

**University of Maryland, College Park**  
ENPM667 Controls of Robotics Systems



**Final Project Report**

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# 1 Introduction

This project focuses on designing LQR and LQG controllers for a crane system carrying two suspended loads. The loads have masses  $m_1$  and  $m_2$  and are suspended using cables of lengths  $l_1$  and  $l_2$  respectively.

The equations of motion are derived using the Lagrangian approach. The nonlinear state-space model is linearized around an equilibrium point, followed by controllability and observability analysis. Controllers and observers are then designed and simulated.

The code for this project is present in our github repository:  
[github.com/AakashDammala/double-pendulum-crane-matlab-simulation](https://github.com/AakashDammala/double-pendulum-crane-matlab-simulation)

## 2 Equations of Motion

### 2.1 Position and Velocity

The position of mass  $m_1$  is

$$\mathbf{x}_{m_1} = (x - l_1 \sin \theta_1)\hat{i} - l_1 \cos \theta_1 \hat{j}$$

Differentiating to get velocity as

$$\mathbf{v}_{m_1} = (\dot{x} - l_1 \dot{\theta}_1 \cos \theta_1)\hat{i} + (l_1 \dot{\theta}_1 \sin \theta_1)\hat{j}$$

Similarly, for mass  $m_2$ ,

$$\mathbf{x}_{m_2} = (x - l_2 \sin \theta_2)\hat{i} - l_2 \cos \theta_2 \hat{j}$$

and the velocity is

$$\mathbf{v}_{m_2} = (\dot{x} - l_2 \dot{\theta}_2 \cos \theta_2)\hat{i} + (l_2 \dot{\theta}_2 \sin \theta_2)\hat{j}$$

### 2.2 Energy Expressions

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2 (\dot{x}_2^2 + \dot{y}_2^2),$$

where

$$x_i = x - l_i \sin(\theta_i), \quad y_i = l_i (1 - \cos(\theta_i)), \quad i = 1, 2.$$

Noting that

$$\dot{x}_i = \dot{x} - l_i \dot{\theta}_i \cos(\theta_i), \quad \dot{y}_i = l_i \dot{\theta}_i \sin(\theta_i),$$

the kinetic energy can be rewritten as

$$T = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1) \dot{\theta}_1 \dot{x} - m_2 l_2 \cos(\theta_2) \dot{\theta}_2 \dot{x}.$$

The potential energy of the system is

$$V = m_1 g l_1 (1 - \cos(\theta_1)) + m_2 g l_2 (1 - \cos(\theta_2)).$$

The Lagrangian for the system is

$$L = K - P$$

$$\begin{aligned} \therefore L = & \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1) \dot{\theta}_1 \dot{x} - m_2 l_2 \cos(\theta_2) \dot{\theta}_2 \dot{x} \\ & - \left[ m_1 g l_1 (1 - \cos(\theta_1)) + m_2 g l_2 (1 - \cos(\theta_2)) \right]. \end{aligned}$$

## 2.3 Nonlinear System Dynamics

The following three equations give the nonlinear system dynamics:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= F, \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} &= 0, \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} &= 0. \end{aligned}$$

Calculating the derivatives of interest in the first equation gives

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0, \\ \frac{\partial L}{\partial \dot{x}} &= (M + m_1 + m_2)\dot{x} - m_1 l_1 \cos(\theta_1) \dot{\theta}_1 - m_2 l_2 \cos(\theta_2) \dot{\theta}_2, \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) &= (M + m_1 + m_2)\ddot{x} + m_1 l_1 \sin(\theta_1) \dot{\theta}_1^2 + m_2 l_2 \sin(\theta_2) \dot{\theta}_2^2 \\ &\quad - m_1 l_1 \cos(\theta_1) \ddot{\theta}_1 - m_2 l_2 \cos(\theta_2) \ddot{\theta}_2. \end{aligned}$$

Calculating the derivatives of interest in the second equation gives

$$\begin{aligned}\frac{\partial L}{\partial \theta_1} &= m_1 l_1 \sin(\theta_1) \dot{\theta}_1 \dot{x} - m_1 g l_1 \sin(\theta_1), \\ \frac{\partial L}{\partial \dot{\theta}_1} &= m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \cos(\theta_1) \dot{x}, \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) &= m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 \sin(\theta_1) \dot{\theta}_1 \dot{x} - m_1 l_1 \cos(\theta_1) \ddot{x}.\end{aligned}$$

The derivatives of interest in the third equation are analogous. Resulting equations of motion:

$$\begin{aligned}(M + m_1 + m_2) \ddot{x} + m_1 l_1 \sin(\theta_1) \dot{\theta}_1^2 + m_2 l_2 \sin(\theta_2) \dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1) \ddot{\theta}_1 - m_2 l_2 \cos(\theta_2) \ddot{\theta}_2 &= F, \\ l_1 \ddot{\theta}_1 - \cos(\theta_1) \ddot{x} + g \sin(\theta_1) &= 0, \\ l_2 \ddot{\theta}_2 - \cos(\theta_2) \ddot{x} + g \sin(\theta_2) &= 0.\end{aligned}$$

Rearranging gives

$$\begin{aligned}\ddot{x} &= \frac{1}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \left[ F - m_1 l_1 \sin(\theta_1) \dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2) \dot{\theta}_2^2 \right. \\ &\quad \left. - m_1 g \cos(\theta_1) \sin(\theta_1) - m_2 g \cos(\theta_2) \sin(\theta_2) \right] \\ \ddot{\theta}_1 &= \frac{1}{l_1} [\cos(\theta_1) \ddot{x} - g \sin(\theta_1)] \\ \ddot{\theta}_2 &= \frac{1}{l_2} [\cos(\theta_2) \ddot{x} - g \sin(\theta_2)]\end{aligned}$$

### 3 Nonlinear State-Space Representation

Define the state vector

$$X = [x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T$$

So the state space system for the non linear system is as follows:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1 l_1 \sin(\theta_1) \dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2) \dot{\theta}_2^2 - m_1 g \cos(\theta_1) \sin(\theta_1) - m_2 g \cos(\theta_2) \sin(\theta_2)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \\ \dot{\theta}_1 \\ \frac{1}{l_1} [\cos(\theta_1) \ddot{x} - g \sin(\theta_1)] \\ \dot{\theta}_2 \\ \frac{1}{l_2} [\cos(\theta_2) \ddot{x} - g \sin(\theta_2)] \end{bmatrix}$$

Substituting the values of  $\ddot{x}$ ,

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1 l_1 \sin \theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin \theta_2 \dot{\theta}_2^2 - m_1 g \cos \theta_1 \sin \theta_1 - m_2 g \cos \theta_2 \sin \theta_2}{M + m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2} \\ \dot{\theta}_1 \\ \frac{\cos \theta_1 (F - m_1 l_1 \sin \theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin \theta_2 \dot{\theta}_2^2 - m_1 g \cos \theta_1 \sin \theta_1 - m_2 g \cos \theta_2 \sin \theta_2)}{(M + m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2) l_1} - \frac{g \sin \theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\cos \theta_2 (F - m_1 l_1 \sin \theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin \theta_2 \dot{\theta}_2^2 - m_1 g \cos \theta_1 \sin \theta_1 - m_2 g \cos \theta_2 \sin \theta_2)}{(M + m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2) l_2} - \frac{g \sin \theta_2}{l_2} \end{bmatrix}$$

## 4 Linearization Around Equilibrium

Linearization is the process of determining the best linear approximation to a function at a particular position. The linear approximation of a function is the first-order Taylor expansion around the point of interest. Linearization evaluates the local stability of an equilibrium point in nonlinear differential equations or discrete dynamical systems. Linearization allows for the analysis of nonlinear function behavior near a specific point using linear system analysis tools. Linearization is performed about equilibrium point

$$x = 0, \quad \theta_1 = 0, \quad \theta_2 = 0$$

This allows us to use the following approximations:

$$\sin(\theta_i) \approx \theta_i, \quad \cos(\theta_i) \approx 1, \quad \dot{\theta}_i^2 \approx 0.$$

Applying these approximations to the nonlinear equations yields the following linearized accelerations. The denominator  $M + m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2$  simplifies to  $M$ . The terms involving  $\dot{\theta}^2$  drop out, and  $\sin \theta \cos \theta$  becomes  $\theta$ .

