How Euler found the eccentricity of planetary orbits

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Abstract

Euler discovered an interesting method for use by astronomers who need to calculate the eccentricity of a celestial body in an elliptical orbit. We describe the mathematics behind Euler's method and show how it can be used by astronomical observers.

Key Words: Eccentricity of elliptical orbits, Euler, Celestial Mechanics

Classifications: 01A50, 51P05, 70F05, 70F15, 70M20

1. Introduction

` In his paper [1], Euler gave astronomers a convenient method for finding the eccentricity of the orbit of a planet or comet from available data on the position of the object as viewed from the sun. It is the purpose of this paper to show in detail Euler's method in a form that is simple for a modern audience. We felt it best to modify Euler's notion for the various angles and parameters that occur. For example, Euler uses n rather than e for the eccentricity of the ellipse.

Euler begins by discussing the fact that planets observed from the earth exhibit a very irregular motion. In general, they move from west to east along the ecliptic. At times however, the motion slows to a stop and the planet even appears to reverse direction and move from east to west. We call this retrograde motion. After some time the planet stops again and resumes its west to east journey.

However, if we observe the planet from the stand point of an observer on the sun, this retrograde motion will not occur, and only a west to east path of the planet is seen.

From the sun, (point O in Figure 1) the planet (point P) is seen to move on an elliptical orbit with the sun at one focus. When the planet is farthest from the sun, we say it is at the “aphelion” and at the perihelion when it is closest. The time for the planet to move from aphelion to perihelion and back is called the period.

The planet’s speed is slowest at the aphelion and fastest at the perihelion. The planet obeys Kepler’s second law. *The radial line from the sun to the planet sweeps out equal areas in equal times.*

The more eccentric the orbit is, the greater is this variation in speed. If the orbit were a circle, the speed would always be constant and equal angles would be swept out in equal times.

We imagine a fictitious companion planet (point X in Figure 1) that circles the sun with the same period as our planet, but with uniform motion. Further, we assume that both the real and the fictitious planet reach the aphelion and perihelion points at the same time. As Euler says:

“After these two planets have passed by the aphelion, the false planet will appear to go faster than the true and the real planet will imperceptibly increase its speed until it will have caught the false one at the perihelion. Then it will pass its partner in speed, and will leave it behind until they rejoin again at the aphelion."

