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# Resolving photon number states in a superconducting circuit

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Final projects for ELE456 at Princeton

May 11, 2017



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# Outline

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- Resolve photon number states in a circuit QED
- System: superconducting qubit + microwave transmission line
- Strong dispersive regime
- Spectroscopic measurements:  
Qubit's spectral lines different for each photon number state



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# The system: circuit QED + cavity QED

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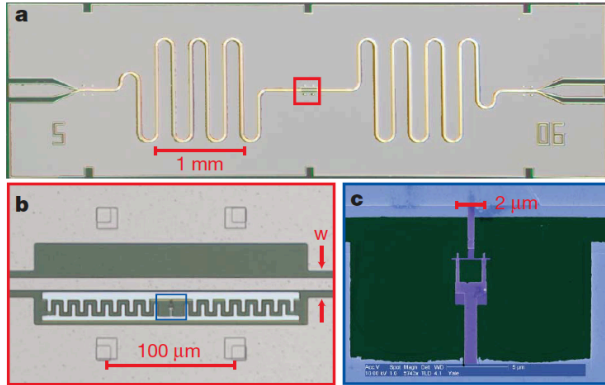


Image from Schuster, D. I., et al. "Resolving photon number states in a superconducting circuit." Nature 445.7127 (2007): 515-518.[3]



# The system: simplified

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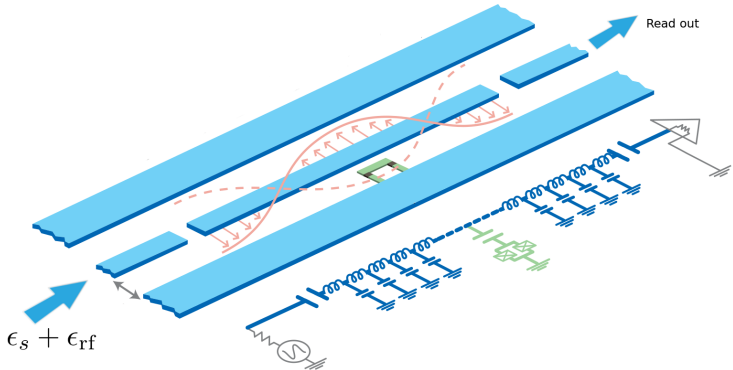


Image from Blais, Alexandre, et al. "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation." Physical Review A 69.6 (2004): 062320.[1]



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# Cavity QED: the Hamiltonian

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## Hamiltonian

$$H = \omega_r \left( a^\dagger a + \frac{1}{2} \right) + \omega_a \frac{\sigma^z}{2} + g \left( a^\dagger \sigma^- + a \sigma^+ \right)$$

- $\omega_r$ : cavity resonance frequency
- $\omega_a$ : qubit transition frequency
- $g$ : strength qubit-photon coupling
- $\Delta = \omega_r - \omega_a$ : detuning between qubit and cavity





# Strong Dispersive Regime

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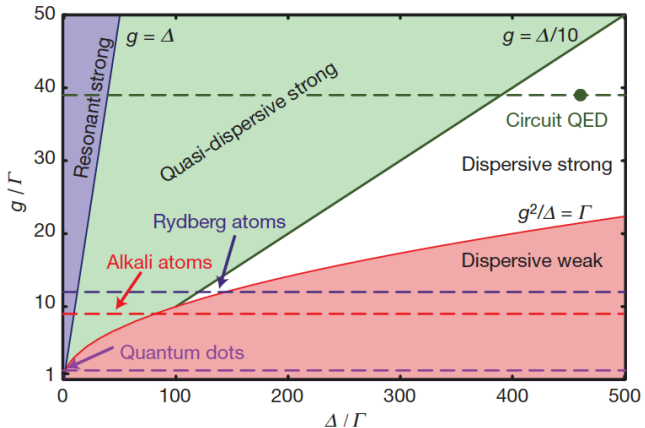


Image from Schuster, D. I., et al. "Resolving photon number states in a superconducting circuit." Nature 445.7127 (2007): 515-518.[3]



# Strong dispersive Regime: Diagonalization

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- Transformation:

$$U = \exp \left( \frac{g}{\Delta} \left( a \sigma^+ - a^\dagger \sigma^- \right) \right)$$

