10.4.ex.2

EE24BTECH11011-PRANAY

Question:
Find the roots of the quadratic equation:

$$(x-2)^2 + 1 = 2x - 3 \tag{0.1}$$

Solution:
Rearranging terms:

$$(x-2)^2 + 1 - (2x-3) = 0 \tag{0.2}$$
$$x^2 - 4x + 4 + 1 - 2x + 3 = 0 \tag{0.3}$$
$$x^2 - 6x + 8 = 0 \tag{0.4}$$

Theoretical solution (Quadratic formula):
The roots are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.5}$$
$$x = \frac{6 \pm \sqrt{36 - 32}}{2} \tag{0.6}$$
$$x = \frac{6 \pm 2}{2} \tag{0.7}$$
$$x = 4 \text{ and } x = 2 \tag{0.8}$$

Computational solution:
To find the eigenvalues of the companion matrix, we use the QR algorithm. The companion matrix (C) is:

$$C = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} \tag{0.9}$$

For the equation $x^2 - 6x + 8 = 0$, we have:

$$b_0 = 8, \quad b_1 = -6 \tag{0.10}$$

$$C = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \tag{0.11}$$

The QR decomposition iteratively computes:

$$C_0 = C \tag{0.12}$$
$$C_{k+1} = R_k Q_k \tag{0.13}$$

where $C_k = Q_k R_k$ is the QR decomposition of C_k.

Starting with:

$$C_0 = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \tag{0.14}$$

we compute the QR steps iteratively until C_k converges to an upper triangular matrix. The eigenvalues are the diagonal entries:

$$C = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \tag{0.15}$$

$$Q_0 = \frac{1}{\sqrt{65}} \begin{pmatrix} -8 & -1 \\ 1 & -8 \end{pmatrix}, \quad R_0 = \begin{pmatrix} \sqrt{65} & \frac{-46}{\sqrt{65}} \\ 0 & \frac{5}{\sqrt{65}} \end{pmatrix} \tag{0.16}$$

$$C_1 = R_0 Q_0 = \begin{pmatrix} 6.707 & -0.707 \\ 0.707 & 3.293 \end{pmatrix} \tag{0.17}$$

$$Q_1 = \begin{pmatrix} -0.99 & -0.14 \\ 0.14 & -0.99 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -6.757 & 0.991 \\ 0 & -3.243 \end{pmatrix} \tag{0.18}$$

$$C_2 = R_1 Q_1 = \begin{pmatrix} 6.988 & -0.012 \\ 0.012 & 3.012 \end{pmatrix} \tag{0.19}$$

$$\lambda_1 = 4, \quad \lambda_2 = 2 \tag{0.20}$$

$$(4-2)^2 + 1 = 2(4) - 3 \tag{0.21}$$
$$5 = 5 \tag{0.22}$$

$$(2-2)^2 + 1 = 2(2) - 3 \tag{0.23}$$
$$1 = 1 \tag{0.24}$$

$$x_1 = 4, \quad x_2 = 2 \tag{0.25}$$

(2) **Newton-Raphson iterative method:**

The function and its derivative are:

$$f(x) = x^2 - 6x + 8 \tag{0.26}$$
$$f'(x) = 2x - 6 \tag{0.27}$$

The difference equation for the Newton-Raphson method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.28}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 6x_n + 8}{2x_n - 6} \tag{0.29}$$

Starting with initial guesses:

$$x_0 = 3 \text{ converges to } 2 \tag{0.30}$$

$$x_0 = 5 \text{ converges to } 4 \tag{0.31}$$

Verification:
Substituting $x = 4$:

$$(4-2)^2 + 1 = 2(4) - 3 \tag{0.32}$$

$$4 + 1 = 8 - 3 \tag{0.33}$$

$$5 = 5 \tag{0.34}$$

Substituting $x = 2$:

$$(2-2)^2 + 1 = 2(2) - 3 \tag{0.35}$$

$$0 + 1 = 4 - 3 \tag{0.36}$$

$$1 = 1 \tag{0.37}$$

Hence, the roots are verified.

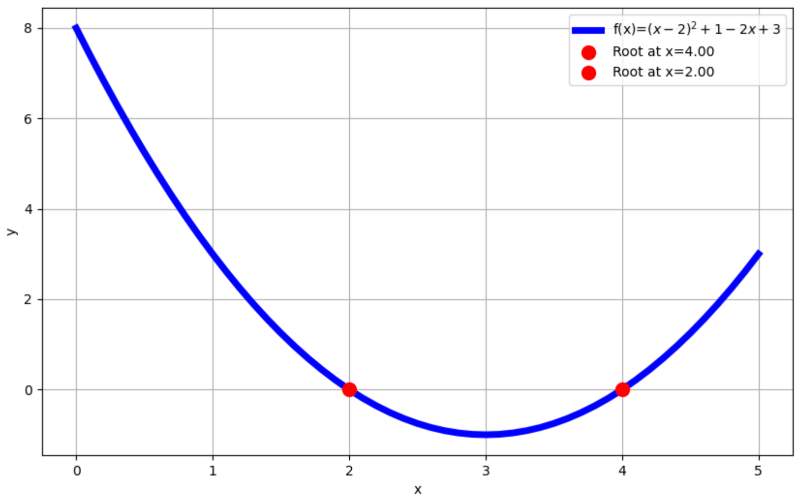

Fig. 0.1: Plot using Newton-Raphson method