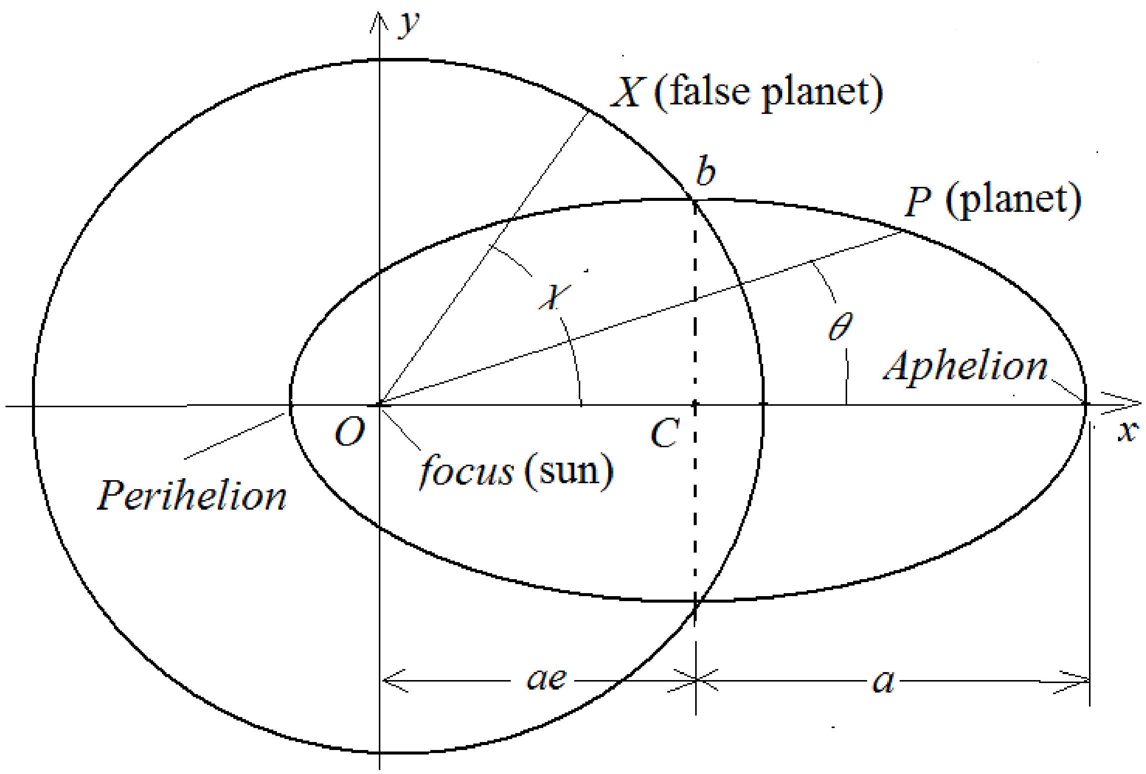

Figure 1: The planet *P* as observed from the sun at *O*,

2. The mean anomaly χ, the true anomaly θ and the "equation of the center"

Astronomers call the angle χ made by the fictitious planet *X* the *mean anomaly*. The angle made by the true planet *P* is *z* and is called the *true anomaly*. The difference of these two angles is $\chi - \theta$ and is called by astronomer's the "equation of the center". (Calling the "quantity " $\chi - \theta$ an "equation" is technically incorrect, however this expression was used by Kepler, and has been retained by astronomers.) We note that

$\chi - \theta$ is zero at the aphelion and gradually increases until it reaches a maximum near *b*, then it decreases to zero again at the perihelion.

We will try to find the maximum of $\chi - \theta$. This maximum value must be a function of the eccentricity of the ellipse *n*. Euler notes that “and inversely, we will have to determine the eccentricity by the biggest equation.” This means that we will observe the maximum of $\chi - \theta$, and from this value, determine the eccentricity of the orbit.

Euler notes that this eccentricity equals the distance between the two foci of the ellipse divided by the length of the major axis. In Figure 2 we see that this is $\frac{2ae}{2a} = e$.