- Hamiltonian to first order in  $\frac{g}{\Delta}$  (dispersive regime):

$$\begin{aligned} H_0 &= U H U^\dagger \\ &\simeq \omega_r \left( a^\dagger a + \frac{1}{2} \right) + \omega_a \frac{\sigma^z}{2} + \chi \left( a^\dagger a + \frac{1}{2} \right) \frac{\sigma^z}{2} \end{aligned}$$

where  $\chi = g/\Delta^2$



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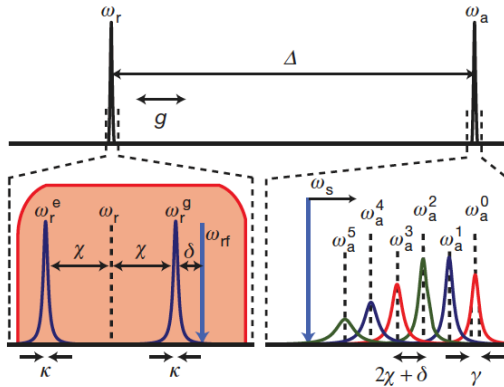


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# Driving terms

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- To conduct a measurement we first drive the cavity:

$$H_{\text{rf}} = \epsilon_{\text{rf}} \left( a^\dagger e^{-i\omega_{\text{rf}}t} + a e^{i\omega_{\text{rf}}t} \right)$$

with  $\omega_{\text{rf}}$  near  $\omega_r$

- The frequency shift of the qubit measured with a sweeping signal

$$H_s = \epsilon_s \left( a^\dagger e^{-i\omega_s t} + a e^{i\omega_s t} \right)$$

with  $\omega_s$  near  $\omega_a$

- Note that relative strength of  $\epsilon_s$  is not mentioned. We treat it as a perturbation.



# Rotating frame and Rotating wave approximation

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- Applying the transformation

$$U = \exp \left[ \frac{g}{\Delta} (a\sigma^+ - a^\dagger\sigma^-) \right]$$

- And moving to the rotating frame:

$$U_I = \exp \left[ it \left( \omega_{\text{rf}} a^\dagger a + \omega_s \sigma^z / 2 \right) \right]$$

Under rotating frame,  $H_{\text{rf}}$  and  $H_s$  are (with RWA):

$$H_{\text{rf}} = \epsilon_{\text{rf}} (a^\dagger + a)$$

$$H_s = \left( \frac{g}{\Delta} \right) \epsilon_s (\sigma^+ + \sigma^-)$$



# Final Hamiltonian and collapse operators

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- Full Hamiltonian:

$$H = \omega_r \left( a^\dagger a + \frac{1}{2} \right) + \omega_a \frac{\sigma^z}{2} + \chi \left( a^\dagger a + \frac{1}{2} \right) \frac{\sigma^z}{2} \\ - \left( \omega_{rf} a^\dagger a + \omega_s \frac{\sigma^z}{2} \right) + \epsilon_{rf} (a^\dagger + a) + \epsilon_s \frac{g}{\Delta} (\sigma^+ + \sigma^-)$$

- Collapse operator:

- Collapse operators cavity:  $\sqrt{\kappa(1+n_{th})}a$ ,  $\sqrt{\kappa n_{th}}a^\dagger$
- Collapse operator qubit:  $\sqrt{\gamma}\sigma^-$
- Dephasing:  $\sqrt{\gamma_\phi}\sigma^z$



- In the experiment, the transmitted amplitude at frequency  $\omega_{\text{rf}}$  is the main observable under steady state.

## Steady state

$$\dot{\rho}_s = 0 = -i[H, \rho_s] + \sum_n \left( 2C_n \rho_s C_n^\dagger - \{\rho_s, C_n^\dagger C_n\} \right)$$

- What they really measure is the expectation of the electrical field  $E \propto \langle a + a^\dagger \rangle$  [2] on a given frequency

$$E \propto \langle a + a^\dagger \rangle = \text{Tr}[\rho_s(a + a^\dagger)]$$



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# Property of the cavity: Analytical

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- Without the qubit, the cavity state is equivalently a damped harmonic oscillator with driving

$$H = \delta a^\dagger a + \epsilon(a + a^\dagger)$$

Collapse operators:  $\sqrt{\kappa(n_{\text{th}} + 1)}a$  and  $\sqrt{\kappa n_{\text{th}}}a^\dagger$

