FPC-BI: Fast Probabilistic Consensus within Byzantine Infrastructures

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Abstract

This paper presents a novel leaderless protocol (FPC-BI: Fast Probabilistic Consensus within Byzantine Infrastructures) with a low communicational complexity and which allows a set of nodes to come to a consensus on a value of a single bit. The paper makes the assumption that part of the nodes are *Byzantine*, and are thus controlled by an adversary who intends to either delay the consensus, or break it. This defines that at least a couple of honest nodes come to different conclusions. We prove that, nevertheless, the protocol works with high probability when its parameters are suitably chosen. Along this the paper also provides explicit estimates on the probability that the protocol finalizes in the consensus state in a given time. This protocol could be applied to reaching consensus in decentralized cryptocurrency systems. A special feature of it is that it makes use of a sequence of random numbers which are either provided by a trusted source or generated by the nodes themselves using some decentralized random number generating protocol. This increases the overall trustworthiness of the infrastructure. A core contribution of the paper is that it uses a *very weak* consensus to obtain a *strong* consensus on the value of a bit, and which can relate to the validity of a transaction.

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1 Introduction

Increasingly, distributed systems need to provide a consensus on the current state of the infrastructure within given time limits, and to a high degree of accuracy. At the core of cryptocurrency transactions, for example, is that miners must achieve a consensus on the current state of transactions. This works well when all the nodes are behaving correctly, but a malicious agent could infect the infrastructure, and try and change the consensus [1].

Suppose that there is a network composed of n nodes, and these nodes need to come to consensus on the value of a bit. Some of these nodes, however, may belong to an adversary, an entity which aims to delay the consensus or prevent it from happening altogether. This paper focuses on this situation - and which is typical in the cryptocurrency applications - when the number n of nodes is large, and where they are possibly (geographically) spread out. This makes the communicational costs important whereas computational complexity and the memory usage are often of a lesser concern.

1.1 Key contributions

The key contribution of this paper is a protocol which allow a larger number of adversarial nodes, and which may be a (fixed) proportion of the total number of nodes, while keeping the communicational complexity low (see Corollary 4.3). It then guarantees fast convergence for all initial conditions. It is important to note that here we do not require that with high probability the consensus should be achieved on the initial majority value. Rather, what we need, is:

(i) if, initially, no *significant majority*[1] of nodes prefer 1, then the final consensus should be 0 whp[2];

(ii) if, initially, a *supermajority*[3] of nodes prefer 1, then the final consensus should be 1 whp.

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[1] loosely speaking, a significant majority is something statistically different from the 50/50 situation; for example, the proportion of 1-opinion is greater than α for some fixed $\alpha > 1/2$

[2] "whp" = "with high probability"

[3] again, this is a loosely defined notion; a supermajority is something already close to consensus, e.g. more than 90% of all nodes have the same opinion

Along with these assumptions, another important assumption that we make is that, among the totality of n nodes, there are qn adversarial (Byzantine) nodes[4], who may not follow the proposed protocol and who may act maliciously in order to prevent the consensus (of the honest nodes) from being achieved.

1.2 Context

To understand the importance of this work to cryptocurrency applications, consider a situation when there are two contradicting transactions. For example, if one transfers all the balance of address A_1 to address A_2, while the other transfers all the balance of address A_1 to address $A_3 \neq A_2$. In this case, neither of the two transactions will be strongly preferred by the nodes of the network, they can then be declared invalid - just in case. On the other hand, it would not be a good idea to *always* declare them invalid, as a malicious actor (Eve) could be able to exploit this. For example, Eve could place a legitimate transaction, such as buying some goods from a merchant. When she receives the goods, she publishes a double-spending transaction - as above - in the hope that *both* will be canceled, and so he would effectively receive her money back (or at least take the money away from the merchant). To avoid this kind of threat, it would be desirable if the first transaction (payment to the merchant) which, by that time, would have probably gained some confidence from the nodes, would stay confirmed, and only the subsequent double-spend gets canceled.

2 Related Work

There is a wide range of classical work on (probabilistic) Byzantine consensus protocols [2, 3, 4, 5, 6, 7]. The disadvantage of the approach of these papers is, however, that they typically require that the nodes exchange $O(n^2)$ messages in each round (which means $O(n)$ messages for each node). In the situation where the communicational complexity matters, this can be a major barrier.

A good deal of work focuses on failures within a network infrastructure, rather than on malicious agents. The work of Liu [8] defines FastBFT, and which is a fast and scalable BFT (Byzantine fault tolerance) protocol. Within this, the work integrates trusted execution environments (TEEs) with lightweight secret sharing, and results in a low latency infrastructure. Crain et al [9] define Democratic Byzantine Fault Tolerance (DBFT) and which is a leaderless Byzantine consensus. This provides a robust infrastructure

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[4]where $q \in [0, 1)$

where there is a failure in the leader of the consensus network. The core contribution is that nodes will process message whenever they receive them, instead of waiting for a co-ordinate to confirm messages. Another Byzantine Fault Tolerant method which does not require a leader node is Honey Badger [10]. This method is asynchronous in its scope and can cope with corrupted nodes. Unfortunately, it does not actually make any commitments around the timing of the delivery of a message, and where even if Eve controls the scheduling of messages, there will be no impact on the overall consensus.

There has also been much research on the probabilistic models where, in each round, a node only contacts a small number of other nodes in order to learn their opinions, and possibly change its own. This type of models is usually called *voter models*, and which were introduced in the 70s by Holley and Liggett [11] and Clifford and Sudbury [12]. A very important observation is that, in most cases, voter models have only two external invariant measures: one concentrated on the "all-0" configuration, and the other one concentrated on the "all-1" – we can naturally call these two configurations "consensus states". Since then, there has been a range of work on voter models; in particular, let us cite [13, 14, 15, 16, 17, 18] which are specifically aimed at reaching consensus and have low communicational complexity (typically, $O(n \ln n)$). However, in these works, the presence of adversarial nodes is usually either not allowed, or is supposed to be very minimal.

3   Model Definition

The developed model assumes that adversarial nodes can exchange information freely between themselves and can agree on a common strategy. In fact, they all may be controlled by a single individual or entity. We also assume that the adversary is *omniscient*: at each moment of time, he is aware of the current opinion of every honest node. While this assumption may seem a bit too extreme, note that the adversarial nodes can query the honest ones a bit more frequently to be aware of the current state of the network; also, even if the "too frequent" queries are somehow not permitted, the adversary can still *infer* (with some degree of confidence) about the opinion of a given honest node by analyzing the history of this node's interactions with all the adversarial nodes.

The remaining $(1 - q)n$ nodes are *honest*, i.e., they follow the recommended protocol. We assume that they are numbered from 1 to $(1 - q)n$; this will enter into several notations below.