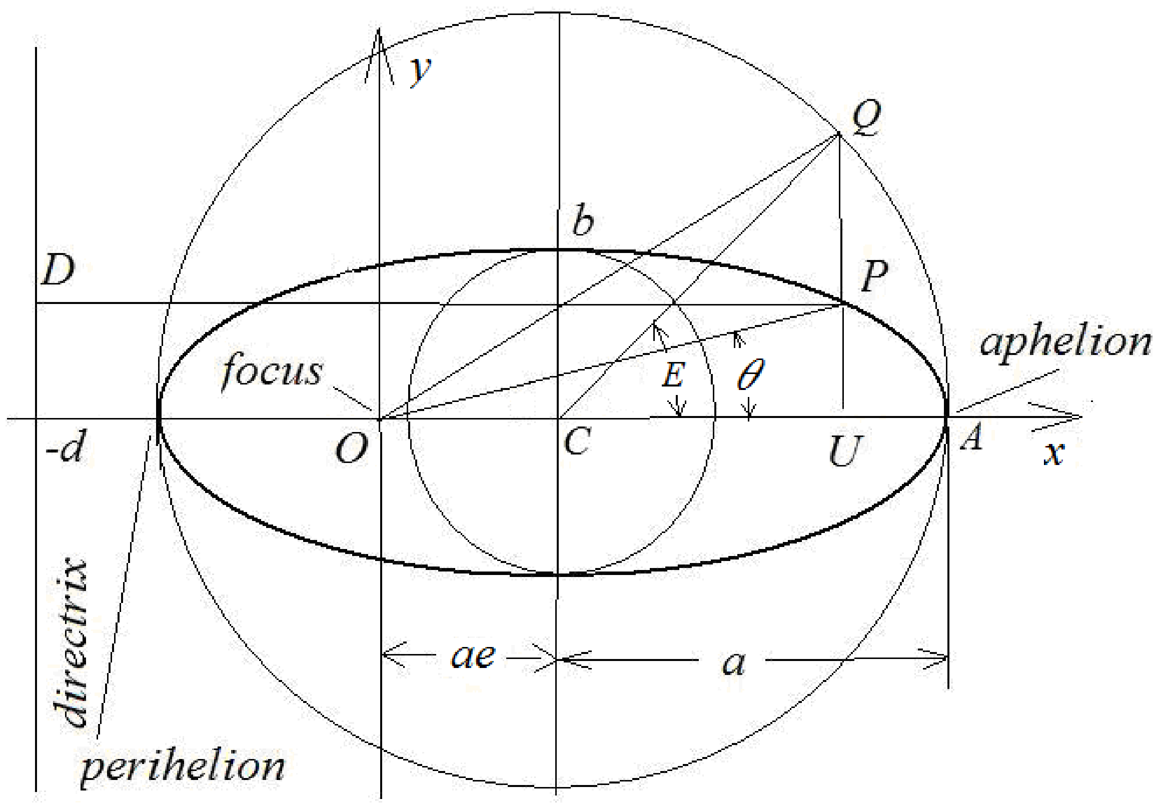

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Figure 2: Features of the ellipse used in E105

When $0 \le e < 1$ the orbit is an ellipse, when $e = 1$ it is a parabola, and when $1 < e$ the orbit is a hyperbola. The distance from the sun to the aphelion is $a + ae$ and the distance from the sun to the perihelion is $a - ae$. The length of the semi-minor axis is $a\sqrt{1-e^2}$. See Appendix I for derivations of these features of the ellipse.

3. The eccentric anomaly *E*, equations of the ellipse and Kepler's equation

In this section we review elementary properties of the ellipse as they relate to planetary orbits. We begin by introducing the "eccentric anomaly *E*" which is shown in Figure 2. This angle E has the property that the equation of the ellipse traced by the planet at *P* can be written parametrically as $x = ae + a\cos\theta$ and $y = b\sin\theta$.
In Figure 2 we see the ellipse with focus at the origin *O* of the *xy*-plane. We imagine that the sun is at point *O* and the planet is at point *P*. *We* will use the variables:

e = eccentricity of the ellipse

θ = true anomaly, (usually the polar angle θ)

E = eccentric anomaly, (often E is used)

a = semi-major axis

b = semi-minor axis.

r = OP (usual polar radius)

t = time for planet to move from A to P

T = period of the planet

$\omega = \dfrac{2\pi}{T}$ = mean angular velocity

$\chi = \omega t$ = mean anomaly

We recall the definition of an ellipse. Let O be a fixed point (focus) and $x = -d$ be a fixed line (directrix). The locus of all points P such that the ratio of the distance from the focus to the distance from the directrix is a constant e (eccentricity) is called a conic section. Thus we have

$$\frac{OP}{DP} = e \quad \text{(a constant).}$$

If $0 < e < 1$ the curve is an ellipse. If $e = 1$ the curve is a parabola. If $e > 1$ the curve is a hyperbola.

