Week 10 Recap

Pranav Rao

Sunday, November 26, 2023

1 Methods of Finding PDF of Functions of Random Variables

1.1 CDF Method of Finding PDF of a Function of a Random Variable

Let U be a function of the random variables Y.

1. Find the region $U \leq u$.
2. Find $F_U(u) = P(U \leq u)$ by integrating $f(y)$ over the region of $U \leq u$.
3. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = \frac{dF_U(u)}{du}$

1.2 Method of Transformations to Find PDF of Function of a Random Variable

- If $g(X)$ is monotonic (i.e. either strictly increasing or decreasing over the range of X) so it is invertible with inverse function $X = g^{-1}(Y)$, then:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

2 Using MGFs to Determine Distribution of Sum of IID Random Variables

- **IID Random Variables:** independent and independently distributed (have the same distribution)
- Let X and Y be independent variables with moment generating functions $M_X(t)$, $M_Y(t)$, respectively. Then the MGF of $X + Y$ can be found as follows:

$$M_{X+Y}(t) = E[e^{tX+tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

- If you calculate the above function and compare it with the MGFs of known functions, you can find the distribution of the sum $X + Y$.

3 Bivariate Transformations Using Jacobians

Suppose that Y_1, Y_2 are continuous random variables with joint density function $f_{Y_1,Y_2}(y_1, y_2)$ and that for all (y_1, y_2) such that $f_{Y_1,Y_2}(y_1, y_2) > 0$.

$$u_1 = h_1(y_1, y_2) \text{ and } u_2 = h_2(y_1, y_2)$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse:

$$y_1 = h^{-1}(u_1, u_2) \text{ and } y_2 = h_2^{-1}(u_1, u_2)$$

If $h^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1, u_2 and the **Jacobian**:

$$J = det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1} \frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_2^{-1}}{\partial u_1} \frac{\partial h_1^{-1}}{\partial u_2} \neq 0$$

Then, the joint density function of U_1 and U_2 is:

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))\, |J|$$

where $|J|$ is the absolute value of the Jacobian.