12.6.6.17

EE24BTECH11011 - Pranay Kumar

QUESTION :
Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume.

SOLUTION :

Theoretical solution :
Let the height of the cylinder be h and the radius of the cylinder base be r. From the geometry of the problem, the relationship between h, r, and R is:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \tag{0.1}$$

The volume of the cylinder is:

$$V = \pi r^2 h \tag{0.2}$$

Substituting r^2 , we get:

$$V(h) = \pi \left(R^2 - \frac{h^2}{4}\right) h \tag{0.3}$$

To maximize $V(h)$, we differentiate with respect to h:

$$\frac{dV}{dh} = \pi \left(R^2 - \frac{3h^2}{4}\right) \tag{0.4}$$

Setting $\frac{dV}{dh} = 0$, we get:

$$R^2 - \frac{3h^2}{4} = 0 \tag{0.5}$$

$$h = \frac{2R}{\sqrt{3}} \tag{0.6}$$

Substituting $h = \frac{2R}{\sqrt{3}}$ into the maximum volume is:

$$V_{\max} = \frac{2\pi R^3}{3\sqrt{3}} \tag{0.7}$$

Computational solution :

Finding the maximum volume can also be done using the **Gradient Ascent method**. The iterative update rule is:

$$h_{n+1} = h_n + \alpha \frac{dV}{dh}\Big|_{h=h_n} \tag{0.8}$$

where α is the learning rate. we have:

$$h_{n+1} = h_n + \alpha\pi \left(R^2 - \frac{3h_n^2}{4} \right) \tag{0.9}$$

Taking:

$$\alpha = 0.001 \tag{0.10}$$

$$h_0 = 1 \tag{0.11}$$

we find:

$$h_{\max} = 1.1547 \quad \text{(approximates } \frac{2R}{\sqrt{3}}\text{)} \tag{0.12}$$

$$V_{\max} = 2.418 \quad \text{(approximates } \frac{2\pi R^3}{3\sqrt{3}}\text{)} \tag{0.13}$$