University of Maryland, College Park

ENPM667 Controls of Robotics Systems

Optimal Control and State Estimation of a Dual-Load Crane System

Students

Pranav Koli

Aakash Dammala

Instructor

Waseem Ansar Malik

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1 Introduction

This project focuses on designing LQR and LQG controllers for a crane system carrying two suspended loads. The loads have masses m_1 and m_2 and are suspended using cables of lengths l_1 and l_2 respectively.

The equations of motion are derived using the Lagrangian approach. The nonlinear state-space model is linearized around an equilibrium point, followed by controllability and observability analysis. Controllers and observers are then designed and simulated.

The code for this project is present in our github repository:
github.com/AakashDammala/double-pendulum-crane-matlab-simulation

2 Equations of Motion

2.1 Position and Velocity

The position of mass m_1 is

$$\mathbf{x}_{m_1} = (x - l_1 \sin\theta_1)\hat{i} - l_1 \cos\theta_1 \hat{j}$$

Differentiating to get velocity as

$$\mathbf{v}_{m_1} = (\dot{x} - l_1\dot{\theta}_1 \cos\theta_1)\hat{i} + (l_1\dot{\theta}_1 \sin\theta_1)\hat{j}$$

Similarly, for mass m_2,

$$\mathbf{x}_{m_2} = (x - l_2 \sin\theta_2)\hat{i} - l_2 \cos\theta_2 \hat{j}$$

and the velocity is

$$\mathbf{v}_{m_2} = (\dot{x} - l_2\dot{\theta}_2 \cos\theta_2)\hat{i} + (l_2\dot{\theta}_2 \sin\theta_2)\hat{j}$$

2.2 Energy Expressions

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right),$$

where

$$x_i = x - l_i\sin(\theta_i), \qquad y_i = l_i\left(1 - \cos(\theta_i)\right), \quad i = 1, 2.$$

Noting that

$$\dot{x}_i = \dot{x} - l_i\dot{\theta}_i \cos(\theta_i), \qquad \dot{y}_i = l_i\dot{\theta}_i \sin(\theta_i),$$

the kinetic energy can be rewritten as

$$T = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1)\dot{\theta}_1\dot{x} - m_2 l_2 \cos(\theta_2)\dot{\theta}_2\dot{x}.$$

The potential energy of the system is

$$V = m_1 g l_1 (1 - \cos(\theta_1)) + m_2 g l_2 (1 - \cos(\theta_2)).$$

The Lagrangian for the system is

$$L = K - P$$

$$\begin{aligned}\therefore\ L = &\frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1)\dot{\theta}_1\dot{x} - m_2 l_2 \cos(\theta_2)\dot{\theta}_2\dot{x} \\ &- \Big[m_1 g l_1(1 - \cos(\theta_1)) + m_2 g l_2(1 - \cos(\theta_2))\Big].\end{aligned}$$

2.3 Nonlinear System Dynamics

The following three equations give the nonlinear system dynamics:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0.\end{aligned}$$

Calculating the derivatives of interest in the first equation gives

$$\begin{aligned}\frac{\partial L}{\partial x} &= 0, \\ \frac{\partial L}{\partial \dot{x}} &= (M + m_1 + m_2)\dot{x} - m_1 l_1 \cos(\theta_1)\dot{\theta}_1 - m_2 l_2 \cos(\theta_2)\dot{\theta}_2, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (M + m_1 + m_2)\ddot{x} + m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 + m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2 \\ &\quad - m_1 l_1 \cos(\theta_1)\ddot{\theta}_1 - m_2 l_2 \cos(\theta_2)\ddot{\theta}_2.\end{aligned}$$

Calculating the derivatives of interest in the second equation gives

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= m_1 l_1 \sin(\theta_1)\dot{\theta}_1 \dot{x} - m_1 g l_1 \sin(\theta_1), \\
\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \cos(\theta_1)\dot{x}, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 \sin(\theta_1)\dot{\theta}_1 \dot{x} - m_1 l_1 \cos(\theta_1)\ddot{x}.
\end{aligned}
$$

The derivatives of interest in the third equation are analogous. Resulting equations of motion:

$$
\begin{aligned}
(M + m_1 + m_2)\ddot{x} + m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 + m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2 - m_1 l_1 \cos(\theta_1)\ddot{\theta}_1 - m_2 l_2 \cos(\theta_2)\ddot{\theta}_2 &= F, \\
l_1 \ddot{\theta}_1 - \cos(\theta_1)\ddot{x} + g \sin(\theta_1) &= 0, \\
l_2 \ddot{\theta}_2 - \cos(\theta_2)\ddot{x} + g \sin(\theta_2) &= 0.
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
\ddot{x} &= \frac{1}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \Big[F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2 \\
&\qquad - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 g \cos(\theta_2)\sin(\theta_2) \Big] \\
\ddot{\theta}_1 &= \frac{1}{l_1}\left[\cos(\theta_1)\ddot{x} - g \sin(\theta_1)\right] \\
\ddot{\theta}_2 &= \frac{1}{l_2}\left[\cos(\theta_2)\ddot{x} - g \sin(\theta_2)\right]
\end{aligned}
$$

3 Nonlinear State-Space Representation

Define the state vector

$$
X = [x,\ \dot{x},\ \theta_1,\ \dot{\theta}_1,\ \theta_2,\ \dot{\theta}_2]^T
$$

So the state space system for the non linear system is as follows:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2 - m_1 g \cos(\theta_1)\sin(\theta_1) - m_2 g \cos(\theta_2)\sin(\theta_2)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \\ \dot{\theta}_1 \\ \dfrac{1}{l_1}\left[\cos(\theta_1)\,\ddot{x} - g\sin(\theta_1)\right] \\ \dot{\theta}_2 \\ \dfrac{1}{l_2}\left[\cos(\theta_2)\,\ddot{x} - g\sin(\theta_2)\right] \end{bmatrix}$$

Substituting the values of $\ddot{x}$,

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{F - m_1 l_1 \sin\theta_1\dot{\theta}_1^2 - m_2 l_2 \sin\theta_2\dot{\theta}_2^2 - m_1 g \cos\theta_1\sin\theta_1 - m_2 g \cos\theta_2\sin\theta_2}{M + m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2} \\ \dot{\theta}_1 \\ \dfrac{\cos\theta_1\left(F - m_1 l_1 \sin\theta_1\dot{\theta}_1^2 - m_2 l_2 \sin\theta_2\dot{\theta}_2^2 - m_1 g \cos\theta_1\sin\theta_1 - m_2 g \cos\theta_2\sin\theta_2\right)}{\left(M + m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2\right) l_1} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \dfrac{\cos\theta_2\left(F - m_1 l_1 \sin\theta_1\dot{\theta}_1^2 - m_2 l_2 \sin\theta_2\dot{\theta}_2^2 - m_1 g \cos\theta_1\sin\theta_1 - m_2 g \cos\theta_2\sin\theta_2\right)}{\left(M + m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2\right) l_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix}$$

4 Linearization Around Equilibrium

Linearization is the process of determining the best linear approximation to a function at a particular position. The linear approximation of a function is the first-order Taylor expansion around the point of interest. Linearization evaluates the local stability of an equilibrium point in nonlinear differential equations or discrete dynamical systems. Linearization allows for the analysis of nonlinear function behavior near a specific point using linear system analysis tools. Linearization is performed about equilibrium point

$$x = 0, \quad \theta_1 = 0, \quad \theta_2 = 0$$

This allows us to use the following approximations:

$$\sin(\theta_i) \approx \theta_i, \qquad \cos(\theta_i) \approx 1, \qquad \dot{\theta_i}^2 \approx 0.$$

Applying these approximations to the nonlinear equations yields the following linearized accelerations. The denominator $M + m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2$ simplifies to M. The terms involving $\dot{\theta}^2$ drop out, and $\sin\theta\cos\theta$ becomes θ.

$$\ddot{x} = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}$$

Substituting the linearized $\ddot{x}$ into the pendulum equations gives

$$\begin{aligned}
\ddot{\theta}_1 &= \frac{1}{l_1}\left(\ddot{x} - g\theta_1\right) = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{Ml_1} - \frac{g\theta_1}{l_1}, \\
\ddot{\theta}_2 &= \frac{1}{l_2}\left(\ddot{x} - g\theta_2\right) = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{Ml_2} - \frac{g\theta_2}{l_2}.
\end{aligned}$$