- When it's off resonant, its steady state is not but approximately a coherent state
- Analytically the photon number expectation value is

$$\bar{n} = \frac{\epsilon^2}{\delta^2 + \kappa^2/4} + n_{\text{th}}$$



# Property of the cavity: Numerical

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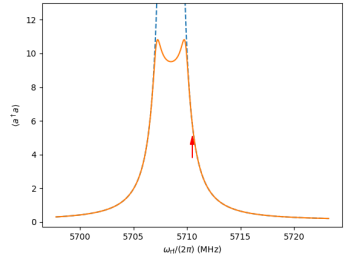
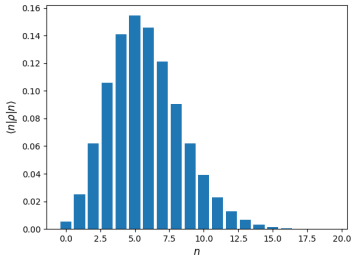
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- Numerically, a truncate on Fock space is needed
- To check the validity of the truncate, we plot the photon distribution and frequency response of the cavity.





# Direct spectroscopic observation of quantized cavity photon number

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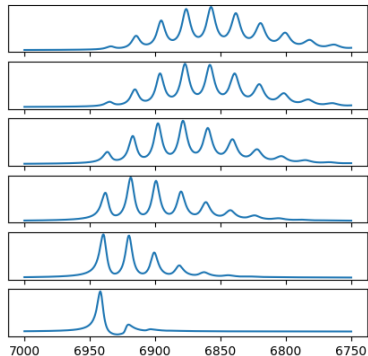
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For a fixed driving  $\epsilon_{\text{rf}}$ ,  
plot the reduction  
 $V_0 - \langle a^\dagger + a \rangle_{ss}$  v.s.  $\omega_s$ .

$\epsilon_{\text{rf}}$  is labeled by  $\bar{n}$  with  
relationship:

$$\bar{n} = n_{\text{th}} + \frac{\epsilon_{\text{rf}}^2}{\delta^2 + \kappa^2/4}$$





# Direct spectroscopic observation of quantized cavity photon number: compare

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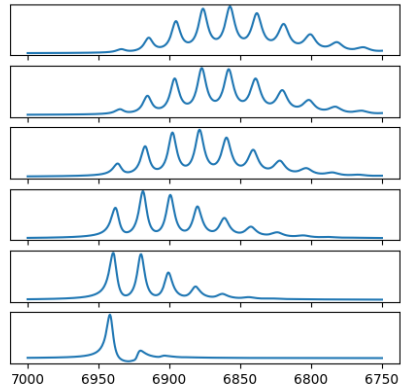
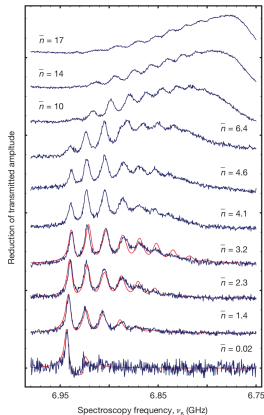
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- Fits well with small  $\bar{n}$ , but other noise becomes significant for larger  $\bar{n}$



# Strengthen?

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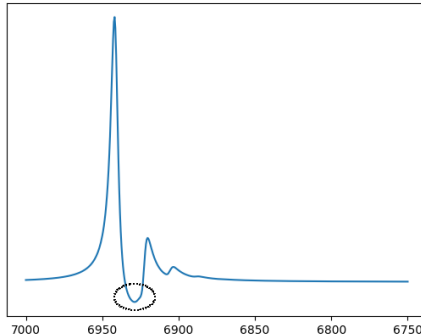
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For small signal, there's a range where the transmitted amplitude is increased. We'll explain it later.



# Thermal Drive

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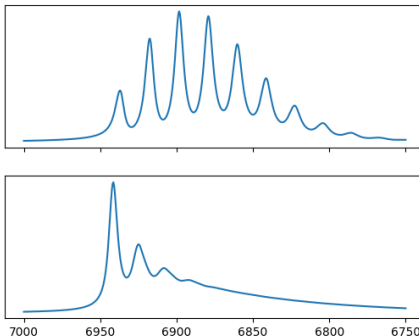
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Conclusion

- Thermal Drive is equivalent to setting  $n_{\text{th}}$  in collapse operator to the driving average, with small  $\epsilon_{\text{rf}}$  to show the phase lock-in at the given frequency.





