Pranav Raikar

23/10/22

Sem III S.E. INFT Branch

College:- VESIT

Q1

(a) → $f(t) = t e^{t} \sin 2t \cos t$

$\sin 2t \cos t = \frac{1}{2}[\sin 3t + \sin t]$

~~L{f(t)}~~

$L\{\frac{1}{2}[\sin 3t + \sin t\} = \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$

$L\{t \sin 2t \cos t\} = \frac{1}{2}(-1)^{1} \frac{d}{ds}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$

$= \frac{-1}{2}\left[\frac{-6s}{(s^2+9)^2} + \frac{-2s}{(s^2+1)^2}\right]$

$$\boxed{L\{e^{t} t \sin 2t \cos t\} = \frac{3(s-1)}{(s^2+9)^2} + \frac{(s-1)}{[(s-1)^2+1]^2}}$$

Q1

(b) ~~F(s) s+2~~ ~~F(s) =~~

$L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}$

$= L^{-1}\left\{\frac{1}{s}\frac{s+2}{s(s+3)}\right\}$

$= L^{-1}\left\{\frac{1}{s}F(s)\right\}$

where $F(s) = \frac{s+2}{s(s+3)}$

Date:
Page:

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{s(s+3)}\right\}$$

$$= L^{-1}\left\{\underbrace{\frac{1}{s+3}}_{F_1(s)} + \underbrace{\frac{2}{s(s+3)}}_{F_2(s)}\right\}$$

$$L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

For

$$L^{-1}\left\{\frac{2}{s(s+3)}\right\} =$$

$$\frac{2}{s(s+3)} = \frac{A}{s} + \frac{B}{(s+3)}$$

$$2 = A(s+3) + Bs$$
$$2 = As + 3A + Bs$$
$$2 = (A+B)s + 3A$$

$$3A = 2$$
$$A = \frac{2}{3}$$

$$A + B = 0$$
$$B = -\frac{2}{3}$$

$$\frac{2}{s(s+3)} = \frac{\frac{2}{3}}{s} + \frac{-\frac{2}{3}}{(s+3)}$$

Date:
Page:

$$L^{-1}\left[\frac{2}{s(s+3)}\right] = L^{-1}\left\{\frac{2}{3s} - \frac{2}{3}\frac{1}{s+3}\right\}$$

$$= \frac{2}{3}(1) - \frac{2}{3}e^{-3t} L^{-1}\left\{\frac{1}{s}\right\}$$

$$= \frac{2}{3} - \frac{2}{3}e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s(s+3)}\right] = e^{-3t} + \frac{2}{3} - \frac{2}{3}e^{-3t}$$

$$L^{-1}\left\{\frac{1}{s}\frac{s+2}{s(s+3)}\right\} = \int_0^t \left(e^{-3t} + \frac{2}{3} - \frac{2}{3}e^{-3t}\right)dt$$

$$= \left(\frac{e^{-3t}}{3} + \frac{2t}{3} - \frac{2}{9}e^{-3t}\right)_0^t$$

$$L^{-1}\left\{\frac{(s+2)}{s^2(s+3)}\right\} = \frac{e^{-3t}}{3} + \frac{2t}{3} - \frac{2}{9}e^{-3t} - \frac{1}{3} + \frac{2}{9}$$

//

Q1

(c) $f(z) = x^2 - y^2 + 2ixy$

comparing with

$f(z) = u + iv$

$\Rightarrow u = x^2 - y^2$

$v = 2xy$

Date:
Page:

$$\frac{\delta u}{\delta x} = 2x$$

$$\frac{\delta u}{\delta y} = -2y$$

$$\frac{\delta v}{\delta x} = 2y$$

$$\frac{\delta v}{\delta y} = \cancel{2y}\ 2x$$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \ \& \ \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$\therefore$ Function $f(z)$ is analytic.

$f(z) = u + iv$

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$ — [partially diff. w.r.t. x]

$f'(z) = 2x + i\,2y$

~~put~~ By milne thompson's method

put $x = z$, $y = 0$

$\cancel{f'z}\ \therefore \boxed{f'(z) = 2x}$

$$\boxed{f(z) = x^2 + c}$$

Date:
Page:

Q1

(d) $f(x) = e^{-|x|}$ — $(-\pi, \pi)$

$f(-x) = e^{-|-x|}$

$= e^{-|x|}$ $\therefore$ function $f(x)$ is even

$\therefore b_n = 0$

Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} e^{-x}\,dx$$

$$= \frac{2}{\pi}\left(\frac{e^{-x}}{(-1)}\right)_0^{\pi}$$

$$a_0 = \frac{-2}{\pi}\left(e^{-\pi} - 1\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx$$