5 Linearized State Space Model

The above equations can be represented in the following state space form as,

$$\dot{X} = AX + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{(M+m_1)g}{Ml_1} & 0 & -\frac{m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{Ml_2} & 0 & -\frac{(M+m_2)g}{Ml_2} & 0 \end{bmatrix}$$

and,

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

6 Controllability Analysis

To determine the controllability of the system using variables, we analyze the Controllability Matrix C for the linearized state-space model. A system is controllable if this matrix has full row rank (n=6). The controllability matrix is

$$C = [B \; AB \; A^2B \; A^3B \; A^4B \; A^5B]$$

The system is controllable if

$$\text{rank}(C) = 6$$

Substituting the values in the matrix C, we get

$$C = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \alpha_2 & 0 & \alpha_1 \\ \frac{1}{M} & 0 & \alpha_2 & 0 & \alpha_1 & 0 \\ 0 & \frac{1}{Ml_1} & 0 & \alpha_6 & 0 & \alpha_4 \\ \frac{1}{Ml_1} & 0 & \alpha_6 & 0 & \alpha_4 & 0 \\ 0 & \frac{1}{Ml_2} & 0 & \alpha_5 & 0 & \alpha_3 \\ \frac{1}{Ml_2} & 0 & \alpha_5 & 0 & \alpha_3 & 0 \end{pmatrix}$$

where

$$\alpha_1 = \frac{\frac{g^2 m_1(M+m_1)}{M^2 l_1} + \frac{g^2 m_1 m_2}{M^2 l_2}}{M l_1} + \frac{\frac{g^2 m_2(M+m_2)}{M^2 l_2} + \frac{g^2 m_1 m_2}{M^2 l_1}}{M l_2},$$

$$\alpha_2 = -\frac{g m_1}{M^2 l_1} - \frac{g m_2}{M^2 l_2},$$

$$\alpha_3 = \frac{\frac{g^2 m_1(M+m_2)}{M^2 l_2^2} + \frac{g^2 m_1(M+m_1)}{M^2 l_1 l_2}}{M l_1} + \frac{\frac{g^2 (M+m_2)^2}{M^2 l_2^2} + \frac{g^2 m_1 m_2}{M^2 l_1 l_2}}{M l_2},$$

$$\alpha_4 = \frac{\frac{g^2 m_2(M+m_1)}{M^2 l_1^2} + \frac{g^2 m_2(M+m_2)}{M^2 l_1 l_2}}{M l_2} + \frac{\frac{g^2 (M+m_1)^2}{M^2 l_1^2} + \frac{g^2 m_1 m_2}{M^2 l_1 l_2}}{M l_1},$$

$$\alpha_5 = -\frac{g(M+m_2)}{M^2 l_2^2} - \frac{g m_1}{M^2 l_1 l_2},$$

$$\alpha_6 = -\frac{g(M+m_1)}{M^2 l_1^2} - \frac{g m_2}{M^2 l_1 l_2},$$

$$\alpha_7 = M^2 l_1 l_2.$$

Determinant of the controllability matrix,

$$det(C) = -\frac{g^6 (l_1 - l_2)^2}{M^6 l_1^6 l_2^6}.$$

If $l_1 \neq l_2$: The determinant is non-zero, the matrix is full rank (rank 6), and the system is fully controllable.

If $l_1 = l_2$: The determinant becomes zero, the rank drops, and the system becomes uncontrollable because the two pendulums behave identically under the same cart acceleration. Hence, the system is controllable if

$$l_1 \neq l_2, \quad l_1 \neq 0, \quad l_2 \neq 0$$

We can compute the controllability test by the following code:

```
% The symbols are
syms M m1 m2 l1 l2 g;
% State space representation of the linearised system
```

```
A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -((M+m2)*g)/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*(l1)); 0; 1/(M*l2)];

% Controllability matrix C
% Symbolic representation
C= [B A*B A*A*B A*A*A*B A*A*A*A*B A*A*A*A*A*B];
disp("The Controllablity matrix C ="); disp(C);
disp("The Determinant of C is "); disp(simplify(det(C)));
disp("The Rank of C is "); disp(rank(C));

% Check Controllablity
if rank(C) == 6
    disp('System is controllable')
else
    disp('System is not controllable')
end

% Substituting l1=l2
disp("When l1 = l2, the controllability matrix is")

% Subs function makes l1 = l2
C1 = subs(C,l1,l2);
disp(C1);
disp("The Determinant of C1 is "); disp(simplify(det(C1)));
disp("The Rank of C1 is "); disp(rank(C1));

% Check Controllablity
if rank(C1) == 6
    disp('System is controllable')
else
    disp('System is not controllable')
end
```

Output when $l_1 \neq l_2$:

```
The Determinant of C is
-(g^6*(l1 - l2)^2)/(M^6*l1^6*l2^6)

The Rank is
     6

System is controllable
```

Output when $l_1 = l_2$:

```
The Determinant of C1 is
0

The Rank of C1 is
     4

System is not controllable
```

7 LQR Controller Design

The LQR cost function is

$$J = \int_0^\infty (X^T Q X + U^T R U)\, dt$$

The optimal control law is

$$u = -KX$$

The gain matrix K is calculated using:

$$K = R^{-1} B^T P$$

The gain K is obtained by solving the Riccati equation, where P is the solution to the Algebraic Riccati Equation:

$$A^T P + PA - PBR^{-1}B^T P = -Q.$$

7.1 System Parameters and Matrices

```
% System Specifications
M_cart = 1000;       % Mass of the cart (kg)
m_load1 = 100;       % Mass of load 1 (kg)
m_load2 = 100;       % Mass of load 2 (kg)
L_cable1 = 20;       % Cable length 1 (m)
L_cable2 = 10;       % Cable length 2 (m)
g = 9.81;            % Gravity (m/s^2)

% Linearized State Space Matrices (A and B)
% State vector: [x; dx; theta1; dtheta1; theta2; dtheta2]
den_M = M_cart;
den_L1 = M_cart * L_cable1;
den_L2 = M_cart * L_cable2;

A_sys = [
    0, 1, 0, 0, 0, 0;
    0, 0, -(m_load1*g)/den_M, 0, -(m_load2*g)/den_M, 0;
    0, 0, 0, 1, 0, 0;
    0, 0, -((M_cart+m_load1)*g)/den_L1, 0, -(m_load2*g)/den_L1, 0;
    0, 0, 0, 0, 0, 1;
    0, 0, -(m_load1*g)/den_L2, 0, -((M_cart+m_load2)*g)/den_L2, 0
];

B_sys = [0; 1/den_M; 0; 1/den_L1; 0; 1/den_L2];

% Verify Controllability
if rank(ctrb(A_sys, B_sys)) == 6
    disp('System is Controllable');
end
```

```
System is controllable
```

7.2 LQR Gain Computation

```
% LQR Weighting Matrices
% Q penalizes state deviation: High penalty on angles (states 3, 5) and
    position (state 1)
```

```
Q_weights = diag([100, 10, 1000, 1, 1000, 1]);
R_weight = 0.01; % R penalizes control effort

% Compute Optimal Gain K using Algebraic Riccati Equation
[K_opt, S_sol, poles_cl] = lqr(A_sys, B_sys, Q_weights, R_weight);

disp('Optimal LQR Gain K:');
disp(K_opt);
```

```
Optimal LQR Gain K:
  100.0000  492.1379   21.4968 -342.1175   37.3513 -160.9723
```

8 Simulation

8.1 Linear System Simulation

```
% Simulation Parameters
t_duration = [0 40];
init_state = [0; 0; deg2rad(5); 0; deg2rad(10); 0]; % Initial
    perturbation

% Create Closed-Loop Linear System
sys_closed = ss(A_sys - B_sys*K_opt, [], eye(6), []);

% Simulate Linear Response
[y_lin, t_lin, x_lin] = initial(sys_closed, init_state, t_duration(end)
    );

% Plot Linear Response
figure;
subplot(2,1,1); plot(t_lin, x_lin(:,1)); title('Linear Cart Position');
     grid on;
subplot(2,1,2); plot(t_lin, rad2deg(x_lin(:,[3,5]))); title('Linear
    Angles'); legend('\theta_1', '\theta_2'); grid on;
```