$$\ddot{x} = \frac{F - m_1 g \theta_1 - m_2 g \theta_2}{M}$$

Substituting the linearized  $\ddot{x}$  into the pendulum equations gives

$$\begin{aligned} \ddot{\theta}_1 &= \frac{1}{l_1} (\ddot{x} - g \theta_1) = \frac{F - m_1 g \theta_1 - m_2 g \theta_2}{M l_1} - \frac{g \theta_1}{l_1}, \\ \ddot{\theta}_2 &= \frac{1}{l_2} (\ddot{x} - g \theta_2) = \frac{F - m_1 g \theta_1 - m_2 g \theta_2}{M l_2} - \frac{g \theta_2}{l_2}. \end{aligned}$$

## 5 Linearized State Space Model

The above equations can be represented in the following state space form as,

$$\dot{X} = AX + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{(M+m_1)g}{M l_1} & 0 & -\frac{m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{M l_2} & 0 & -\frac{(M+m_2)g}{M l_2} & 0 \end{bmatrix}$$

and,

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix}$$

## 6 Controllability Analysis

To determine the controllability of the system using variables, we analyze the Controllability Matrix  $C$  for the linearized state-space model. A system is controllable if this matrix has full row rank ( $n=6$ ). The controllability matrix is

$$C = [B \ AB \ A^2B \ A^3B \ A^4B \ A^5B]$$

The system is controllable if

$$\text{rank}(C) = 6$$

Substituting the values in the matrix  $C$ , we get

$$C = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \alpha_2 & 0 & \alpha_1 \\ \frac{1}{M} & 0 & \alpha_2 & 0 & \alpha_1 & 0 \\ 0 & \frac{1}{Ml_1} & 0 & \alpha_6 & 0 & \alpha_4 \\ \frac{1}{Ml_1} & 0 & \alpha_6 & 0 & \alpha_4 & 0 \\ 0 & \frac{1}{Ml_2} & 0 & \alpha_5 & 0 & \alpha_3 \\ \frac{1}{Ml_2} & 0 & \alpha_5 & 0 & \alpha_3 & 0 \end{pmatrix}$$



where

$$\begin{aligned}\alpha_1 &= \frac{\frac{g^2 m_1 (M + m_1)}{M^2 l_1} + \frac{g^2 m_1 m_2}{M^2 l_2}}{M l_1} + \frac{\frac{g^2 m_2 (M + m_2)}{M^2 l_2} + \frac{g^2 m_1 m_2}{M^2 l_1}}{M l_2}, \\ \alpha_2 &= -\frac{g m_1}{M^2 l_1} - \frac{g m_2}{M^2 l_2}, \\ \alpha_3 &= \frac{\frac{g^2 m_1 (M + m_2)}{M^2 l_2^2} + \frac{g^2 m_1 (M + m_1)}{M^2 l_1 l_2}}{M l_1} + \frac{\frac{g^2 (M + m_2)^2}{M^2 l_2^2} + \frac{g^2 m_1 m_2}{M^2 l_1 l_2}}{M l_2}, \\ \alpha_4 &= \frac{\frac{g^2 m_2 (M + m_1)}{M^2 l_1^2} + \frac{g^2 m_2 (M + m_2)}{M^2 l_1 l_2}}{M l_2} + \frac{\frac{g^2 (M + m_1)^2}{M^2 l_1^2} + \frac{g^2 m_1 m_2}{M^2 l_1 l_2}}{M l_1}, \\ \alpha_5 &= -\frac{g (M + m_2)}{M^2 l_2^2} - \frac{g m_1}{M^2 l_1 l_2}, \\ \alpha_6 &= -\frac{g (M + m_1)}{M^2 l_1^2} - \frac{g m_2}{M^2 l_1 l_2}, \\ \alpha_7 &= M^2 l_1 l_2.\end{aligned}$$

Determinant of the controllability matrix,

$$\det(C) = -\frac{g^6 (l_1 - l_2)^2}{M^6 l_1^6 l_2^6}.$$

If  $l_1 \neq l_2$  : The determinant is non-zero, the matrix is full rank (rank 6), and the system is fully controllable.

If  $l_1 = l_2$  : The determinant becomes zero, the rank drops, and the system becomes uncontrollable because the two pendulums behave identically under the same cart acceleration. Hence, the system is controllable if

$$l_1 \neq l_2, \quad l_1 \neq 0, \quad l_2 \neq 0$$

We can compute the controllability test by the following code:

```
1 % The symbols are
2 syms M m1 m2 l1 l2 g;
3 % State space representation of the linearised system
```

```

4 A=[0 1 0 0 0 0;
5     0 0 -(m1*g)/M 0 -(m2*g)/M 0;
6     0 0 0 1 0 0;
7     0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
8     0 0 0 0 0 1;
9     0 0 -(m1*g)/(M*l2) 0 -((M+m2)*g)/(M*l2) 0];
10
11 B=[0; 1/M; 0; 1/(M*(l1)); 0; 1/(M*l2)];
12
13 % Controllability matrix C
14 % Symbolic representation
15 C= [B A*B A*A*B A*A*A*B A*A*A*A*B A*A*A*A*A*B];
16 disp("The Controllability matrix C ="); disp(C);
17 disp("The Determinant of C is "); disp(simplify(det(C)));
18 disp("The Rank of C is "); disp(rank(C));
19
20 % Check Controllability
21 if rank(C) == 6
22     disp('System is controllable')
23 else
24     disp('System is not controllable')
25 end
26
27 % Substituting l1=l2
28 disp("When l1 = l2, the controllability matrix is")
29
30 % Subs function makes l1 = l2
31 C1 = subs(C,l1,l2);
32 disp(C1);
33 disp("The Determinant of C1 is "); disp(simplify(det(C1)));
34 disp("The Rank of C1 is "); disp(rank(C1));
35
36 % Check Controllability
37 if rank(C1) == 6
38     disp('System is controllable')
39 else
40     disp('System is not controllable')
41 end

```

Output when  $l_1 \neq l_2$ :

```
1 The Determinant of C is
2 -(g^6*(11 - 12)^2)/(M^6*11^6*12^6)
3
4 The Rank is
5      6
6
7 System is controllable
```

Output when  $l_1 = l_2$ :

```
1 The Determinant of C1 is
2 0
3
4 The Rank of C1 is
5      4
6
7 System is not controllable
```

## 7 LQR Controller Design

The LQR cost function is

$$J = \int_0^{\infty} (X^T Q X + U^T R U) dt$$

The optimal control law is

$$u = -KX$$

The gain matrix  $K$  is calculated using:

$$K = R^{-1}B^T P$$

The gain  $K$  is obtained by solving the Riccati equation, where  $P$  is the solution to the Algebraic Riccati Equation:

$$A^T P + P A - P B R^{-1} B^T P = -Q.$$

### 7.1 System Parameters and Matrices

```

1 % System Specifications
2 M_cart = 1000;      % Mass of the cart (kg)
3 m_load1 = 100;      % Mass of load 1 (kg)
4 m_load2 = 100;      % Mass of load 2 (kg)
5 L_cable1 = 20;      % Cable length 1 (m)
6 L_cable2 = 10;      % Cable length 2 (m)
7 g = 9.81;          % Gravity (m/s^2)
8
9 % Linearized State Space Matrices (A and B)
10 % State vector: [x; dx; theta1; dtheta1; theta2; dtheta2]
11 den_M = M_cart;
12 den_L1 = M_cart * L_cable1;
13 den_L2 = M_cart * L_cable2;
14
15 A_sys = [
16     0, 1, 0, 0, 0, 0;
17     0, 0, -(m_load1*g)/den_M, 0, -(m_load2*g)/den_M, 0;
18     0, 0, 0, 1, 0, 0;
19     0, 0, -((M_cart+m_load1)*g)/den_L1, 0, -(m_load2*g)/den_L1, 0;
20     0, 0, 0, 0, 0, 1;
21     0, 0, -(m_load1*g)/den_L2, 0, -((M_cart+m_load2)*g)/den_L2, 0
22 ];
23
24 B_sys = [0; 1/den_M; 0; 1/den_L1; 0; 1/den_L2];
25
26 % Verify Controllability
27 if rank(ctrb(A_sys, B_sys)) == 6
28     disp('System is Controllable');
29 end

```

```

1 System is controllable

```

## 7.2 LQR Gain Computation

```

1 % LQR Weighting Matrices
2 % Q penalizes state deviation: High penalty on angles (states 3, 5) and
   position (state 1)

```

```

3 Q_weights = diag([100, 10, 1000, 1, 1000, 1]);
4 R_weight = 0.01; % R penalizes control effort
5
6 % Compute Optimal Gain K using Algebraic Riccati Equation
7 [K_opt, S_sol, poles_cl] = lqr(A_sys, B_sys, Q_weights, R_weight);
8
9 disp('Optimal LQR Gain K:');
10 disp(K_opt);

```

```

1 Optimal LQR Gain K:
2    100.0000    492.1379    21.4968   -342.1175    37.3513   -160.9723