Our protocol will be divided into epochs which we call rounds. The basic feature of it is that, in each round, each node may query k other nodes about

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their current opinion (i.e., the preferred value of the bit). We allow k to be relatively large (say, $k = 50$ or so), but still assume that $k \ll n$. We also assume that the complete list of the nodes is known to all the participants, and any node can *directly* query any other node. For the sake of clarity of the presentation, for now we assume that all nodes (honest and adversarial) always respond to the queries; in Section 6 we deal with the general situation when nodes can possibly remain silent. This, by the way, will result in a new "security threshold" $\phi^{-2} \approx 0.38$ (where ϕ is the Golden Ratio), different from the "usual" security thresholds $\frac{1}{2}$ and $\frac{1}{3}$.

With respect to the *behavior* of the adversarial nodes, there are two important cases to be distinguished:

- *Cautious adversary*[5]: any adversarial node must maintain the same opinion in the same round, i.e., respond the same value to all the queries it receives in that round.

- *Berserk adversary*: an adversarial node may respond differently to things for different queries in the same round.

To explain the reason why the adversary may choose to be cautious, first note that we also assume that nodes have identities and sign all their messages; this way, one can always *prove* that a given message originates from a given node. Now, if a node is not cautious, this may be detected by the honest nodes (e.g., two honest nodes may exchange their query history and verify that the same node passed contradicting information to them). In such a case, the offender may be penalized by *all* the honest nodes (the nodes who discovered the fraud would pass that information along, together with the relevant proof). Since, in the sequel, we will see that the protocol provides more security and converges faster against a cautious adversary, it may be indeed a good idea for the honest nodes to adopt additional measures in order to detect the "berserk" behavior. Also, since k would be typically large and each node is queried k times on average during each round, we make a further simplifying assumption that a cautious adversary just chooses (in some way) the opinions of all his nodes *before* the current round starts and then communicates these opinions to whoever asks.

3.1 Generating random numbers

The protocol we are going to describe requires the system to generate, from time to time (more precisely, once in each round), a random number available

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[5]also know as *covert adversary*, cf. [19]

to all the participants (this is very similar to the "global-coin" approach used in many works on Byzantine consensus, see e.g. [2]). For the sake of cleanness of the presentation and the arguments, in this paper we mainly assume that these random numbers are provided by a trusted source, not controlled by the adversary[6]. We observe that such random number generation can be done in a decentralized way as well (provided that the proportion q of the adversarial nodes is not too large), see e.g. [20, 21, 22, 23, 24]. If a "completely decentralized" solution proves to be too expensive (from the point of view of computational and/or communicational complexity), one can consider "intermediate" ones, such as using a smaller committee for this, and/or making use of many publicly available RNGs. It is important to observe that (as we will see from the analysis below), even if from time to time the adversary can get (total or partial) control of the random number, this can only lead to delayed consensus, but he cannot convince different honest nodes of different things, i.e., safety is not violated. Also, it is not necessary that really *all* honest nodes agree on the same number; if most of them do, this is already fine. This justifies the idea that, in our context, both decentralization and "strong consensus" are not of utter importance for the specific task of random number generation. We postpone the rest of this discussion to Section 6, since the methods we employ for proving our results are relevant for it.

Before actually describing our protocol, it is important to note that we assume that there is no central entity that "supervises" the network and can somehow know that the consensus was achieved and therefore it is time to stop. This means that each node must decide when to stop using a *local* rule, i.e., using only the information locally available to it.

3.2 Parameter setup

The protocol depends on a set of integer and real parameters:

- $1/2 < a \leq b < 1$, the threshold limits in the first round (they are needed to assure (i)–(ii) on page 2);

- $\beta \in (0, 1/2)$, the threshold limit parameter in the subsequent rounds;

- $m_0 \in \mathbb{N}$, the cooling-off period;

- $\ell \in \mathbb{N}$, the number of consecutive rounds (when the cooling-off period is over) with the same opinion after which it becomes final, for one node.

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[6]i.e., the adversary may be omniscient (knows all information that exists *now*), but he is not *prescient* (cannot know the future)

Now, let us describe our protocol. First, we assume that each node decides on the initial value of the bit, according to any reasonable rule[7]. Then, we describe the *first round* of the protocol in the following way:

- in the first round, each honest node j randomly queries other nodes k times (repetitions and self-queries are allowed[8]) and records the number $\eta_1(j)$ of 1-opinions it receives;

- *after that*, the value of the random variable $X_1 \sim U[a,b]$ is made available to the nodes[9];

- then, each honest node uses the following decision rule: if $k^{-1}\eta_1(j) \geq X_1$, it adopts opinion 1, otherwise it adopts opinion 0.

In the subsequent rounds, the dynamics is almost the same, we only change the interval where the uniform random variable lives:

- in the round $m \geq 2$, each honest node j randomly queries other nodes k times, and records the number $\eta_m(j)$ of 1-opinions it receives;

- *after that*, the value of the random variable $X_m \sim U[\beta, 1-\beta]$ is made available to the nodes;

- then, each honest node which does not yet have *final* opinion uses the following decision rule: if $k^{-1}\eta_m(j) \geq X_m$, it adopts opinion 1, otherwise it adopts opinion 0.

As mentioned above, if an honest node has the same opinion during ℓ consecutive rounds after the cooling-off period (i.e., counting from time $m_0 + 1$ on) this opinion becomes final.

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[7] for example, if a node sees a valid transaction x (which does not contradict to prior transactions) at time t, and during the time interval $[t, t+\Delta]$ it does not see any transactions that contradict to x it may initially decide that x is *good*, setting the value of the corresponding bit to 1

[8] we have chosen this mainly to facilitate the subsequent analysis; of course, e.g. querying k *other* nodes chosen uniformly at random can be analyzed in a similar way, with some more technical complications because the relevant random variables will not be exactly Binomial, but only approximately so (it will be Hypergeometric, in fact). In any case, we will see below that k typically will be much less than $\sqrt{n}$, which makes querying the same node in the same round quite unlikely.

[9] $U[a,b]$ stands for the uniform probability distribution on interval $[a,b]$

3.3 Consensus mechanism

Let us now explain informally what makes our protocol converge fast to the consensus even in the Byzantine setting. The general idea is the following: if the adversary (Eve) knows the decision rules that the honest nodes use, she can then predict their behaviour and adjust her strategy accordingly, in order to be able to delay the consensus and further mess with the system. Therefore, let us make these rules unknown to all the participants, including Eve. Specifically, even though Eve's nodes can control (to some extent) the expected proportion of 1-responses among the k queries, she cannot control the value that the "threshold" random variable assumes. As a consequence, the decision threshold X_1 will likely be "separated" from that typical proportion.

When this separation happens, the opinions of the honest nodes would tend very strongly in one of the directions whp. Then, it will be extremely unlikely that the system leaves this "pre-consensus" state, due to the fact that the decision thresholds, however random, are always uniformly away from 0 and 1. Also, we mention that a similar protocol was considered in [25]. However, there only "fixed thresholds" were used, which gives Eve much more control, so that, in particular, then she could delay the consensus a great deal. As a last remark, it is important to note that having "independently random thresholds" (i.e., each node independently chooses its own decision threshold) is not enough to achieve the effect described above — these "locally random" decisions will simply average out; that is, having common random numbers is indeed essential.

