Math 2331, Spring 2020

Midterm 1 practice

Below is a reminder of some important topics that are covered on midterm 1.

(1) Matrix algebra.

(2) Find the solution to a system of equations $A\vec{x} = \vec{b}$ parametrically; this means find the free variables (or *parameters*) and write the general solution in terms of these.

(3) Row operations and reduced row echelon form.

(4) The rank of a matrix, recognizing when a square matrix is invertible, finding the inverse, using the inverse to solve equations.

(5) Vector spaces and linear transformations (how to recognize when a function $T : V \to W$ between vector spaces is a linear transformation, and when it isn't).

(6) The matrix of a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

(7) Finding the inverse of a linear transformation.

(8) Special linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ (shears, rotations, reflections, etc).

(9) Subspaces, span, linear relations, linear independence, basis, dimension. You should know how to decide if a collection of vectors is linearly independendent, or when a collection of vectors spans the whole vector space in which they live.

(10) Writing vectors in terms of a basis.

(11) Kernels and images of linear transformations; how to find bases of these.

(12) The rank nullity theorem.

(13) The $\mathcal{B}$-matrix of a linear transformation.

(14) The dot product, orthogonality, length, and othornormal sets of vectors.

(15) Orthogonal projection onto a subspace $V \subset \mathbb{R}^n$.

Below are some practice problems; more may be added later.

(1) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ -10 & -8 & 12 \\ 7 & 5 & -10 \end{bmatrix}$.

(a) Find A^{-1}. Check your answer using matrix multiplication.

(b) Use your answer from part A to solve the matrix equation $AX + C = B$, where

$$B = \begin{bmatrix} -1 & 3 & 5 \\ 0 & -4 & 7 \\ 5 & 8 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 3 & 9 \\ 6 & -2 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

(2) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2z - 4x + y \\ 3y - z \\ x + y + z \end{bmatrix}.$$

(a) Find the standard matrix of T.

(b) Find the $\mathcal{B}$-matrix of T, where $\mathcal{B}$ is the basis of $\mathbb{R}^3$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(3) Let $T : M_{4\times 4}(\mathbb{R}) \to \mathbb{R}$ be the linear transformation defined by

$$T(A) = A_{11} + A_{22} - A_{33} - A_{44}.$$

Given that the image of T is all of $\mathbb{R}$, find the dimension of $\ker(T)$.

(4) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & -5 & 0 & 3 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & -1 & -2 & 0 & -3 \\ -1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 2 & 3 & 5 \end{bmatrix}$$

(a) Find a basis for the image of A.

(b) Without actually computing the kernel of A, what is its dimension?

(5) (a) Show that the following vectors form a basis for $\mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 16 \\ 0 \end{bmatrix}, \qquad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

(b) Write the vector $\begin{bmatrix} 3 \\ 0 \\ -11 \\ 8 \end{bmatrix}$ as a linear combination of these vectors.

(6) Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{bmatrix}$$

(a) Find RREF(A).

(b) Using your answer from part (a), solve the equation $A\vec{x} = 0$ parametrically.

(c) Using your answer from part (b), find a basis for the kernel of A.

(d) Find a basis for the image (that is, the column space) of A.

(7) Let V be the subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Let $\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Find vectors $\vec{x}^{\perp}, \vec{x}^{\parallel} \in \mathbb{R}^2$ such that

- $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel}$
- $\vec{x}^{\parallel} \in V$,
- $\vec{x}^{\perp}$ orthogonal to L.

(8) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Find a nonzero vector $\vec{w}$ which is orthogonal to $\vec{v}_1$ and $\vec{v}_2$.