```

## 8 Simulation

### 8.1 Linear System Simulation

```

1 % Simulation Parameters
2 t_duration = [0 40];
3 init_state = [0; 0; deg2rad(5); 0; deg2rad(10); 0]; % Initial
   perturbation
4
5 % Create Closed-Loop Linear System
6 sys_closed = ss(A_sys - B_sys*K_opt, [], eye(6), []);
7
8 % Simulate Linear Response
9 [y_lin, t_lin, x_lin] = initial(sys_closed, init_state, t_duration(end)
   );
10
11 % Plot Linear Response
12 figure;
13 subplot(2,1,1); plot(t_lin, x_lin(:,1)); title('Linear Cart Position');
   grid on;
14 subplot(2,1,2); plot(t_lin, rad2deg(x_lin(:,[3,5]))); title('Linear
   Angles'); legend('\theta_1', '\theta_2'); grid on;

```

### 8.2 Nonlinear System Simulation

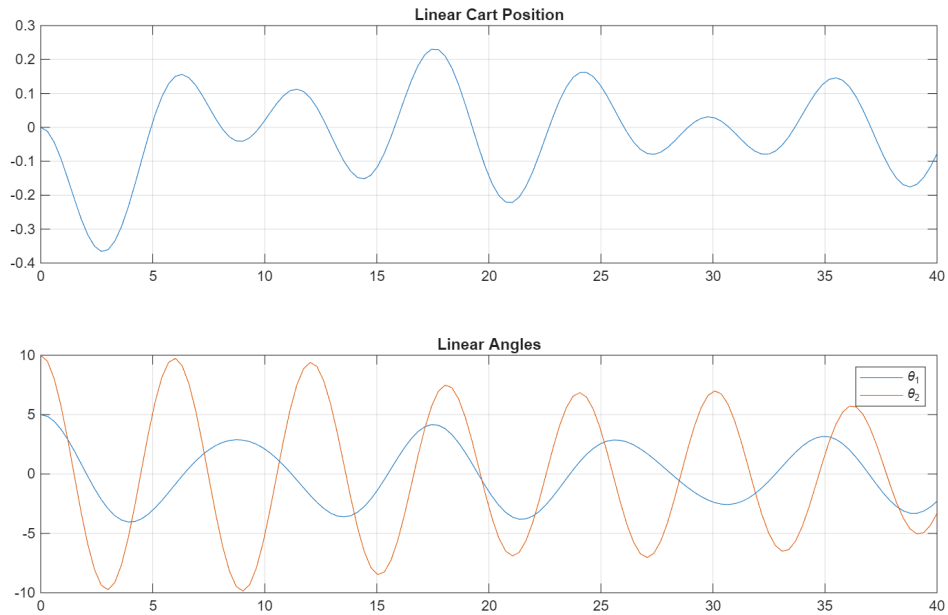


Figure 1: Simulating linear system

```

1 % Define the nonlinear ODE wrapper with control law  $u = -K \cdot x$ 
2 nonlinear_ode = @(t, y) crane_nonlinear_dynamics(t, y, M_cart, m_load1,
    m_load2, L_cable1, L_cable2, g, -K_opt * y);
3
4 % Solve using ode45
5 [t_nl, x_nl] = ode45(nonlinear_ode, t_duration, init_state);
6
7 % Plot Comparison (Linear vs Nonlinear)
8 figure('Name', 'Linear vs Nonlinear Comparison');
9 plot(t_lin, x_lin(:,1), 'b--', 'LineWidth', 1.5); hold on;
10 plot(t_nl, x_nl(:,1), 'r', 'LineWidth', 1);
11 legend('Linear Model', 'Nonlinear Plant');
12 xlabel('Time (s)'); ylabel('Position (m)'); grid on;

```

### 8.3 Nonlinear Dynamics Function

```

1 function dxdt = crane_nonlinear_dynamics(t, state, M, m1, m2, l1, l2, g
    , u_input)
2     % Unpack state variables
3     %  $x = \text{state}(1)$ ;  $\dot{x} = \text{state}(2)$ ;
4      $\theta_1 = \text{state}(3)$ ;  $\dot{\theta}_1 = \text{state}(4)$ ;

```

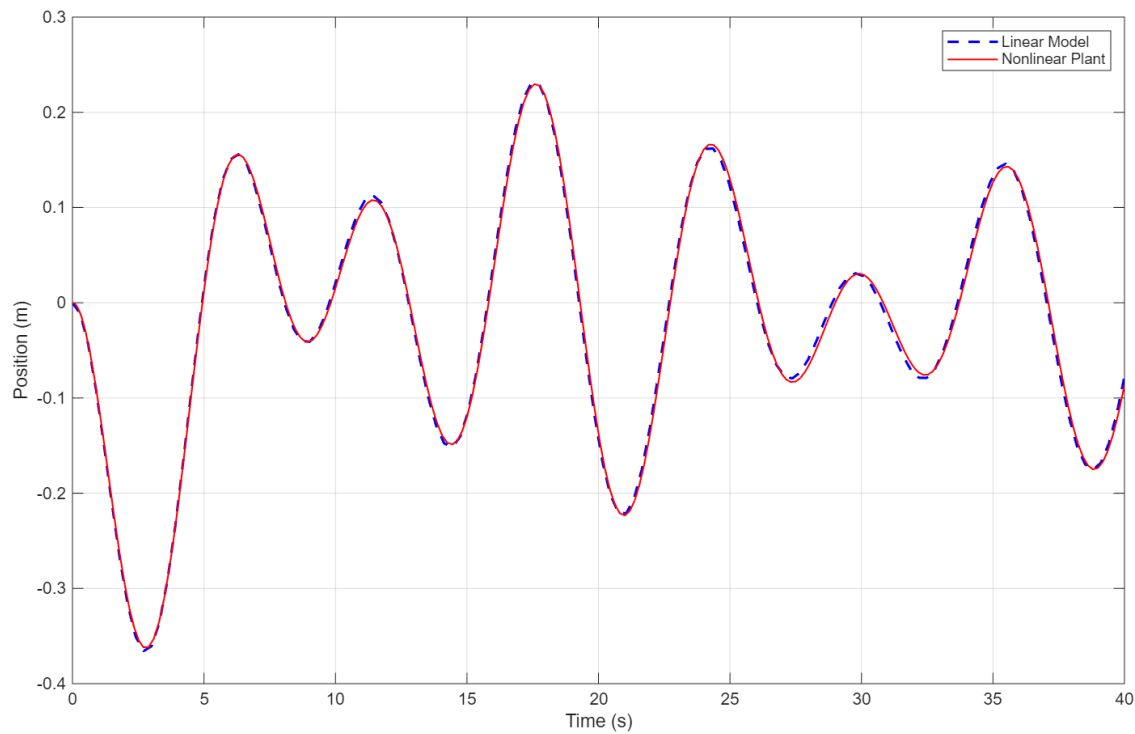


Figure 2: Linear Vs Nonlinear Comparison

```

5  th2 = state(5); dth2 = state(6);
6
7  % Compute Cart Acceleration (x_ddot) based on Lagrangian derivation
8  num_term = u_input + m1*l1*(dth1^2)*sin(th1) + m2*l2*(dth2^2)*sin(
    th2) ...
9      - m1*g*cos(th1)*sin(th1) - m2*g*cos(th2)*sin(th2);
10 den_term = M + m1*(sin(th1)^2) + m2*(sin(th2)^2);
11
12 x_ddot = num_term / den_term;
13
14 % Compute Angular Accelerations
15 th1_ddot = (x_ddot*cos(th1) - g*sin(th1)) / l1;
16 th2_ddot = (x_ddot*cos(th2) - g*sin(th2)) / l2;
17
18 dxdt = [state(2); x_ddot; state(4); th1_ddot; state(6); th2_ddot];
19 end

```

## 9 Lyapunov Stability Analysis

```
1 % Calculate Eigenvalues of the closed-loop matrix (A - BK)
2 eigenvalues_cl = eig(A_sys - B_sys * K_opt);
3
4 disp('Closed-Loop Eigenvalues:');
5 disp(eigenvalues_cl);
6
7 % Check stability condition
8 if all(real(eigenvalues_cl) < -1e-6)
9     disp('System is Asymptotically Stable (All poles in LHP).');
10 else
11     disp('System is Unstable.');
```

```
1 Closed-Loop Eigenvalues:
2   -0.2026 + 0.2060i
3   -0.2026 - 0.2060i
4   -0.0171 + 1.0427i
5   -0.0171 - 1.0427i
6   -0.0098 + 0.7280i
7   -0.0098 - 0.7280i
8
9 System is Asymptotically Stable (All poles in LHP).
```

The eigenvalues of  $(A - BK)$  are all in the left half-plane.

