HW #2

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1) There are 2 relevant properties to confirm. We need to show that $W \subset V$ is closed under $+$ and $\times$.

a) We consider whether if $\det A = 0$ and $\det B = 0$

if i) $\det(A+B) \neq 0$. Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

So it's not a vector space.

b) If trace $A = 0$, trace $B = 0$, then

$\text{trace}(A+B) = 0 = \sum_{i=1}^{n} (a_{i,i} + b_{i,i}) = \sum_{i=1}^{n} a_{i,i} + \sum_{i=1}^{n} b_{i,i} = 0$

$\text{trace } cA = \sum_{i=1}^{n} c\, a_{i,i} = c \sum_{i=1}^{n} a_{i,i} = 0$

Also, the zero matrix (all zeros) is in W since trace $0 = 0$

so W is a subspace

c) If f, g are continuous on $[0,1]$ and $f(0)f(1) = g(0)g(1) = 1$

$f(0) = 2 \qquad g(0) = 3$

$f(1) = \frac{1}{2} \qquad g(1) = \frac{1}{3}$

$h(x) = f(x)\, g(x)$

$h(0) = f(0) + g(0) = 2 + 3 = 5$

$h(1) = f(1) + g(1) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$h(0)\, h(1) = \frac{25}{6} \neq 1$

We have shown that W is <u>not</u> a subspace of V.

d) Let $h(x) = f(x) + g(x)$

$f(\frac{1}{2}) = \int_0^1 f(t)\,dt$; $g(\frac{1}{2}) = \int_0^1 g(t)\,dt$

$h(\frac{1}{2}) = f(\frac{1}{2}) + g(\frac{1}{2}) = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = \int_0^1 [f(t) + g(t)]\,dt = \int_0^1 h(t)\,dt$

so the 1st property is satisfied.

Let $h(x) = c\, g(x)$

$h(\frac{1}{2}) = c\, g(\frac{1}{2}) = c \int_0^1 g(t)\,dt = \int_0^1 c\, g(t)\,dt = \int_0^1 h(t)\,dt$

Also, let $f(x) = 0$ be the zero vector, which clearly is in W.

so all properties are satisfied and W is <u>a subspace</u>.

e) Let $v = a + b$; $\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} = 0$; $\frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y} = 0$

$\nabla \cdot v = \frac{\partial}{\partial x}(a_1 + b_1) + \frac{\partial}{\partial y}(a_2 + b_2) = \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y}\right) + \left(\frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y}\right)$

$= 0$

Let $v = c\,a$

$\nabla \cdot v = \frac{\partial}{\partial x}(c\,a_1) + \frac{\partial}{\partial y}(c\,a_2) = c\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y}\right) = 0$ as derived

Also, let $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the zero vector, which has $\nabla \cdot v = 0$, so it's in W.

so W is a subspace.

2. a. $a(x^2 - 3) + b(2 - x) + c(x^2 + 2x + 1) = 0$

$(a+c)x^2 + (2c - b)x + (c + 2b - 3a) = 0$

$a = -c$; $b = 2c$; $c + (4c + 3c) = 0$

$c = 0$; $b = 0$; $a = 0$

Using polynomial equivalence, we see that this only 0 when the weights a, b, c are 0, so they are linearly independent.

b. Consider a generic $p(x) = \ell x^2 + m x + n$

$\ell x^2 + mx + n = (a+c)x^2 + (2c - b)x + (c + 2b - 3a)$

$\ell = a + c$

$m = -b + 2c$

$n = -3a + 2b + c$

$3\ell + n = 2b + 4c$

$(3\ell + n) + 2m = 8c$

$c = \frac{3\ell + 2m + n}{8}$ $\quad b = \frac{(3\ell + n) - 4c}{2} = \frac{(3\ell + n) - \left(\frac{3\ell + 2m + n}{2}\right)}{2}$

$b = \frac{\frac{3\ell}{2} - m + \frac{n}{2}}{2} = \frac{9\ell + 2m + n}{4}$

$a = 1 - c = \frac{5\ell - 2m - n}{8}$

Since we can find a set of weights for any $p(x) \in P^{(2)}$, these span $P^{(2)}$.

c) By the previous two properties, they form a basis for $P^{(2)}$.

The coords are $a = \frac{0 - 0 - 1}{8} = -\frac{1}{8}$

$b = \frac{0 + 0 + 1}{4} = \frac{1}{4}$

$c = \frac{0 + 0 + 1}{8} = \frac{1}{8}$

3) We will use the property [1] that if $f, g \in F$ have the same first two elements, denoted f_1 & f_2 and g_1 & g_2, then the entire sequence is the same.

Base case: $f_1 = f_2$ & $g_1 = g_2$. Strong induction, $f_i = g_i \;\forall\, i : i < n$. $f_n = f_{n-1} + f_{n-2} = g_{n-1} + g_{n-2} = g_n$

a) Let $x_{i,j}$ mean x_j of f_i, or the jth element of f_i

1) We first claim the 0 vector $f = 0$

$f = 0$ if $x_i = 0 \;\forall\, i$. This is in F because $x_n = x_{n-1} + x_{n-2} = 0$

Then $f_1 + 0 = f_1$ because $x'_{1,1} = x_{1,1} + 0$, $x'_{1,2} = x_{1,2} + 0$ so the entire series must be the same [1]

2) Then we claim $f+g = h \in F$

$$\left.\begin{aligned} h_1 &= f_1 + g_1 \\ h_2 &= f_1 + g_2 \end{aligned}\right\} \text{base case}$$