Since $DP = d + r\cos\theta$ we have

$$\frac{OP}{DP} = \frac{r}{d + r\cos\theta} = e \;,$$

and solving for r we get

$$r = \frac{de}{1 - e\cos\theta}. \tag{3.1}$$

At the perihelion we have from the definition $\frac{r}{d-r} = e$, and solving for r we get

$$r_{perihilion} = \frac{ed}{1+e}. \tag{3.2}$$

At the aphelion we have $\frac{r}{d+r} = n$ so we get

$$r_{aphelion} = \frac{ed}{1-e}. \tag{3.3}$$

Since the major axis of our ellipse is

$$2a = r_{perihilion} + r_{aphelion} = \frac{ed}{1+e} + \frac{ed}{1-e} = \frac{2ed}{1-e^2},$$

Thus it follows that

$$ed = a(1-e^2),$$

and so our equation for the ellipse (3.1) becomes

(3.4) $$r = \frac{a(1-e^2)}{1-e\cos\theta}.$$

Returning to (3.2) we now have

$$r_{perihilion} = \frac{ed}{1+e} = \frac{a(1-e^2)}{1+e} = a - ae.$$

This last relation tells us that the distance from the focus to the center of the ellipse is ae as shown on the Figure 2.

From Figure 2 we can now calculate the distance *OU*.

(3.5) $$OU = r\cos\theta = ae + a\cos E.$$

We can rewrite (3.4) as

$$r - er\cos\theta = a(1-e^2),$$

and using (3.5) to eliminate $r\cos\theta$ we get

$$r - e(ae + a\cos E) = a(1-e^2)$$

which simplifies to

(3.6) $$r = a(1+e\cos E).$$

From this we infer that when $E = \pi/2$, then $r = a$. It follows then from Figure 2 that we have a right triangle *OCb* with legs *ae*, and *b* and hypotenuse *a*. It follows that the semi minor axis is given by

(3.7) $$b = a\sqrt{1-e^2}.$$

Notice also that

$$\cos\theta = \frac{OU}{OP} = \frac{ae + a\cos E}{r},$$

and replacing r by (3.6) we get

(3.8) $$\cos\theta = \frac{e + \cos E}{1 + e\cos E}.$$

Also

$$\sin\theta = \frac{PU}{OP} = \frac{b\sin E}{r},$$

and using (3.6) and (3.7) this becomes

(3.9) $$\sin\theta = \frac{\sqrt{1-e^2}\,\sin E}{1 + e\cos E}$$

From (3.8) and (3.9) we get

(3.10) $$\tan\theta = \frac{\sqrt{1-e^2}\,\sin E}{e + \cos E}$$

4. Using calculus find when the maximum of $\chi - \theta$ occurs in terms of *e* and *E*.

We will now find the maximum of " the equation of the center" $\chi - \theta$**.** Our analysis closely follows Euler's reasoning in [1].

From Kepler's equation $\chi = E + e\sin E$ we have

(4.1) $$d\chi = (1 + e\cos E)dE.$$

Differentiating (3.8) and simplifying we get

(4.2) $$\sin\theta\, d\theta = \frac{(1-e^2)\sin E}{(1 + e\cos E)^2}\, dE.$$

But by (3.9) $\sin\theta = \dfrac{\sqrt{1-e^2}\sin E}{1+e\cos E}$, so (4.2) becomes

(4.3) $$d\theta = \frac{\sqrt{1-e^2}}{1+e\cos E}dE\,.$$

Since at the maximum of $\chi-\theta$, we have $d\chi = d\theta$, then from (4.1) and (4.3) we have

$1+e\cos E = \dfrac{\sqrt{1-e^2}}{1+e\cos E}$. Thus we have

(4.3a) $$1+e\cos E = \sqrt[4]{1-e^2}$$

and therefore

(4.4) $$\cos E = \frac{\sqrt[4]{1-e^2}-1}{e}\,.$$

Note that this last result gives the exact value of E for which $\chi-\theta$ is a maximum. Note that for small eccentricity e, $\cos E \approx -\dfrac{e}{4}$, and so $E \approx \dfrac{\pi}{2}$. Now we let λ be that small change in the angle by writing $E = \dfrac{\pi}{2}+\lambda$ and thus using $\sin\lambda = \sin\left(E-\pi/2)\right) = -\cos E$ and (4.4) we get