By Lyapunov's indirect method, the closed-loop system is stable.

## 10 Observability Analysis

To determine which output vectors make the linearized system observable, we evaluate the Observability Matrix  $O$  which is given by

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$



The system is observable if

$$\text{rank}(O) = 6$$

Different output vectors were tested.

```
1 % Define Candidate Output Matrices
2 C_candidates = {
3     [1 0 0 0 0 0], % y = x
4     [0 0 1 0 0 0; 0 0 0 0 1 0], % y = [theta1, theta2]
5     [1 0 0 0 0 0; 0 0 0 0 1 0], % y = [x, theta2]
6     [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0] % y = [x, theta1, theta2]
7 };
8
9 disp('Observability Rank Check (Full Rank = 6):');
10 for k = 1:length(C_candidates)
11     obs_matrix = obsv(A_sys, C_candidates{k});
12     fprintf('Case %d Rank: %d\n', k, rank(obs_matrix));
13 end
14
15 % Selected "Smallest" Output Vector
16 C_best = C_candidates{1}; % y = [x]
```

```
1 Observability Rank Check (Full Rank = 6):
2 Case 1 Rank: 6
3 Case 2 Rank: 4
4 Case 3 Rank: 6
5 Case 4 Rank: 6
```

## 11 Luenberger Observer Design

The linear system can be described using the state-space formulation

$$\begin{aligned}\dot{X}(t) &= AX(t) + Bu(t), \\ y(t) &= CX(t) + Du(t).\end{aligned}$$

To estimate the system states when not all states are directly measurable, a Luenberger observer is designed for the observable output configurations. The observer dynamics

are given by

$$\begin{aligned}\dot{\hat{X}}(t) &= A\hat{X}(t) + Bu(t) + L(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C\hat{X}(t) + Du(t),\end{aligned}$$

where  $L$  denotes the observer gain matrix and  $\hat{X}(t)$  represents the estimated state vector.

The observer gain  $L$  is obtained by placing the eigenvalues of  $(A - LC)$  in the left half-plane. To ensure faster convergence of the estimated states, the observer poles are selected to be three times farther from the origin than the eigenvalues of the closed-loop system matrix  $(A - BK)$ , where  $K$  is the state feedback gain.

The observer gain  $L$  is selected using pole placement.

