8.3.9

EE24BTECH11011 - Pranay Kumar

Question: Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

Solution:

The equation of the ellipse is given by:

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{1}$$

where:

$$\mathbf{V} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -1. \tag{2}$$

The equation of the line is given by:

$$\frac{x}{a}+\frac{y}{b} = 1, \tag{3}$$

which can be rewritten in parametric form as:

$$\mathbf{x} = \kappa \begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{4}$$

The intersection of the line and the ellipse is given by:

$$\kappa_i = \frac{-\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^\top\mathbf{V}\mathbf{m})}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}, \tag{5}$$

where:

$$\mathbf{h} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} b \\ a \end{pmatrix}. \tag{6}$$

After solving, we find the intersection points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. These points will be used to compute the enclosed area.

The area between the curves can be expressed as:

$$\text{Area} = \int_{x_1}^{x_2} [\text{Line} - \text{Ellipse}]\, dx. \tag{7}$$

Theoretical Solution:

For the line:

$$y_{\text{line}} = b\left(1 - \frac{x}{a}\right), \tag{8}$$

and for the ellipse:

$$y_{\text{ellipse}} = b\sqrt{1 - \frac{x^2}{a^2}}. \tag{9}$$

Thus, the area becomes:

$$\text{Area} = \int_{x_1}^{x_2} \left[b\left(1 - \frac{x}{a}\right) - b\sqrt{1 - \frac{x^2}{a^2}} \right] dx. \tag{10}$$

Solving this integral yields the exact value of the area.

Simulated Solution:

Using numerical methods, we divide the interval $[x_1, x_2]$ into n intervals and apply the trapezoidal rule: align A = h $\sum_{i=1}^{n-1} \left[f(x_i) + f(x_{i+1}) \right]$,

where $f(x) = y_{\text{line}} - y_{\text{ellipse}}$.

The numerical solution matches the theoretical value within an acceptable margin of error.