8.2 Nonlinear System Simulation

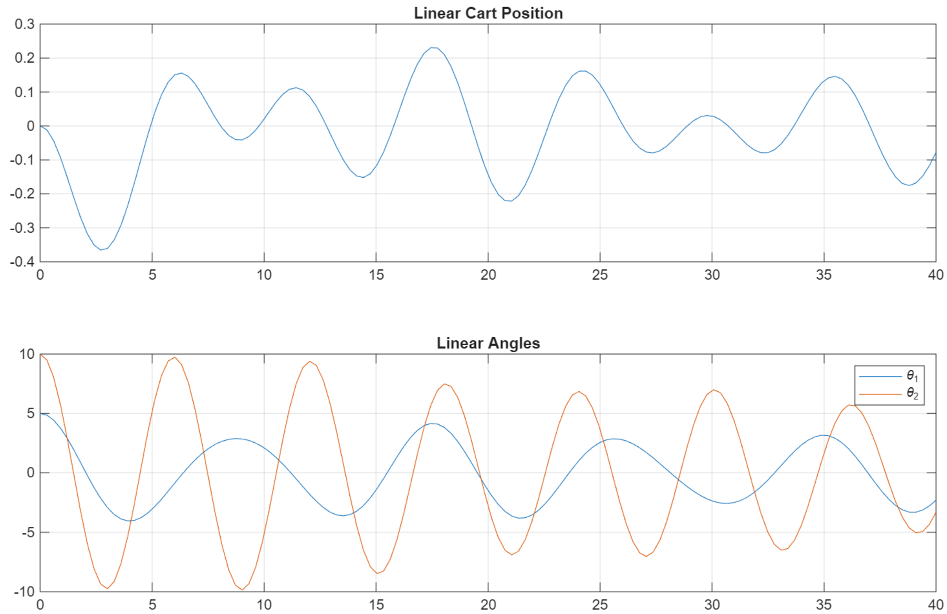

Figure 1: Simulating linear system

```matlab
% Define the nonlinear ODE wrapper with control law u = -K*x
nonlinear_ode = @(t, y) crane_nonlinear_dynamics(t, y, M_cart, m_load1, m_load2, L_cable1, L_cable2, g, -K_opt * y);

% Solve using ode45
[t_nl, x_nl] = ode45(nonlinear_ode, t_duration, init_state);

% Plot Comparison (Linear vs Nonlinear)
figure('Name', 'Linear vs Nonlinear Comparison');
plot(t_lin, x_lin(:,1), 'b--', 'LineWidth', 1.5); hold on;
plot(t_nl, x_nl(:,1), 'r', 'LineWidth', 1);
legend('Linear Model', 'Nonlinear Plant');
xlabel('Time (s)'); ylabel('Position (m)'); grid on;
```

8.3 Nonlinear Dynamics Function

```matlab
function dxdt = crane_nonlinear_dynamics(t, state, M, m1, m2, l1, l2, g, u_input)
    % Unpack state variables
    % x = state(1); dx = state(2);
    th1 = state(3); dth1 = state(4);
```

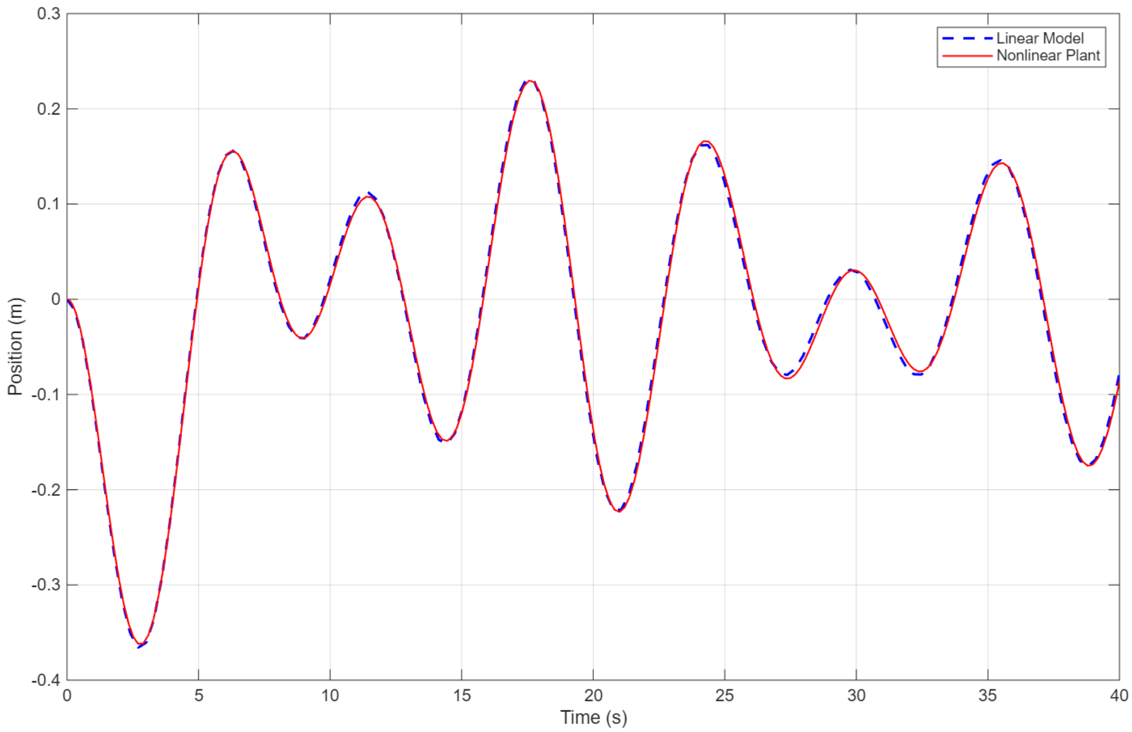

Figure 2: Linear Vs Nonlinear Comparison

```
    th2 = state(5); dth2 = state(6);

    % Compute Cart Acceleration (x_ddot) based on Lagrangian derivation
    num_term = u_input + m1*l1*(dth1^2)*sin(th1) + m2*l2*(dth2^2)*sin(
       th2) ...
               - m1*g*cos(th1)*sin(th1) - m2*g*cos(th2)*sin(th2);
    den_term = M + m1*(sin(th1)^2) + m2*(sin(th2)^2);

    x_ddot = num_term / den_term;

    % Compute Angular Accelerations
    th1_ddot = (x_ddot*cos(th1) - g*sin(th1)) / l1;
    th2_ddot = (x_ddot*cos(th2) - g*sin(th2)) / l2;

    dxdt = [state(2); x_ddot; state(4); th1_ddot; state(6); th2_ddot];
end
```

9 Lyapunov Stability Analysis

```
% Calculate Eigenvalues of the closed-loop matrix (A - BK)
eigenvalues_cl = eig(A_sys - B_sys * K_opt);

disp('Closed-Loop Eigenvalues:');
disp(eigenvalues_cl);

% Check stability condition
if all(real(eigenvalues_cl) < -1e-6)
    disp('System is Asymptotically Stable (All poles in LHP).');
else
    disp('System is Unstable.');
end
```

```
Closed-Loop Eigenvalues:
  -0.2026 + 0.2060i
  -0.2026 - 0.2060i
  -0.0171 + 1.0427i
  -0.0171 - 1.0427i
  -0.0098 + 0.7280i
  -0.0098 - 0.7280i

System is Asymptotically Stable (All poles in LHP).
```

The eigenvalues of $(A - BK)$ are all in the left half-plane.
By Lyapunov's indirect method, the closed-loop system is stable.

10 Observability Analysis

To determine which output vectors make the linearized system observable, we evaluate the Observability Matrix O which is given by

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The system is observable if

$$\mathrm{rank}(\mathcal{O}) = 6$$

Different output vectors were tested.

```
% Define Candidate Output Matrices
C_candidates = {
    [1 0 0 0 0 0],                          % y = x
    [0 0 1 0 0 0; 0 0 0 0 1 0],             % y = [theta1, theta2]
    [1 0 0 0 0 0; 0 0 0 0 1 0],             % y = [x, theta2]
    [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0] % y = [x, theta1, theta2]
};

disp('Observability Rank Check (Full Rank = 6):');
for k = 1:length(C_candidates)
    obs_matrix = obsv(A_sys, C_candidates{k});
    fprintf('Case %d Rank: %d\n', k, rank(obs_matrix));
end

% Selected "Smallest" Output Vector
C_best = C_candidates{1}; % y = [x]
```

```
Observability Rank Check (Full Rank = 6):
Case 1 Rank: 6
Case 2 Rank: 4
Case 3 Rank: 6
Case 4 Rank: 6
```

11 Luenberger Observer Design

The linear system can be described using the state–space formulation

$$\dot{X}(t) = AX(t) + Bu(t),$$
$$y(t) = CX(t) + Du(t).$$

To estimate the system states when not all states are directly measurable, a Luenberger observer is designed for the observable output configurations. The observer dynamics are given by

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(t) + L\big(y(t) - \hat{y}(t)\big),$$
$$\hat{y}(t) = C\hat{X}(t) + Du(t),$$

where L denotes the observer gain matrix and $\hat{X}(t)$ represents the estimated state vector.