# Thermal Drive: compare

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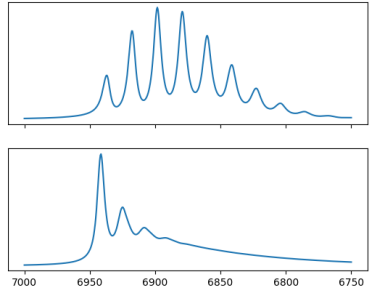
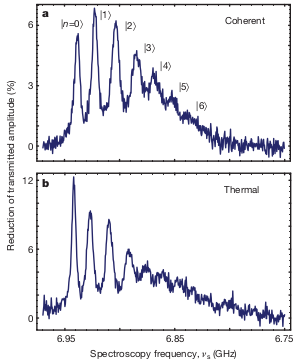
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- Note that there's no thermal drive theory fitting. Our results tracks fewer peaks, but this depends on how they do the measurement, which is not mentioned in the paper.



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# Discussion: The picture of what happens

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- The peaks shows discreteness in the photon state in the cavity.
- Exciting the qubit making the cavity off-resonance, which results in the reduction?



# Discussion: The picture of what happens

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- The peaks shows discreteness in the photon state in the cavity.
- Exciting the qubit making the cavity off-resonance, which results in the reduction? **NOT TRUE**



# Discussion: The picture of what happens

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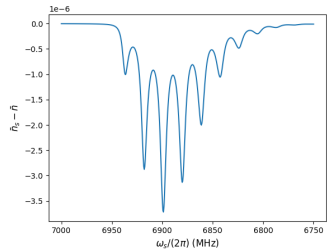
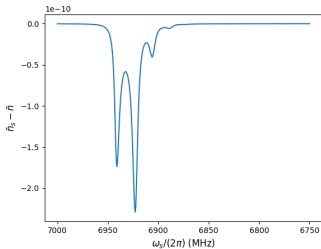
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- The peaks shows discreteness in the photon state in the cavity.
- Exciting the qubit making the cavity off-resonance, which results in the reduction? **NOT TRUE**
- Expected photon number increases at the peaks!





# What happens

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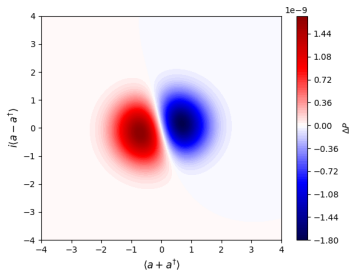
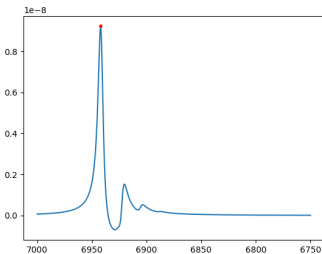
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- Excitation of the qubit is not the dominant effect, but the polarization of the qubit, which twists the cavity photon state.
- This can be shown from the difference of the Wigner function (quasiprobability distribution on phase diagram) with/without the signal field.





# What happens

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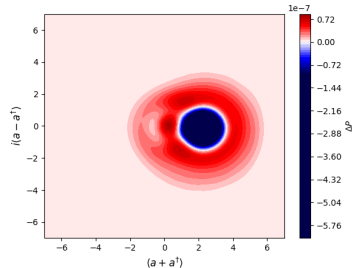
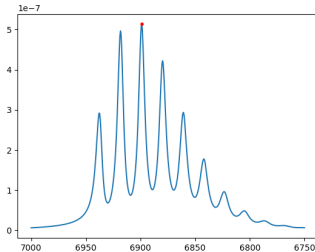
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# Conclusion

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- Existence of photons in the cavity shifts the qubit frequency, which can be read out by applying the sweeping signal to see the qubit spectrum
- The way the qubit state affects the cavity state is trick: more like polarization of qubit affect the wave function



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- Existence of photons in the cavity shifts the qubit frequency, which can be read out by applying the sweeping signal to see the qubit spectrum
- The way the qubit state affects the cavity state is trick: more like polarization of qubit affect the wave function
- “Approximately” the peak height can be interpreted as the photon number distribution: “Resolving” photon number
- Potential application of quantum nondemolition measurement (QND)





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- ▶ Alexandre Blais, Ren-Shou Huang, Andreas Wallraff, Steven M Girvin, and R Jun Schoelkopf.

Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation.

*Physical Review A*, 69(6):062320, 2004.

- ▶ David Isaac Schuster.

*Circuit quantum electrodynamics.*

Yale University, 2007.

- ▶ DI Schuster, AA Houck, JA Schreier, A Wallraff, JM Gambetta, A Blais, L Frunzio, J Majer, B Johnson, MH Devoret, et al.

