Thoughts on OLGs

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Introduction

As some context beforehand, overlapping generations models (OLGs) at a very basic sense with only groups of consumers looks at the trends in savings and consumption for households who live for only a finite number of time periods. From this standpoint we can then incorporate firms with production functions that determine the amount of goods available for households to consume who offer labor to firms in exchange, and also governments who can tax and offer other consumption services. Let's put the basic OLG model into math language before we discuss how we can extend the problem.

Specifically, let N many households be born in each time period $t = 1, 2, 3, \ldots$. Each household lives for two periods; the period in which they were born and the next consecutive period in which they consume a single type of a non-storable good. Let there also be N many "initially old" households at time $t = 1$: this means that at time $t = 1$ we also populate it with a group of households that act as if they were born in $t = 0$ but are not able to consume anything in that period. The utility for an household born in period t who values consumption in a current period with the function $u(c)$ is

$$U\left(c_t^t, c_{t+1}^t\right) = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right).$$

Here $\beta \in (0, 1]$ is a discount factor, and c_i^j is the consumption for a household in period i who was born in period j. Households are meant to optimize their utility with respect to the conditions that

$$c_t^t \leq w_t^t,$$
$$c_{t+1}^t \leq w_{t+1}^t.$$

Here, similarly to the indexing on consumption, w_i^j is the endowment of the consumption good for a household in period i who was born in period j. In a market in which each of these households interact with eachother we allow the existence of consumption bonds, or agreements between households that in exchange for one household giving another household units of the consumption good in the current period, in the next period the household who recieved the goods must then give some number of consumption goods in the future period to the household that originally gave the goods. Note that this restricts households to trading bonds only with other households born in the same time period since if they were to trade with households in different periods then in the next period the previously old generation of households would no longer be alive and would not be able to fufill their debt obligations. Furthermore, it is also logically consistent that if all the households born in a certain time period are homogenous in both utility functions and endowment then it would make no sense for them to trade bonds with eachother since there is no scenario in which a trade between two households would benefit both parties.

With this setup in mind in order to solve one such scenario we can setup the following lagrangian maximization problem for a consumer at time t

$$\mathcal{L}(c_t^t, c_{t+1}^t, \mu_1, \mu_2) = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right) + \mu_1\left(w_t^t - c_t^t\right) + \mu_2\left(w_{t+1}^t - c_{t+1}^t\right),$$

and the following first order conditions:

$$\begin{aligned} c_t^t &: u'\left(c_t^t\right) = \mu_1, \\ c_{t+1}^t &: u'\left(c_{t+1}^t\right) = \mu_2, \\ \mu_1 &: c_t^t = w_t^t, \\ \mu_2 &: c_{t+1}^t = w_{t+1}^t. \end{aligned}$$

Which would then very clearly imply that $c_t^t = w_t^t$ and $c_{t+1}^t = w_{t+1}^t$ meaning each household consumes exactly their in period endowment. This is not a very exciting outcome but introducing even a little bit of complexity can lead to more interesting scenarios.

In class we discussed a few variations of the setup above:

1. Added in-period heterogeneity that incentivizes consumption bond trading.
2. Added in the concept of fiat money.
3. Added in governments that tax households and also consume some amount goods as well.
4. Added in varying population sizes over time.
5. We discussed a very basic version of adding firms with production functions

There are some things that I want to explore that we didn't explicitly discuss in class:

1. Households that live for N many periods
2. Add in stochasticity
3. Discuss a generalized sense of heterogeneity

4. Develop a more involved model with firms
5. Consider a continuum of firms and/or firms
6. What other financial instruments can we consider other than fiat money and consumption bonds
7. Add a more complicated government (basically a government that does more than just taxes and consumes goods)
8. Multiple goods

I'm sure that each of these concepts have been picked apart and discussed in fine detail since whenever the first theory on OLGs was developed but I want to work through and model some of them myself without having been influenced by any of the papers or textbooks that discuss these topics (which also leaves a lot of room for errors on my end so be aware of that) as a means to work out my - econ related - creative muscles. In addition, to preface, with the foresight of having already done a bit of work on each of these questions, in order to come to some level of a meaningful answer that relates a trend of savings or an actual equilibrium we have to impose more restrictions on the structure of the endowments (if not actual quantities then at least relationships between values in the form of inequalities). All that being said, here are my thoughts on a few of these items.

Multiple Goods

The basic difference here is that we add dimensionality to the utility function, endowments, and consumption choices. In this case let's just assume that there are two goods. Let's proactively introduce a little bit of heterogeneity (otherwise we have the degenerate case where everyone just consume their endowment) and say that each period an equal number of two types of workers are born. A worker who gets endowment $(0,0)$ when they are young and (w_1^o, w_2^o), and another worker who gets endowment (w_1^y, w_2^y) when they are young and $(0,0)$. With this in mind (and by normalizing both the young income and old income populations to $N = 1$), we can write the household's optimization problem as

$$\max_{(c_{1,t}^t, c_{1,t+1}^t), (c_{2,t}^t, c_{2,t+1}^t), (s_{1,1}^t, s_{1,2}^t, s_{2,1}^t, s_{2,2}^t)} u(c_{1,t}^t, c_{2,t}^t) + \beta u(c_{1,t+1}^t, c_{2,t+1}^t)$$

such that the individual budget constraints are satisfied respectively for the young income and the old income worker are

$$\begin{aligned} c_{1,t}^t + s_{1,1}^t + s_{1,2}^t &\leq w_1^y & \qquad c_{2,t}^t + s_{2,1}^t + s_{2,2}^t &\leq w_2^y \\ c_{1,t+1}^t &\leq R_{1,1}^t s_{1,1}^t + R_{1,2}^t s_{1,2}^t & \qquad c_{2,t+1}^t &\leq R_{2,1}^t s_{2,1}^t + R_{2,2}^t s_{2,2}^t \end{aligned}$$

and

$$c_{1,t}^t + s_{1,1}^t + s_{1,2}^t \leq 0 \qquad\qquad c_{2,t}^t + s_{2,1}^t + s_{2,2}^t \leq 0$$
$$c_{1,t+1}^t \leq R_{1,1}^t s_{1,1}^t + R_{1,2}^t s_{1,2}^t + w_1^o \qquad\qquad c_{2,t+1}^t \leq R_{2,1}^t s_{2,1}^t + R_{2,2}^t s_{2,2}^t + w_2^o$$