Date:
Page:

$$=\frac{2}{\pi}\int_{0}^{\pi} e^{-x}\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(\cancel{-}\cos nx+n\sin nx\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(\cancel{n\sin nx}-\cos nx\right)\right]_0^{\pi}$$

$$=\frac{-2}{\pi}\left[\frac{e^{-x}}{1+n^2}\cos nx\right]_0^{\pi}$$

$$=\frac{-2}{\pi}\left[\frac{e^{-\pi}}{1+n^2}\cos n\pi-\frac{1}{1+n^2}\right]$$

$$=\frac{-2}{\pi}\left[\frac{e^{-\pi}(-1)^n}{1+n^2}-\frac{1}{1+n^2}\right]$$

Fourier series is

$$f(x)=\frac{-(e^{-\pi}-1)}{\pi}+\sum_{n=1}^{\infty}\frac{-2}{\pi}\left[\frac{e^{-\pi}(-1)^n-1}{1+n^2}\right]$$

Q2 (a) $I=\int_0^{\infty}\frac{e^{-t}-\cos t}{te^{4t}}\,dt$

$$=\int_0^{\infty}e^{-4t}\left(\frac{e^{-t}-\cos t}{t}\right)dt$$

$$L\{e^{-t}\}=\frac{1}{s+1}\quad \& \quad L\{\cos t\}=\frac{s}{s^2+1}$$

Date: ______
Page: ______

$$L\left\{e^{-t} - \cos t\right\} = \frac{1}{s+1} - \frac{s}{s^2+1}$$

$$L\left\{\frac{e^{-t} - \cos t}{t}\right\} = \int_s^\infty \frac{1}{s+1} - \frac{1}{2}\frac{2s}{s^2+1}\, ds$$

$$= \Big[\log(s+1) - \frac{1}{2}\log(s^2+1)\Big]_s^\infty$$

$$= \left(\log \frac{s+1}{\sqrt{s^2+1}}\right)_s^\infty$$

$$= \left(0 - \log \frac{s+1}{\sqrt{s^2+1}}\right)$$

$$L\left\{\frac{e^{-t} - \cos t}{t}\right\} = \log \frac{\sqrt{s^2+1}}{s+1}$$

$$\int_0^\infty e^{-4t}\left(\frac{e^{-t} - \cos t}{t}\right) dt$$

compare $e^{-st} = e^{-4t}$

$s = 4$

$$= \log \frac{\sqrt{16+1}}{4+1}$$

$$\boxed{I = \log \frac{\sqrt{17}}{5}}$$

Q2
(c) $u = 2x(1-y) = 2x - 2xy$

$\frac{\delta u}{\delta x} = 2(1-y)$ | $\frac{\delta u}{\delta y} = -2x$

$\frac{\delta^2 u}{\delta x^2} = 0$ | $\frac{\delta^2 u}{\delta^2 y} = 0$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

$\therefore$ u is Harmonic function

$u = 2x - 2xy$

$\frac{\delta u}{\delta x} = \cancel{2(1-y)}\ 2 - 2y$

$\frac{\delta u}{\delta y} = -2x$

$f(z) = u + iv$

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$ — (partially diff. w.r.t. x)

$$= \frac{\delta u}{\delta x} - i\frac{\delta u}{\delta y}$$

$f'(z) = 2 - 2y + i2x$

using milne thompson

put $x = z$, $y = 0$

$f'(z) = 2 + i2z$

Date:
Page:

$$f(z) = \int 2 + i2z\, dz$$

$$= 2z + iz^2 + c$$

$$= 2(x+iy) + i(x+iy)^2$$

$$= 2x + 2iy + i(x^2 - y^2 + i2xy)$$

$$= 2x + 2iy + x^2 i - y^2 i - 2xy$$

$$\boxed{f(z) = (2x - 2xy) + i(2y + x^2 - y^2)}$$

$\Rightarrow$ ~~the~~ harmonic conjugate is
$v = 2y + x^2 - y^2$

Q2
(b) $f(x) = kx^2 e^{-x} \qquad 0 \le x \le 1$

(a) $\int_0^1 f(x)\, dx = 1$

$$\int_0^1 kx^2 e^{-x}\, dx = 1$$

$$k\left[\frac{x^2 e^{-x}}{(-1)} - \frac{2x e^{-x}}{1} + \frac{2e^{-x}}{-1}\right]_0^1 = 1$$

$$k\left[-e^{-1} - 2e^{-1} - 2e^{-1} + 2\right] = 1$$

Date:
Page:

$$k = \frac{-13}{50}$$

$$f(x) = \frac{-13}{50} x^2 e^{-x}$$

mean

$$\mu_1' = \int x f(x) dx$$

$$= \int_0^1 x \frac{-13}{50} x^2 e^{-x} dx$$

$$= \frac{-13}{50} \int_0^1 x^3 e^{-x} dx$$

$$= \frac{-13}{50} \left[\frac{x^3 e^{-x}}{-1} - \frac{3x^2 e^{-x}}{1} + \frac{6x e^{-x}}{-1} - \frac{6 e^{-x}}{1} \right]_0^1$$

$$= \frac{-13}{50} \left(\frac{e^{-1}}{-1} - \frac{3e^{-1}}{1} + \frac{6e^{-1}}{-1} - 6e^{-1} + \right.$$