Resolving photon number states in a superconducting circuit.

*Nature*, 445(7127):515–518, 2007.

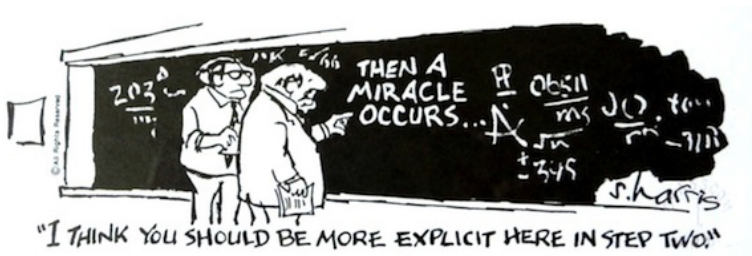


# The End...

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## Conclusion

Thank you for listening!



## Q & A



# Josephson junction and superconducting circuit

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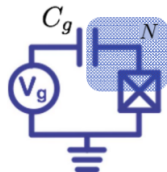
- The Hamiltonian

$$H = E_c(N - N_g)^2 - E_J \cos \delta$$

- Commutation relationship:  $[\delta, N] = i$ , this means  $e^{\pm i\delta} |n\rangle = |n \pm 1\rangle$
- Approximately two-level system:  $0 \leq N_g \leq 1$ ,  $N = 0, 1$ :

$$H = -E_c(1 - 2N_g)\sigma^z - \frac{1}{2}E_J\sigma^x$$

- With coupling,  $N_g \rightarrow N_g + CV_0(a + a^\dagger)/2e$
- At degeneracy point ( $N_g = 1/2$ ) and use eigen-basis, we can have JC model up to some constants.





# Energy levels

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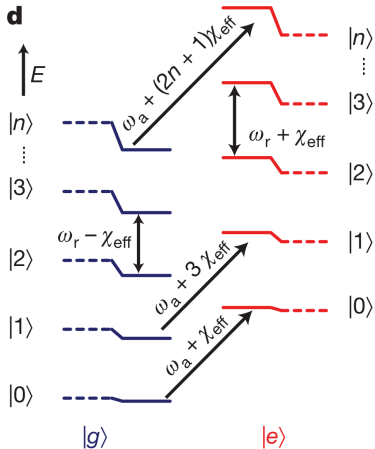


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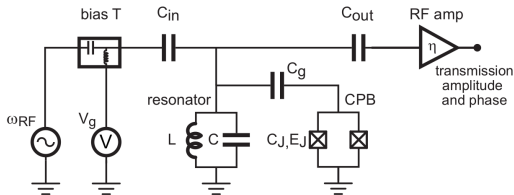


# Measurement

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In the experiment, the transmitted amplitude at frequency  $\omega_{\text{rf}}$  is the main observable. The exact way to measure can be found in Schuster's thesis [2]:



- What we really measure is the expectation of the voltage, or electrical field  $E \propto \langle a + a^\dagger \rangle$



# Wigner function (Wigner quasiprobability distribution)

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- Wigner function is an analogue of classical probability distribution on phase space

Definition: Wigner function

$$\begin{aligned} P(x, p) &\equiv \frac{1}{(2\pi\hbar)^n} \int d^n y \, \psi(x - y/2) \psi^*(x + y/2) e^{ip \cdot y/\hbar} \\ &= \frac{1}{(2\pi\hbar)^n} \int d^n y \, \langle x - y/2 | \rho | x + y/2 \rangle e^{ip \cdot y/\hbar} \end{aligned}$$

- Marginals:

$$\begin{aligned} \int d^n p \, P(x, p) &= \langle x | \rho | x \rangle \\ \int d^n x \, P(x, p) &= \langle p | \rho | p \rangle \end{aligned}$$



# Wigner function: properties

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- Inner product  $\rightarrow$  overlap:

$$|\langle\psi|\varphi\rangle|^2 = 2\pi\hbar \int d^n x d^n p P_\psi(x, p) P_\varphi(x, p)$$

- Operator Wigner transformation and expectation values:

$$g(x, p) \equiv \int d^n y \langle x - y/2 | G | x + y/2 \rangle e^{ip \cdot y/\hbar}$$

$$\text{Tr}[\rho G] = \int d^n x d^n p P(x, p) g(x, p)$$

- Cauchy inequality for pure state

$$-\frac{2}{\hbar} \leq P(x, p) \leq \frac{2}{\hbar}$$