Linear Transformation Overview

A **linear transformation** is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication. It can often be represented in a matrix form, making it easier to visualize or compute.

Here's a **chart overview** of linear transformations, including common examples:

Chart of Linear Transformations

Type of Transformation	Description	Matrix Representation	Example
Scaling	Multiplies vectors by a scalar, enlarging or shrinking.	$c \cdot I$ (I: identity)	Scale by 2: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
Reflection	Flips vectors across a line or plane.	Depends on axis or plane.	Reflect in x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Rotation	Rotates vectors around the origin by an angle θ.	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Rotate 90°: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Shear	Slants vectors in one direction, distorting the shape.	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (Horizontal)	Horizontal shear by 2: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
Projection	Projects vectors onto a line or plane.	Dependent on projection axis.	Onto x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Example: Rotation by 90°

Let's consider a rotation transformation in a 2D space:

1. **Matrix Representation:**

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2. **Input Vector:**

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. **Apply the Transformation:**

$$A \cdot v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

4. **Interpretation:** The vector $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, initially aligned with the x-axis, is rotated 90° counterclockwise to become $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, now aligned with the y-axis.

This simple framework generalizes to other transformations as well!

Understanding Linear Transformation with an Example

A **linear transformation** maps vectors from one space to another while preserving vector addition and scalar multiplication. Let's break it down step by step with an example:

Definition Recap

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if for all vectors $u, v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$:

1. $T(u + v) = T(u) + T(v)$ (Preserves addition)
2. $T(c \cdot u) = c \cdot T(u)$ (Preserves scalar multiplication)

Example: Rotation Transformation in 2D

1. The Transformation

We will rotate vectors in a 2D plane by 90° counterclockwise. The rotation transformation $T(x, y)$ can be represented as:

$$T(x, y) = \begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For 90°, the cosine and sine values are:

$$\cos(90^\circ) = 0, \quad \sin(90^\circ) = 1$$

So the transformation matrix becomes:

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

---

2. Applying the Transformation

Let's apply T to a vector $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Using the transformation:

$$T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Perform the matrix multiplication:

$$T(v) = \begin{bmatrix} (0 \cdot 2) + (-1 \cdot 3) \\ (1 \cdot 2) + (0 \cdot 3) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

---

3. Interpretation

- The original vector $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ pointed to the upper right quadrant.
- After the transformation, $T(v) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ points to the upper left quadrant, rotated 90° counterclockwise.

Verification of Linearity

To confirm this is a linear transformation:

1. **Addition Property**: For $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$T(v_1 + v_2) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$$

   Compute:

$$T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

   Separately:

$$T(v_1) + T(v_2) = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

   Since $T(v_1 + v_2) = T(v_1) + T(v_2)$, addition is preserved.

2. **Scalar Multiplication Property**: For $c = 2$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$:

$$T(c \cdot v) = T\left(2 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$$

   Compute:

$$T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

   Separately:

$$c \cdot T(v) = 2 \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

   Since $T(c \cdot v) = c \cdot T(v)$, scalar multiplication is preserved.

Conclusion

The rotation transformation is a linear transformation. It satisfies the properties of linearity, and the matrix representation simplifies computation. This example demonstrates how a linear transformation maps vectors while preserving the structure of the vector space.

Types of Linear Transformations (with Examples)

Linear transformations can perform various operations on vectors, such as scaling, rotation, reflection, shear, and projection. Each type has a unique matrix representation. Let's explore these with examples:

---

1. Scaling

Definition:

Scaling stretches or shrinks vectors by a scalar factor.

Matrix Representation:

$$T(x, y) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- a: Scaling factor for x-axis.
- b: Scaling factor for y-axis.

Example:

Scaling by 2 in both directions:

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

---

2. Rotation

Definition:

Rotates vectors by an angle θ about the origin.

Matrix Representation:

$$T(x, y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example:

Rotation by 90°:

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

---

3. Reflection

Definition:

Flips vectors across a line or plane.

Matrix Representation:

Reflection depends on the axis or line:

- Reflection across the x-axis:

$$T(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example:

Reflecting $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ across the x-axis:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

---

4. Shear

Definition:

Shear slants vectors in one direction, changing the shape but preserving the area (in 2D).

Matrix Representation:

- Horizontal shear:

$$T(x,y) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Vertical shear:

$$T(x,y) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example:

Horizontal shear with $k = 2$:

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

---

5. Projection

Definition:

Projects vectors onto a line or plane, reducing dimensions.

Matrix Representation:

- Projection onto the x-axis:

$$T(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example:

Projecting $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ onto the x-axis:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Summary Table

Type	Matrix Representation	Example
Scaling	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$	Scale by 2: $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$
Rotation	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Rotate $90°$: $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$
Reflection	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflect in x-axis: $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$
Shear	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	Horizontal shear: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
Projection	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Onto x-axis: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

These types of linear transformations form the building blocks for operations in geometry, computer graphics, and machine learning.