The observer gain L is obtained by placing the eigenvalues of $(A - LC)$ in the left half–plane. To ensure faster convergence of the estimated states, the observer poles are selected to be three times farther from the origin than the eigenvalues of the closed–loop system matrix $(A - BK)$, where K is the state feedback gain.

The observer gain L is selected using pole placement.

11.1 Linear and Nonlinear Plots

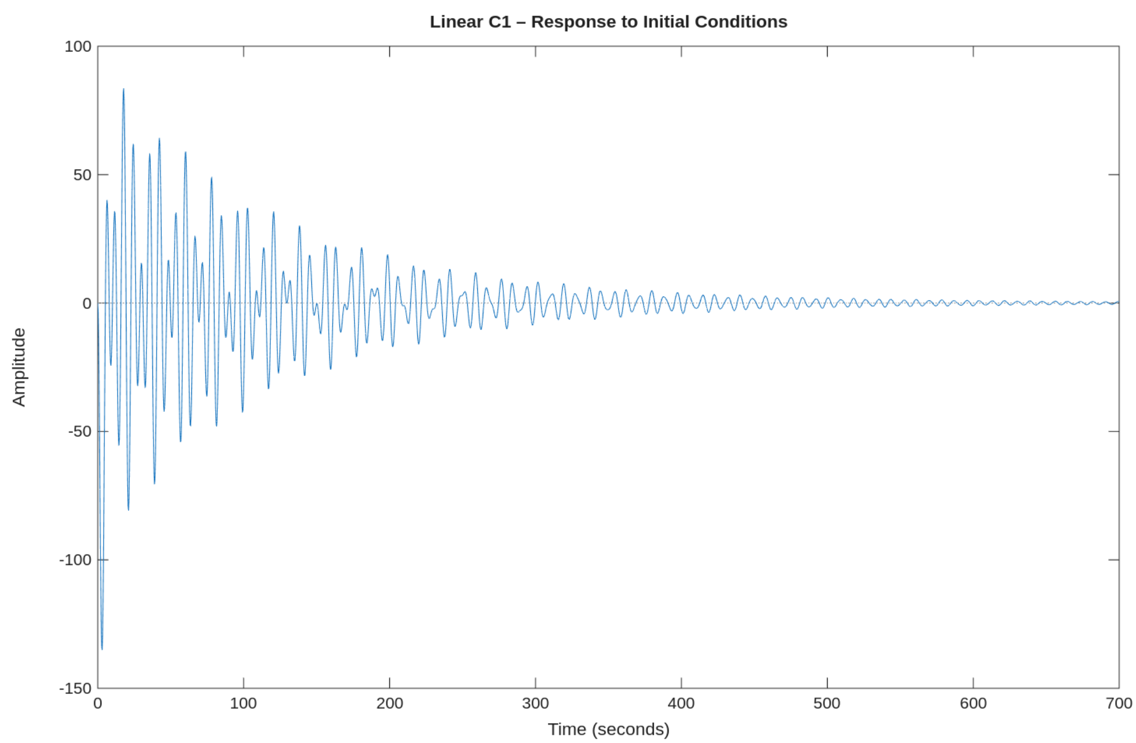

Figure 3: Linear Initial Conditions for C1

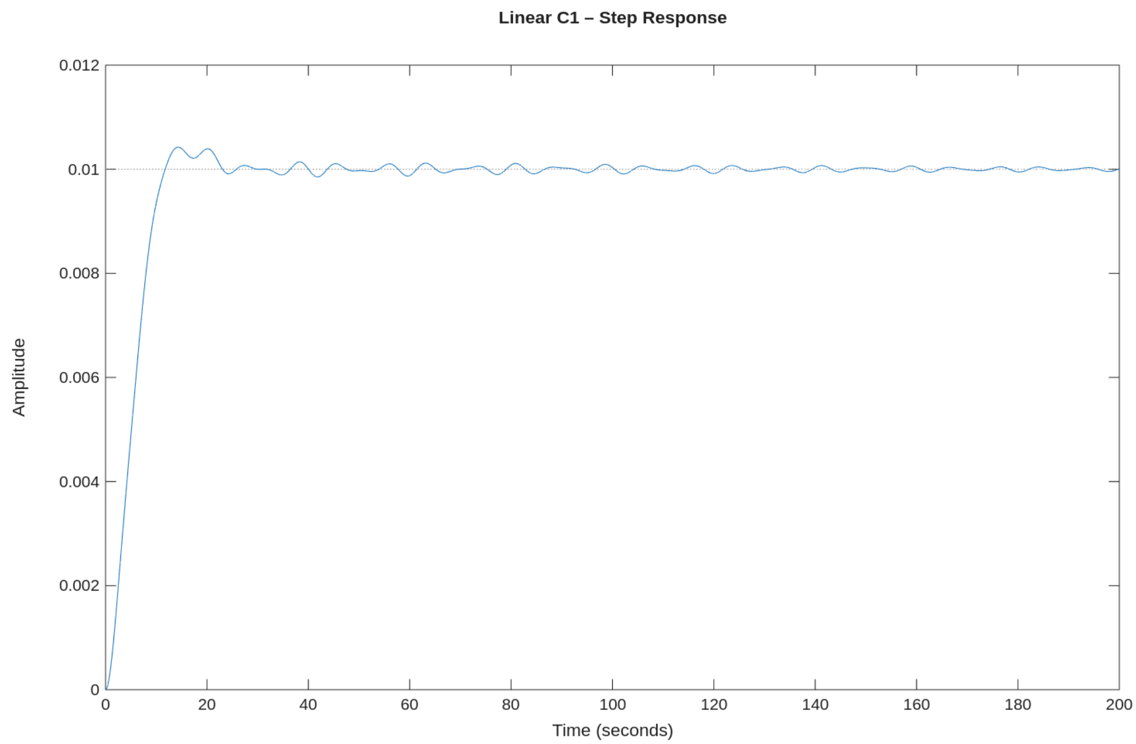

Figure 4: Linear Step Response for C1

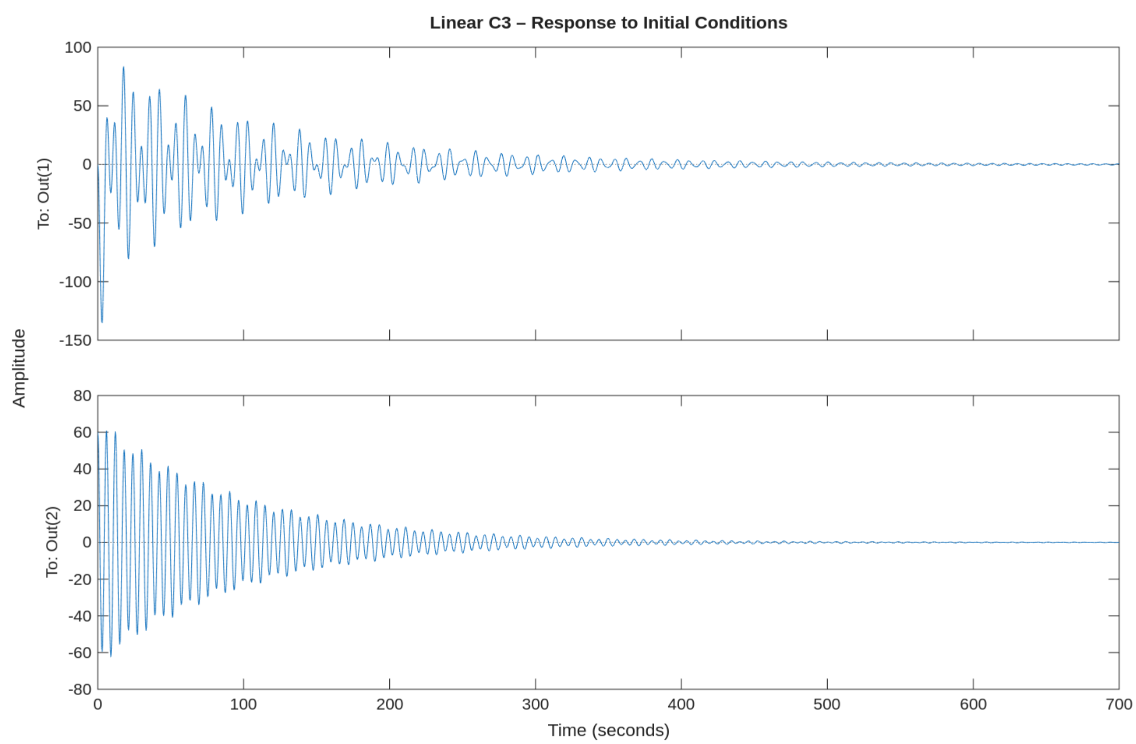

Figure 5: Linear Initial Conditions for C3

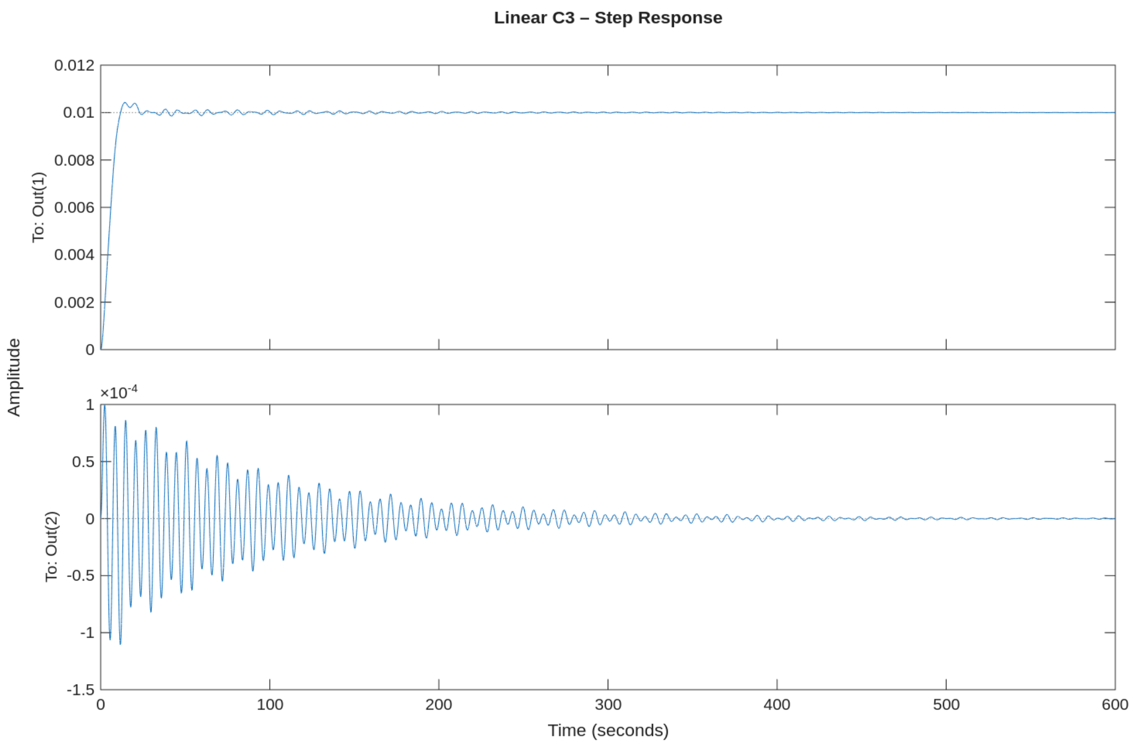

Figure 6: Linear Step Response for C3

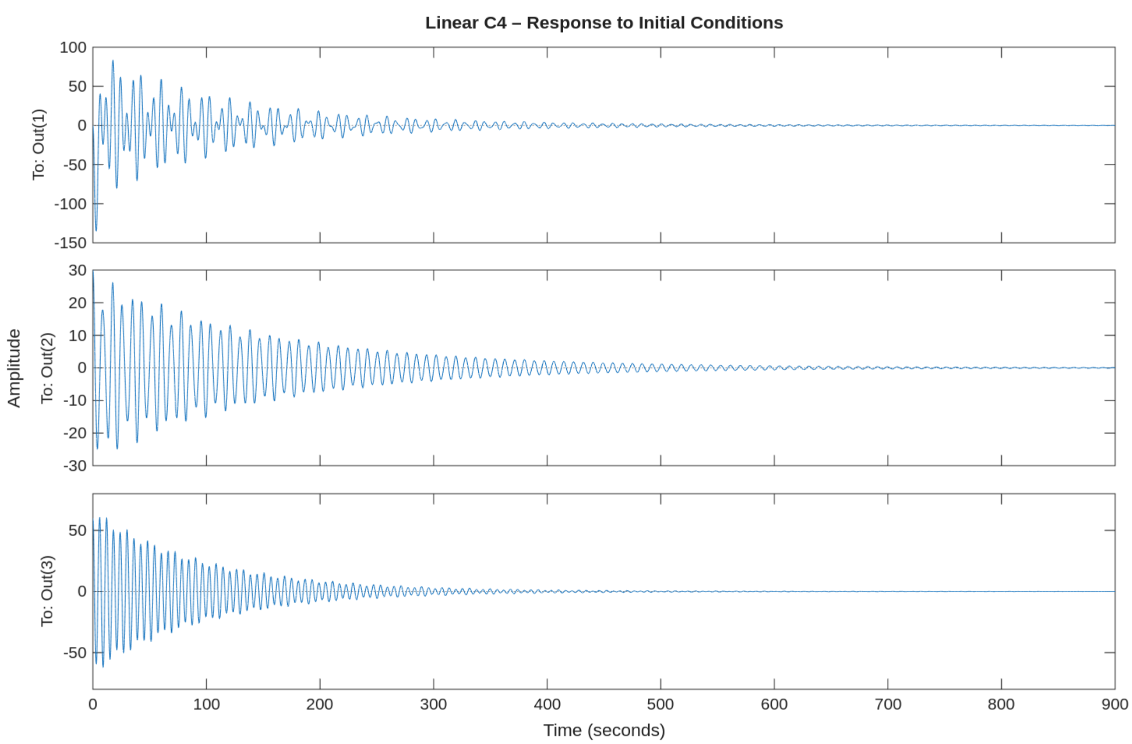

Figure 7: Linear Initial Conditions for C4

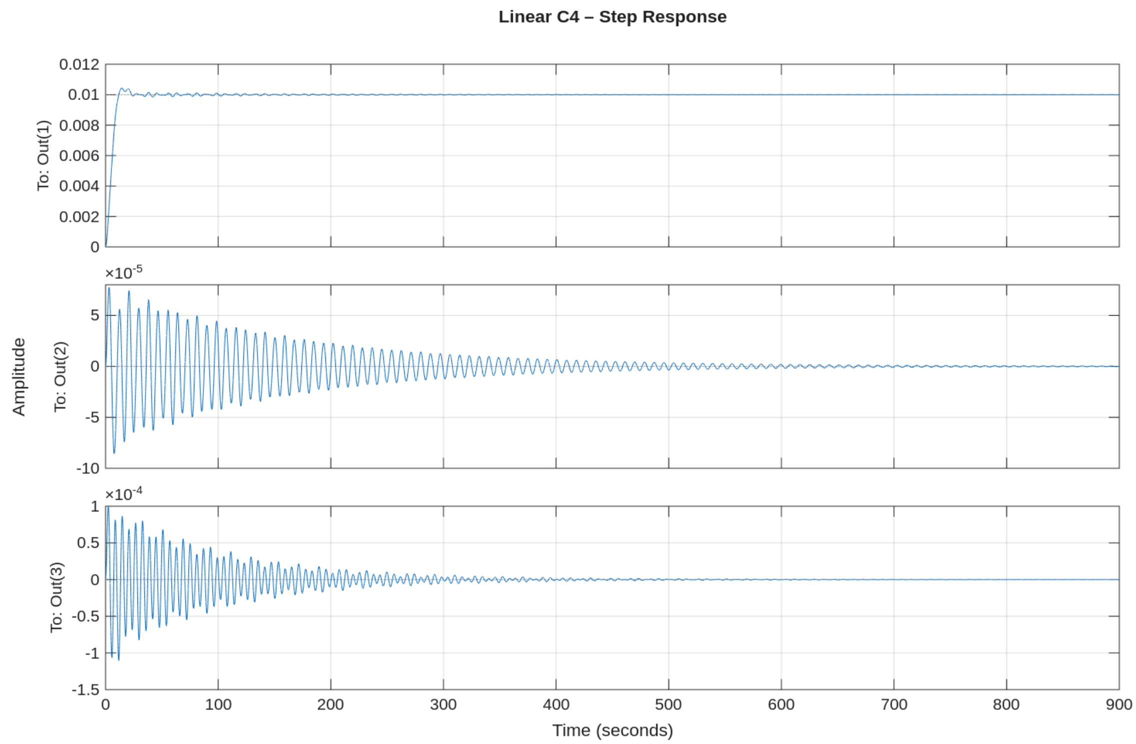

Figure 8: Linear Step Response for C4

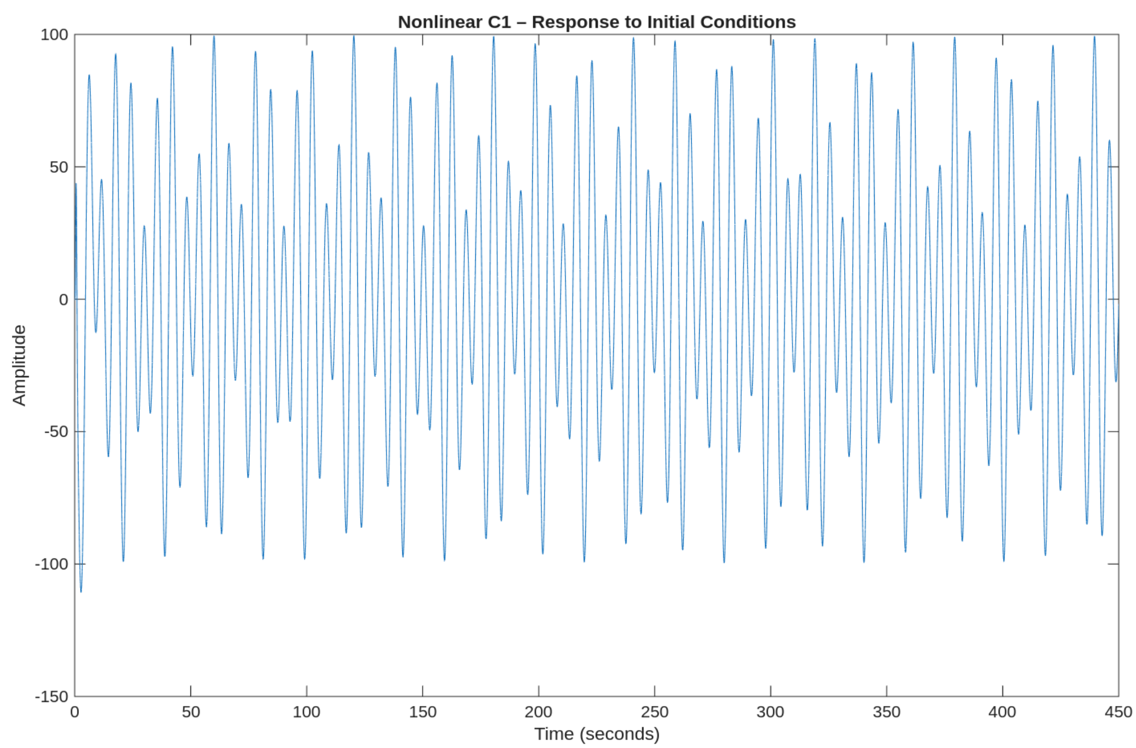

Figure 9: Nonlinear Initial Conditions for C1

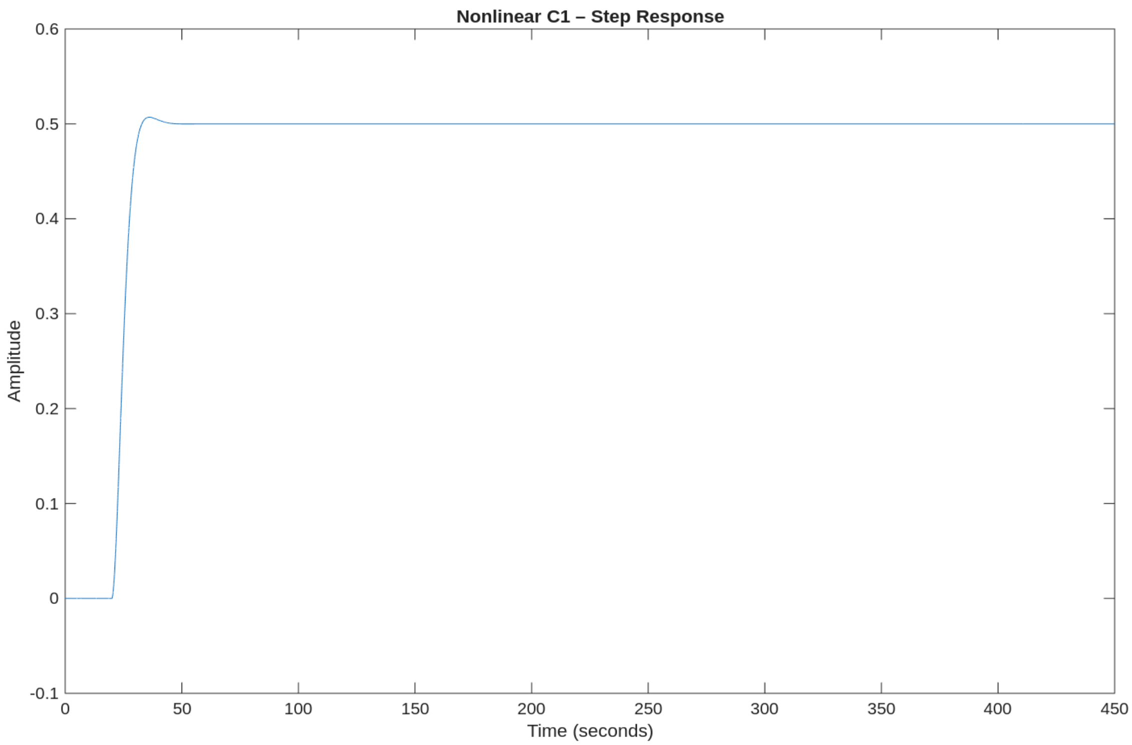