4 Results

We define two events relative to the final consensus value:

$$H_i = \{\text{all honest nodes eventually reach final opinion } i\}, \quad i = 0, 1. \quad (1)$$

Thus, the union $H_0 \cup H_1$ stands for the event that all honest nodes agree on the same value, i.e., that the consensus was achieved.

For $0 < q < \beta < \frac{1}{2}$, abbreviate

$$\varphi_{\beta,q,k} = \frac{\beta - q}{2(1 - q)} - e^{-\frac{1}{2}k(\beta - q)^2}.$$

In the following, we assume that k is large enough so that $\varphi_{\beta,q,k} > 0$ (indeed, the first term in the above display is strictly positive, and the second one

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converges to 0 as $k \to \infty$). Let us also denote

$$W(n, k, m_0, \ell, u) = (1-q)n\Big(\big(1 - \big(1 - e^{-\frac{1}{2}k(\beta-q)^2}\big)^\ell\big)^u + \Big(\frac{e^{-\frac{1}{2}k(\beta-q)^2}}{1 - e^{-\frac{1}{2}k(\beta-q)^2}}\Big)^{\ell-1}\Big)$$
$$+ (m_0 + \ell u)e^{-2(1-q)n\varphi_{\beta,q,k}^2}, \tag{2}$$

and

$$\psi_{cau}(n, k) = 2\exp\big(-\tfrac{1}{8}n\tfrac{(\beta-q)^2}{4(1-q)}\big) + (1-2\beta)^{-1}\sqrt{2k^{-1}\ln\tfrac{4(1-q)}{\beta-q}}, \tag{3}$$

$$\psi_{ber}(n, k) = 2\exp\big(-\tfrac{1}{8}n\tfrac{(\beta-q)^2}{4(1-q)}\big) + \frac{q + \sqrt{2k^{-1}\ln\tfrac{4(1-q)}{\beta-q}}}{1 - 2\beta}. \tag{4}$$

(in the above notation, we omit the dependence on q and β). As it will become clear shortly, we will need $W(n, k, m_0, \ell, u)$ to be small, and ψ's (which, as the reader probably have noted, relate to cautious and berserk adversaries) to be strictly less than 1. It is not difficult to see (we elaborate more on that below) that (recall that $q < \beta$) the value of the expression in (2) will be small indeed if n is large and k is at least $C \ln n$ for a large C. Then, the first term in the expression in (3) will be very small for large n, while the second term will also be small for a sufficiently large k. As for (4), it shares the same first term with (3); the second term, however, will be of constant order, and if we want it to be strictly less than 1 for a large k, we need the constraint $q < 1 - 2\beta$ to hold.

Now, we begin formulating our main results. Let $\mathcal{N}$ be the number of rounds until *all* honest nodes achieve their final opinions. The next result controls both the number of necessary rounds and the probability that the final consensus is achieved (i.e., the event $H_0 \cup H_1$ occurs):

Theorem 4.1. *(i) For any strategy of a cautious adversary, it holds that*

$$\mathbb{P}\big[(H_0 \cup H_1) \cap \{\mathcal{N} \le m_0 + \ell u\}\big] \ge 1 - W(n, k, m_0, \ell, u) - \big(\psi_{cau}(n, k)\big)^{m_0}. \tag{5}$$

(ii) For any strategy of a berserk adversary, we have

$$\mathbb{P}\big[(H_0 \cup H_1) \cap \{\mathcal{N} \le m_0 + \ell u\}\big] \ge 1 - W(n, k, m_0, \ell, u) - \big(\psi_{ber}(n, k)\big)^{m_0}. \tag{6}$$

Note that the only difference between (5) and (6) is in the second terms of (3) and (4). As we will see in the proofs, these terms enter into the part which is "responsible" for the estimates on the time moment when the adversary loses control on the situation which permits one of the opinions to reach a supermajority; from that moment on, there is essentially no difference if the adversary is cautious or berserk.

Corollary 4.2. *For a cautious adversary we need that $q < \beta$, while for a berserk adversary we also need that $q < 1 - 2\beta$. Recalling also that β must belong to $(0, 1/2)$, it is not difficult to see that*

- *for a cautious adversary, for any $q < 1/2$ and all large enough n we are able to adjust the parameters k, β, m_0, ℓ in such a way that the protocol works whp (in particular, a β-value sufficiently close to $1/2$ would work);*

- *however, for a berserk adversary, we are able to do the same only for $q < 1/3$ (here, $\beta = 1/3$ would work).*

Corollary 4.3. *One may be interested in asymptotic results, for example, of the following kind: assume that the number of nodes n is fixed (and large), and the proportion of Byzantine nodes q is acceptable (i.e., less than $1/2$ for the case of cautious adversary, or less than $1/3$ for the case of berserk adversary, as discussed above). We then want to choose the parameters of the protocol in such a way that the probabilities in (5) and (6) are at least $1 - \varepsilon(n)$, where $\varepsilon(n)$ is polynomially small in n (i.e., $\varepsilon(n) = O(n^{-h})$ for some $h > 0$).*

First, $\beta = 1/3$ works in both cases; then, a quick analysis of (5)–(6) shows that one possibility is: chose $k = C \ln n$ (with a sufficiently large constant in front), ℓ of constant order, and $m_0 = O\big(\frac{\ln n}{\ln \ln n}\big)$ for cautious adversary or $m_0 = O(\ln n)$ for a berserk one.

That is, the overall communicational complexity will be at most $O\big(\frac{n \ln^2 n}{\ln \ln n}\big)$ for a cautious adversary and $O(n \ln^2 n)$ for a berserk one.

Next, let $\hat{p}_0$ be the initial proportion of 1-opinions among the honest nodes. Our second result shows that if, initially, no significant majority of nodes prefer 1, then the final consensus will be 0 whp, and if the supermajority of nodes prefer 1, then the final consensus will be 1 whp (recall (i)–(ii) on page 2), and it is valid in the general case (i.e., for both cautious and berserk adversaries).

Theorem 4.4. *(i) First, suppose that $\hat{p}_0(1-q) + q < a$, and assume that k is sufficiently large so that*

$$e^{-2k(a - \hat{p}_0(1-q) - q)^2} \leq \frac{\beta - q}{4(1 - q)}.$$

Then, we have

$$\mathbb{P}\big[H_0 \cap \{\mathcal{N} \leq m_0 + \ell u\}\big] \geq 1 - \exp\big(-\tfrac{1}{8} n \tfrac{(\beta - q)^2}{4(1-q)}\big) - W(n, k, m_0, \ell, u).$$
(7)

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(ii) Now, suppose that $\hat{p}_0(1-q) > b$, and assume that k is sufficiently large so that

$$e^{-2k(\hat{p}_0(1-q)-b)^2} \leq \frac{\beta - q}{4(1-q)}.$$

Then, the same estimate (7) holds for $\mathbb{P}[H_1 \cap \{\mathcal{N} \leq m_0 + \ell u\}]$.

