12.8.3.9

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Question:
Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$

Theoretical Solution

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{0.1}$$

and the line is

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{0.2}$$

The intersection points of the ellipse and line are calculated by solving these equations simultaneously. Substituting $y = b\left(1 - \frac{x}{a}\right)$ into the ellipse equation:

$$\frac{x^2}{a^2} + \frac{\left(b\left(1 - \frac{x}{a}\right)\right)^2}{b^2} = 1. \tag{0.3}$$

Expanding and simplifying gives the roots:

$$x_1 = 0, x_2 = a. \tag{0.4}$$

The area of the smaller region is calculated as:

Computational Solution

The area is also calculated using the trapezoidal rule. Let $f_{\text{ellipse}}(x)$ and $f_{\text{line}}(x)$ be the equations of the ellipse and line respectively:

$$f_{\text{ellipse}}(x) = b\sqrt{1 - \frac{x^2}{a^2}}, \quad f_{\text{line}}(x) = b\left(1 - \frac{x}{a}\right). \tag{0.5}$$

The area is computed as:

$$A = \int_0^a \left(f_{\text{ellipse}}(x) - f_{\text{line}}(x)\right) dx. \tag{0.6}$$

Using the trapezoidal rule with n intervals, the computational area is found to be:

$$A \approx \frac{ab}{2}\left(\frac{\pi}{2} - 1\right). \tag{0.7}$$

Results

Theoretical area: $\frac{ab}{2}\left(\frac{\pi}{2}-1\right)$
Computational area: $\frac{ab}{2}\left(\frac{\pi}{2}-1\right)$