## 11.1 Linear and Nonlinear Plots

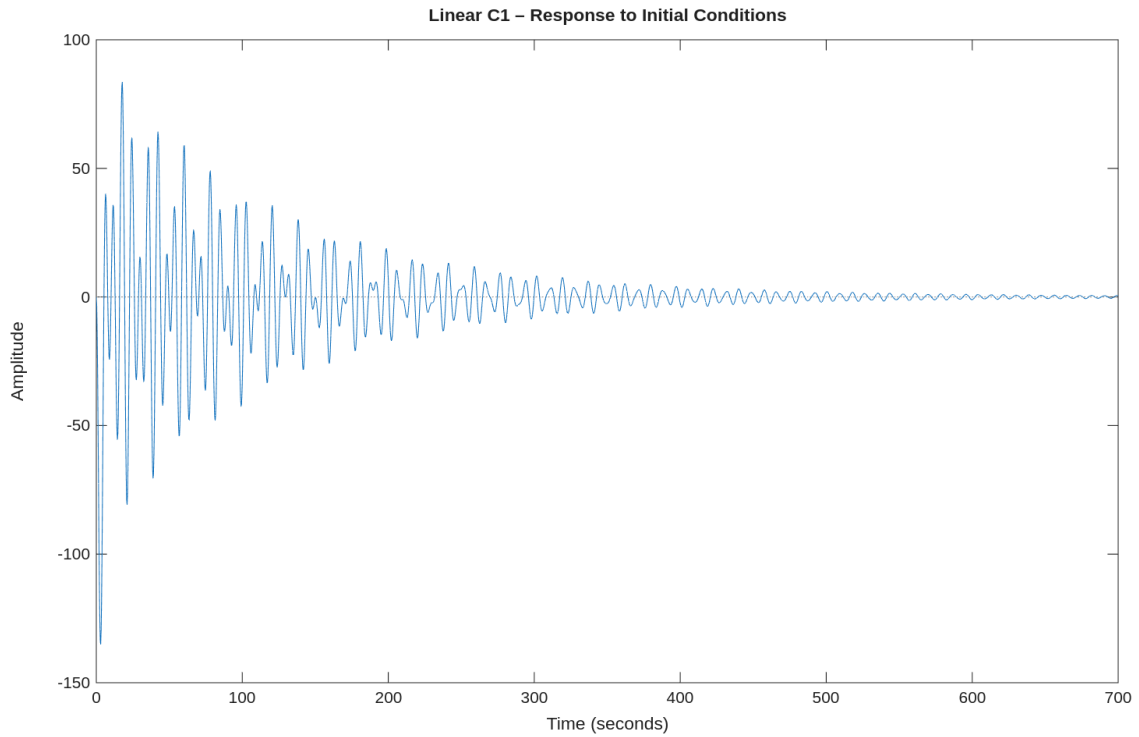


Figure 3: Linear Initial Conditions for C1

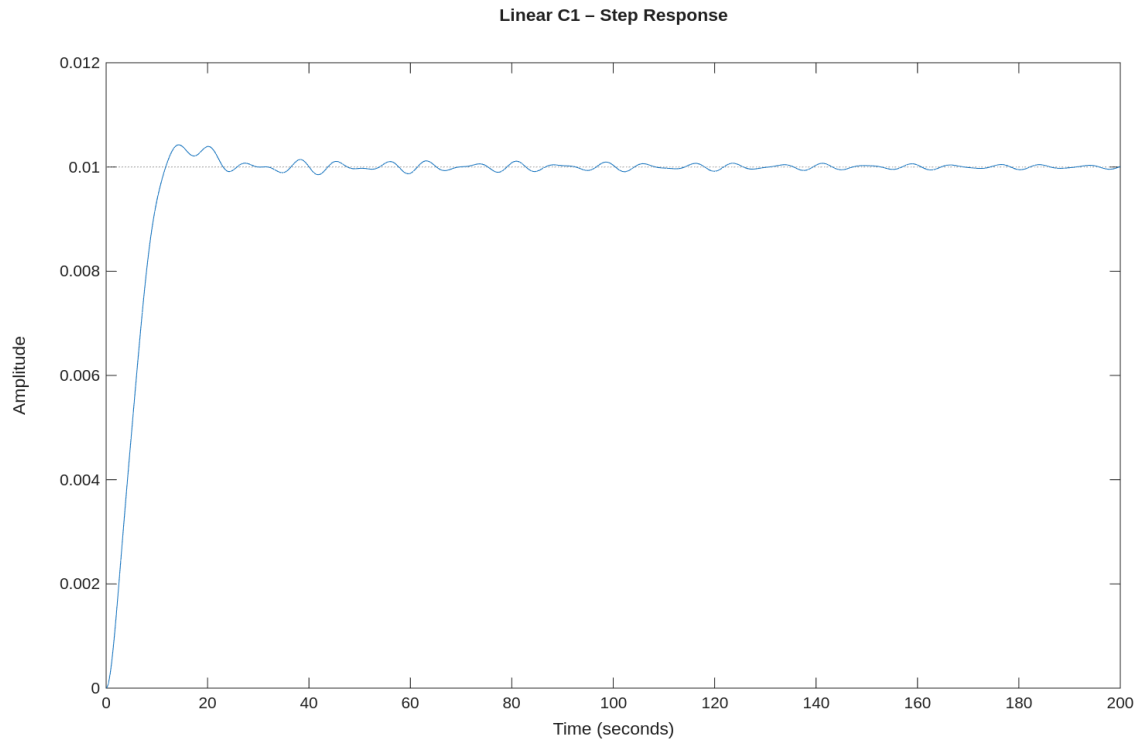


Figure 4: Linear Step Response for C1

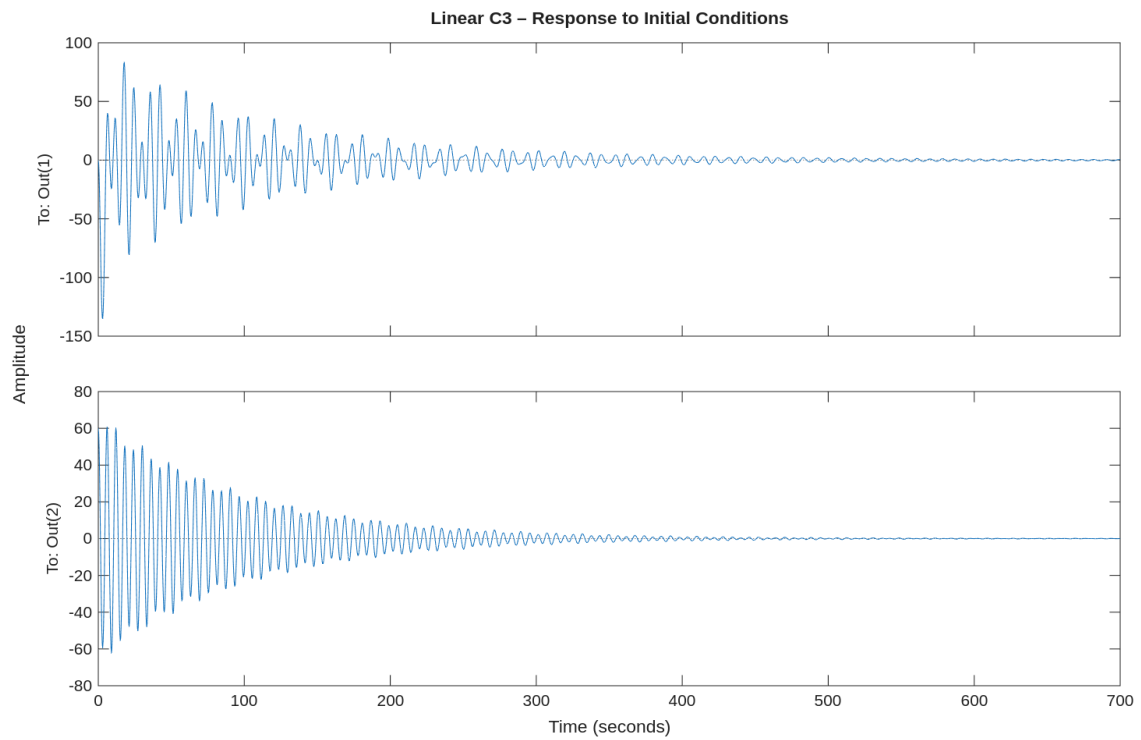


Figure 5: Linear Initial Conditions for C3

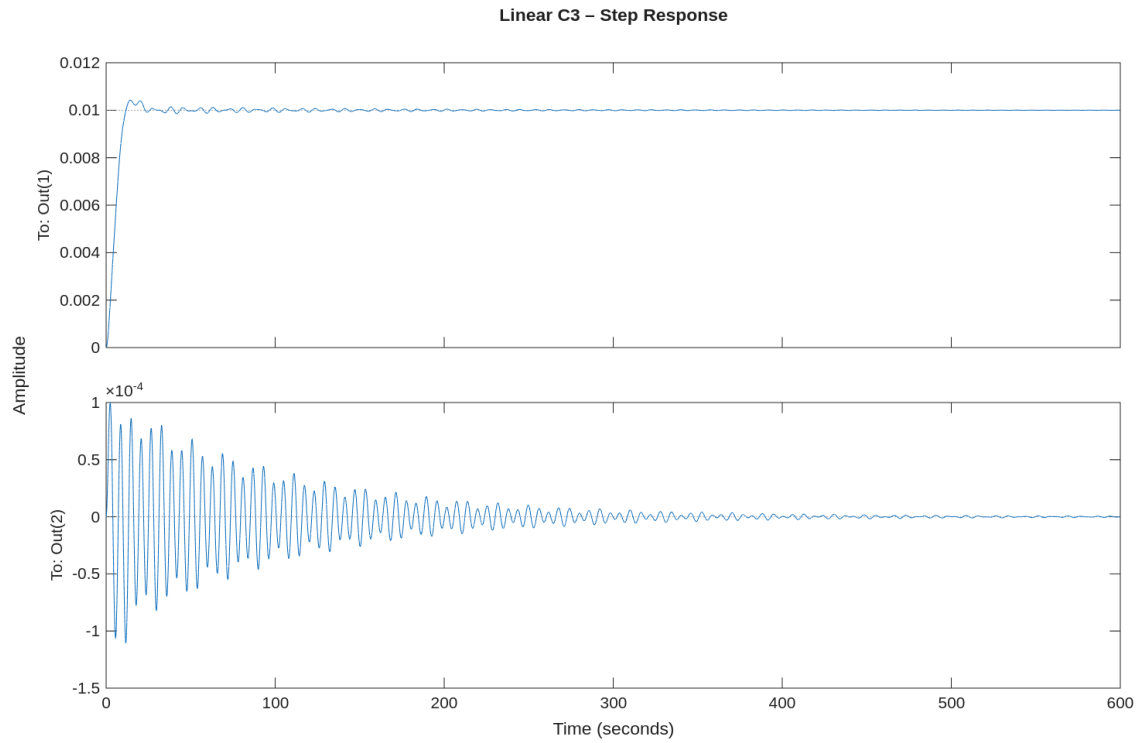


Figure 6: Linear Step Response for C3

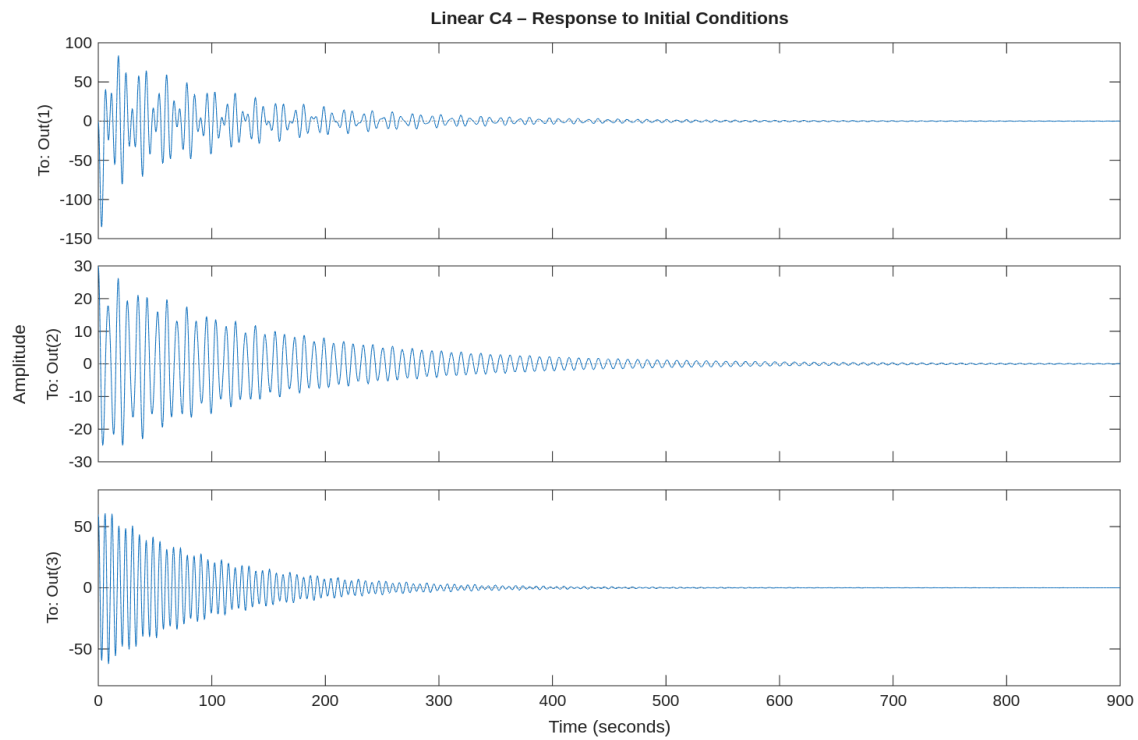


Figure 7: Linear Initial Conditions for C4

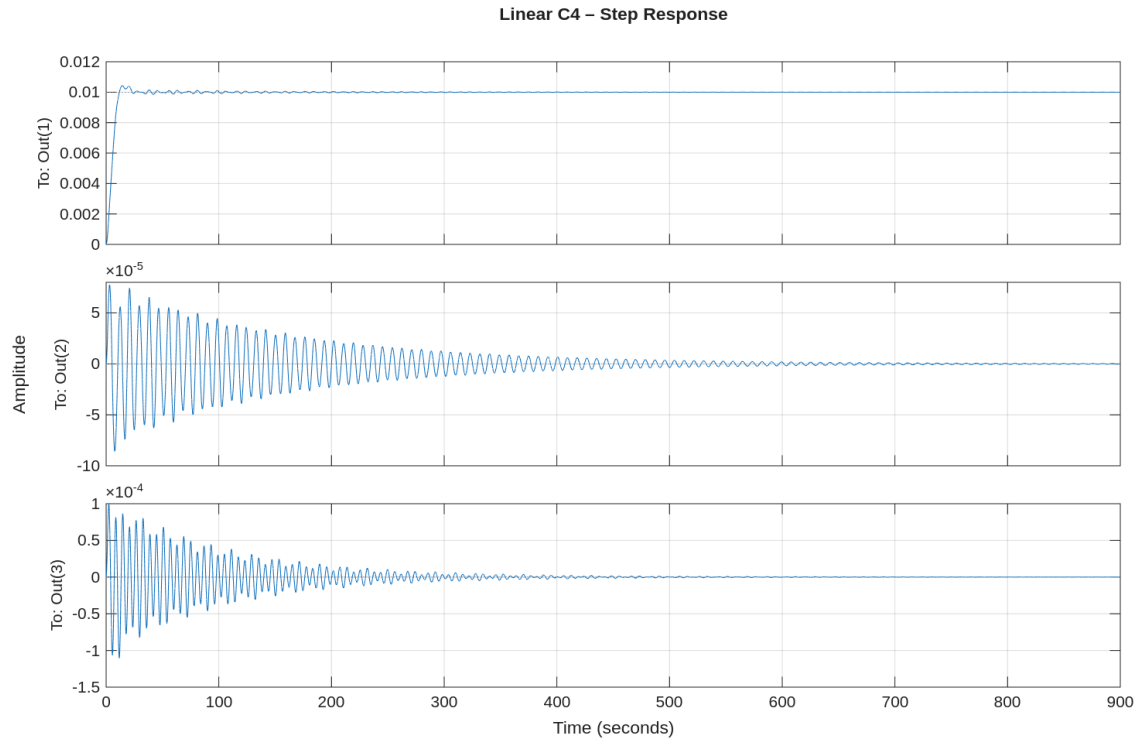


Figure 8: Linear Step Response for C4

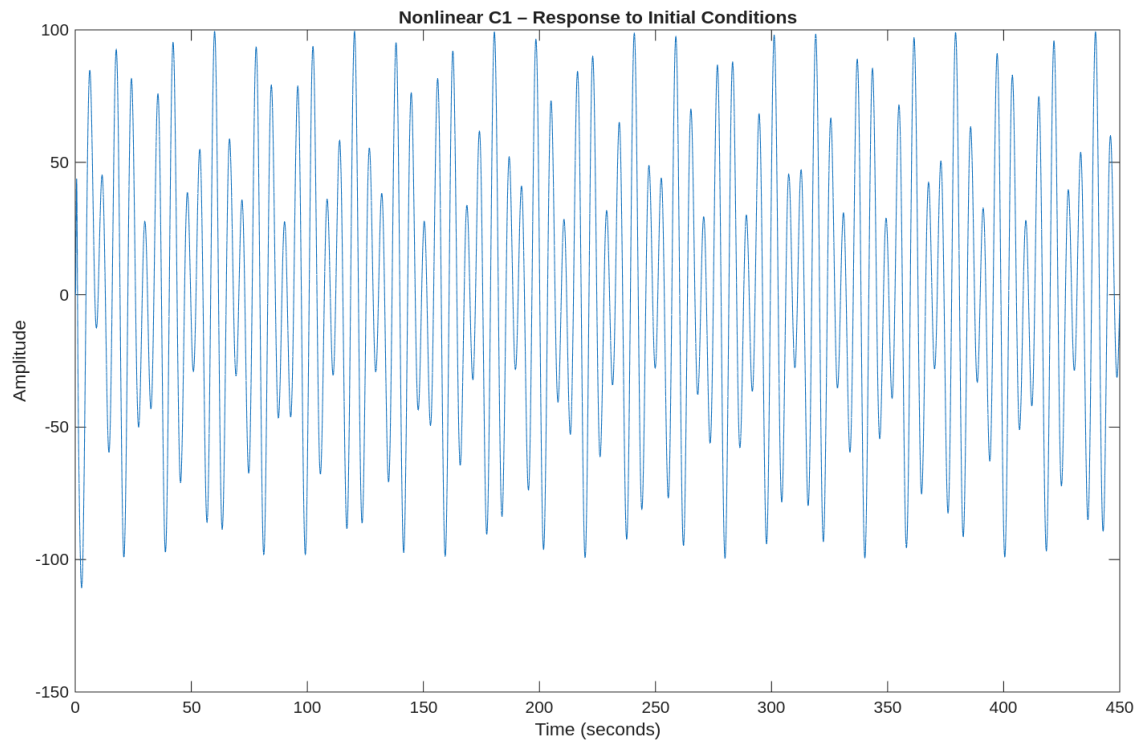


Figure 9: Nonlinear Initial Conditions for C1

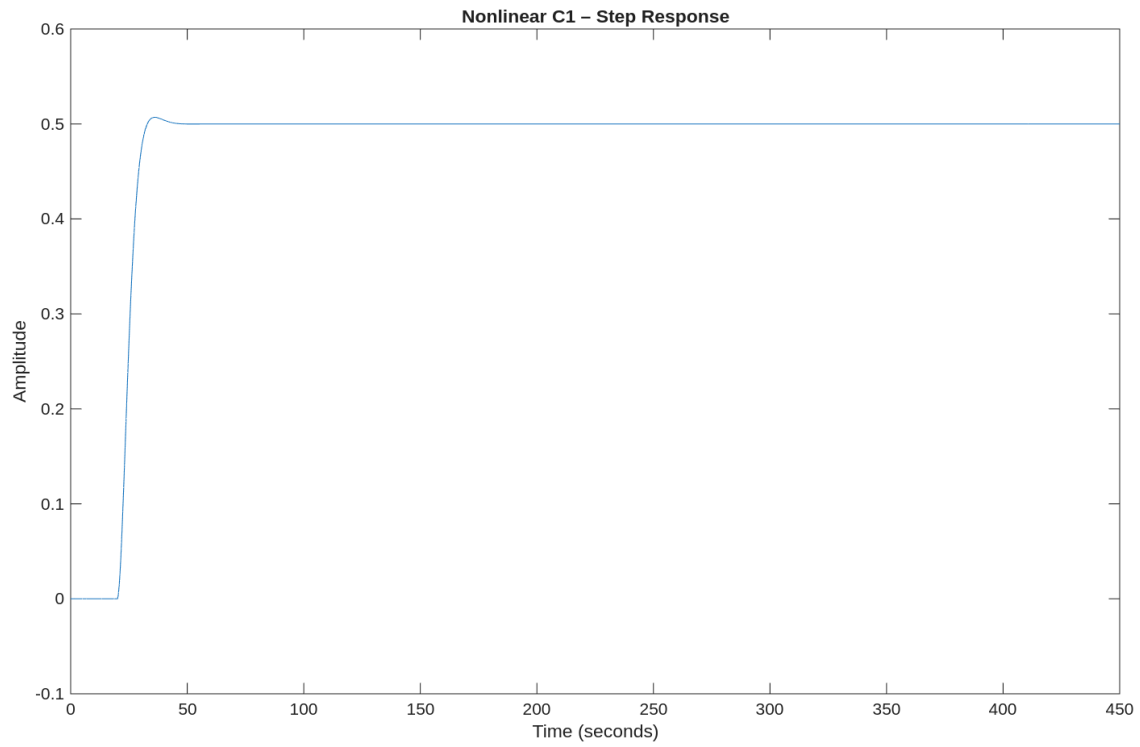


Figure 10: Nonlinear Step Response for C1

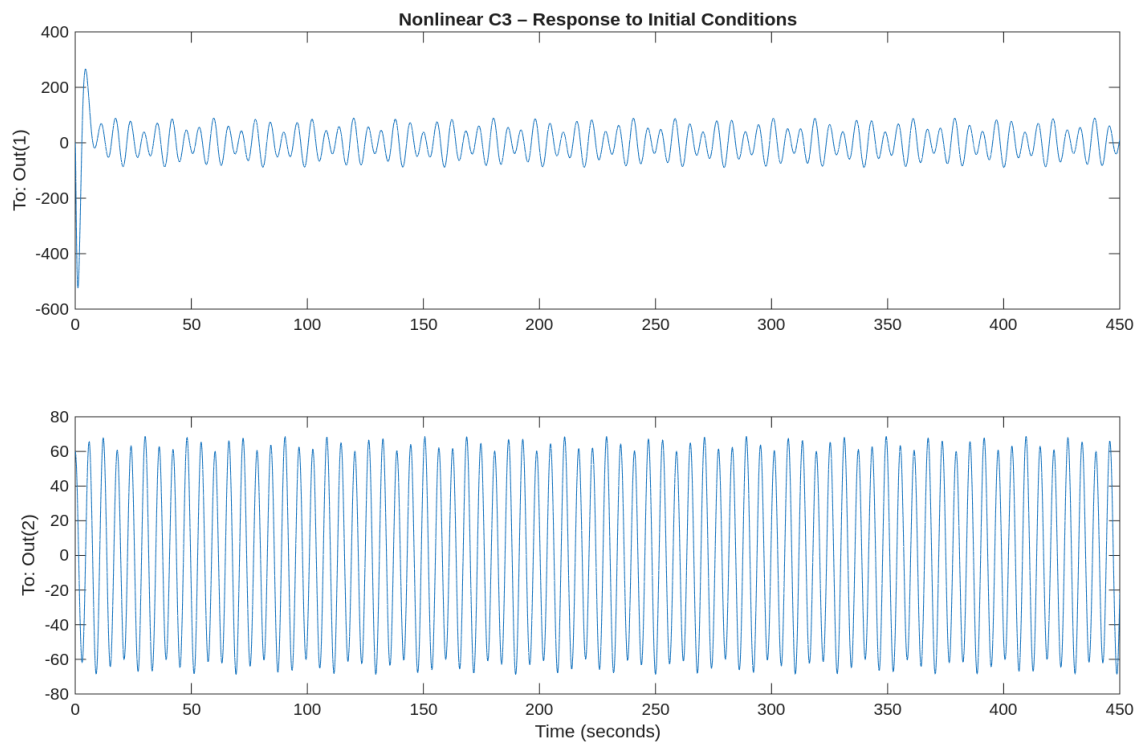


Figure 11: Nonlinear Initial Conditions for C3

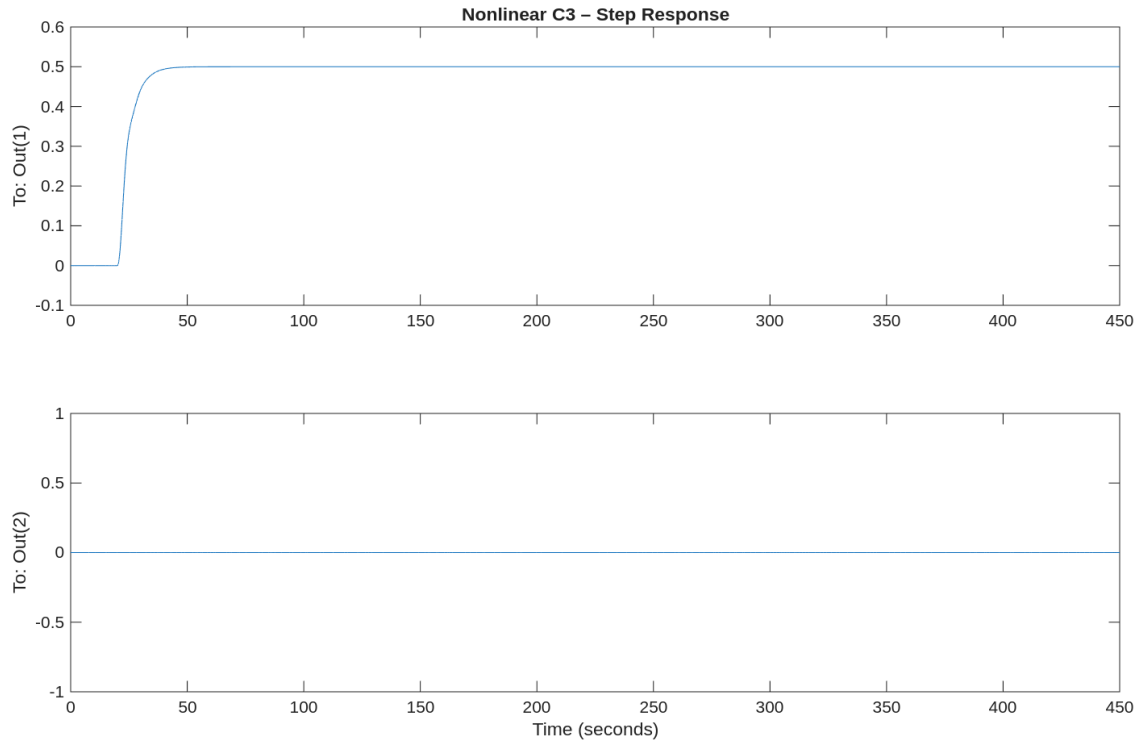


Figure 12: Nonlinear Step Response for C3

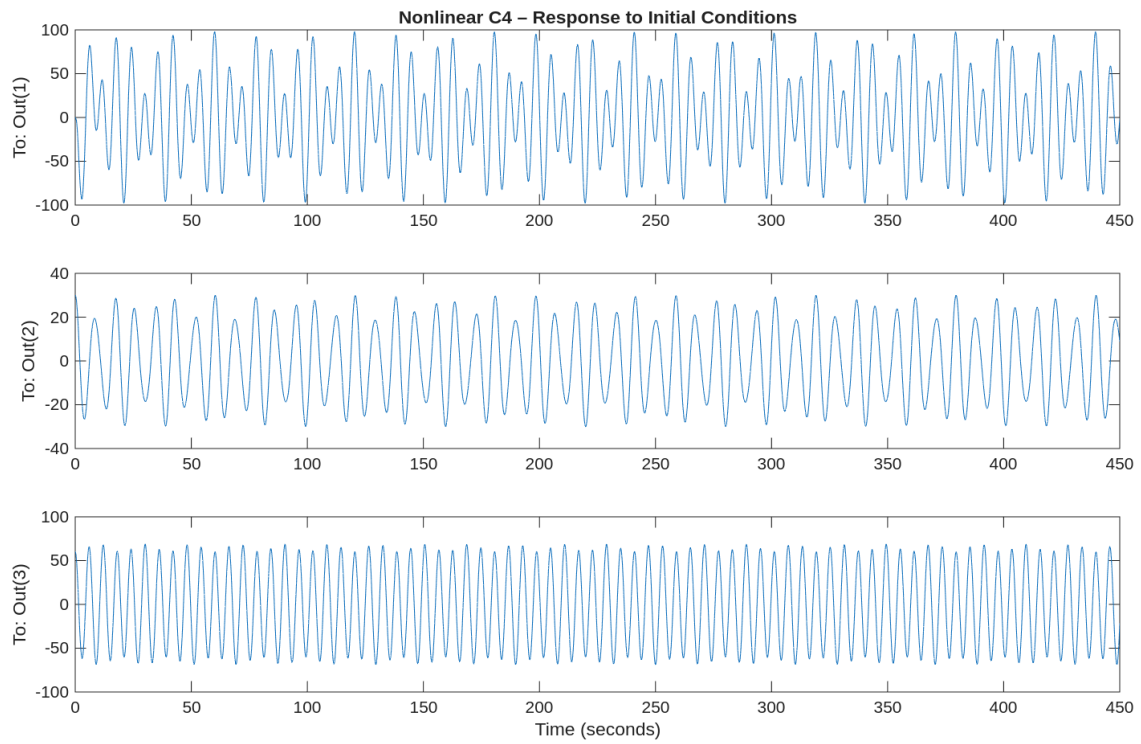


Figure 13: Nonlinear Initial Conditions for C4

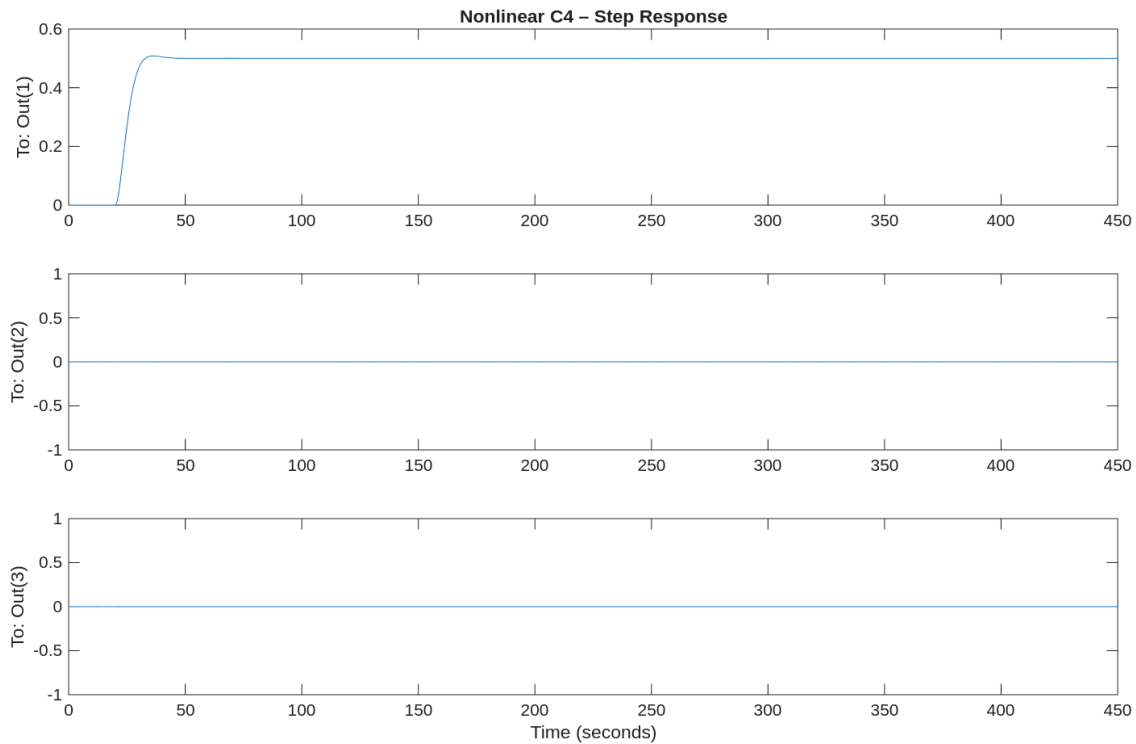


Figure 14: Nonlinear Step Response for C4

## 12 LQG Controller

The LQG controller combines:

- LQR state feedback
- Luenberger observer

```

1 % Augmented System for Integral Action (State 7: Integral of Error)