We also mention that the estimates (5) and (6) are probably not quite sharp because we have used some union bounds and other "worst-case" arguments when proving them. For example, for $n = 1000$, $k = 20$, $\beta = 1/3$, $\ell = m_0 = 10$, $q = 0.1$, the system was simulated[10] five thousand times, with all of them resulting in consensus after not more than 44 rounds; for these parameter values, the bounds provided by (5) and (6) are not quite useful. For more concrete results on the number of necessary rounds until consensus (with different parameters), see Figure 1. It is interesting to observe that, in most cases, the protocol finalizes after the minimal number $m_0 + \ell = 10$ of rounds and the probability that it lasts for more than 20 rounds seems to be very small.

5 Proofs

We start with some preliminaries. Let us recall the Hoeffding's inequality [26]: if $0 < \gamma < p < 1$, then

$$\mathbb{P}[k^{-1}S_k \leq \gamma] \leq \exp\{-2k(p - \gamma)^2\}, \tag{8}$$

and the same estimate also holds for $\mathbb{P}[k^{-1}S_k \geq \gamma]$ in the case $0 < p < \gamma < 1$.

To better understand the difference between cautious and berserk adversaries, look at Figure 2. Here, $\hat{p}$ is the initial proportion of 1-opinions between the honest nodes, and the crosses mark the proportion of 1-responses to the k queries that the honest nodes obtain. The cautious adversary can choose any $\tilde{p} \in [\hat{p}(1-q), \hat{p}(1-q) + q]$ (by adjusting the opinions of his nodes appropriately, so that the overall proportion of 1-opinions would be $\tilde{p}$), and then those crosses will be (mostly) concentrated in the interval of length of order $k^{-1/2}$ around $\tilde{p}$. On the other hand, the berserk adversary can cause the crosses to be distributed in any way on the whole interval $[\hat{p}(1-q), \hat{p}(1-q) + q]$,

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[10]with the simple adversarial strategy "vote for the weakest" aiming to prevent the honest nodes from achieving supermajority of one of the opinions for as long as possible; however, we do not believe that the adversary can invent something radically better since, as we will see below, the adversary loses control completely after such a supermajority is achieved
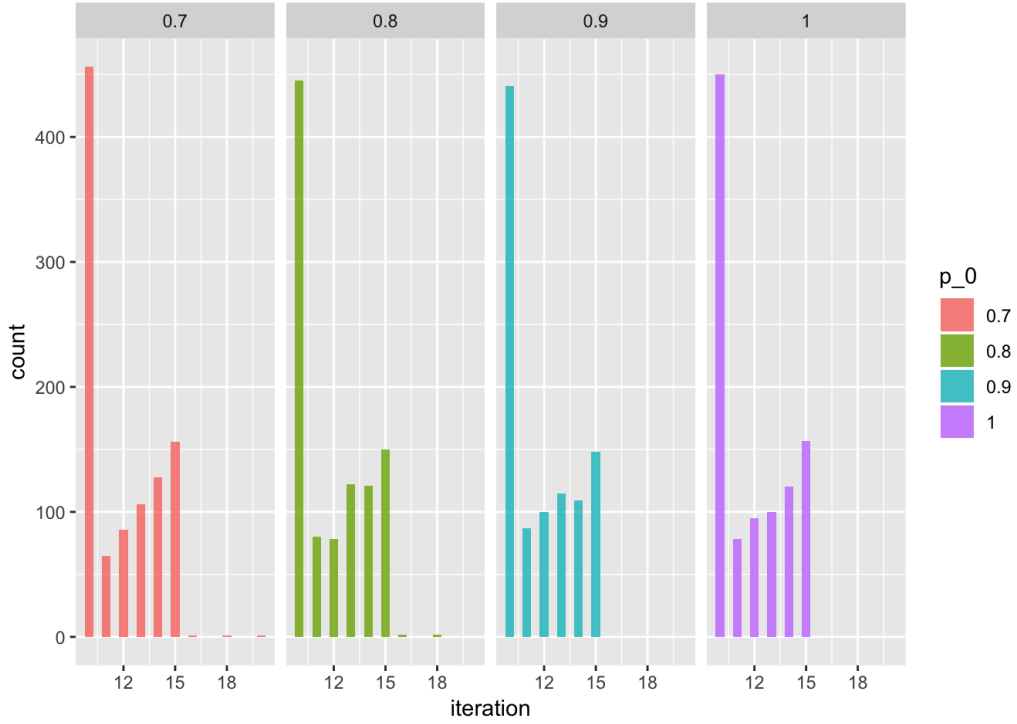
Figure 1: Number of rounds till the protocol finalizes, with $n = 1000$, $a = 0.75$, $b = 0.85$, $m_0 = \ell = 5$, $k = 20$, and $q = 0.1$.
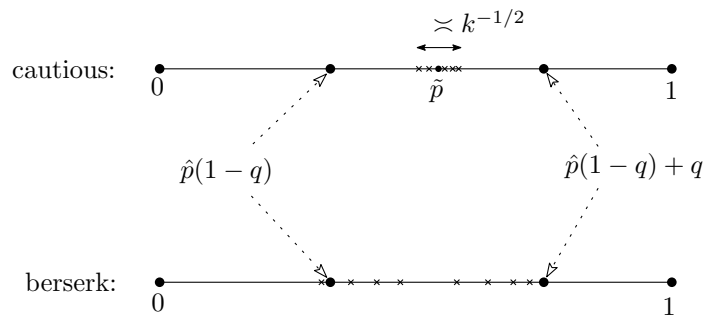

Figure 2: What cautious and berserk adversaries can achieve

with some of them even going a bit out of it (on the distance of order $k^{-1/2}$ again).

Next, we need an auxiliary result on a likely outcome of a round in the case when the adversary cannot make the typical proportion of 1-responses to be close to the decision threshold. Let $\eta(j)$ be the number of 1-responses among k queries that jth honest node receives; in general, the random variables $(\eta(j), j = 1, \ldots, (1-q)n)$ are not independent, but they are *conditionally independent* given the adversary's strategy. (Note that $\eta(j) \sim \mathcal{B}(k, \tilde{p})$ with some possibly random $\tilde{p}$ if the adversary is cautious, but the situation may be more complicated for a berserk one.) For a fixed $\lambda \in (0, 1)$, define a random variable

$$\hat{p} = \frac{1}{(1-q)n} \sum_{j=1}^{(1-q)n} \mathbf{1}\{\eta(j) \geq \lambda k\};$$

so that $\hat{p}$ is the new proportion of 1-opinions among the honest nodes, given that the "decision threshold" equals λ. Then, the following result holds:

Lemma 5.1. *(i) Assume that, conditioned on any adversarial strategy, there are some positive c and θ such that $\eta(j)$ is stochastically dominated by $\mathcal{B}(k, \lambda - c)$ for all $j = 1, \ldots, (1-q)n$, and $\mathbb{P}[\mathcal{B}(k, \lambda - c) \geq \lambda k] \leq \theta$. Then, for any $v > 0$*

$$\mathbb{P}[\hat{p} > \theta + v] \leq e^{-2(1-q)nv^2}. \tag{9}$$

(ii) Assume that, conditioned on any adversarial strategy, $\eta(j)$ stochastically dominates $\mathcal{B}(k, \lambda+c)$ for all $j = 1, \ldots, (1-q)n$, and $\mathbb{P}[\mathcal{B}(k, \lambda+c) \leq \lambda k] \leq \theta$. Then, for any $v > 0$

$$\mathbb{P}[\hat{p} < 1 - \theta - v] \leq e^{-2(1-q)nv^2}. \tag{10}$$

Proof. For (i), we observe that $(1-q)n\hat{p}$ is stochastically dominated by $\mathcal{B}(n, \theta)$, and then (9) follows from (8). The proof of the part (ii) is completely analogous. $\square$

Note that, by (8), $\mathbb{P}[\mathcal{B}(k, \lambda - c) \geq \lambda k] \leq e^{-2kc^2}$ (and the same holds for $\mathbb{P}[\mathcal{B}(k, \lambda + c) \leq \lambda k]$), so we will normally use Lemma 5.1 with $\theta = e^{-2kc^2}$.