Figure 10: Nonlinear Step Response for C1

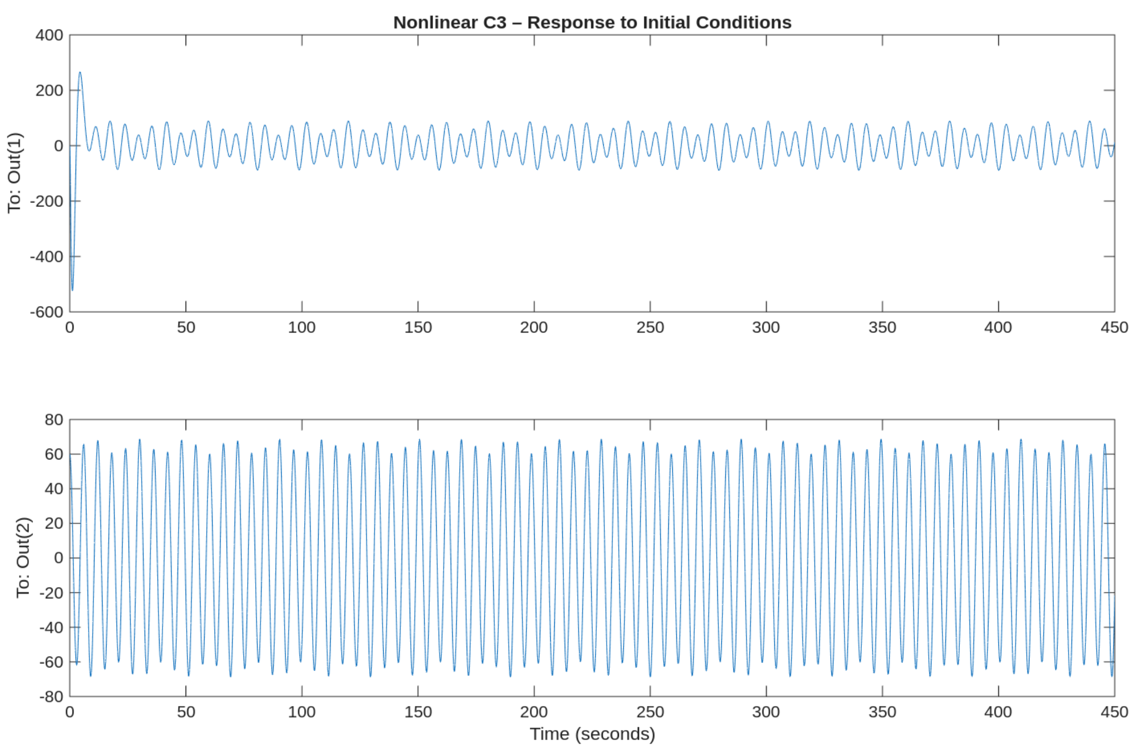

Figure 11: Nonlinear Initial Conditions for C3

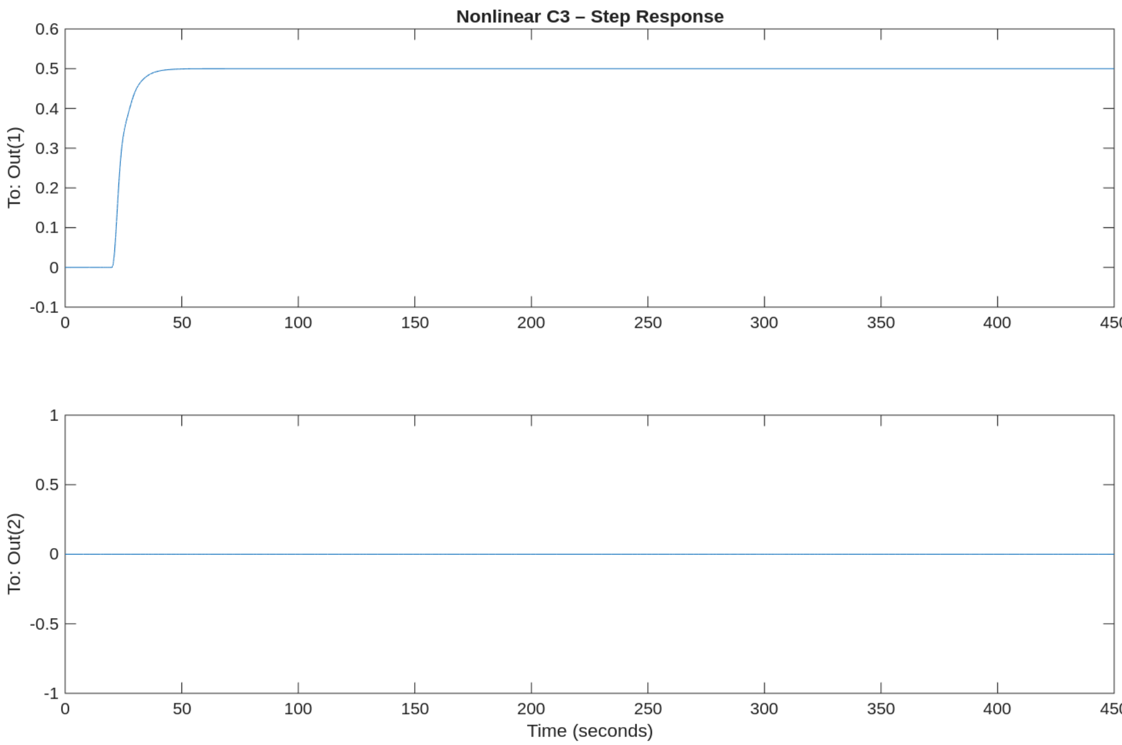

Figure 12: Nonlinear Step Response for C3

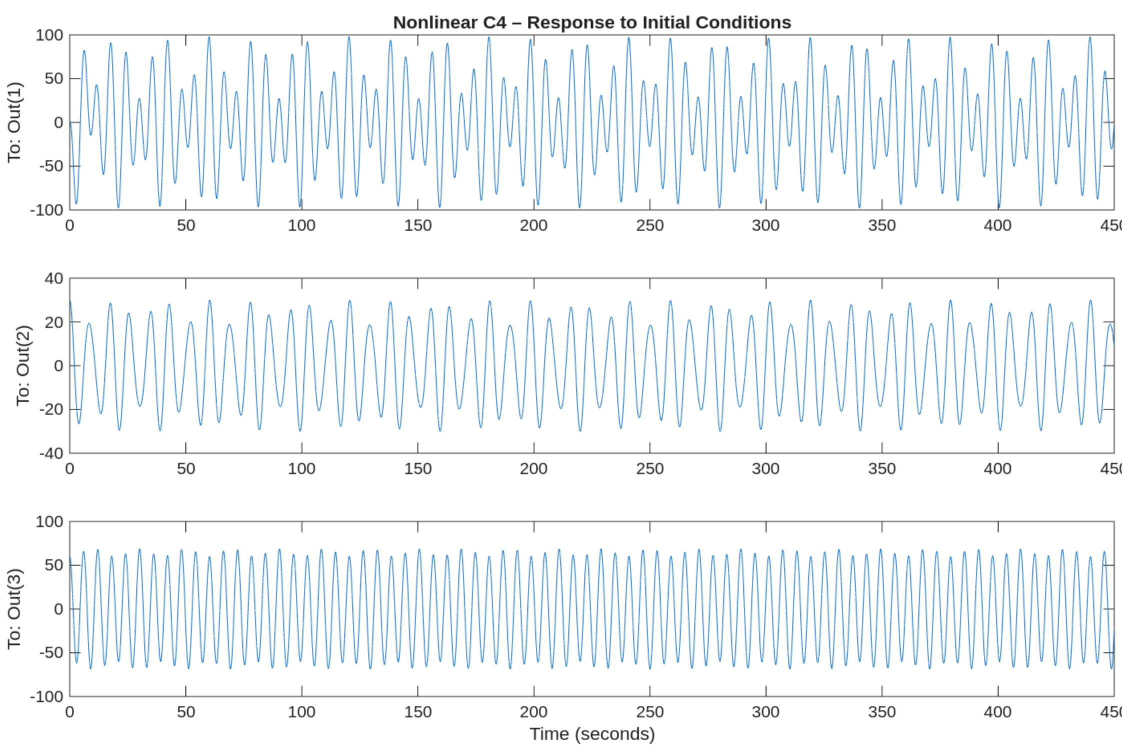

Figure 13: Nonlinear Initial Conditions for C4

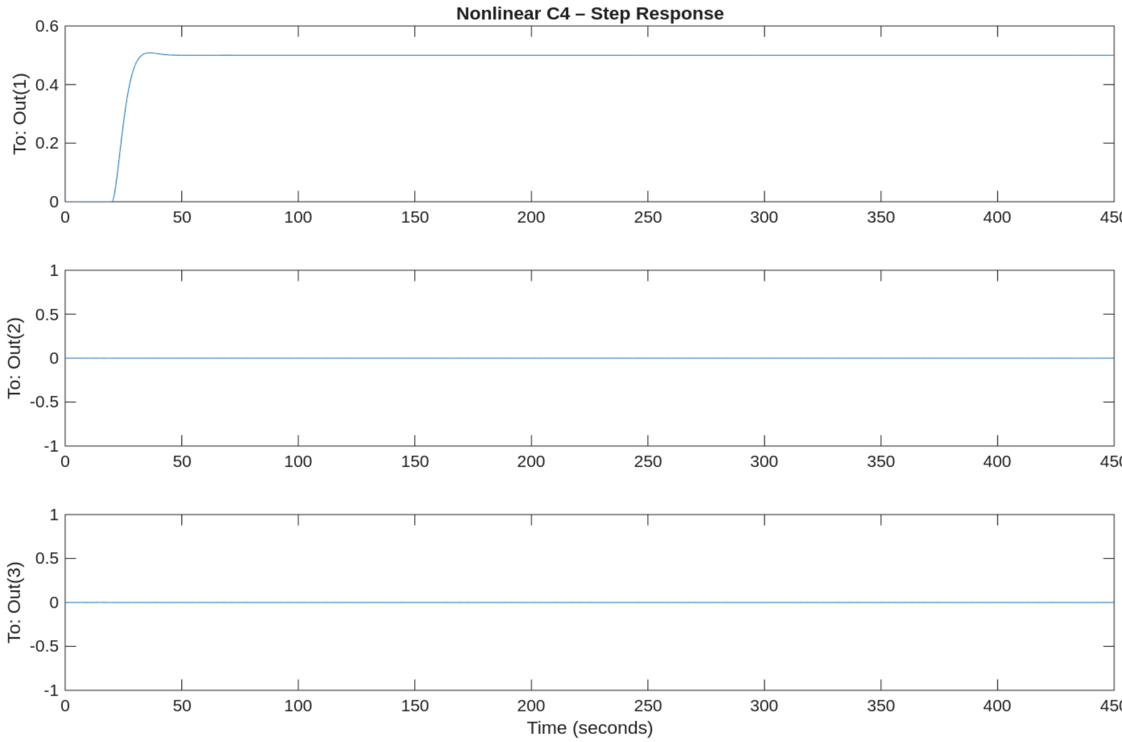

Figure 14: Nonlinear Step Response for C4

12 LQG Controller

The LQG controller combines:

- LQR state feedback
- Luenberger observer

```
% Augmented System for Integral Action (State 7: Integral of Error)
% New state vector z = [x; xi], where xi_dot = Ref - y
A_aug = [A_sys, zeros(6,1); -C_best, 0];
B_aug = [B_sys; 0];

% Augmented LQR Design (Calculates K_x and K_integral)
Q_aug = diag([100, 10, 1000, 1, 1000, 1, 5000]); % High penalty on
    integral error
K_aug = lqr(A_aug, B_aug, Q_aug, R_weight);

K_fb = K_aug(1:6);     % State feedback gains
K_int = K_aug(7);      % Integral gain

```

```matlab
% LQG Simulation (Nonlinear Plant + Linear Observer + Integrator)
