**Key attributes of an algorithm:**

1. Correctness
2. Performance:
3. Hardware:

Cache. Be careful of cache usage, GPUs, parallel processing.

1. Software:

**Definition of algorithm:**

Step by step description of a solution of a problem.

**Steps to solve an algorithm:**

1. Come up with a concise problem statement.
2. Present a solution.
3. Prove the correctness of your solution.
4. Analyze its performance (complexity analysis).

**STABLE MATCHING**

Definition:

A matching S is a set of ordered pairs.

A perfect matching S’ is a matching with the property that each member of M and each member of W appear in exactly one pair in S’. There is no singlehood or polygamy.

Problem: How do you match n men with n women so they stay together ever after?

Ans:

Step 1: Come up with a concise problem statement.

We have set of n men

M = [m1, m2, m3, …, mn]

We have set of n women

M = [w1, w2, w3, …, wn]

Add notion of preferences.

Each man m ranks all women. M prefers w to w’ if m ranks w higher than w’.

Ordered ranking of m is his preferred list.

Similarly, each woman w ranks all men.

Now, define instability.

A pair (m, w’) where m prefers w’ over w and w’ prefers m’ over m is an instability.

Next,

Define stable matching:

A matching is stable is:

1. It is a perfect matching
2. There are no instabilities w.r.t. S.

Define input and output.

Input: Preference lists for a set of n men and n women.

Output: A set of n matches (ordered pairs) with no instability.

Step 2: Solution

Use Gayle – Shapley algorithm.

We have n free men and n free women at the start.

Gayle-Shapley algorithm says that:

Men propose to women (any man).

This man goes and proposes to his top choice woman.

If the woman accepts (if he is on the top of her list as well).

If not,

Woman temporarily accepts the proposal by getting engaged. This can be broken if a better proposal comes along. If a better man does not come along, she sticks with the original man.

If a better man comes along, she accepts her preferred man, and the original man goes back to the pool of free man. Now this man proposes to the next highly ranked woman.

This continues until everybody is engaged.

This will always give you a stable matching.

Stable matching is not unique. Gayle-Shapley algorithm can find you two stable matching:

1. When men propose
2. When women propose

High level algorithm:

Initially all m ϵ M and w ϵ W are free.

While there is a free man:

Choose a free man

Let w be m’s highest ranked woman to whom he has not proposed yet.

If w is free then:

(m, w) become engaged.

Else, say w is engaged to m’:

If w prefers m’ to m then

M remains free

Else

(m, w) become engaged

M’ becomes free

End if

End if

End while

Return the set of engaged pairs

A valid partner is one who is paired in any possible stable matching. The best valid partner is one who is the highest preferred valid partner.

Step 3: Proof of correctness

1. Prove that while loop terminates in at most n2 iterations.

Observations:

1. Each iteration of the while loop consists of a free man proposing to a woman
2. After her first engagement, a woman will always stay engaged.
3. Prove that at termination we have a perfect matching

The algorithm terminates when everyone is engaged, which is the definition of perfect matching.

1. Prove that there are no instabilities in our perfect matching

Observations:

1. From the woman’s perspective, she starts single, and once she gets engaged, she gets into better engagements.
2. From the man’s perspective, he starts single with highest ranking woman, and once he gets engaged, he gets into worse engagements (he might get dropped repeatedly by women only to settle for a lower ranked woman in his list).

Prove by showing contradiction. Take a situation where the contradicting situation doesn’t exist, making the answer right.

1. Complexity Analysis:

Operations at each iteration:

1. Find a free man  
   Stack: O(1)

Queue: O(1)

Linked list: O(1)

Array: O(1)

1. For a man m, identify the highest ranked woman whom he has not yet proposed.

O(1) (Locate the preferred woman in the list)

1. For a woman w, identify if she’s engaged or not.

O(1) (to find the status of woman)

1. If she is, decide if w wants to engage with which man (according to the preference).

Since it is not in sorted order, it will take linear time to search the man.

O(n).

Hence, we take the rankings of all men. Now this will take O(1)

1. Put a man back in the list of free men.   
   Stack: O(1)

Queue: O(1)

Linked list: O(1)

Array: O(1)

Preparation:

O(n) for one woman. To search n long list, it takes O(n2) time.

Hence,

Preparation takes: O(n2)

G-S iterations takes: O(n2)

Total Complexity: O(n2)

Question 1

The proposing group in Gale-Shapley algorithm always end up with their best valid partners.

Ans:

Proof by contradiction.

Assume a stable matching S where a man m is not paired with his best valid partner w.

Let us consider the point of execution where the first rejection occurs.

At this juncture, w rejects m either due to being paired with a better partner or a more preferred partner proposes to w.

Let m’ be the man preferred by w over m.

Also, since m is the first man to be rejected, m’ does not have any rejections.

Hence, m’ prefers w over another woman, say w’.

Since m and w are valid partners, assume a stable matching S’ where (m,w) are paired.

However, since m’ is preferred over m by w and m’ prefers w over w’, the pair (m’,w) is an instability.

This is a contradiction to our assumption that S’ is a stable matching.

Hence, men are always paired with their best valid partners.

Question 2

The group accepting or rejecting the proposals always end up with their worst valid matching partners.

Ans:

Proof by Contradiction.

Let (m, w) be the pair where m is the worst valid matching partner of w in a stable matching S.

Suppose there exist another stable matching S’ where w is not matched with its worst matching partner.

Let m’ be the man paired with w in S’. This implies that the man m’ prefers w over any other woman w’.

Also, w prefers m’ over m since m’ is not the worst valid matching partner.

However, the pair (m’, w) is a instability in S. This is a contradiction.