Another elementary fact we need is

Lemma 5.2. *Let $(\xi_m^{(j)}, m \geq 1), j = 1, \ldots, N$ be N sequences of independent Bernoulli trials[11] with success probability $h \in (0, 1)$. For $j = 1, \ldots, N$ define*

$$\tau_j^{(1)} = \min\left\{m \geq \ell : \xi_m^{(j)} = \xi_{m-1}^{(j)} = \ldots = \xi_{m-\ell+1}^{(j)} = 1\right\}$$

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[11]the sequences themselves are *not* assumed to be independent between each other

and
$$\tau_j^{(0)} = \min\left\{m \geq \ell : \xi_m^{(j)} = \xi_{m-1}^{(j)} = \ldots = \xi_{m-\ell+1}^{(j)} = 0\right\}$$
to be the first moments when runs of ℓ ones (respectively, zeros) are observed in jth sequence. Then, for all $u \in \mathbb{N}$,

$$\mathbb{P}[\tau_j^{(1)} \leq \ell u, \tau_j^{(1)} < \tau_j^{(0)}] \geq 1 - (1 - h^\ell)^u - \left(\frac{1-h}{h}\right)^{\ell-1} \tag{11}$$

for all $j = 1, \ldots, N$, and

$$\mathbb{P}[\tau_j^{(1)} \leq \ell u, \tau_j^{(1)} < \tau_j^{(0)}, \forall j = 1, \ldots, N] \geq 1 - N\left((1 - h^\ell)^u + \left(\frac{1-h}{h}\right)^{\ell-1}\right). \tag{12}$$

Proof. First, it is clear that

$$\mathbb{P}[\tau_j^{(1)} \leq \ell u] \geq 1 - (1 - h^\ell)^u \tag{13}$$

(divide the time interval $[1, \ell u]$ into u subintervals of length ℓ and note that each of these subintervals is all-1 with probability h^ℓ). Then, the following is an easy exercise on computing probabilities via conditioning (for the sake of completeness, we prove this fact in the Appendix):

$$\mathbb{P}[\tau_j^{(1)} < \tau_j^{(0)}] = 1 - \frac{(1-h)^{\ell-1}(1-h^\ell)}{h^{\ell-1} + (1-h)^{\ell-1} - (h(1-h))^{\ell-1}}. \tag{14}$$

Observe that (14) implies that (since $1 - h^\ell \leq 1$ and $(1-h)^{\ell-1} - (h(1-h))^{\ell-1} \geq 0$)

$$\mathbb{P}[\tau_j^{(1)} < \tau_j^{(0)}] \geq 1 - \left(\frac{1-h}{h}\right)^{\ell-1},$$

and so, using the above together with (13) and the union bound, we obtain (11). The relation (12) is then a direct consequence of (11) (again, with the union bound). $\square$

To prove our main results, we need some additional notation. Let $\varrho(j)$ be the round when the jth (honest) node finalizes its opinion. Denote

$$R_m = \{j : \varrho(j) \leq m\}$$

to be the subset of honest nodes that finalized their opinions by round m. Let also $\hat{\xi}_m(j)$ be the opinion of jth node after the mth round and

$$\hat{p}_m = \frac{1}{(1-q)n} \sum_{j=1}^{(1-q)n} \hat{\xi}_m(j) \tag{15}$$

be the proportion of 1-opinions among the honest nodes after the jth round in the original system.

Proof of Theorem 4.1. Let us define the random variable

$$\Psi = \min\left\{ m \geq 1 : \hat{p}_m \leq \frac{\beta - q}{2(1-q)} \text{ or } \hat{p}_m \geq 1 - \frac{\beta - q}{2(1-q)} \right\} \tag{16}$$

to be the round after which the proportion of 1-opinions among the honest nodes either becomes "too small", or "too large". We now need the following fact:

Lemma 5.3. *For all $s \leq m_0 + \ell$, it holds that (recall (3) and (4))*

$$\mathbb{P}[\Psi > s] \leq \begin{cases} \left(\psi_{cau}(n, k)\right)^{s-1}, & \text{for cautious adversary,} \\ \left(\psi_{ber}(n, k)\right)^{s-1}, & \text{for berserk adversary.} \end{cases} \tag{17}$$

Proof. Observe that $s \leq m_0 + \ell$ implies that a node cannot finalize its opinion before round s. Consider first the case of a cautious adversary. Abbreviate (for this proof) $\mu = \frac{\beta - q}{4(1-q)}$. Let $m \geq 2$ and observe that, for *any* fixed $h \in [0, 1]$ we have (recall that $X_m \sim U[\beta, 1 - \beta]$)

$$\mathbb{P}\left[e^{-2k(X_m - h)^2} \geq \mu\right] = \mathbb{P}\left[(X_m - h)^2 \leq \frac{\ln \mu^{-1}}{2k}\right]$$

$$= \mathbb{P}\left[h - \sqrt{\frac{\ln \mu^{-1}}{2k}} \leq X_m \leq h + \sqrt{\frac{\ln \mu^{-1}}{2k}}\right]$$

$$\leq (1 - 2\beta)^{-1}\sqrt{\frac{2\ln \mu^{-1}}{k}}. \tag{18}$$

Now, assume that $\tilde{p}_{m-1} = h$. Under this, using (8) and (18), we obtain by conditioning on the value of X_m

$$\mathbb{P}[\hat{p}_m \in (2\mu, 1 - 2\mu)] = \mathbb{E}\mathbb{P}[\hat{p}_m \in (2\mu, 1 - 2\mu) \mid X_m]$$

$$= \mathbb{E}\big(\mathbb{P}[\hat{p}_m \in (2\mu, 1 - 2\mu) \mid X_m]\mathbf{1}\{e^{-2k(X_m - h)^2} < \mu\}$$

$$+ \mathbb{P}[\hat{p}_m \in (2\mu, 1 - 2\mu) \mid X_m]\mathbf{1}\{e^{-2k(X_m - h)^2} \geq \mu\}\big)$$

$$\leq \mathbb{E}\big(\mathbb{P}[\hat{p}_m > 2\mu \mid X_m]\mathbf{1}\{e^{-2k(X_m - h)^2} < \mu, h < X_m\}$$

$$+ \mathbb{P}[\hat{p}_m < 1 - 2\mu \mid X_m]\mathbf{1}\{e^{-2k(X_m - h)^2} < \mu, h > X_m\}$$

$$+ \mathbf{1}\{e^{-2k(X_m - h)^2} \geq \mu\}\big)$$

$$\leq 2\mathbb{P}\big[((1-q)n)^{-1}\mathcal{B}((1-q)n, e^{-2k(1-\mu)}) < 1 - 2\mu\big]$$

$$+ \mathbb{P}[e^{-2k(X_m - h)^2} \geq \mu]$$

$$\leq \psi_{cau}(n, k), \tag{19}$$

recall (3). This implies the first comparison in (17).

