

# Linear Combination in REF

In the context of solving a system of linear equations represented in **row echelon form** (REF), the nature of the solution—**unique**, **infinite**, or **inconsistent**—depends on the properties of the matrix and augmented column. Let's break this down:

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## 1. Unique Solution:

- **Condition:**
  - The system has **exactly one solution** if the matrix in REF has a pivot (leading non-zero entry) in every column corresponding to a variable.
  - In other words, there are no free variables.
- **Key Indicators:**
  - The number of pivot rows = the number of variables (columns of the coefficient matrix).
  - The augmented column does not create any contradictions.
- Example in REF:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right].$$

This corresponds to a unique solution  $x = a, y = b, z = c$ .

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## 2. Infinite Solutions:

- **Condition:**
  - The system has **infinitely many solutions** if:
    - There are **fewer pivots than variables**, meaning at least one column (variable) is not associated with a pivot (a **free variable** exists).
    - The system is consistent (no contradictory rows).

- **Key Indicators:**
  - The number of pivot rows < the number of variables.
  - No contradictory rows like  $[0 \ 0 \ \dots \ | \ b]$ , where  $b \neq 0$ .
- Example in REF:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 3 & b \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Here,  $x_3$  is a free variable, leading to infinitely many solutions.

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### 3. Inconsistent System:

- **Condition:**
  - The system is **inconsistent** if there is a row in REF where all the coefficients are zero, but the augmented column has a non-zero entry (a contradiction).
- **Key Indicators:**
  - A row of the form  $[0 \ 0 \ \dots \ 0|b]$ , where  $b \neq 0$ , exists in the matrix.
- Example in REF:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 3 & b \\ 0 & 0 & 0 & c \ (c \neq 0) \end{array} \right].$$

This system is inconsistent because the last row is contradictory.

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### Summary Table:

Condition	Number of Pivots	Free Variables	Contradictions
Unique Solution	Pivots = Variables	None	None
Infinite Solutions	Pivots < Variables	Exists	None
Inconsistent System	Any number of pivots	Irrelevant	Contradiction exists

Let me know if you'd like help solving an example!