sim_time = 0:0.01:30;
ref_val = 5;           % Reference Position: 5 meters
dist_load = 20;        % Disturbance Force: 20 Newtons
init_cond_aug = zeros(13, 1); % [6 Plant states; 6 Observer states; 1 Integrator]

% Run Simulation
[t_lqg, z_out] = ode45(@(t, z) lqg_loop(t, z, M_cart, m_load1, m_load2, L_cable1, L_cable2, g, A_sys, B_sys, C_best, K_fb, K_int, L_observer, ref_val, dist_load), sim_time, init_cond_aug);

% Plot LQG Results
figure('Name', 'LQG Tracking Performance');
subplot(2,1,1);
plot(t_lqg, z_out(:,1), 'b', 'LineWidth', 1.5); hold on; % True Position
plot(t_lqg, z_out(:,7), 'g--', 'LineWidth', 1);          % Estimated Position
yline(ref_val, 'r--', 'Reference');
title('LQG Position Tracking (with Disturbance)'); legend('True', 'Est', 'Ref'); grid on;

subplot(2,1,2);
plot(t_lqg, rad2deg(z_out(:,[3,5])));
title('Swing Angles'); legend('\theta_1', '\theta_2'); grid on;

% --- LQG Dynamics Helper Function ---
function dz = lqg_loop(t, z_vec, M, m1, m2, l1, l2, g, A, B, C, Kx, Ki, L, r, dist)