(4.5) $$\sin\lambda = \frac{1-\sqrt[4]{1-e^2}}{e}\,.$$

From Kepler's equation $\chi = E - e\sin E$, and $E = \dfrac{\pi}{2}+\lambda$ we have

(4.6) $$\chi = \frac{\pi}{2}+\lambda+e\cos\lambda\,,$$ with (4.5). Define μ by

(4.7) $$\theta = \frac{\pi}{2} - \mu .$$

From (3.8) $\cos\theta = \dfrac{e+\cos E}{1+e\cos E}$ and (4.7) we get

$$\sin\mu = \frac{e-\sin\lambda}{1-e\sin\lambda} .$$

Replacing $\sin\lambda$ with (4.5) and simplifying we get

(4.8) $$\sin\mu = \frac{1}{e} - \frac{1}{e}\sqrt[4]{\left(1-e^2\right)^3} .$$

From (4.6) and (4.7) we see that the maximum of $\chi - \theta$ is

(4.9) $$\chi - \theta = \lambda + \mu + e\cos\lambda .$$

This completes our derivation of the greatest value of the equation of the center.

5. Euler's Table

Euler ends his paper by giving a table which astronomers can use to find the eccentricity of a planetary orbit when the maximum of $\chi - \theta$ is known. In the first column we see the eccentricity (Euler calls it n) ranging from 0 to 1 in increments of 0.01. In the second column we see the corresponding maximum of $\chi - \theta$ as given by (4.9).

The astronomer can then use interpolation to find the closest value to the eccentricity from the first column corresponding to his observed maximum in the second column. The remaining columns were used by Euler as intermediate calculations to obtain the numbers in column two. Finally only columns one and two are needed for an astronomer to find the eccentricity of an elliptical orbit of a celestial body.

Euler's Table for finding the eccentricity of celestial bodies

Note that Euler uses n for eccentricity rather than our *e*. in the table.

Excentricité. n	Plus grande equation $\lambda + \mu + n\cos\lambda$	Anomalie excentrique $3^s + \lambda$ λ	Anomalie moyenne $3^s+\lambda+n\cos\lambda$ $\lambda+n\cos\lambda$	Log. dist. au Soleil $la+l\sqrt[4]{(1-nn)}$ $l\sqrt[4]{(1-nn)}$	$n\cos\lambda$	μ
0, 00	0, 0, 0	0, 0, 0	0, 0, 0	0,0000000	0, 0, 0	0, 0, 0
0, 01	1, 8,45	0, 8, 36	0, 42, 59	9,9999891	0, 34,23	0,25,47
0, 02	2,17,31	0, 17, 12	1, 25, 58	9,9999565	1, 8,46	0,51,34
0, 03	3,26,17	0, 25, 48	2, 8, 56	9,9999022	1,43, 8	1,17,22
0, 04	4,35, 4	0, 34, 24	2, 51, 54	9,9998261	2,17,30	1,43,10
0, 05	5,43,52	0, 43, 1	3, 34, 53	9,9997282	2,51,52	2, 8,59
0, 06	6,52,41	0, 51, 38	4, 17, 52	9,9996084	3,26,14	2,34,49
0, 07	8, 1,32	1, 0, 16	5, 0, 52	9,9994667	4, 0,36	3, 0,40
0, 08	9,10,26	1, 8, 55	5, 43, 53	9,9993029	4,34,57	3,26,33
0, 09	10,19,22	1, 17, 35	6, 26, 54	9,9991170	5, 9,19	3,52,28
0, 10	11,28,20	1, 26, 16	7, 9, 56	9,9989088	5,43,40	4,18,24
0, 11	12,37,21	1, 34, 59	7, 52, 59	9,9986782	6,18, 0	4,44,22
0, 12	13,46,26	1, 43, 43	8, 36, 3	9,9984252	6,52,20	5,10,23
0, 13	14,55,34	1, 52, 28	9, 19, 8	9,9981494	7,26,40	5,36,26
0, 14	16, 4,46	2, 1, 15	10, 2, 14	9,9978508	8, 0,59	6, 2,32
0, 15	17, 4, 1	2, 10, 3	10, 45, 20	9,9975292	8,35,17	6,28,41
0, 16	18,23,21	2, 18, 53	11, 28, 28	9,9971843	9, 9,35	6,54,53
0, 17	19,32,45	2, 27, 45	12, 11, 37	9,9968160	9,43,52	7,21, 8
0, 18	20,42,15	2, 36, 39	12, 54, 48	9,9964240	10,18, 9	7,47,27
0, 19	21,51,51	2, 45, 36	13, 38, 1	9,9960080	10,52,25	8,13,50
0, 20	23, 1,32	2, 54, 35	14, 21, 15	9,9955678	11,26,40	8,40,17
0, 21	24,11,19	3, 3, 37	15, 4, 31	9,9951031	12, 0,54	9, 6,48
0, 22	25,21,12	3, 12, 41	15, 47, 48	9,9946136	12,35, 7	9,33,24
0, 23	26,31,13	3, 21, 49	16, 31, 8	9,9942135	13, 9,19	10, 0, 5
0, 24	27,41,20	3, 31, 0	17, 14, 30	9,9935588	13,43,30	10,26,50
0, 25	28,51,35	3, 40, 14	17, 57, 54	9,9929928	14,17,40	10,53,41