For a berserk adversary, the calculation is quite analogous (recall Figure 2), so we omit it. $\square$
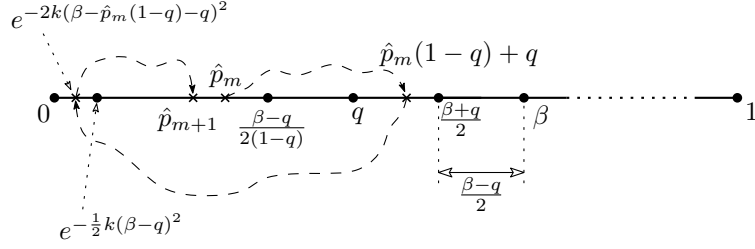
Figure 3: Transition from $\hat{p}_m$ to $\hat{p}_{m+1}$: after mth round, being $\hat{p}_m \leq \frac{\beta-q}{2(1-q)}$, the adversary may "grow" the proportion of 1s to $\hat{p}_m(1-q)+q \leq \frac{\beta+q}{2}$. Then, since the difference between that and "the least possible threshold" β is at least $\frac{\beta-q}{2}$, the probability that an undecided node would have opinion 1 in the next round is at most $e^{-\frac{1}{2}k(\beta-q)^2}$. Then, with overwhelming probability $\hat{p}_{m+1}$ will be at most $\frac{\beta-q}{2(1-q)}$, and so it goes.

Next, we need a result that shows that if one of the opinions has already reached a supermajority, then this situation is likely to be preserved.

Lemma 5.4. *Let $m \geq 2$; in the following, A will denote a subset of $\{1, \ldots, (1-q)n\}$.*

(i) *Let G_0 be the event that $\hat{p}_m \leq \frac{\beta-q}{2(1-q)}$, $R_{m-1} = A$, and $\hat{\xi}_{m-1}(j) = 0$ for all $j \in A$. Then*

$$\mathbb{P}\Big[\hat{p}_{m+1} \leq \frac{\beta-q}{2(1-q)} \mid G_0\Big] \geq 1 - e^{-2(1-q)n\varphi_{\beta,q,k}^2}. \tag{20}$$

(ii) *Let G_1 be the event that $\hat{p}_m \geq 1 - \frac{\beta-q}{2(1-q)}$, $R_{m-1} = A$, and $\hat{\xi}_{m-1}(j) = 1$ for all $j \in A$. Then*

$$\mathbb{P}\Big[\hat{p}_{m+1} \geq 1 - \frac{\beta-q}{2(1-q)} \mid G_1\Big] \geq 1 - e^{-2(1-q)n\varphi_{\beta,q,k}^2}. \tag{21}$$

Proof. We prove only part (i), the proof of the other part is completely analogous. Now, look at Figure 3: essentially, this is a direct consequence of Lemma 5.1 with $\theta = e^{-2k(\frac{\beta-q}{2})^2} = e^{-\frac{1}{2}k(\beta-q)^2}$ and $v = \varphi_{\beta,q,k}$. Observe also that, if some honest nodes already decided on 0 definitely, it holds that $(1-q)n\hat{p}_{m+1}$ is stochastically dominated by $\mathcal{B}\big((1-q)n, e^{-\frac{1}{2}k(\beta-q)^2}\big)$. $\square$

Now, we are able to conclude the proof of Theorem 4.1. Let us introduce

the random variable

$$Z = \begin{cases} \min\left\{m > \Psi : \hat{p}_m > \frac{\beta-q}{2(1-q)}\right\} & \text{on } \hat{p}_\Psi \leq \frac{\beta-q}{2(1-q)}, \\ \min\left\{m > \Psi : \hat{p}_m < 1 - \frac{\beta-q}{2(1-q)}\right\} & \text{on } \hat{p}_\Psi \geq 1 - \frac{\beta-q}{2(1-q)} \end{cases} \tag{22}$$

to be the first moment after Ψ when the honest nodes' opinion has drifted away from supermajority. Denote also

$$\hat{\tau}_j^{(1)} = \min\left\{m \geq m_0 + \ell : \hat{\xi}_m^{(j)} = \hat{\xi}_{m-1}^{(j)} = \ldots = \hat{\xi}_{m-\ell+1}^{(j)} = 1\right\}$$

and

$$\hat{\tau}_j^{(0)} = \min\left\{m \geq m_0 + \ell : \hat{\xi}_m^{(j)} = \hat{\xi}_{m-1}^{(j)} = \ldots = \hat{\xi}_{m-\ell+1}^{(j)} = 0\right\}.$$

Next, observe that

$$(H_0 \cup H_1) \cap \{\mathcal{N} \leq m_0 + \ell u\} \subset D_1 \cap D_2 \cap D_3,$$

where

$$D_1 = \{\Psi \leq m_0\},$$
$$D_2 = \{Z \geq m_0 + \ell u\},$$
$$D_3 = \{\text{there is } i \in \{0, 1\} \text{ such that } \hat{\tau}_j^{(i)} \leq m_0 + \ell u, \hat{\tau}_j^{(i)} < \hat{\tau}_j^{(1-i)}$$
$$\text{for all } j = 1, \ldots, (1-q)n\}.$$

To obtain the estimates (5) and (6), it is enough to note that the lower bounds on, respectively, $\mathbb{P}[D_1]$, $\mathbb{P}[D_2]$, and $\mathbb{P}[D_3]$, follow from, respectively, Lemma 5.3, Lemma 5.4, and Lemma 5.2 (and also the union bound). $\qquad\square$

Proof of Theorem 4.4. We prove only the part (i); the proof of the other part is completely analogous. In fact, to obtain the proof it is enough to observe that, if $a - \hat{p}_0(1-q) - q > 0$ and $e^{-2k(a-\hat{p}_0(1-q)-q)^2} \leq \frac{\beta-q}{4(1-q)}$, then, by (8), with probability at least $1 - \exp\left(-\frac{1}{8}n\frac{(\beta-q)^2}{4(1-q)}\right)$ it happens that $\hat{p}_1 \leq \frac{\beta-q}{2(1-q)}$ (so, in particular, $\Psi = 1$); next, the same argument as in the proof of Theorem 4.1 does the work. $\qquad\square$

6 Further generalizations

In this section we argue that our protocol is *robust*, that is, it is possible to adapt it in such a way that it is able to work well in more "practical" situations. Specifically, observe that nodes may not always respond queries, and the adversarial nodes sometimes may do so *deliberately*. The protocol described in Section 3 is not designed to handle this, so it needs to be amended. There are at least two natural ways to deal with this situation:

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(i) let each node to take the decision based on the responses that it effectively received (i.e., instead of $k^{-1}\eta_m(j)$ use $\eta_m(j)/\zeta_m(j)$, where $\zeta_m(j)$ is the number of responses that the jth node received in the mth round);

(ii) each node queries more than k nodes, say, $2k$ or more; since whp the number of responses received will be at least k (for definiteness, let us assume that the probability that a query is left unresponded is less than $\frac{1}{2}$), the node then keeps exactly k responses and discards the rest;

and it is of course also possible to combine them. The practical difference between these two options is probably not so big; for the sake of formulating the results in a more clean way, let us assume that a node simply issues queries sequentially until getting exactly k responses.