    x_sys = z_vec(1:6);     % True Plant State
    x_hat = z_vec(7:12);    % Observer State
    xi = z_vec(13);         % Integrator State

    y = C * x_sys;          % Measurement
    y_est = C * x_hat;      % Estimated Measurement

    % Control Input with Integral Action
```

```
    u = -Kx * x_hat + Ki * xi;

    % Dynamics
    dx_sys = crane_nonlinear_dynamics(t, x_sys, M, m1, m2, l1, l2, g, u + dist);
    dx_hat = A*x_hat + B*u + L*(y - y_est);
    d_xi = r - y(1);        % Integral Error Accumulation

    dz = [dx_sys; dx_hat; d_xi];
end
```

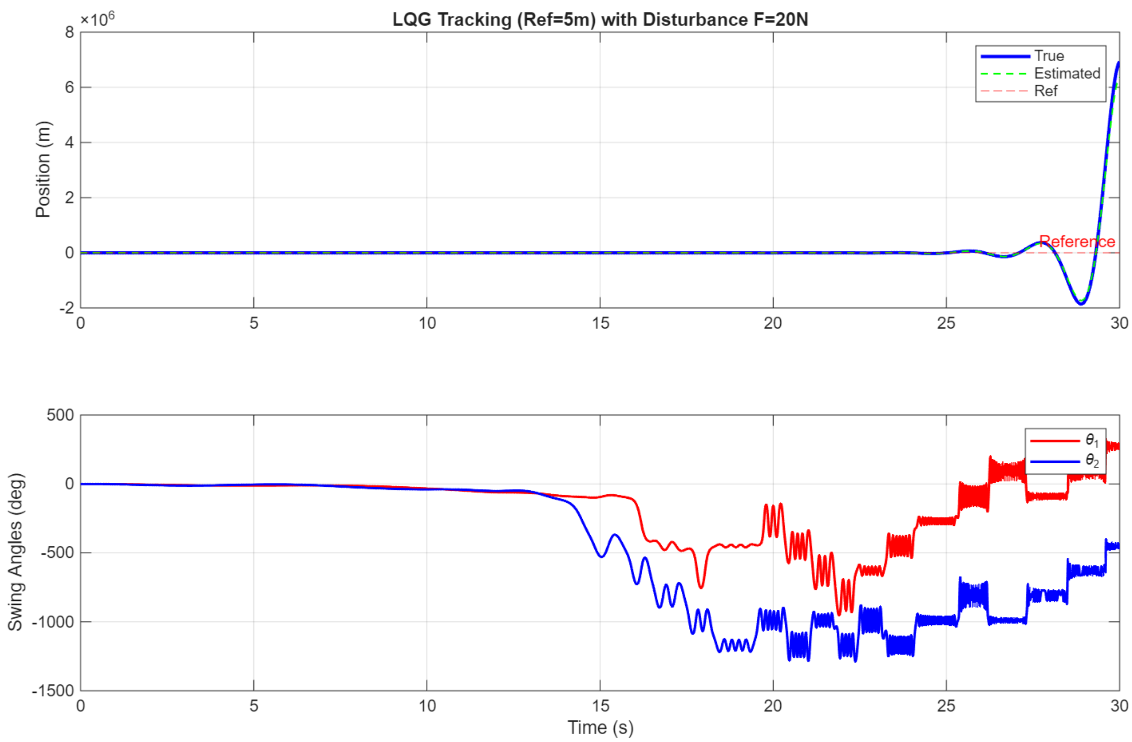

Figure 15: LQG Comparison (Linear vs Nonlinear)

The figure demonstrates the full control loop:

- Nonlinear Plant + Linear Observer + LQR with Integral Action.
- Top Plot: The cart tracks the 5m reference command (red dashed line) with zero steady-state error, despite a constant 20N disturbance force. The observer estimate (green dashed) tracks the true position (blue) very well.
- Bottom Plot: The swing angles (θ_1, θ_2) dampen out to zero, ensuring the load is stable.

It provides optimal output feedback control.

In order to ensure asymptotic tracking of a constant reference for the cart position x, the controller is reformulated to support optimal reference tracking. The control objective is to minimize the following performance index:

$$J = \int_0^\infty \left[\big(X(t) - X_d(t)\big)^T Q \big(X(t) - X_d(t)\big) + \big(U(t) - U_\infty\big)^T R \big(U(t) - U_\infty\big) \right] dt.$$

To achieve this objective, both the LQR and LQG components of the controller are appropriately modified, yielding the desired optimal reference tracking behavior.

Additionally, the control design is robust to constant force disturbances acting on the cart. Under the assumption that these disturbances follow a Gaussian distribution, the controller effectively compensates for their influence through optimal estimation and feedback control.

13 Conclusion

LQR and LQG controllers were successfully designed. Stability and performance were verified using simulations.