n	$\lambda+\mu$ $+n\,\mathrm{cof}\,\lambda$	λ	$\lambda+n\,\mathrm{cof}\,\lambda$	$l\sqrt[4]{(1-nn)}$	$n\,\mathrm{cof}\,\lambda$	μ
0, 25	28,51,35	3, 40, 14	17, 57, 54	9,9929928	14,17,40	10,53,41
0, 26	30, 1,57	3, 49, 31	18, 41, 20	9,9924006	14,51,49	11,20,37
0, 27	31,12,28	3, 58, 52	19, 24, 49	9,9917816	15,25,57	11,47,40
0, 28	32,23, 7	4, 8, 16	20, 8, 19	9,9911356	16, 0, 3	12,14,48
0, 29	33,33,57	4, 17, 45	20, 51, 53	9,9904620	16,34, 8	12,42, 4
0, 30	34,44,57	4, 27, 18	21, 35, 30	9,9897603	17, 8,12	13, 9,27
0, 31	35,56, 6	4, 36, 55	22, 19, 9	9,9890301	17,42,14	13,36,56
0, 32	37, 7,24	4, 46, 36	23, 2, 51	9,9882707	18,16,15	14, 4,33
0, 33	38,18,55	4, 56, 22	23, 46, 36	9,9874816	18,50,14	14,32,19
0, 34	39,30,37	5, 6, 13	24, 30, 24	9,9866622	19,24,11	15, 0,12
0, 35	40,42,30	5, 16, 9	25, 14, 16	9,9858118	19,58, 7	15,28,14
0, 36	41,54,35	5, 26, 10	25, 58, 11	9,9849297	20,32, 1	15,56,24
0, 37	43, 6,53	5, 36, 17	26, 42, 10	9,9840153	21, 5,53	16,24,43
0, 38	44,19,25	5, 46, 30	27, 26, 13	9,9830677	21,39,43	16,53,12
0, 39	45,32,12	5, 56, 50	28, 10, 20	9,9820861	22,13,30	17,21,52
0, 40	46,45,13	6, 7, 16	28, 54, 31	9,9810698	22,47,15	17,50,42
0, 41	47,58,28	6, 17, 48	29, 38, 46	6,9800178	23,20,58	18,19,42
0, 42	49,12, 0	6, 28, 28	30, 23, 69	9,9789291	23,54,38	18,48,54
0, 43	50,25,49	6, 39, 15	31, 7, 31	9,9778027	24,28,16	19,18,18
0, 44	51,39,55	6, 50, 10	31, 52, 1	9,9766376	25, 1,51	19,47,54
0, 45	52,54,19	7, 1, 12	32, 36, 35	9,9754327	25,35,22	20,17,43
0, 46	54, 9, 0	7, 12, 23	33, 21, 14	9,9741866	26, 8,51	20,47,45
0, 47	55,24, 2	7, 23, 43	34, 6, 0	9,9728983	26,42,17	21,18, 2
0, 48	56,39,26	7, 35, 13	34, 50, 53	9,9715663	27,15,40	21,48,33
0, 49	57,55,10	7, 46, 52	35, 35, 51	9,9701891	27,48,59	22,19,19
0, 50	59,11,15	7, 58, 40	36, 20, 54	9,9687653	28,22,14	22,50,21

