. **In the context of a stable roommate problem involving four students (a, b, c, d), each student ranks the others in a strict order of preference. A matching involves forming two pairs of students, and it is considered stable if no two separated students would prefer each other over their current roommates. The question is whether a stable matching always exists in this scenario. If it does, provide proof; if not, present an example of roommate preferences where no stable matching is possible.**

   Ans.
   False.

   Counterexample: -
   Consider this preference list:
   a = [b, c, d]
   b = [c, d, a]
   c = [a, b, d]
   d = [b, a, c]

   Here, if we pair (a, c) and (b, d),
   According to the preference list, c prefers a over b, but a prefers b over c. Here, a and c are paired as a is c’s preferred choice, but a prefers b over c (current roommate).

Similarly,
d prefers b over a, but b prefers c over d. Here, b and d are paired as b is d's preferred choice, but b prefers c over d (current roommate).
Both a and b are not paired with their preferred choice. Hence, these pairs are not stable.
Hence, a stable matching does not always exist in this scenario.

3. **Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.**

   Ans.
   False.
   Counterexample:
   Consider 3 men m, m', m'' and 3 women w, w', w''
   Preference list for men:
   m = [w, w', w'']
   m' = [w', w'', w]
   m'' = [w'', w, w']

   Preference list for women:
   w = [m', m, m'']
   w' = [m, m', m'']
   w'' = [m, m'', m']

   If m proposes to w, w accepts it as m is her preferred choice.
   m' proposes to w', w' accepts it as it's the next best choice after m.
   m'' proposes to w'', w'' accepts it.
   Final pairs that we get are: (m, w), (m', w'), (m'', w'').

   Here, there is no pair (m, w) where m is ranked first on the preference list of w, and w is ranked first on the preference list of m. Hence, the statement "In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m" is proven false through this counterexample.

4. **Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.**

   Ans:
   True.

   In a stable matching, no man and woman (who are not currently paired together) would prefer each other over their assigned partners. In every stable matching S for this instance, the pair (m, w) will belong to S as neither m nor w would prefer other options over their current partners. If (m, w) is a mutual first choice, there is no incentive for either m or w to deviate from this pairing. Hence, if there exists a man m and a woman w such that both are ranked first on each other's preference list, then in every stable matching S for this instance, the pair (m, w) belongs to S.

5. **Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation.**
   For some n ≥ 2, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

   Ans:
   True
   Example:
   Consider 3 men m, m′, m′′ and 3 women w, w′, w′′
   Preference list for men:
   m = [w, w′′, w′]
   m′ = [w′′, w′′, w′]
   m′′ = [w, w′′, w′]

   Preference list for women:
   w = [m, m, m′′]
   w′ = [m′′, m′, m]
   w′′ = [m′, m′′, m]

   If m proposes to w, w accepts as m is her preferred choice.
   m′ proposes to w′′, w′′ accepts it as m′ is her preferred choice.
   m′′ proposes to w, but is rejected as w is already with m (her preferred choice). Hence, m′′ is paired with w′.
   Final pairs that we get are: (m, w), (m′, w′′), (m′′, w′).

   Here, all the women match with their preferred men, even though m′′ did not match with its preferred woman w. Hence, the statement "For some n ≥ 2, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most." is proved to be true through this example.

6. **Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.**
   For all n ≥ 2, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man.

   Ans:
   True

Consider a situation where all the men have put different women as their first preferences.
Eg: -
m = [w, w', w'']
m' = [w', w'', w]
m'' = [w'', w, w']

Here, every man has a different woman as his first choice.
Now, according to G-S algorithm, women start with their worst preferred man, until a better choice comes up.
Consider this preference list for women:
w = [m', m'', m]
w' = [m'', m, m']
w'' = [m, m', m'']

As you can see in this example, all the women that are first choices of men have those respective men as their last preferences. But as every man proposes to a different woman, the women have no choice but to stick with their first choice (the least preferred one), as they don't have any better options coming their way. This works for all n>=2.

Hence, the statement "For all n ≥ 2, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man" is proven true through this counterexample.

7. **For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m" that she truly prefers to both m and m'? (We can ask the same question for men, but will focus on the case of women for purposes of this question.)**
Resolve this question by doing one of the following two things:
(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or
(b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

Ans.
(a)
To prove: For any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm.

Proof:
Here, we consider a women w who prefers man m to m'.
In Gale-Shapley algorithm, women start by being paired with their top ranked partner.
If m proposes to w, she would accept it as it would be her preferred choice. If m' proposes later, w would reject the proposal as w is already with her preferred choice.

Now, if we switch the position of m and m’ in w’s preferred list by falsely claiming that w prefers m’ to m. Now, if m’ proposes to w, she would accept it as it would be her first proposal. But since m is higher on her true preference list, if m proposes to w later in the algorithm, she would accept m’s proposal over m’. Either way, w ends up with her true preferred man m.

Hence, for any set of preference lists, switching the order of a pair on the list cannot improve a woman’s partner in the Gale-Shapley algorithm.

8. There are six students, Harry, Ron, Hermione, Ginny, Draco, and Cho. This class requires them to pair up and work on pair programming. Each has preferences over who they want to be paired with. The preferences are:
Harry: Cho > Ron > Hermione > Ginny > Draco
Ron: Ginny > Harry > Hermione > Cho > Draco
Hermione: Ron > Harry > Ginny > Cho > Draco
Ginny: Harry > Cho > Hermione > Ron > Draco
Draco : Cho > Ron > Ginny > Hermione > Harry
Cho: Hermione > Harry > Ron > Ginny > Draco
Show that there is no stable matching. That means showing that no matter who you put together, there will always be two potential partners who are not matched but prefer each other to the current partner.

Ans:
For a stable matching, there won’t be any potential partners who are not matched but prefer each other to the current partner.
Hence, to prove that there is no stable matching, we have to check that no matter who you put together, there will always be two potential partners who are not matched but prefer each other to the current partner.

Let’s start by pairing Harry and Cho.
Cho is the most preferred choice by Harry.
But Cho prefers Hermoine over Harry.
This creates instability in the pairing of Harry and Cho.

If Ron is paired with Ginny:
Ginny is the most preferred choice by Ron.
But Ginny prefers Harry over Ron.
This creates instability in the pairing of Ron and Ginny.

If Hermione is paired with Ron:
Ron is the most preferred choice by Hermoine.
But Ron prefers Ginny over Hermione.
This creates instability in the pairing of Hermione and Ron.

If Ginny is paired with Harry:
Harry is the most preferred choice by Ginny.
But Harry prefers Cho over Ginny.
This creates instability in the pairing of Ginny and Harry.

If Draco is paired with Cho:
Cho is the most preferred choice by Draco.
But Cho prefers Hermoine over Draco.
This creates instability in the pairing of Draco and Cho.

If Cho is paired with Hermione:
Hermione is the most preferred choice by Cho.
But Hermoine prefers Ron over Cho.
This creates instability in the pairing of Cho and Hermione.

In each case, there is a pair that prefers each other over their current partners, leading to instability. Therefore, there is no stable matching in this scenario.