Q3 To prove:

(b) $L\{\int_0^\infty e^{-t} \int_0^t e^{u} u^{-1} \sin u \, du\} = \frac{\pi}{2}$

Sol:

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty \frac{1}{s^2+1} = \frac{1}{1}\left(\tan^{-1}\frac{s}{1}\right)_s^\infty$$

Date:
Page:

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$L\left\{\frac{\sin u}{u}\right\} = \tan^{-1} \frac{1}{s}$$

$$L\left\{e^{u}\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{(s-1)}$$

$$L\left\{\int_0^t e^{u}\frac{\sin u}{u}\right\} = \frac{1}{s}\tan^{-1}\frac{1}{(s-1)}$$

~~comparing with~~

$$\int_0^\infty e^{-t}\int_0^t \frac{e^{u}\sin u}{u}$$

compare ~~with~~ $e^{-st} = e^{-t}$

$s = 1$

$$\therefore \int_0^\infty e^{-t}\int_0^t e^{u}\frac{\sin u\,du}{u} = \frac{1}{1}\tan^{-1}\frac{1}{0}$$

$$= \frac{\pi}{2} \;//$$

Hence proved

Q 5

(b) $u = y^3 - 3x^2 y$

$$\frac{\delta u}{\delta x} = -6xy \quad \Big| \quad \frac{\delta^2 u}{\delta x^2} = -6y$$

Date:
Page:

$$\frac{\delta u}{\delta y} = 3y^2 - 3x^2$$

$$\frac{\delta^2 u}{\delta y^2} = 6y$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 6y - 6y$$
$$= 0$$

$\therefore$ u is harmonic function.

~~δ~~ $f(z) = u + iv$

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$ — diff. partially w.r.t. x

$$f'(z) = \frac{\delta u}{\delta x} - i\frac{\delta u}{\delta y}$$

$$f'(z) = -6xy - i3y^2 + 3x^2$$

By milne thompson method.

put $x = z$, $y = 0$

$$f'(z) = -6z + 3z^2$$

$$f(z) = -6\frac{z^2}{2} + \cancel{3}\frac{z^3}{}$$

$$\boxed{f(z) = -3z^2 + z^3}$$

~~$= -3(x+iy)^2$~~

~~$= -3()x^2 - 3y^2$~~

~~$f(z) = -3z(-3x^2 + 3y^2 + i6xy$~~

~~$= z(3z + z^2)$~~

$$\cancel{f(z) = -z(3z+z^2)}$$

$$\cancel{= -[(x+iy)(3x+3iy+x^2-y^2-i2xy)]}$$

$$\cancel{= -[3x^2+i3xy+x^3}$$

$$f(z) = -3z^2+z^3$$

$$= -3(x^2-y^2+i2xy)+(x+iy)^3$$

$$= -3x^2+3y^2-i6xy+(x^3+i3x^2y-3xy^2-iy^3)$$

$$= -3x^2+3y^2+x^3$$

Q5

(c)

x :	-2	-1	0	1	2	3
$P(X)$:	0.1	k	0.2	$2k$	0.3	k

$$\Sigma P(X) = 1$$

$$\cancel{4 \cdot 6k = 1}$$

$$k = 0.1$$

Date:
Page:

X:	-2	-1	0	1	2	3
$p(x) =$	0.1	0.1	0.2	0.2	0.3	0.1

(5) mean (μ_1')

$$\mu_1' = \sum x\, p(x)$$

$$= (-2\times 0.1) + (-1\times 0.1) + 0 + (1\times 0.2) + (2\times 0.3) + (3\times 0.1)$$

$$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3$$

$$\boxed{\mu_1' = 0.8}$$

Variance $V(x) = \mu_2' - (\mu_1')^2$

$$\mu_2' = \sum x^2 p(x)$$

$$= (4\times 0.1) + (1\times 0.1) + 0 + (1\times 0.2) + (4\times 0.3) + (9\times 0.1)$$

$$= 0.4 + 0.1 + 0.2 + 1.2 + 0.9$$

$\mu_2' = 2.8$

Variance $= 2.8 - 0.64$

$\boxed{\text{Variance} = 2.16}$