In addition, the following markets must clear

- **Consumption Bonds**

  Specifically that for all t we have that

$$\begin{aligned} s_{1,1}^{y,t} + s_{1,1}^{o,t} &= 0 \\ s_{1,2}^{y,t} + s_{1,2}^{o,t} &= 0 \\ s_{2,1}^{y,t} + s_{2,1}^{o,t} &= 0 \\ s_{2,2}^{y,t} + s_{2,2}^{o,t} &= 0 \end{aligned}$$

- **Goods Market**

  Specifically that for all t we have that

$$\begin{aligned} c_{1,t}^{y,t} + c_{1,t}^{o,t} + c_{1,t}^{y,t-1} + c_{1,t}^{o,t-1} &= w_1^y + w_1^o \\ c_{2,t}^{y,t} + c_{2,t}^{o,t} + c_{2,t}^{y,t-1} + c_{2,t}^{o,t-1} &= w_2^y + w_2^o \end{aligned}$$

Now with all this in mind, I'm just going to leave it after writing out the first order conditions for the young agent. They are

$$
\begin{aligned}
c_{1,t}^t &: u_1(c_{1,t}^t, c_{2,t}^t) - \lambda_{1,t} = 0 \\
c_{2,t}^t &: u_2(c_{1,t}^t, c_{2,t}^t) - \lambda_{2,t} = 0 \\
c_{1,t+1}^t &: \beta u_1(c_{1,t+1}^t, c_{2,t+1}^t) - \lambda_{1,t+1} = 0 \\
c_{2,t+1}^t &: \beta u_2(c_{1,t+1}^t, c_{2,t+1}^t) - \lambda_{2,t+1} = 0 \\
s_{1,1}^t &: \lambda_{1,t} + \lambda_{1,t+1} R_{1,1}^t = 0 \\
s_{1,2}^t &: \lambda_{1,t} + \lambda_{1,t+1} R_{1,2}^t = 0 \\
s_{2,1}^t &: \lambda_{2,t} + \lambda_{2,t+1} R_{2,1}^t = 0 \\
s_{2,2}^t &: \lambda_{2,t} + \lambda_{2,t+1} R_{2,2}^t = 0 \\
\lambda_{1,t} &: c_{1,t}^t + s_{1,1}^t + s_{1,2}^t = w_1^y \\
\lambda_{2,t} &: c_{2,t}^t + s_{2,1}^t + s_{2,2}^t = w_2^y \\
\lambda_{1,t+1} &: R_{1,1}^t s_{1,1}^t + R_{1,2}^t s_{1,2}^t = c_{1,t+1}^t \\
\lambda_{2,t+1} &: R_{2,1}^t s_{2,1}^t + R_{2,2}^t s_{2,2}^t = c_{2,t+1}^t
\end{aligned}
$$

From this discussion we can see a few key takeaways:

- There is a new interaction between individuals in that in the vanilla model I discussed at the very top where in the name of consumption smoothing, since there is no means of transforming one good into another, agents can make agreements to trade one good for another. This means that as the number of goods increases linearly the dimentionality of the problem increases exponentially. Again, there might be some tricks to reduce the dimensionality that I'm missing or some other issues with my setup but as of now I see how there are often issues in models when we increase the number of goods just from one to two.
- The general FOCs structure is similar but one important detail is that the FOCs for consumption are dependent on both foods irregarless which one we are taking the derivative with respect to.
- We can assume that for N many goods, the FOCs and setup is very similar except we have exponentially more conditions and constraints.

N Period Lives

In this case, individuals live for longer than just two periods. The case we will work out is if an individual lives for three periods and there some easy generalizations to make after this example. There are many reasons why we would want to develop a model where agents can live for multiple periods, we might want to represent a more clear distinction between more parts of a household's life (for example, an education period where household's are not making any money, an initial job period where household's recieve a wage but that is not as high as the income they recieve in their next period where they are in the peak of their career, and finally a retirement period where they make no income once again). We can also introduce another dimension of heterogeneity here as well in creating households that live for varying time periods.

That all being said, this case will cover the scenario where agents live for three periods, when they are young, middle aged, and old. Each agent is the same and they recieve endowments w_y, w_m, w_o in each of the respective periods. There will not be any sources of heterogeneity

From this discussion there are a few key takeaways:

- A cool new interaction that happens here is that even though each of the agents are the exact same, consumption bond trading still occurs. Specifically in the two period case people didn't want to trade with others in a different generation since old people don't want to make trade with the young generation as they won't be alive to complete the transaction in the next period; and if you are a young agent then you have no other choice than to try to trade with an old agent given that all the people in the same generation are homogenous. Now, however, even if every single agent in a generation is homogenous, in a model where agents live for more than two periods non-old agents still have other generations who want to trade. Specifically in this case ...

Stochasticity

Finally, we had discussed stochasticity in the setting of