We use strong induction, $\forall j : j < i$, $h_j = h_{j-1} + h_{j-2}$

$$h_i = f_i + g_i = (f_{i-1} + g_{i-1}) + (f_{i-2} + g_{i-2})$$
$$= h_{i-1} + h_{i-2}$$

3) Finally we claim $cf + g \in F$

$$\left.\begin{aligned} g_1 &= cf_1 \\ g_2 &= cf_2 \end{aligned}\right\} \text{base case}$$

We use strong induction, $\forall j : j < i$, $g_j = g_{j-1} + g_{j-2}$

$$g_i = cf_i = c(f_{i-1} + f_{i-2}) = g_{i-1} + g_{i-2} \text{ as desired}$$

Thus F is a vector space.

b) Let $f_1 = (1, 0, 1, 1, 2, 3, 5, \dots)$
$f_2 = (0, 1, 1, 2, 3, 5, 8, \dots)$

and we claim $\{f_1, f_2\}$ is a basis.

FIRST) f_1 & f_2 are linearly independent

since $c_1 f_1 + c_2 f_2 = (c_1, c_2, \dots)$ only is $0 \text{ or } (0, 0, 0, \dots)$

if $c_1 = c_2 = 0$

NEXT) f_1 & f_2 span F.

Consider $g_1 = (x_1, x_2, x_1 + x_2, \dots)$

We let $c_1 = x_1$ and $c_2 = x_2$

Then $(g_1)_1 = c_1 (f_1)_1 + c_2 (f_2)_1 = c_1 = x_1$

$(g_1)_2 = c_1 (f_1)_2 + c_2 (f_2)_2 = c_2 = x_2$ (base case)

By strong induction, we have that $(g_1)_i = c_1 (f_1)_i + c_2 (f_2)_i \ \forall i : i < n$

$$(g_1)_n = (g_1)_{n-1} + (g_1)_{n-2} = (c_1 (f_1)_{n-1} + c_2 (f_2)_{n-1})$$
$$+ (c_1 (f_1)_{n-2} + c_2 (f_2)_{n-2})$$
$$= c_1 (f_1)_n + c_2 (f_2)_n,$$

so they are the same vectors and we found the

c) f^* has coords $c_1 = c_2 = 1$.

4) We first use the result from PS1 that the first two rows are linearly independent and $b^k = b^1 + (k-1)(b^2 - b^1)$

KERNEL) $Ax = 0$. $A = \{b_j^i\}$

We consider b^1 and $(b^2 - b^1) = c^2$ where $c_i^2 = (n+i) - i(n+1)$
$= n + i - in - i = n(1-i)$ $\rightarrow \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -1 & -2 & \cdots & (1-n) \end{bmatrix}$

which we can further simplify to $c_{1,i} = 1 - i$

$$x_2 = \sum_{k=3}^{n} (1-k) x_k$$
$$c_i^1 = b_i^1 + c_i^2 \times 2 = i + 2(1-i) = 2 - i$$
$$x_1 = \sum_{k=3}^{n} (k-2) x_k$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 & \cdots \\ 0 & 1 & +2 & 3 & 4 & \cdots \end{bmatrix}$$

$$x = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ \vdots \end{bmatrix} + \dots \quad \text{where } (\omega_k)_1 = k-2, \ (\omega_k)_2 = 1-k$$
$$(\omega_k)_i = 1 \text{ if } i = k, \ 0 \text{ if } i \neq k \ \forall i > 2$$

IMAGE) We take the pivot columns, which is the first two columns as the basis

$$\begin{bmatrix} 1 \\ n+1 \\ 2n+1 \\ \vdots \\ n(n-1)+1 \end{bmatrix} \ \& \ \begin{bmatrix} 2 \\ n+2 \\ 2n+2 \\ \vdots \\ n(n-1)+2 \end{bmatrix}$$

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$$\begin{matrix} 1 & n+1 & 2n+1 & \cdots & n(n-1)+1 \\ 2 & n+2 & 2n+2 & \cdots & n(n-1)+2 \end{matrix} \Longrightarrow [(i-1)n + j]$$
$$- j[(i-1)n+1]$$
$$= (j-1)n(1-i)$$

We see that rank $(A^T) = 2$ as expected.

$$a_i^1 = (i-1)n + 1$$
$$b_i^2 = \underbrace{a_i^2 - a_i^1}_{r} = (1-i)$$

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \cdots \\ 0 & -n & -2n & \cdots \end{bmatrix}$$

$$b_i^1 = b_i^2(n+1) + a_i^1 = (i-1)n + 1 + (n+1)(1-i) = \cancel{n} - \cancel{n} + 1 + \cancel{n} + 1 - \cancel{n} - i = 2 - i$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 & \cdots \\ 0 & -1 & -2 & \cdots \end{bmatrix}$$

$$x_1 = \sum_{k=3}^{n} (k-2) x_k \qquad (1 \ 2 \ 3 \ \cdots \ n-2)^T$$
$$x_2 = \sum_{k=3}^{n} (2-k) x_k \qquad (-2 \ -3 \ -4 \ \cdots \ (2-n))^T$$

$$x = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ \vdots \end{bmatrix} + \dots \quad \text{where } (\omega_k)_1 = k-2, \ (\omega_k)_2 = 1-k$$
$$(\omega_k)_i = 1 \text{ if } i = k, \ 0 \text{ if } i \neq k \ \forall i > 2$$

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We take the pivot columns (which is rows of A):

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix} \ \& \ \begin{bmatrix} n+1 \\ n+2 \\ n+3 \\ \vdots \\ 2n \end{bmatrix}$$