Now, we define the notion of a *semi-cautious* adversary: every node it controls will not give *contradicting* responses (i.e., 0 to one node and 1 to another node in the same round) but can sometimes remain silent; since it does not make sense for a node to remain silent altogether in a given round (that would just reduce the fraction of the adversarial nodes in the network), there are two possible adversarial node behaviours:

- a node answers "0" to some queries and does not answer other queries;

- a node answers "1" to some queries and does not answer other queries.

Here is the result we have for a semi-cautious adversary:

Theorem 6.1. *If the adversary is semi-cautious, assume that* $\frac{1}{2-q} - \beta + \sqrt{2k^{-1}\ln\frac{4(1-q)}{\beta-q}} < 1 - 2\beta$. *Then, for any adversarial strategy, we have*

$$\mathbb{P}\big[(H_0 \cup H_1) \cap \{\mathcal{N} \leq m_0 + \ell u\}\big] \geq 1 - \big(\psi_{semi}(n,k)\big)^{m_0} - W(n,k,m_0,\ell,u), \tag{23}$$

where

$$\psi_{semi}(n,k) = 2\exp\big(-\tfrac{1}{8}n\tfrac{(\beta-q)^2}{4(1-q)}\big) + \frac{\frac{1}{2-q} - \beta + \sqrt{2k^{-1}\ln\frac{4(1-q)}{\beta-q}}}{1 - 2\beta}. \tag{24}$$

In this situation, the fact corresponding to Corollary 4.2 will be the following (in particular, note the new "security threshold" $\phi^{-2} \in (\frac{1}{3}, \frac{1}{2})$ that we obtain here):

Corollary 6.2. *For a semi-cautious adversary, we need that* $\frac{1}{2-q} - \beta < 1 - 2\beta$, *or, equivalently,* $q < 2 - \frac{1}{1-\beta}$ *(it is only in this case that we will be able to find*

large enough k such that the hypothesis of Theorem 6.1 is satisfied). Since we also still need that $q < \beta$, solving $\beta = 2 - \frac{1}{1-\beta}$, we obtain that q must be less than $\frac{3-\sqrt{5}}{2} = \frac{1}{1+\phi} = \phi^{-2} \approx 0.38$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio. Then, as before, it is straightforward to show that, for a semi-cautious adversary, for any $q < \frac{1}{1+\phi}$ and all large enough n we are able to adjust the parameters k, β, m_0, ℓ in such a way that the protocol works whp (in particular, a β-value sufficiently close to $\frac{1}{1+\phi}$ would work).

Proof of Theorem 6.1. As observed before, an "always-silent" strategy is not interesting for an adversarial node, since this will, in practice, only reduce their quantity. Now, assume that, for some $\gamma \in [0,1]$,

- γqn adversarial nodes reply "0" or remain silent;

- $(1 - \gamma)qn$ adversarial nodes reply "1" or remain silent.

Then, if the adversary wants to decrease a honest node's confidence in the 1-opinion, those nodes who may answer "1" will remain silent, and so with probability $\frac{1-q}{1-q+\gamma q}$ the response will be obtained from a honest node, while with probability $\frac{\gamma q}{1-q+\gamma q}$ the response will be obtained from an adversarial node. This gives

$$\hat{p}\frac{1-q}{1-q+\gamma q} = \frac{\hat{p}(1-q)}{1-(1-\gamma)q}$$

as the "lower limit" for the (expected) proportion of 1s in the queries. Analogously, if the adversary wants to increase an honest node's confidence in the 1-opinion those nodes who may answer "0" will remain silent, and so with probability $\frac{1-q}{1-q+(1-\gamma)q}$ the response will be obtained from a honest node, while with probability $\frac{(1-\gamma)q}{1-q+(1-\gamma)q}$ the response will be obtained from an adversarial node. This gives

$$\hat{p}\frac{1-q}{1-q+(1-\gamma)q} + \frac{(1-\gamma)q}{1-q+(1-\gamma)q} = \frac{\hat{p}(1-q)+(1-\gamma)q}{1-\gamma q}$$

as the corresponding "upper limit". So, analogously to Figure 2, the semi-cautious adversary can achieve the "crosses" to be distributed on the interval

$$\mathcal{I}_\gamma := \left[\frac{\hat{p}(1-q)}{1-(1-\gamma)q}, \frac{\hat{p}(1-q)+(1-\gamma)q}{1-\gamma q}\right] \tag{25}$$

in any way. Now, it is elementary to see that both endpoints of the above interval decrease when γ increases; if we want to make it symmetric (around $\frac{1}{2}$), we need to solve

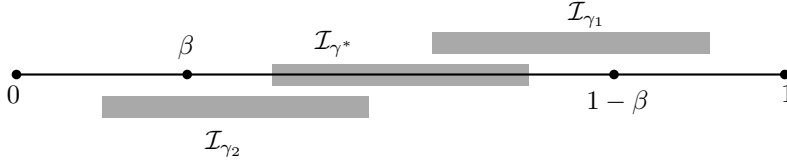$$\frac{\hat{p}(1-q)}{1-(1-\gamma)q} = 1 - \frac{\hat{p}(1-q)+(1-\gamma)q}{1-\gamma q},$$

Figure 4: The "intervals of control" of a semi-cautious adversary, for $\gamma_1 < \gamma^* < \gamma_2$

or, equivalently

$$\frac{\hat{p}}{1 - (1 - \gamma)q} = \frac{1 - \hat{p}}{1 - \gamma q}$$

for γ. This gives the solution $\gamma^* = q^{-1}(2\hat{p} - 1) + (1 - \hat{p})$. After substituting γ^* to (25), the symmetrized interval becomes

$$\mathcal{I}_{\gamma^*} = \Big[\frac{1 - q}{2 - q}, \frac{1}{2 - q}\Big]$$

(somewhat unexpectedly, because it doesn't depend on $\hat{p}$ anymore). It is actually worth noting that γ^* does not necessarily belongs to $[0, 1]$ (so it is not always possible to make this interval symmetric), but it does not pose a problem due to the following. Look at Figure 4: due to the monotonicity,

$$\max_{\gamma \in [0,1]} \big|[\beta, 1 - \beta] \cap \mathcal{I}_\gamma\big| \leq \frac{1}{2 - q} - \beta : \tag{26}$$

indeed, for all γ we see that either the interval $[\beta, \frac{1-q}{2-q})$ or the interval $(\frac{1}{2-q}, 1 - \beta]$ is a subset of $[\beta, 1 - \beta] \setminus \mathcal{I}_\gamma$.