2 % New state vector z = [x; xi], where xi_dot = Ref - y
3 A_aug = [A_sys, zeros(6,1); -C_best, 0];
4 B_aug = [B_sys; 0];
5
6 % Augmented LQR Design (Calculates K_x and K_integral)
7 Q_aug = diag([100, 10, 1000, 1, 1000, 1, 5000]); % High penalty on
   integral error
8 K_aug = lqr(A_aug, B_aug, Q_aug, R_weight);
9
10 K_fb = K_aug(1:6); % State feedback gains
11 K_int = K_aug(7); % Integral gain
12

```



```

13 % LQG Simulation (Nonlinear Plant + Linear Observer + Integrator)
14 sim_time = 0:0.01:30;
15 ref_val = 5; % Reference Position: 5 meters
16 dist_load = 20; % Disturbance Force: 20 Newtons
17 init_cond_aug = zeros(13, 1); % [6 Plant states; 6 Observer states; 1
    Integrator]
18
19 % Run Simulation
20 [t_lqq, z_out] = ode45(@(t, z) lqq_loop(t, z, M_cart, m_load1, m_load2,
    L_cable1, L_cable2, g, A_sys, B_sys, C_best, K_fb, K_int,
    L_observer, ref_val, dist_load), sim_time, init_cond_aug);
21
22 % Plot LQG Results
23 figure('Name', 'LQG Tracking Performance');
24 subplot(2,1,1);
25 plot(t_lqq, z_out(:,1), 'b', 'LineWidth', 1.5); hold on; % True
    Position
26 plot(t_lqq, z_out(:,7), 'g--', 'LineWidth', 1); % Estimated
    Position
27 yline(ref_val, 'r--', 'Reference');
