Week 11 and 12 Recap

Pranav Rao

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1 Chebyshev's Inequality

- For any random variable X and any $\varepsilon > 0$:

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} Var(X)$$

- Also note this other result: denote $Var(X)$ by σ^2 and consider the probability that X is within a few standard deviations from its expectation μ:

$$P(|X - E(X)| < k\sigma) \geq 1 - \frac{1}{k^2}$$

  where k is a small integer.

- You can use this inequality to bound probabilities.

2 New Random Variable Notation

- Let n be the number of indepdendent observations from the chosen from the probability probability distribution of a random variable X.

- Consider $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ where X_i are iid with mean μ_X and variance σ_X^2. Then:

$$\mu_{\overline{X}} = E[\overline{X}] = E(\frac{X_1 + X_2 + \ldots + X_n}{n})$$

$$\sigma_{\overline{X}}^2 = Var(\overline{X}) = Var(\frac{X_1 + X_2 + \ldots + X_n}{n})$$

- $\mu_{\overline{X}} = \mu_X$

- $\sigma_{\overline{X}}^2 = \frac{\sigma_X^2}{n}$, regardless of n.

- As n gets large, $\overline{X}$ approaches μ_X (by Law of Large numbers - see below).

3 Law of Large Numbers

- **Idea:** if you have a hotel with infinitely many rooms which contain guests that are flipping coins forever, the strong law of large numbers says that, in virtually every room of the hotel, the sequence of averages will converge to $\frac{1}{2}$ and stay close to $\frac{1}{2}$ for all remaining terms.

- **Weak Law of Large Numbers:** Let $X_1, X_2, \ldots$ be an i.i.d sequence of random variables with finite mean μ and variance σ^2. For $n = 1, 2, \ldots$, let $S_n = X_1 + \ldots + X_n$. Then:
$$\lim_{n\to\infty} P(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon) = 0$$

- **Strong Law of Large Numbers:** Let $X_1, X_2, \ldots$ be an i.i.d sequence of random variables with finite mean μ. For $n = 1, 2, \ldots$, let $S_n = X_1 + \ldots + X_n$. Then:
$$P(\lim_{n\to\infty} \frac{S_n}{n} = \mu) = 1$$
We say that $\frac{S_n}{n}$ converges to μ with probability 1.

- Basically this law means that for some $\overline{X}, X, \mu_X$, as $n \to \infty$, $\overline{X} \to \mu_X$.

4 Central Limit Theorem

Let X_1, X_2 be any sequence of i.i.d. random variables with finite positive variance. Let μ be the expected value and σ^2 the variance of each of the X_i. For $n \geq 1$, let Z_N be defined by:

$$Z_n = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma}$$

Then, for any number a:

$$\lim_{n\to\infty} F_{Z_n}(a) = \phi(a)$$

where ϕ is the distribution function of $N(0, 1)$. In other words, the distribution function of Z_n converges to the distribution function ϕ of the standard normal distribution as $n \to \infty$.