This essentially takes care of the argument in the proof of Lemma 5.3 (since we now understand what is the minimal length of the interval that the adversary cannot control), and the rest of the proof is completely analogous to that of Theorem 4.1: indeed, as observed before, the adversary loses control after the random time Ψ. $\qquad\square$

We also observe that Theorem 4.4 remains valid also for a semi-cautious adversary.

Next, let us discuss what do we really need from the (decentralized) random number generator. In fact, it is not so much: we need that, regardless of the past, with probability at least θ (where $\theta > 0$ is a fixed parameter) the next outcome is a uniform random variable which is "unpredictable" for the adversary; this random number is seen by at least (1-δ) proportion of honest nodes, where δ is reasonably small. What we can prove in such a

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situation depends on what the remaining $\delta(1-q)n$ honest nodes use as their decision thresholds: they can use some "second candidate" (in case there is an alternative source of common randomness), or they can choose their thresholds independently and randomly, etc. Each of such situations would need to be treated separately, which is certainly doable, but left out of this paper. Let us note, though, that the "worst-case" assumption is that the adversary can "feed" the (fake) decision thresholds to those $\delta(1-q)n$ honest nodes. This would effectively mean that these nodes would behave as cautious adversaries in the next round (which matters if the random time Ψ did not yet occur). Therefore, for the sake of obtaining bounds like (5)–(6) and (23) we can simply pretend that the value of q is increased by δ.

Now, assuming that $\delta = 0$, it is easy to figure out how this will affect our results: indeed, in our proofs, all random thresholds matter only until Ψ. It is then straightforward to obtain the following fact:

Proposition 6.3. *Assume the above on the random number generation (with $\theta \in (0,1)$ and $\delta = 0$). Then, the estimates (5)–(6) and (23) remain valid with $1 - \theta + \theta\psi_*(n,k)$ on the place of $\psi_*(n,k)$ (with $* \in \{cau, ber, semi\}$).*

In view of the above result, let us stress that one of the main ideas of this paper is: we use a "rather weak" consensus (on the random numbers, as above) to obtain a "strong" consensus on the value of a bit (i.e., validity of a transaction). Also, let us observe that a partial control of the random numbers does not give access to a lot of power (in the worst case the adversary would delay the consensus a bit, but that is all), so there is not much need to be restrictive on the degree of decentralization for that part[12]: a smaller subcommittee can take care of the random numbers' generation, and some VDF-based random number generation scheme (such as [21]) may be used to further prevent this subcommittee from leaking the numbers before the due time).

7   Conclusions and Future Work

In this paper we described a consensus protocol which is able to withstand a substantial proportion of Byzantine nodes, and obtained some explicit estimates on its safety and liveness. A special feature of our protocol is that it uses a sequence of random numbers (produced by some external source or by the nodes themselves) in order to have a "randomly moving decision threshold" which quickly defeats the adversary's attempts to mess with the

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[12]in other words, it may make sense that different parts of the system are decentralized to a different degree, there is nothing a priory wrong with it

consensus. It is also worth noting that the "quality" of those random numbers is not critically important – only the estimates on Ψ (Lemma 5.3) will be affected in a non-drastic way. In particular, one can permit that the random numbers might be biased, or even that the adversary might get control of these numbers from time to time. Also, it is clear from the proofs that there is no need for the honest nodes to achieve consensus on the actual values of these random numbers: if some (not very large) proportion of honest nodes does not see the same number as the others, this will not cause problems. All this is due to the fact that, when the proportion of 1-opinions among the honest nodes becomes "too small" or "too large" (i.e., less than $\frac{\beta-q}{2(1-q)}$ or greater than $1 - \frac{\beta-q}{2(1-q)}$ in our proofs), the adversary does not have any control anymore.

We need to comment on anti-Sybil measure in practical implementations: indeed, it would be quite unfortunate if the adversary is able to deploy an excessively large number of nodes, thus inflating the value of q. One of the possible approaches is using a variant of Proof-of-Stake; with it, when querying, one needs to choose the node proportionally to its weight (stake).

Of course, one may consider also further modifications of the protocol. For example, one can get rid of the cooling-off period or make its length depend on the "strength" of the prevailing opinion in the first rounds (but then ℓ should probably be increased), consider different final acceptance rules (for example, at least $\ell - y$ node's opinions among the last ℓ must be the same), and so on.

8 References

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Acknowledgments

A   Appendix

Here we prove a simple fact about runs of zeros and ones in a sequence of Bernoulli trials, which will imply (14). Namely, let $\xi_1, \xi_2, \xi_3, \ldots$ be i.i.d. random variables with $\mathbb{P}[\xi_i = 1] = 1 - \mathbb{P}[\xi_i = 0] = h$, and let, for $r, s \in \mathbb{N}$

$$\tau = \min\{m \geq r : \xi_m = \xi_{m-1} = \ldots = \xi_{m-r+1} = 1\}$$

and

$$\sigma = \min\{m \geq s : \xi_m = \xi_{m-1} = \ldots = \xi_{m-s+1} = 0\}$$

to be the first moments when we see runs of r ones (respectively, s zeros).

Proposition A.1. *It holds that*

$$\mathbb{P}[\tau < \sigma] = \frac{h^{r-1}(1 - (1-h)^s)}{h^{r-1} + (1-h)^{s-1} - h^{r-1}(1-h)^{s-1}}. \tag{27}$$

Proof. To prove (27), we use conditioning. Abbreviate $p_0 = \mathbb{P}[\tau < \sigma \mid \xi_1 = 0]$ and $p_1 = \mathbb{P}[\tau < \sigma \mid \xi_1 = 1]$. Then, conditioning on the number of consecutive zeros in the beginning, we write

$$p_0 = \sum_{j=1}^{s-1}(1-h)^{j-1}h p_1 = (1 - (1-h)^{s-1})p_1, \tag{28}$$

and, conditioning on the number of consecutive ones in the beginning, we obtain that

$$p_1 = \sum_{j=1}^{r-1} h^{j-1}(1-h)p_0 + \sum_{j=r}^{\infty} h^{j-1}(1-h) \times 1 = (1 - h^{r-1})p_0 + h^{r-1}. \tag{29}$$

Solving (28)–(29) for $p_{0,1}$ yields

$$p_0 = \frac{h^{r-1}(1 - (1-h)^{s-1})}{h^{r-1} + (1-h)^{s-1} - h^{r-1}(1-h)^{s-1}},$$

$$p_1 = \frac{h^{r-1}}{h^{r-1} + (1-h)^{s-1} - h^{r-1}(1-h)^{s-1}},$$

and we then obtain (27) by using the obvious relation $\mathbb{P}[\tau < \sigma] = (1-h)p_0 + hp_1$. $\qquad\square$