n	$\lambda + \mu + n \cos\lambda$	λ	$\lambda + n \cos\lambda$	$l\sqrt[4]{(1-nn)}$	$n \cos\lambda$	μ
0, 50	59, 11, 15	7, 58, 40	36, 20, 54	9,9687653	28, 22, 14	22, 50, 21
0, 51	60, 27, 44	8, 10, 39	37, 6, 4	9,9672932	28, 55, 25	23, 21, 40
0, 52	61, 44, 36	8, 22, 49	37, 51, 21	9,9657712	29, 28, 32	23, 53, 15
0, 53	63, 1, 56	8, 35, 12	38, 36, 47	9,9641973	30, 1, 35	24, 25, 9
0, 54	64, 19, 41	8, 37, 47	39, 12, 20	9,9625696	30, 34, 33	24, 57, 21
0, 55	65, 37, 52	8, 50, 34	39, 58, 0	9,9608860	31, 7, 26	25, 29, 52
0, 56	66, 56, 30	9, 13, 33	40, 53, 47	9,9591443	31, 40, 14	26, 2, 43
0, 57	68, 15, 42	9, 26, 49	41, 39, 46	9,9573420	32, 12, 57	26, 35, 56
0, 58	69, 35, 25	9, 40, 18	42, 25, 52	9,9554766	32, 45, 34	27, 9, 33
0, 59	70, 55, 43	9, 54, 2	43, 12, 6	9,9535452	33, 18, 4	27, 43, 37
0, 60	72, 16, 32	10, 8, 2	43, 58, 30	9,9515450	33, 50, 28	28, 18, 2
0, 61	73, 37, 58	10, 22, 20	44, 45, 5	9,9494726	34, 22, 45	28, 52, 53
0, 62	75, 0, 4	10, 36, 58	45, 31, 53	9,9473246	34, 54, 55	29, 28, 11
0, 63	76, 22, 51	10, 51, 55	46, 18, 52	9,9450973	35, 25, 57	30, 3, 59
0, 64	77, 46, 18	11, 7, 11	47, 6, 2	9,9427866	35, 58, 51	30, 40, 16
0, 65	79, 10, 28	11, 22, 49	47, 53, 25	9,9403880	36, 30, 36	31, 17, 3
0, 66	80, 35, 30	11, 38, 51	48, 41, 2	9,9378967	37, 2, 11	31, 54, 28
0, 67	82, 1, 18	11, 55, 16	49, 28, 53	9,9353076	37, 33, 37	32, 32, 25
0, 68	83, 27, 53	12, 12, 6	50, 16, 57	9,9326148	38, 4, 51	33, 10, 56
0, 69	84, 55, 28	12, 29, 25	51, 5, 19	9,9298121	38, 35, 54	33, 50, 9
0, 70	86, 24, 2	12, 47, 13	51, 53, 57	9,9268925	39, 6, 44	34, 30, 5
0, 71	87, 53, 37	13, 5, 32	52, 42, 53	9,9238485	39, 37, 21	35, 10, 44
0, 72	89, 24, 21	13, 24, 26	53, 32, 9	9,9206716	40, 7, 43	35, 52, 12
0, 73	90, 56, 15	13, 43, 56	54, 21, 45	9,9173525	40, 37, 49	36, 34, 30
0, 74	92, 29, 23	14, 4, 5	55, 11, 42	9,9138806	41, 7, 37	37, 17, 41
0, 75	94, 3, 53	14, 24, 55	56, 2, 3	9,9102445	41, 37, 8	38, 1, 50