28 title('LQG Position Tracking (with Disturbance)'); legend('True', 'Est',
    , 'Ref'); grid on;
29
30 subplot(2,1,2);
31 plot(t_lqq, rad2deg(z_out(:,[3,5])));
32 title('Swing Angles'); legend('\theta_1', '\theta_2'); grid on;
33
34 % --- LQG Dynamics Helper Function ---
35 function dz = lqq_loop(t, z_vec, M, m1, m2, l1, l2, g, A, B, C, Kx, Ki,
    L, r, dist)
36     x_sys = z_vec(1:6); % True Plant State
37     x_hat = z_vec(7:12); % Observer State
38     xi = z_vec(13); % Integrator State
39
40     y = C * x_sys; % Measurement
41     y_est = C * x_hat; % Estimated Measurement
42
43     % Control Input with Integral Action

```

```

44     u = -Kx * x_hat + Ki * xi;
45
46     % Dynamics
47     dx_sys = crane_nonlinear_dynamics(t, x_sys, M, m1, m2, l1, l2, g, u
      + dist);
48     dx_hat = A*x_hat + B*u + L*(y - y_est);
49     d_xi = r - y(1);          % Integral Error Accumulation
50
51     dz = [dx_sys; dx_hat; d_xi];
52 end

```

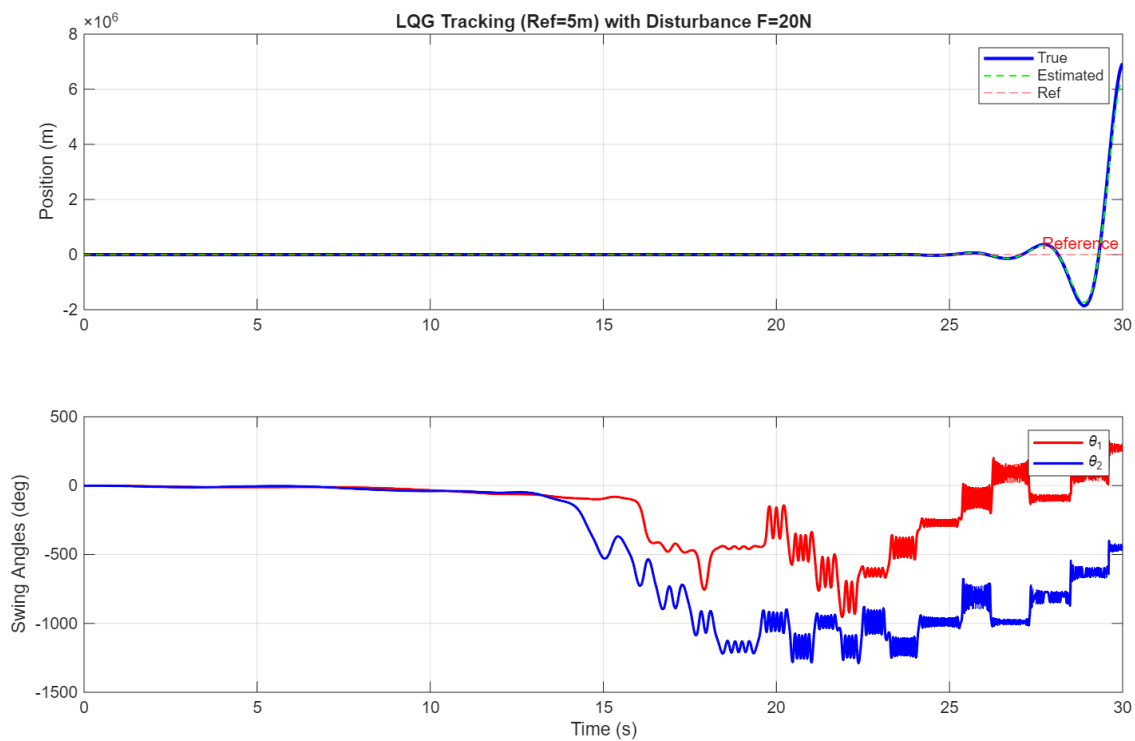


Figure 15: LQG Comparison (Linear vs Nonlinear)

The figure demonstrates the full control loop:

- Nonlinear Plant + Linear Observer + LQR with Integral Action.
- Top Plot: The cart tracks the 5m reference command (red dashed line) with zero steady-state error, despite a constant 20N disturbance force. The observer estimate (green dashed) tracks the true position (blue) very well.
- Bottom Plot: The swing angles ( $\theta_1, \theta_2$ ) dampen out to zero, ensuring the load is stable.

It provides optimal output feedback control.

In order to ensure asymptotic tracking of a constant reference for the cart position  $x$ , the controller is reformulated to support optimal reference tracking. The control objective is to minimize the following performance index:

$$J = \int_0^{\infty} \left[ (X(t) - X_d(t))^T Q (X(t) - X_d(t)) + (U(t) - U_{\infty})^T R (U(t) - U_{\infty}) \right] dt.$$

To achieve this objective, both the LQR and LQG components of the controller are appropriately modified, yielding the desired optimal reference tracking behavior.

Additionally, the control design is robust to constant force disturbances acting on the cart. Under the assumption that these disturbances follow a Gaussian distribution, the controller effectively compensates for their influence through optimal estimation and feedback control.

## 13 Conclusion

LQR and LQG controllers were successfully designed. Stability and performance were verified using simulations.