248

n	$\lambda + \mu + n \cos \lambda$	λ	$\lambda + n \cos \lambda$	$l\sqrt[4]{(1-nn)}$	$n \cos \lambda$	μ
0.75	94. 3.53	14.24.55	56. 2. 3	9.9102445	41.37. 8	38. 1.50
0.76	95.39.51	14. 46. 32	56. 52. 50	9.9064310	42. 6.18	38.47. 1
0.77	97.17.19	15. 8.57	57.44. 2	9.9024253	42.35. 5	39.33.17
0.78	98.56.26	15. 32. 15	58. 35. 41	9.8982107	43. 3.26	40.20.45
0.79	100.37.21	15. 56. 31	59. 27. 51	9.8937681	43.31.20	41. 9.30
0.80	102.20.17	16. 21. 53	60. 20. 39	9.8890756	43.58.46	41.59.38
0.81	104. 5.23	16. 48. 26	61. 14. 4	9.8841080	44.25.38	42.51.19
0.82	105.52.41	17. 16. 16	62. 8. 7	9.8788360	44.51.51	43.44.40
0.83	107.42.42	17. 45. 33	63. 2. 53	9.8732250	45.17.20	44.39.49
0.84	109.45.27	18. 16. 27	63. 58. 28	9.8672344	45.42. 1	45.46.59
0.85	111.31.31	18. 49. 14	64. 55. 5	9.8608157	46. 5.51	46.36.26
0.86	113.31.59	19. 24. 1	65. 52. 38	9.8539102	46.28.37	47.38.21
0.87	115.34.16	20. 1. 8	66. 51. 16	9.8464462	46.50. 8	48 43. 0
0.88	117.42.10	20. 41. 1	67. 51. 14	9.8383348	47.10.13	49.50.56
0.89	119.55.28	21. 24. 3	68. 52. 41	9.8294636	47.28.38	51. 2.47
0.90	122.14.47	22. 10. 54	69. 55. 53	9.8196884	47.44.59	52.18.54
0.91	124.41.47	23. 2. 12	71. 1. 5	9.8088189	47.58.53	53.39.42
0.92	127.15.18	23. 59. 7	72. 8. 42	9.7965978	48. 9.35	55. 6.36
0.93	130. 0.21	25. 2. 51	73. 19. 17	9.7826638	48.16.26	56.41. 4
0.94	132.59.48	26. 15. 49	74. 33. 48	9.7664882	48.17.59	58.25. 0
0.95	136.13.59	27. 40. 23	75. 52. 40	9.7472511	48.12.17	60.20.19
0.96	139.50.41	29. 22. 17	77. 18. 18	9.7235790	47.56 1	62.32.23
0.97	144. 1.57	31. 30. 30	78. 53. 24	9.6928969	47.22.54	65. 8.33
0.98	149. 8.43	34. 25. 1	80. 44. 15	9.6494238	46.19.14	68,24.28
0.99	156.10.30	39. 6. 11	83. 7. 14	9.5747133	44. 1. 3	73. 3.16
1.00	180 0. 0	90. 0 0	90. 0. 0	$-\infty$	0. 0. 0	90. 0. 0
9375645	132.13.33	25. 56. 55	74. 15. $5\frac{2}{3}$	9.7706712$\frac{1}{2}$	48.18.10$\frac{2}{3}$ maximum.	57.58.27$\frac{2}{3}$

References

[1] Euler, L., *Memoire sur la plus grande equation des planets,* (Memoir on the Maximum value of an Equation of the Planets), (E105), on the web at the Euler Archive, http://www.math.dartmouth.edu/~euler/.